DECAY ESTIMATES FOR A PERTURBED TWO-TERMS SPACE-TIME FRACTIONAL DIFFUSIVE PROBLEM

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ABSTRACT. In the present paper we consider the Cauchy-type problem associated to the space-time fractional differential equation

$$\partial_t u + \partial_t^\beta (-\Delta)^{1-\beta} u - \Delta u = g(t,x), \quad t > 0, \; x \in \mathbb{R}^n$$

with $\beta \in (0,1)$, where the fractional derivative ∂_t^β is in Caputo sense and $(-\Delta)^{1-\beta}$ is the fractional Laplace operator of order $1-\beta$. We provide sufficient conditions on the perturbation g which guarantees that the solution satisfies the same long-time decay estimates of the case $g = 0$, assuming initial datum in $H^{s,m}$ for some $s > 0$ and $m \in (1,\infty)$. We apply the obtained results to study the existence of global-in-time solutions to the associated nonlinear problems,

$$\partial_t u + \partial_t^\beta (-\Delta)^{1-\beta} u - \Delta u = \begin{cases} |u|^p, \\ \nabla(u|u|^{p-1}), \end{cases}$$

assuming small initial datum in $H^{s,m}$ and supercritical or critical powers.

1. INTRODUCTION

We consider the Cauchy-type problem for a fractional differential equation

$$\begin{cases} \partial_t u + (-\Delta)^{1-\beta} \partial_t^\beta u - \Delta u = g(t,x), \quad t > 0, \; x \in \mathbb{R}^n, \\ u(0,x) = u_0(x); \end{cases} \tag{1}$$

with $\beta \in (0,1)$. Here $\partial_t^\beta u$ denotes the Caputo (left-sided) fractional derivative of order β, with starting time 0, with respect to the time variable. Namely, for any given $x \in \mathbb{R}^n$, we put $\partial_t^\beta u(t,x) = (D_{0+}^\beta u(\cdot, x))(t)$, as defined in [18, Section 2.4], that is,

$$(D_{0+}^\beta y)(t) = (J_{0+}^{1-\beta} y')(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{y'(s)}{(t-s)^\beta}\, ds,$$

for any $t > 0$ and $y \in$, where $J_{0+}^{1-\beta}$ denotes the Riemann-Liouville integral of order $1-\beta$ and Γ is the Euler gamma function. Moreover, for any $\alpha > 0$ we define the fractional Laplace operator $(-\Delta)^\alpha : H^{s,m} \to H^{s-2\alpha,m}$, $m \in (1,\infty)$, as $(-\Delta)^\alpha f = \mathscr{F}^{-1}(|\xi|^{2\alpha} \hat{f})$, where $\mathscr{F}$ is the Fourier transformation (in $\mathcal{S}'$), and $\hat{f} = \mathscr{F}(f)$ denotes the Fourier transform of f (see [12] for an introduction to fractional Laplace operator).

The fractional differential operator $L := \partial_t + \partial_t^\beta (-\Delta)^{1-\beta} - \Delta$ is scale-invariant, or "quasi-homogeneous" of type $(1,1,1/2)$ (see [11, Definition 2.2]), that is,

$$L\big(u\big(\lambda\cdot, \lambda^{\frac{1}{2}}\cdot\big)\big)(t,x) = \lambda(Lu)\big(\lambda t, \lambda^{\frac{1}{2}} x\big),$$

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for any $\lambda > 0$, $t > 0$ and $x \in \mathbb{R}^n$. This property of L implies a lack of oscillations in the fundamental solution to the homogeneous problem

$$\begin{cases} \partial_t u + (-\Delta)^{1-\beta}\partial_t^\beta u - \Delta u = 0, & t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{2}$$

that is, (1) with $g = 0$. In turn, this provides the L^m well-posedness of the problem. On the other hand, the fundamental solution gains some limited smoothing effect by its parabolic nature, so that the solution $u \in \mathcal{C}([0,\infty), H^{s,m})$ verifies the following decay estimate (see later, Proposition 3.1)

$$\|u(t,\cdot)\|_{\dot{H}^{s,m}} \le C\, (1+t)^{-\frac{s}{2}} \|u_0\|_{H^{s,m}} \quad \text{for any } s \in (0, 2\beta]. \tag{3}$$

The restriction $s \le 2\beta$ is due to the fact that the smoothing effect for the fundamental solution to (2) is limited to 2β derivatives (see later, (8)). This smoothing effect appears in many other evolution models (see, for instance, damped evolution models in [5,26]). However, in those cases no restriction on s appears: more derivatives always brings more decay rate in time, as per the heat equation.

The fractional nature of the problem makes less obvious to deal with the perturbation term on the right-hand side $g(t,x)$, since Duhamel's principle does not apply in the standard way (see, for instance, [18, Example 4.9] for a simpler equation, see also [29]).

Having this in mind, we look for sufficient conditions on $g(t,x)$ which guarantee that the solution to (1) remains in $\mathcal{C}([0,\infty), H^{s,m})$ and that $\|u(t,\cdot)\|_{\dot{H}^{s,m}}$ has a decay rate $t^{-\frac{s}{2}}$ as t goes to infinity.

As an example of an application of the obtained decay estimates for the perturbed equation (1), we investigate the semilinear problem

$$\begin{cases} \partial_t u + (-\Delta)^{1-\beta}\partial_t^\beta u - \Delta u = f(u), & t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{4}$$

where $f(u) = |u|^p$ with $p > 1$, or, more in general,

$$|f(u) - f(v)| \le C\,|u - v|(|u|^{p-1} + |v|^{p-1}), \tag{5}$$

for some constant $C > 0$ independent on u and v. It is well-known that such nonlinear perturbation may cause the solution to blow-up in finite time, when p is smaller than some critical power, usually called Fujita exponent. Nevertheless, for p larger than Fujita exponent, global-in-time solutions exist, provided that initial data are sufficiently small, in some space. Fujita exponents have been determined for fractional partial differential equations in several papers, see for instance [8] for the fractional wave-diffusive equation and [7] for the fractional subdiffusive equation.

1.1. **Main Results.** We here summarize the main results which we will prove in the paper, outlining the ideas of the proofs, for the ease of reading. The solution to (1) may be written in the form

$$u(t,x) = K_0(t,\cdot) *_{(x)} u_0 + \int_0^t K_1(t-\tau,\cdot) *_{(x)} g(\tau,\cdot)\,d\tau, \tag{6}$$

where the expression of $\mathscr{F}K_j$, $j = 0, 1$, may be explicitly obtained, see later, (26).

We stress that the classical Duhamel's principle does not apply to fractional equations (see, for instance, [29]); as a consequence, the kernel K_1 related to the right-hand side term in (1) is different from the kernel K_0 related to the initial datum u_0; in particular, the two different kernels have different smoothing properties (see Remark 2.2).

To estimate u, we get an integral representation formula for $\mathscr{F}K_j$, $j = 0, 1$. In particular, we derive the sharp estimate (see (27) and (28)):

$$|\mathscr{F}K_j(t,\xi)| \approx \langle t|\xi|^2 \rangle^{-\beta-j}, \qquad j = 0, 1. \tag{7}$$

In order to apply the Fourier multiplier theory, in particular, Mikhlin-Hörmander theorem (see later, Corollary 2.4), we also prove the following estimate which involves the derivatives of $\mathscr{F}K_j(t,\xi)$, with respect to ξ.

Lemma 1.1. *Let K_0 and K_1 be as in (6). Then it holds*

$$|\partial_\xi^\gamma \hat{K}_0(t,\xi)| \lesssim \langle \sqrt{t}\,\xi \rangle^{-2\beta} |\xi|^{-|\gamma|} \tag{8}$$

and

$$|\partial_\xi^\gamma \hat{K}_1(t,\xi)| \lesssim \langle \sqrt{t}\,\xi \rangle^{-2\beta-2} |\xi|^{-|\gamma|} \tag{9}$$

for any $\gamma \in \mathbb{N}^n$, with $\gamma \geq 0$.

Thanks to (8) in Lemma 1.1, it is immediate to prove the L^m well-posedness of the homogeneous problem (2), and get the decay rate $\|u(t,\cdot)\|_{\dot{H}^{s,m}} \leq C \left(1+t\right)^{-\min\left\{\frac{s}{2},\beta\right\}} \|u_0\|_{H^{s,m}}$ for any $s > 0$ (see later, Proposition 3.1).

However, in view of (6), the fact that K_1 has different properties with respect to K_0, in particular, it has better smoothing properties (see Remark 2.2), makes interesting to study sufficient conditions on g which guarantee that this perturbation does not influence the behavior of the solution.

A sufficient condition on the perturbation g which guarantees that the solution u to (1) remains in $\mathcal{C}([0,\infty), H^{s,m})$ may be easily given.

Proposition 1.2. *Let $n \geq 1$, $m \in (1,\infty)$, $s \in \mathbb{R}$, and $u_0 \in H^{s,m}$. Assume that $g \in L^1_{\text{loc}}([0,\infty), H^{s,m})$ or that $g \in L^r_{\text{loc}}([0,\infty), H^{s-2b,m})$ with $b \in (0,1)$, for some $r > 1/(1-b)$. Then $u \in \mathcal{C}([0,\infty), H^{s,m})$.*

Remark 1.1. The mechanism which regulates the interplay between integrability in time and regularity in space assumed for g in Proposition 1.2 may be understood noticing that, in view of (9), "a gain of $2b$ derivatives in space may be obtained paying a singularity in time t^{-b} as $t \to 0$".

The strict inequality in the condition $r > 1/(1-b)$ in Proposition 1.2 is related to the lack of Hardy-Littlewood-Sobolev inequality for L^1. Indeed, the Hardy-Littlewood-Sobolev inequality (see [18, Lemma 2.1(b)])

$$\|J_{0+}^{1-b} h\|_{L^{r^*}([0,t])} \leq C(r,b,t) \|h\|_{L^r([0,t])}, \qquad \text{with } \frac{1}{r^*} = \frac{1}{r} + b,$$

holds for any $h \in L^r([0,t])$, with $r \in (1,\infty)$, provided that $r^* > 1$, that is, $r > 1/(1-b)$. The inclusion $L^{r^*}([0,t]) \subset L^1([0,t])$ guarantees the integrability of $J_{0+}^{1-b} h$ over $[0,t]$ for any $t > 0$.

To understand how Lemma 1.1 comes into play, it is useful to split the integral in (6) in two intervals. In §3, using Corollary 2.4, we will be able to prove the following.

Lemma 1.3. *Let $b \in [0,1)$. Assume that $g \in L^1_{\text{loc}}([0,\infty), H^{s,m})$ if $b = 0$, or that $g \in L^r_{\text{loc}}([0,\infty), H^{s-2b,m})$, for some $r > 1/(1-b)$, if $b \in (0,1)$. Then*

$$\int_{(t-1)_+}^t \|K_1(t-\tau,\cdot) * g(\tau,\cdot)\|_{H^{s,m}} \, d\tau \leq C \int_{(t-1)_+}^t (t-\tau)^{-b} \|g(\tau,\cdot)\|_{H^{s-2b,m}} \, d\tau, \tag{10}$$

$$\int_0^{t-1} \|K_1(t-\tau,\cdot) * g(\tau,\cdot)\|_{H^{s,m}} \, d\tau \leq C \int_0^{t-1} \|g(\tau,\cdot)\|_{H^{s-2-2\beta,m}} \, d\tau, \quad \text{for } t > 1, \tag{11}$$

where $C > 0$ is independent on g and t.

The proof of Proposition 1.2 easily follows from Lemma 1.3 (see §3 for the details).

Due to the diffusive nature of the equation, it is expected that when $s > 0$, the homogeneous quantity $\|u(t,\cdot)\|_{\dot{H}^{s,m}}$ decays as $t \to \infty$. We show that this decay rate is $t^{-\min\left\{\frac{s}{2},\beta\right\}}$ under suitable assumptions on g. We stress that the fact that the decay rate is not faster than $t^{-\beta}$ is related to the partial, polynomial, smoothing effect that appears for this equation, see (7), if we compare it with the heat equation, for instance (whose fundamental solution has exponential decay, $e^{-t|\xi|^2}$). This phenomenon also appears in the sub-diffusive case (19), treated in [7], even replacing the Caputo derivative ∂_t^β with the Riemann-Liouville fractional derivative; in [7] the homogeneity properties of problem (19) allows to investigate more general $L^p - L^q$ decay estimates for the solution $\tilde{v}$, for $1 \leq p \leq q \leq \infty$; however, due to the limited smoothing effect the restriction $n(1/p - 1/q) < 2$ appears, and then the decay rate $t^{-\frac{n\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$ of $\|\tilde{v}(t,\cdot)\|_{L^q}$ can not be faster than $t^{-\alpha}$. We mention that problem (19) with $\alpha \in (1,2)$ is studied in [8].

In order to obtain the desired decay rate for the solution to (1), we may replace (11) by an estimate of the integral for the homogeneous quantity

$$\|K_1(t-\tau,\cdot)*g(\tau,\cdot)\|_{\dot{H}^{s,m}},$$

when $s > 0$. We have the following.

Lemma 1.4. *Let $s > 0$ and assume that $g \in L^1_{\text{loc}}([0,\infty), H^{s-2-2\beta,m})$. Then*

$$\int_0^{t-1} \|K_1(t-\tau,\cdot)*g(\tau,\cdot)\|_{\dot{H}^{s,m}}\, d\tau \le C \int_0^{t-1} (t-\tau)^{-\min\left\{\frac{s}{2},1+\beta\right\}} \|g(\tau,\cdot)\|_{H^{s-2-2\beta,m}}\, d\tau, \qquad (12)$$

where $C > 0$ is independent on t and g.

Moreover, assume that $s \in (0, 2+2\beta)$ and let $q \in (1,m]$ and $g \in L^1_{\text{loc}}([0,\infty), H^{s+a-2-2\beta,q})$, where

$$a = n\left(\frac{1}{q} - \frac{1}{m}\right), \quad \text{is such that } s + a \le 2 + 2\beta. \qquad (13)$$

Then, additional decay may be produced replacing (12) by the estimate

$$\int_0^{t-1} \|K_1(t-\tau,\cdot)*g(\tau,\cdot)\|_{\dot{H}^{s,m}}\, d\tau \le C \int_0^{t-1} (t-\tau)^{-\min\left\{\frac{s+a}{2},1+\beta\right\}} \|g(\tau,\cdot)\|_{H^{s+a-2-2\beta,q}}\, d\tau, \qquad (14)$$

We stress that (12) corresponds to (14) with $a = 0$. We always fix $a = 0$ in (14) if $s \ge 2+2\beta$.

Thanks to Lemmas 1.3 and 1.4, we are able to prove our main result (see §3 for the details).

Theorem 1.5. *Let $n \ge 1$, $m \in (1,\infty)$ and $s \ge 0$. Assume that $g \in L^1_{\text{loc}}([0,\infty), H^{s,m})$ or that $g \in L^r_{\text{loc}}([0,\infty), H^{s-2b,m})$ with $b \in (0,1)$, for some $r > 1/(1-b)$. If $s < 2+2\beta$, possibly also assume that $g \in L^1_{\text{loc}}([0,\infty), H^{s-2-2\beta+a,q})$ for some $q \in (1,m]$ with $a = a(n,m,q)$ defined by (13) such that $s + a \le 2 + 2\beta$, otherwise fix $a = 0$. Assume that*

$$A = \sup_{t \ge 0}(1+t)^{\min\left\{\frac{s}{2},\beta\right\}} \int_{(t-1)_+}^t (t-\tau)^{-b} \|g(\tau,\cdot)\|_{H^{s-2b,m}}\, d\tau, \qquad (15)$$

and

$$B = \sup_{t \ge 1} t^{\min\left\{\frac{s}{2},\beta\right\}} \int_0^{t-1} (t-\tau)^{-\frac{s+a}{2}} \|g(\tau,\cdot)\|_{H^{s-2-2\beta+a,q}}\, d\tau, \qquad (16)$$

are finite. Then the unique solution $u \in \mathcal{C}([0,\infty), H^{s,m})$ verifies the decay estimate

$$\|u(t,\cdot)\|_{\dot{H}^{s,m}} \le C(1+t)^{-\min\left\{\frac{s}{2},\beta\right\}}(\|u_0\|_{H^{s,m}} + A + B), \qquad (17)$$

for any $t \ge 0$, where $C > 0$ is independent of t, u_0, g, A and B.

The fact that the solution is in $\mathcal{C}([0,\infty), H^{s,m})$ in Theorem 1.5 is guaranteed by Proposition 1.2.

1.2. **Application of Theorem 1.5.** In §4 we will apply Theorem 1.5 to obtain global-in-time existence results for the correspondent semilinear Cauchy-type problem (4). In particular, we have in mind to apply Theorem 1.5 for functions g for which some decay estimate holds. In view of this, we provide some concrete examples of assumptions on a polynomial decay rate on g to estimate the quantities A and B in (15) and (16). Estimates of integrals as in the forthcoming Examples 1.2 and 1.3 are standard in dealing with integral terms coming from the application of Duhamel's principle, especially in the application to nonlinear problems. An earlier version of these estimates goes back to [25]. For the ease of reading, we collect those integral estimates in Lemmas 3.2 and 3.3, in §3.

Example 1.1. Assume that the estimate

$$\|g(\tau,\cdot)\|_{H^{s-2b,m}} \le C\,(1+\tau)^{-\min\left\{\frac{s}{2},\beta\right\}}$$

holds for any $\tau \geq 0$, for some $b \in (0,1)$ and $C > 0$. Then the quantity A in (15) is finite. More precisely, one may estimate $A \leq 2C/(1-b)$. Indeed, if $t \geq 1$, we may estimate

$$\int_{t-1}^{t} (t-\tau)^{-b} (1+\tau)^{-\min\{s/2,\beta\}} \, d\tau \leq t^{-\min\{s/2,\beta\}} \int_{t-1}^{t} (t-\tau)^{-b} d\tau$$

$$= \frac{1}{1-b} \, t^{-\min\{s/2,\beta\}} \leq \frac{2}{1-b} \, (1+t)^{-\min\{\frac{s}{2},\beta\}}.$$

On the other hand, if $t \leq 1$, then we may estimate

$$\int_{0}^{t} (t-\tau)^{-b} (1+\tau)^{-\min\{s/2,\beta\}} \, d\tau \leq \int_{0}^{t} (t-\tau)^{-b} \, d\tau = \frac{1}{1-b} \, t^{1-b}$$

$$\leq \frac{1}{1-b} \leq \frac{2}{1-b} \, (1+t)^{-\min\{\frac{s}{2},\beta\}}.$$

Example 1.2. Assume that $s \in [0,2)$ and that the estimate, possibly singular at $\tau = 0$,

$$\|g(\tau,\cdot)\|_{H^{s-2-2\beta+a,q}} \leq \begin{cases} C\,\tau^{-1+\frac{a}{2}} & \text{if } s \leq 2\beta, \\ C\,\tau^{-1+\frac{s+a}{2}-\beta} & \text{if } s \in (2\beta,2), \end{cases}$$

holds for any $\tau > 0$, for some $C > 0$ and $a \in (0, 2-s)$. Then the quantity B in (16) is finite. Indeed (see later, Lemma 3.2),

$$\int_{0}^{t-1} (t-\tau)^{-\frac{s+a}{2}} \tau^{-1+\frac{a}{2}} \, d\tau \leq C_1 \, t^{-\frac{s}{2}},$$

and

$$\int_{0}^{t-1} (t-\tau)^{-\frac{s+a}{2}} \tau^{-1+\frac{s+a}{2}-\beta} \, d\tau \leq C_1 \, t^{-\beta};$$

here, $C_1 > 0$ depends only on s and a.

Example 1.3. Assume that the estimate

$$\|g(\tau,\cdot)\|_{H^{s-2-2\beta,m}} \leq C \, (1+\tau)^{-d},$$

holds for any $\tau \geq 0$, for some $C > 0$ and $d > 1$. Then, the quantity B in (16) is finite. Indeed (see later, Lemma 3.3),

$$\int_{0}^{t-1} (t-\tau)^{-\frac{s}{2}} (1+\tau)^{-d} \, d\tau \leq C_1(d) \, t^{-\frac{s}{2}}.$$

Background. A limited smoothing effect phenomenon appears in the following perturbed two-terms diffusive problem

$$\begin{cases} \partial_t v + \partial_t^{\alpha} v - \Delta v = f(t,x) & t > 0, \ x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), \end{cases} \tag{18}$$

with $\alpha \in (0,1)$, recently studied by the authors in [10]. However, this model is deeply different from (1). First of all, the related homogeneous problem it is not quasi-homogeneous, and it can be shown that its asymptotic profile is described by the solution to the same problem where $\partial_t v$ is stroken (diffusion phenomenon), studied in [7],

$$\begin{cases} \partial_t^{\alpha} \tilde{v} - \Delta \tilde{v} = 0 & t > 0, \ x \in \mathbb{R}^n, \\ \tilde{v}(0,x) = \tilde{v}_0(x), \end{cases} \tag{19}$$

Another crucial difference is that the smoothing effect for the solution operator to (18) is independent of the order α of the time fractional derivative: it amounts on 2 derivatives, independently on α.

Differential equations with fractional in time derivatives are increasingly used to model physical phenomena in which some memory effect or hereditary process appear, for instance in areas like rheology, biology, engineering, mathematical physics, etc. (see for instance [20–23] and the reference given therein). One can refer to [18] or [23] for a deep study about the theory of time fractional derivatives. Also nonlocal in space operators are experiencing many applications in different subjects, such as, among others,

crystal dislocation [14, 24], fluid mechanics [2, 4, 30]. The use of fractional calculus introduced several new mathematical challenges; an open problem in this field is finding some effective tools for writing explicit solutions to fractional ordinary differential equations. This latter issue becomes even more challenging if the equation contains multiple fractional in time derivatives. In the literature other authors have investigated the existence of solutions to the Cauchy-type problem associated to some multi-terms fractional partial differential equations, in suitable functional spaces. For instance, the study of the following two-term time fractional diffusion-wave equation was already faced

$$b_1 \partial_t^{\delta_1} w + b_2 \partial_t^{\delta_2} w - c^2 \Delta w = F(t, x, w), \quad t > 0, \, x \in \mathbb{R}^n, \tag{20}$$

for $b_1, b_2 \in \mathbb{R}$, $\delta_1, \delta_2 > 0$ and $F \equiv 0$ or F nonlinear, under given assumptions on the exponents δ_1 and δ_2 and on the function F. An extended review about this problem can be found for instance in [27, 28], where the existence of upper viscosity solutions to (20) is discussed. Also the study of the H^s well-posedness for multi-point value problems for fractional partial differential equations like (1) was already treated, for instance, in [16]. Some results about the well-posedness and regularity of solutions to (20) and more general models in bounded domains are discussed for instance in [1, 3, 6, 13, 32]. The problem of finding a suitable representation of solution is strictly related to solving fractional ordinary differential equations in the form

$$\partial_t^{\delta_1} w + \lambda \partial_t^{\delta_2} w + \mu w = 0, \tag{21}$$

for λ, $\mu \in \mathbb{R}$. An exact solution to the initial value problems associated to (21) can be expressed in terms of generalized Mittlag-Leffler type functions (see [19]). In some special case the analytical solutions to such equations can be derived by using the Laplace transform method (see, for instance [15]).

In the study of our problem, we can rely on a representation formula for the solution to (1); in particular, we have an integral representations of the kernels which allows us to investigate suitable pointwise estimates, essential for applying tools from Fourier multipliers theory.

Notation. For any $s \in \mathbb{R}$ and $q \in (1, \infty)$ we define the Bessel potential space

$$H^{s,q} = \left\{ f \in \mathcal{S}' : \, \langle \xi \rangle^s \hat{f} \in L^q \right\},$$

equipped with the norm $\|f\|_{H^{s,q}} = \|\mathscr{F}^{-1}(\langle \xi \rangle^s \hat{f})\|_{L^q}$. Here the symbol $\langle \xi \rangle$ denotes the quantity $\sqrt{1 + |\xi|^2}$. Namely, $H^{s,q}$ is the image of L^q via the application of the Bessel potential, with its induced norm. For integer values of $s \geq 1$ and for any $q \in (1, \infty)$, $H^{s,q} = W^{s,q}$, the classic Sobolev space of functions in L^m with their derivatives up to order s. For all $s \geq 0$, $q \in (1, \infty)$, and $f \in H^{s,q}$, we define the homogeneous quantity $\|f\|_{\dot{H}^{s,q}} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^q}$.
In this paper, $f \lesssim g$ means that $f \leq Cg$ for some constant $C > 0$, and $f \approx g$ means that $f \lesssim g \lesssim f$.

2. Proof of Lemma 1.1

In order to prove Lemma 1.1, we first need to obtain an appropriate expression for the kernels K_0 and K_1 in (6). Applying the Fourier transform to problem (1) we obtain the following Cauchy-type problem for a parameter dependent fractional differential equation

$$\begin{cases} \partial_t \hat{u} + |\xi|^{2-2\beta} \partial_t^\beta \hat{u} + |\xi|^2 \hat{u} = \hat{g}(t, \xi), \quad t > 0, \\ \hat{u}(0, \xi) = \hat{u}_0(\xi), \end{cases} \tag{22}$$

with $\xi \in \mathbb{R}^n$. Since the left-hand side of the equation in (22) is scale-invariant, we make the change of variable $r = t|\xi|^2$, setting $y(t|\xi|^2) = \hat{u}(t, \xi)$, so that the equation becomes

$$|\xi|^2 (y' + D_{0+}^\beta y + y) = \hat{g}(r|\xi|^{-2}, \xi).$$

Letting $c_0 = \hat{u}_0(\xi)$ and $f(r) = |\xi|^{-2} \hat{g}(r|\xi|^{-2}, \xi)$ for any $\xi \neq 0$, problem (22) gives us

$$\begin{cases} y' + D_{0+}^\beta y + y = f(r), \quad r > 0, \\ y(0) = c_0. \end{cases} \tag{23}$$

We are now in the position to apply the following result.

Lemma 2.1 (see Lemma 1 in [10]). *Assume that $y = y(r)$ solves the Cauchy-type problem (23). Then*

$$y(r) = c_0 G_0(r) + \int_0^r G_1(r - \rho) f(\rho) \, d\rho,$$

where G_0 and G_1 have the following integral representations:

$$G_0(r) = \frac{\sin(\beta\pi)}{\beta\pi} \int_0^\infty e^{-xr} \, x^{\beta-1} \, \varphi(x) \, dx \tag{24}$$

$$G_1(r) = \frac{\sin(\beta\pi)}{\beta\pi} \int_0^\infty e^{-xr} \, x^\beta \, \varphi(x) \, dx \tag{25}$$

for any $r \geq 0$, with

$$\varphi(x) = \frac{1}{(1-x)^2 + x^{2\beta} + 2(1-x)x^\beta \cos(\beta\pi)} \, .$$

Taking $f(r) = |\xi|^{-2} \hat{g}(r|\xi|^{-2}, \xi)$ as in (23), by the change of variable $\rho = \tau|\xi|^2$, we get

$$\int_0^r G_1(r - \rho) f(\rho) \, d\rho = |\xi|^{-2} \int_0^r G_1(r - \rho) \hat{g}(\rho|\xi|^{-2}, \xi) \, d\rho$$

$$= \int_0^{r|\xi|^{-2}} G_1(r - \tau|\xi|^2) \hat{g}(\tau, \xi) \, d\tau$$

$$= \int_0^t G_1((t - \tau)|\xi|^2) \hat{g}(\tau, \xi) \, d\tau,$$

where in the last equality we replaced $r = t|\xi|^2$. We conclude that formula (6) holds for problem (1) with K_0 and K_1 satisfying

$$\hat{K}_0(t, \xi) = G_0(t|\xi|^2), \qquad \text{and} \qquad \hat{K}_1(t, \xi) = G_1(t|\xi|^2), \tag{26}$$

where G_0 and G_1 are as in (24) and (25). In particular, we may write $\hat{K}_0(t, \xi)$ and $\hat{K}_1(t, \xi)$ in integral form.

In order to prove Lemma 1.1, we first look for an asymptotic estimate for the integral in (24) and (25), as $r \to \infty$. We employ the following version of Watson's lemma ([31, p.133]) for nonsmooth functions, whose straightforward proof we provide for the ease of reading.

Lemma 2.2. *Let $\varphi \in L^1_{\text{loc}}([0, +\infty))$, with $\varphi(x) \, e^{-Mx}$ in L^1, for some $M \geq 0$. Assume that φ is continuous at 0, with $\varphi(0) \neq 0$. Then*

$$\int_0^\infty e^{-xr} \, x^{\beta-1} \, \varphi(x) \, dx = \Gamma(\beta) \, r^{-\beta}(\varphi(0) + o(1)), \quad \text{as } r \to \infty,$$

for any $\beta > 0$, where Γ is the Euler gamma function. The integral above is defined for any $r \geq M$.

Proof. We preliminarily notice that the integral is defined for any $r \geq M$, due to the fact that φ is continuous at 0 and $\beta > 0$, so that $e^{-x(r-M)} \, x^{\beta-1} \, \varphi(x)$ is in L^1.

We fix $\varepsilon > 0$. Let $\delta > 0$ be such that $|\varphi(x) - \varphi(0)| < \varepsilon$ for any $x \in (0, \delta)$. We first notice that

$$\int_0^\delta e^{-xr} \, x^{\beta-1} \, dx = r^{-\beta} \int_0^{\delta r} e^{-x} \, x^{\beta-1} \, dx = r^{-\beta} \Gamma(\beta) - r^{-\beta} \int_{\delta r}^\infty e^{-x} \, x^{\beta-1} \, dx = r^{-\beta} \, \Gamma(\beta) + O(e^{-\delta r}).$$

Similarly,

$$\int_\delta^\infty e^{-xr} \, x^{\beta-1} \, |\varphi(x)| \, dx = \int_\delta^\infty e^{-x(r-M)} \, x^{\beta-1} \, \psi(x) \, dx = O(e^{-\delta r}),$$

where we put $\psi(x) = |\varphi(x)| \, e^{-Mx}$, which we assumed to be in L^1. Therefore, we proved that

$$\left| \int_0^\infty e^{-xr} \, x^{\beta-1} \, \varphi(x) \, dx - \varphi(0) \, \Gamma(\beta) \, r^{-\beta} \right| \leq \varepsilon \, \Gamma(\beta) \, r^{-\beta} + O(e^{-\delta r}),$$

and this concludes the proof. $\square$

It is clear that G_0 and G_1 in (24) and (25) verify the assumptions of Lemma 2.2 with $M = 0$, since φ is continuous and in L^1, due to

$$\varphi(x) \leq \frac{1}{(1 - \cos(\beta\pi))((1-x)^2 + x^{2\beta})} \,,$$

and $\varphi(0) = 1$. Therefore, $G_0(r)$ and $G_1(r)$ are bounded and

$$G_j(r) = r^{-\beta-j}\,\Gamma(\beta+j)\,\frac{\sin(\beta\pi)}{\beta\pi}, \quad \text{as } r \to \infty, \qquad j = 0, 1.$$

In particular,

$$G_0(r) \approx \langle r \rangle^{-\beta}, \tag{27}$$

$$G_1(r) \approx \langle r \rangle^{-1-\beta}. \tag{28}$$

We can now prove Lemma 1.1.

Proof. [Proof of Lemma 1.1] By the homogeneity of $\hat{K}_0$ and $\hat{K}_1$, it is sufficient to prove (8) and (9) for $t = 1$.

For $\gamma = 0$ the proof follows by Lemma 2.2, see (27), (28). In order to prove the result for $|\gamma| \geq 1$ we notice that

$$G_j^{(k)}(r) = \frac{\sin(\beta\pi)}{\beta\pi} \int_0^\infty e^{-xr} x^{\beta+k+j-1} \varphi(x)\,dx, \quad j = 0, 1,$$

for all $k \in \mathbb{N}$. Thus, by applying again Lemma 2.2 we get

$$r^k G_j^{(k)}(r) = r^{-\beta-j}\,\Gamma(\beta+k+j)\,\frac{\sin(\beta\pi)}{\beta\pi}, \quad \text{as } r \to \infty, \qquad j = 0, 1.$$

In particular,

$$r^k |G_j^{(k)}(r)| \lesssim \langle r \rangle^{-\beta-j}, \quad j = 0, 1,$$

and then,

$$|\partial_\xi^\gamma \hat{K}_0(1, \xi)| \lesssim \sum_{k=\lceil |\gamma|/2 \rceil}^{|\gamma|} G_j^{(k)}(|\xi|^2)|\xi|^{2k-|\gamma|} \lesssim \langle \xi \rangle^{-2\beta-2j}|\xi|^{-|\gamma|}.$$

This concludes the proof. $\quad\square$

2.1. **Multiplier estimates.** Thanks to Lemma 1.1, we may apply Mikhlin-Hörmander multiplier theorem.

Definition 2.3. For any $1 \leq p \leq q \leq \infty$ we denote by M_p^q the space of the Fourier transforms $\hat{T}$ of tempered distributions T which satisfies

$$\|T * f\|_{L^q} \lesssim \|f\|_{L^p},$$

for all f in the Schwartz space $S(\mathbb{R}^n)$. The space M_p^q is endowed with the "multiplier norm"

$$\|\hat{T}\|_{M_p^q} := \sup\left\{ \|\mathcal{F}^{-1}(\hat{T}\mathcal{F}(f))\|_q : f \in S(\mathbb{R}^n),\ \|f\|_p = 1 \right\}.$$

In particular, we set $M_p := M_p^p$. The elements in M_p^q are called multipliers of type (p, q).

The multiplier norm is invariant by translation and has the following behavior with respect to dilations:

$$\|m(t\cdot)\|_{M_p^q} = t^{-n\left(\frac{1}{p} - \frac{1}{q}\right)} \|m\|_{M_p^q}. \tag{29}$$

Thanks to representation (26) we can use the homogeneity of the kernels G_0 and G_1; then, for any $1 \leq p \leq q \leq \infty$ and $\sigma \in \mathbb{R}$, we get

$$\|\hat{K}_i(t, \cdot)|\xi|^\sigma\|_{M_p^q} = t^{-\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{\sigma}{2}} \|\hat{K}_i(1, \cdot)|\xi|^\sigma\|_{M_p^q}, \quad i = 0, 1, \tag{30}$$

for any $t \geq 0$. The Mikhlin-Hörmander theorem in its simplest form states that if

$$|\partial_\xi^\gamma m(\xi)| \leq C\,|\xi|^{-|\gamma|}, \quad |\gamma| \leq 1 + [n/2],$$

then $m \in M_p$ for any $p \in (1, \infty)$.

As an immediate consequence of Lemma 1.1 and identity (30), by applying Mikhlin-Hörmander theorem, we get the following result.

Corollary 2.4. Let $p \in (1, \infty)$ and $b_1, b_2 \geq 0$. Then

$$\|\hat{K}_0(t, \cdot)|\xi|^{b_1}\langle\xi\rangle^{b_2}\|_{M_p} \leq Ct^{-\frac{b_1}{2}}(1 + t^{-\frac{b_2}{2}}), \quad \text{if } b_1 + b_2 \leq 2\beta, \tag{31}$$

$$\|\hat{K}_1(t, \cdot)|\xi|^{b_1}\langle\xi\rangle^{b_2}\|_{M_p} \leq Ct^{-\frac{b_1}{2}}(1 + t^{-\frac{b_2}{2}}), \quad \text{if } b_1 + b_2 \leq 2\beta + 2, \tag{32}$$

for some constant $C > 0$.

Proof. Since M_p multiplier norms are invariant by dilation and $\hat{K}_j(t, \xi) = \hat{K}_j(1, \xi\sqrt{t})$, we get

$$\|\hat{K}_j(t, \cdot)|\xi|^b\|_{M_p} = t^{-\frac{b}{2}}\|\hat{K}_j(1, \cdot)|\xi|^b\|_{M_p}. \tag{33}$$

However, $\langle\xi\rangle^{b_2}$ is not homogeneous, so to use (33), we first estimate

$$\|\hat{K}_j(t, \cdot)|\xi|^{b_1}\langle\xi\rangle^{b_2}\|_{M_p} \leq C\left(\|\hat{K}_j(t, \cdot)|\xi|^{b_1}\|_{M_p} + \|\hat{K}_j(t, \cdot)|\xi|^{b_1+b_2}\|_{M_p}\right),$$

where we used that

$$m(\xi) = \frac{\langle\xi\rangle^{b_2}}{1 + |\xi|^{b_2}}$$

is in M_p for any $p \in (1, \infty)$ (for instance, by Mikhlin-Hörmander theorem). Therefore, we obtain

$$\|\hat{K}_j(t, \cdot)|\xi|^{b_1}\langle\xi\rangle^{b_2}\|_{M_p} \leq C\,t^{-\frac{b_1}{2}}\left(\|\hat{K}_j(1, \cdot)|\xi|^{b_1}\|_{M_p} + t^{-\frac{b_2}{2}}\|\hat{K}_j(1, \cdot)|\xi|^{b_1+b_2}\|_{M_p}\right),$$

and the proof follows by Mikhlin-Hörmander theorem. $\square$

In order to obtain estimates in the M_p^q norm, with $1 < p < q < \infty$, one may combine Mikhlin-Hörmander theorem with Hardy-Littlewood-Sobolev theorem.

Definition 2.5. Let $\sigma \in (0, n/2)$. We define the *Riesz potential* of order 2σ as

$$I_{2\sigma}f(x) = \mathscr{F}^{-1}\left(|\xi|^{-2\sigma}\hat{f}(\xi)\right)(x) \equiv C_{n,\sigma}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2\sigma}}\,dy.$$

The Hardy-Littlewood-Sobolev theorem states that if $f \in L^p$ for some $p \in (1, n/2\sigma)$, then $I_{2\sigma}f \in L^{p^*}$ where

$$\frac{1}{p} - \frac{1}{p^*} = \frac{2\sigma}{n}, \quad \text{and} \quad \|I_{2\sigma}f\|_{L^{p^*}} \lesssim \|f\|_{L^p}. \tag{34}$$

2.2. Additional remarks on the kernels K_0 and K_1.
In the following, we compare the kernels of our two-terms problem with the kernels of the fractional diffusion problem.

Remark 2.1. Let $w = w(t, x)$ be the solution to the linear Cauchy-type problem

$$\begin{cases} (-\Delta)^{1-\beta}\partial_t^\beta w - \Delta w = g(t, x) & t > 0, \ x \in \mathbb{R}^n, \\ w(0, x) = w_0(x). \end{cases} \tag{35}$$

and $K_0^\dagger, K_1^\dagger$ the corresponding kernels. As in [7] it is easy to show that $\hat{K}_0^\dagger(t, \xi) = E_{\beta,1}(-|\xi|^{2\beta}t^\beta)$, where $E_{\beta,1}$ is the Mittag-Leffler function of indexes β and 1 (see [18]). Thus, the kernel $\hat{K}_0^\dagger$ has the same scaling properties of $\hat{K}_0$, that is $\hat{K}_0^\dagger(t, \xi) = \hat{K}_0^\dagger(t|\xi|^2, 1)$. As a consequence, if we consider the difference of the two kernels $K_0(t, \cdot)$ and $K_0^\dagger(t, \cdot)$ we do not gain any additional decay for $t \to \infty$; namely, for $\beta > 1/4$ in low space dimension $n < 4\beta$ we have

$$\||\xi|^s(\hat{K}_0(t, \cdot) - \hat{K}_0^\dagger(t, \cdot))\|_{M_1^2} = t^{-\frac{n}{4} - \frac{s}{2}}\|K_0(1, \cdot) - K_0^\dagger(1, \cdot)\|_{\dot{H}^s} \approx t^{-\frac{n}{4} - \frac{s}{2}} \approx \||\xi|^s K_0(t, \cdot)\|_{M_1^2},$$

for any $s \in [0, 2\beta - n/2)$, since the difference $K_0(1, \cdot) - K_0^\dagger(1, \cdot)$ is not trivial. In the previous line we used the property $M_1^2 = M_2^\infty = (L^2)' = L^2$. It is easy to show that if $g = 0$, $u_0 \in L^1 \cap L^2$ and the moment condition

$$M = \int_{\mathbb{R}^n} u_0(x)\,dx \neq 0,$$

holds, following as in [10, Theorem 2], the asymptotic profiles of the solution to (2), and to (35) with $w_0 = u_0$, are described by $M K_0(t,x)$ and $M K_0^\dagger(t,x)$, respectively, in the sense that

$$\|(K_0(t,\cdot) * u_0) - M K_0(t,\cdot)\|_{L^2} = o(t^{-\frac{n}{4}}), \qquad \|(K_0^\dagger(t,\cdot) * u_0) - M K_0^\dagger(t,\cdot)\|_{L^2} = o(t^{-\frac{n}{4}}).$$

As a consequence, we obtain that

$$\begin{aligned}
\lim_{t \to \infty} t^{\frac{n}{4}} \|u(t,\cdot) - w(t,\cdot)\|_{L^2} &= \lim_{t \to \infty} t^{\frac{n}{4}} \|(K_0(t,\cdot) * u_0) - (K_0^\dagger(t,\cdot) * u_0)\|_{L^2} \\
&= |M| \, t^{\frac{n}{4}} \, \|\hat{K}_0(t,\cdot) - \hat{K}_0^\dagger(t,\cdot)\|_{L^2} \\
&= |M| \, \|\hat{K}_0(1,\cdot) - \hat{K}_0^\dagger(1,\cdot)\|_{L^2} \neq 0.
\end{aligned}$$

Thus, u does not behave asymptotically like w. This fact supports the idea that the presence of the integer order derivative $\partial_t u$ in the Cauchy-type problem (1) does not produce the same effects as in the fractional damped heat equation (18).

Remark 2.2. We stress that the difference in the smoothing properties of the two kernels K_0 and K_1 can be motivated looking at the smoothing effects for the Cauchy-type problem (35): it is easy to derive the identities $K_0^\dagger = H_0$ and $K_1^\dagger = I_{2(1-\beta)} H_1$, where H_0 and H_1 are the kernels of the following subdiffusive problem

$$\begin{cases} \partial_t^\beta v + (-\Delta)^\beta v = I_{2(1-\beta)} g(t,x) \\ v(0,x) = w_0(x), \end{cases} \tag{36}$$

which is obtained by problem (35) applying the Riesz potential $I_{2(1-\beta)}$ of order $2(1-\beta)$ to both sides of the equation. In particular, for any $p \in (1,\infty)$ we have that $|\xi|^\alpha \hat{K}_0^\dagger = |\xi|^\alpha \hat{H}_0 \in M_p$ for any $\alpha \in [0, 2\beta]$; whereas, being $|\xi|^\alpha \hat{H}_1 \in M_p$ for all $\alpha \in [0 \, 4\beta]$, we derive $|\xi|^\alpha \hat{K}_1^\dagger \in M_p$ for any $\alpha \in [0, 2 + 2\beta]$. The study of problem (36) may be tackled more in details following the approach used in [7].

3. Proofs of Proposition 1.2 and Theorem 1.5

We may now employ Corollary 2.4 to prove the desired estimates for the solution to (1).

We first consider the homogeneous problem (2).

Proposition 3.1. *Let $n \geq 1$ and $s \in \mathbb{R}$. Assume that $u_0 \in H^{s,m}$ for some $m \in (1,\infty)$. Then the solution $u \in \mathcal{C}([0,\infty), H^{s,m})$ to (2) verifies the estimate*

$$\|u(t,\cdot)\|_{H^{s,m}} \leq C \, \|u_0\|_{H^{s,m}}, \tag{37}$$

for some $C > 0$ independent of u_0 and t. Moreover, if $s > 0$ then we have the decay estimate

$$\|u(t,\cdot)\|_{\dot{H}^{s,m}} \leq C \, t^{-\min\{\frac{s}{2},\beta\}} \|u_0\|_{H^{s,m}}, \tag{38}$$

for any $t \geq 1$.

Proof. Applying (31) with $b_1 = b_2 = 0$, we immediately derive (37) by

$$\|u(t,\cdot)\|_{H^{s,m}} \leq \|\hat{K}_0(t,\cdot)\|_{M_m} \|u_0\|_{H^{s,m}} \leq C \, \|u_0\|_{H^{s,m}}.$$

Let $s > 0$ and $t \geq 1$. Applying (31) with $b_2 = 0$, and setting $b_1 = s$ if $s \leq 2\beta$, or $b_1 = 2\beta$ otherwise, we get

$$\|u(t,\cdot)\|_{\dot{H}^{s,m}} \leq \| |\xi|^s \hat{K}_0(t,\cdot)\|_{M_m} \|u_0\|_{L^m} \leq C \, t^{-\frac{s}{2}} \|u_0\|_{L^m}, \quad \text{if } s \leq 2\beta,$$

$$\|u(t,\cdot)\|_{\dot{H}^{s,m}} \leq \| |\xi|^{2\beta} \hat{K}_0(t,\cdot)\|_{M_m} \| |\xi|^{s-2\beta} u_0\|_{L^m} \leq C \, t^{-\beta} \|u_0\|_{H^{s-2\beta,m}}, \quad \text{if } s > 2\beta.$$

The proof of (38) follows. The proof of the continuity is standard. $\square$

To deal with the solution to (1) we shall now also consider the integral containing K_1 in (6) and prove Lemmas 1.3 and 1.4.

Proof of Lemma 1.3. We first prove (10). Due to $t - \tau \leq 1$, using (32) with $b_1 = 0$ and $b_2 = 2b$, we estimate

$$\|K_1(t - \tau, \cdot) *_{(x)} g(\tau, \cdot)\|_{H^{s,m}} \leq \|\langle \xi \rangle^{2b} \hat{K}_1(t - \tau, \cdot)\|_{M_m} \|g(\tau, \cdot)\|_{H^{s-2b,m}}$$
$$\leq C(t - \tau)^{-b} \|g(\tau, \cdot)\|_{H^{s-2b,m}},$$

for any $b \in [0, 1)$. The assumption $b < 1$ guarantees that $(t - \tau)^{-b}$ is integrable over $[t - 1, t]$. Moreover, $2b < 2 < 2 + 2\beta$. This proves (10).

To prove (11), we use (32) with $b_1 = 0$ and $b_2 = 2 + 2\beta$, so that

$$\|K_1(t - \tau, \cdot) *_{(x)} g(\tau, \cdot)\|_{H^{s,m}} \leq \|\langle \xi \rangle^{2+2\beta} \hat{K}_1(t - \tau, \cdot)\|_{M_m} \|g(\tau, \cdot)\|_{H^{s-2-2\beta,m}}$$
$$\leq C \|g(\tau, \cdot)\|_{H^{s-2b,m}},$$

since $t - \tau \geq 1$. This proves (11). $\qquad\square$

The proof of Proposition 1.2 follows combining Proposition 3.1 and Lemma 1.3.

Proof of Proposition 1.2. By Hölder inequality, we estimate

$$\int_{(t-1)_+}^t (t - \tau)^{-b} \|g(\tau, \cdot)\|_{H^{s-2b,m}} \, d\tau \leq \begin{cases} C \, t^{-b} \|g\|_{L^r([t-1,t], H^{s-2b,m})} & \text{if } t \geq 1, \\ C \, t^{1-\frac{1}{r}-b} \|g\|_{L^r([0,t], H^{s-2b,m})} & \text{if } t \in (0, 1). \end{cases}$$

Moreover, for $t > 1$, we just estimate

$$\int_0^{t-1} \|g(\tau, \cdot)\|_{H^{s-2-2\beta,m}} \, d\tau \leq \|g\|_{L^1([0,t-1], H^{s-2-2\beta,m})}.$$

The proof of the continuity is a standard consequence of the fact that $g \in L^r_{\text{loc}}(\mathbb{R}_+, H^{s-2b,m})$; indeed, for a sequence t_h in $\mathbb{R}_+$ we get:

$$t_h \to t \Rightarrow \int_{t_h}^t (t - \tau)^{-b} \|g(\tau, \cdot)\|_{H^{s-2b,m}} \, d\tau \leq C \, |t^{1-\frac{1}{r}-b} - t_h^{1-\frac{1}{r}-b}| \, \|g\|_{L^r([t_h,t], H^{s-2b,m})} \to 0,$$

if $b \in (0, 1)$, and

$$t_h \to t \Rightarrow \int_{t_h}^t \|g(\tau, \cdot)\|_{H^{s,m}} \, d\tau \leq C \, \|g\|_{L^1([t_h,t], H^{s,m})} \to 0,$$

if $b = 0$, due to the absolute continuity of the Lebesgue measure. If $t_h > t$, we just replace $[t_h, t]$ by $[t, t_h]$. $\qquad\square$

Now we fix $s > 0$ and we prove Lemma 1.4.

Proof of Lemma 1.4. First, let $s + a \leq 2 + 2\beta$. By using (32) with $b_1 = s + a$ and $b_2 = 2 + 2\beta - s - a$, we may estimate

$$\|K_1(t - \tau, \cdot) *_{(x)} g(\tau, \cdot)\|_{\dot{H}^{s,m}} \leq \|\langle \xi \rangle^{2\beta+2-s-a} |\xi|^{s+a} \hat{K}_1(t - \tau, \cdot)\|_{M_m} \|\mathscr{F}^{-1}(|\xi|^{-a} \langle \xi \rangle^{s-2\beta-2+a} \hat{g}(\tau, \cdot))\|_{L^m}$$
$$\leq C(t - \tau)^{-\frac{s+a}{2}} \|\mathscr{F}^{-1}(|\xi|^{-a} \langle \xi \rangle^{s-2\beta-2+a} \hat{g}(\tau, \cdot))\|_{L^m}.$$

We stress that the fact that $s + a$ is the power of $|\xi|$ and not of $\langle \xi \rangle$ is crucial to produce the decay $(t - \tau)^{-\frac{s+a}{2}}$, since $t - \tau \geq 1$. Finally, we may estimate

$$\|\mathscr{F}^{-1}(|\xi|^{-a} \langle \xi \rangle^{s-2\beta-2+a} \hat{g}(\tau, \cdot))\|_{L^m} = \|I_a f\|_{L^m} \leq C \|f\|_{L^q} = C \|g(\tau, \cdot)\|_{H^{s-2\beta-2+a,q}},$$

where $I_a f = \mathscr{F}^{-1}(|\xi|^{-a} f)$ is the Riesz potential of $f = \mathscr{F}^{-1}(\langle \xi \rangle^{s-2\beta-2+a} \hat{g}(\tau, \cdot))$, thanks to Hardy-Littlewood-Sobolev theorem, see (34). This proves (14) when $s + a \leq 2 + 2\beta$.

In the case $s > 2 + 2\beta$, using (32) with $b_1 = 2 + 2\beta$ and $b_2 = 0$, we may estimate

$$\|K_1(t - \tau, \cdot) *_{(x)} g(\tau, \cdot)\|_{\dot{H}^{s,m}} \leq \||\xi|^{2+2\beta} \hat{K}_1(t - \tau, \cdot)\|_{M_m} \|\mathscr{F}^{-1}(|\xi|^{s-2-2\beta} \hat{g}(\tau, \cdot))\|_{L^m}$$
$$\leq C(t - \tau)^{-1-\beta} \|g(\tau, \cdot)\|_{\dot{H}^{s-2-2\beta,m}}.$$

This concludes the proof of (12). $\qquad\square$

Remark 3.1. Following the proof of (12), we see that $\|g(\tau,\cdot)\|_{H^{s-2\beta-2+a,q}}$ may be replaced by the homogeneous quantity $\|g(\tau,\cdot)\|_{\dot{H}^{s-2\beta-2,m}}$ when $s > 2 + 2\beta$. Similarly, if $s > 2b$, we may estimate

$$\|K_1(t-\tau,\cdot) *_{(x)} g(\tau,\cdot)\|_{\dot{H}^{s,m}} \leq \||\xi|^{2b}\hat{K}_1(t-\tau,\cdot)\|_{M_m}\|g(\tau,\cdot)\|_{\dot{H}^{s-2b,m}}$$
$$\leq C(t-\tau)^{-b}\|g(\tau,\cdot)\|_{\dot{H}^{s-2b,m}},$$

and using this when dealing with $\|u(t,\cdot)\|_{\dot{H}^{s,m}}$, instead of relying on (10).

Theorem 1.5 may then be modified accordingly, using the homogeneous quantity $\|g(\tau,\cdot)\|_{\dot{H}^{s-2b,m}}$ in (15), when $s > 2b$, and the homogeneous quantity $\|g(\tau,\cdot)\|_{\dot{H}^{s-2\beta-2,m}}$ in (16), when $s > 2 + 2\beta$.

The proof of Theorem 1.5 verbatim follows combining Propositions 3.1 and 1.2 with (10) and (14).

Proof of Theorem 1.5. For any $t \geq 1$, using (6), we obtain

$$\|u(t,\cdot)\|_{\dot{H}^{s,m}} \leq C\,t^{-\min\{\frac{s}{2},\beta\}}\|u_0\|_{H^{s,m}} + C\int_{t-1}^{t}(t-\tau)^{-b}\|g(\tau,\cdot)\|_{H^{s-2b,m}}\,d\tau$$
$$+ C\int_0^{t-1}(t-\tau)^{-\frac{s+a}{2}}\|g(\tau,\cdot)\|_{H^{s-2-2\beta+a,q}}\,d\tau$$
$$\leq C_1\,(1+t)^{-\min\{\frac{s}{2},\beta\}}\left(\|u_0\|_{H^{s,m}} + A + B\right).$$

For $t \leq 1$, using (6), we obtain

$$\|u(t,\cdot)\|_{\dot{H}^{s,m}} \leq C\,\|u_0\|_{H^{s,m}} + C\int_0^t(t-\tau)^{-b}\|g(\tau,\cdot)\|_{H^{s-2b,m}}\,d\tau$$
$$\leq C_1\left(\|u_0\|_{H^{s,m}} + A\right).$$

This concludes the proof. $\square$

Remark 3.2. We stress that many other estimates for the solution to (1) may be proved using (10), (11) and (14), which are consequences of Lemma 1.1. For instance, one may be interested into have a bounded solution to (1), in the sense that $\|u(t,\cdot)\|_{H^{s,m}}$ is bounded with respect to t. In this case, the assumption of Theorem 1.5 may be relaxed. Assuming that

$$A_0 = \sup_{t \geq 0}\int_{(t-1)_+}^{t}(t-\tau)^{-b}\,\|g(s,\cdot)\|_{H^{s-2b,m}}\,d\tau, \tag{39}$$

and

$$B_0 = \sup_{t \geq 1}\int_0^{t-1}(t-\tau)^{-\frac{s+a}{2}}\,\|g(\tau,\cdot)\|_{H^{s-2-2\beta+a,q}}\,d\tau, \tag{40}$$

are finite, the following estimate immediately follows by (10) and (11):

$$\|u(t,\cdot)\|_{H^{s,m}} \leq C(\|u_0\|_{H^{s,m}} + A_0 + B_0), \tag{41}$$

for any $t \geq 1$, where $C > 0$ is independent of t, u_0, A_0 and B_0.

3.1. **Proof of the integral estimates in Examples 1.2 and 1.3.** The estimates provided in Examples (1.2) and (1.3) are consequences of variants of a well-known result for integrals related to the application of Duhamel's principle (an earlier version of this estimate goes back to [25]). For the ease of reading, we provide statements and proofs.

Lemma 3.2. *Let $a_1, a_2 \in [0, 1)$. Then the following estimate holds*

$$\int_0^t(t-\tau)^{-a_1}\tau^{-a_2}\,d\tau \leq C\,t^{1-a_1-a_2},$$

for any $t \geq 1$, where the constant $C = C(a_1, a_2) > 0$ is independent of $t \geq 1$.

Proof. It is convenient to split the integral into

$$I_1 = \int_0^{\frac{t}{2}} (t-\tau)^{-a_1} \tau^{-a_2} \, d\tau,$$

$$I_2 = \int_{\frac{t}{2}}^t (t-\tau)^{-a_1} \tau^{-a_2} \, d\tau.$$

The first integral may be estimated as

$$I_1 \leq 2^{a_1} t^{-a_1} \int_0^{\frac{t}{2}} \tau^{-a_2} \, d\tau = \frac{2^{a_1+a_2-1}}{1-a_2} t^{1-a_1-a_2},$$

and by a change of variable we also find

$$I_2 = \int_0^{\frac{t}{2}} (t-\tau)^{-a_2} \tau^{-a_1} \, d\tau \leq \frac{2^{a_1+a_2-1}}{1-a_1} t^{1-a_1-a_2}.$$

This concludes the proof. □

The function τ^{-a_2} shall be replaced by $(1+\tau)^{-a_2}$ to avoid the singularity at $\tau = 0$, when $a_2 \geq 1$. In particular, we have the following.

Lemma 3.3. *Let $a_1 \in [0,1)$ and $a_2 > 1$. Then the following estimate holds*

$$\int_0^t (t-\tau)^{-a_1} (1+\tau)^{-a_2} \, d\tau \leq C \, t^{-a_1},$$

for any $t \geq 1$, where the constant $C = C(a_1, a_2) > 0$ is independent of $t \geq 1$.

Proof. As in the proof of Lemma 3.2, we split the integral into

$$I_1 = \int_0^{\frac{t}{2}} (t-\tau)^{-a_1} (1+\tau)^{-a_2} \, d\tau,$$

$$I_2 = \int_{\frac{t}{2}}^t (t-\tau)^{-a_1} (1+\tau)^{-a_2} \, d\tau.$$

The integral I_1 may now be estimated as

$$I_1 \leq 2^{a_1} t^{-a_1} \int_0^{\frac{t}{2}} (1+\tau)^{-a_2} \, d\tau = \frac{2^{a_1}}{a_2-1} t^{-a_1} \left(1 - (2/(t+2))^{a_2-1}\right) \leq \frac{2^{a_1}}{a_2-1} t^{-a_1}.$$

To estimate the integral I_2 we proceed as in the proof of Lemma 3.2, and we get

$$I_2 \leq 2^{a_2} (t+2)^{-a_2} \int_0^{\frac{t}{2}} \tau^{-a_1} \, d\tau = \frac{2^{a_1+a_2-1}}{1-a_1} (t+2)^{-a_2} t^{1-a_1} \leq \frac{2^{a_2}}{1-a_1} t^{-a_1}.$$

This concludes the proof. □

4. APPLICATION TO A NONLINEAR PROBLEM

In order to look for the existence of global-in-time solution to (4), we first provide a nonexistence result. Namely, according to the integrability of the initial datum u_0, we find that no global-in-time solution to (4) exist, even in a weak sense, if the power nonlinearity in (4) is too small.

Theorem 4.1. *Let $u_0 \in L^m$ for some $m \in (1, \infty)$ which satisfies*

$$u_0(x) \geq \varepsilon_0 (1+|x|)^{-\frac{n}{m}} (\ln(e+|x|))^{-1}, \tag{42}$$

for some $\varepsilon_0 \in (0,1)$. Then, if there exists a global-in-time weak solution to (4) with $f = |u|^p$, then $p \geq 1 + 2m/n$.
On the other hand, if $u_0 \in L^1$ satisfies

$$\int_{\mathbb{R}^n} u_0(x) \, dx > 0, \tag{43}$$

1 *and problem* (4), *with* $f = |u|^p$, *admits a global-in-time weak solution, then* $p > 1 + 2/n$ *if* $\beta \in (0, 2/(n+2)]$,
2 *or* $p \geq 1 + 2/n$ *if* $\beta \in (2/(n+2), 1)$.

3   We postpone the proof of Theorem 4.1 to §5.

4   The fact that global-in-time solutions may exist for critical nonlinearities $p = 1 + 2m/n$, when initial
5 datum is in L^m with $m > 1$, is expected and confirmed by the forthcoming Theorem 4.2, which provides
6 some existence result in this critical case, under additional assumptions on the space dimension n and
7 on β. In the case of L^1 initial datum, the existence of global-in-time solutions is excluded by Theorem 4.2,
8 but only in the case $\beta \in (0, 2/(n+2)]$. The existence of global-in-time solutions for supercritical powers
9 is guaranteed again by Theorem 4.2 (see Remark 4.1). In the case $\beta \in (2/(n+2), 1)$, we are not aware if
10 the case of critical power nonlinearity belongs to the existence or to the nonexistence range.

11 **Theorem 4.2.** *Let* $m \in (1, \infty)$ *and fix a space dimension* $n < 2m$, *and a regularity* $s \in (n/m, 2)$.
12 *Let* $p \geq 1 + 2m/n$ *if* $1 + 2m/n \geq 1/\beta$, *and* $p > 1/\beta$ *otherwise. Then, there exists* $\varepsilon_0 > 0$ *such that if*

$$u_0 \in H^{s,m}, \quad \text{with } \|u_0\|_{H^{s,m}} \leq \varepsilon_0, \tag{44}$$

13 *there is a uniquely determined solution* $u \in \mathcal{C}([0, \infty), H^{s,m})$ *to* (4) *with* $f = |u|^p$. *Moreover, the solution* u
14 *satisfies the following decay estimates:*

$$\|u(t, \cdot)\|_{\dot{H}^{\kappa,m}} \leq C \, (1+t)^{-\min\left\{\frac{\kappa}{2}, \beta\right\}} \|u_0\|_{H^{s,m}}, \qquad \kappa \in [0, s], \tag{45}$$

15 *where* $C > 0$ *is independent of* $t > 0$.

16 *Remark* 4.1. As a corollary of Theorem 4.2, if the initial datum is in $H^{s,1}$ for a sufficiently large s, then
17 Theorem 4.2 may be applied for any $m \in (1, \infty)$ in space dimension $n = 1, 2$, so that the global-in-time
18 small data solutions exist if $p > \max\{1 + 2/n, 1/\beta\}$ in space dimension $n = 1, 2$.

19   We stress that we do not expect that the existence exponent $\max\{1 + 2m/n, 1/\beta\}$ in Theorem 4.2 is
20 critical when $1 + 2m/n < 1/\beta$. More precisely, we expect that for any $m \in (1, \infty)$ it is possible to prove the
21 existence of global-in-time solutions with small datum in $H^{s,m}$ for $p > \tilde{p}(n, m, \beta)$ with $\tilde{p}(n, m, \beta) < 1/\beta$ if

$$1 + \frac{2m}{n} < \frac{1}{\beta}.$$

22 However, proving a general result would be rather technical, so we only discuss a simple scenario in
23 Proposition 4.3.

24   This very peculiar effect shows that working with L^1 regularity of the datum or, more in general, L^m
25 regularity, with m close to 1, is not the best possible choice to find global-in-time small data solutions.
26 This is due to the asymmetry in the Duhamel's principle for fractional equations, that is, K_1 has better
27 smoothing properties than K_0 (see also Remark 2.2).

28 **Proof of Theorem 4.2.** We consider the evolution space $X = C([0, \infty), H^{s,m})$ and the norm

$$\|u\|_X = \sup_{t \geq 0} \sup_{\kappa \in [0,s]} (1+t)^{\min\left\{\frac{\kappa}{2}, \beta\right\}} \|u(t, \cdot)\|_{\dot{H}^{\kappa,m}}.$$

29 Applying the fractional Sobolev embedding or Gagliardo-Nirenberg inequality (see, for instance, [17]) we
30 conclude that if $u \in X$ then

$$\|u(\tau, \cdot)\|_{L^r} \lesssim (1+\tau)^{-\min\left\{\frac{n}{2}\left(\frac{1}{m} - \frac{1}{r}\right), \beta\right\}} \|u\|_X, \tag{46}$$

31 for any $r \in [m, \infty)$, since

$$H^{\kappa,m} \hookrightarrow L^r, \qquad \|h\|_{L^r} \leq C \, \|h\|_{\dot{H}^{\kappa,m}}, \quad \kappa = n\left(\frac{1}{m} - \frac{1}{r}\right),$$

32 and $n/m < s$. Let us define the operator

$$N : u \in X(T) \mapsto Nu(t, x) = K_0(t, \cdot) *_{(x)} u_0 + Fu(t, x), \qquad Fu = \int_0^t K_1(t - \tau, \cdot) *_{(x)} f(u(\tau, \cdot)) \, d\tau. \tag{47}$$

Applying Proposition 3.1, we find that

$$\|K_0 *_{(x)} u_0\|_X \leq C_1 \|u_0\|_{H^{s,m}},$$

for some $C_1 > 0$. We will prove that

$$\|Fu - Fv\|_X \leq C_2 \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right), \tag{48}$$

for some $C_2 > 0$. We now want to prove that N is a contraction on a closed ball $B_R = \{u \in X : \|u\|_X \leq R\}$, provided that R is sufficiently small. Let us fix $R = 2C_1 \|u_0\|_{H^{s,m}}$. Estimate (48) proves that F is a contraction with constant $L < 1/2$ on B_R if R is a sufficiently small, namely, $2C_2 R^{p-1} < 1/2$. This provides the smallness condition on $\|u_0\|_{H^{s,m}} = R/(2C_1)$. Now N maps B_R onto B_R and is a contraction, as well. Therefore there exists a unique fixed point, that is, a unique weak solution u, in X. Moreover, $u \in B_R$, that is, $\|u\|_X \leq R = 2C_1 \|u_0\|_{H^{s,m}}$, and this implies estimates (45).

It remains to prove (48). For any u and v in $X(T)$ we define

$$g(\tau, x) = f(u(\tau, x)) - f(v(\tau, x)).$$

As a consequence of (46) and (5) we obtain

$$\|g(\tau, \cdot)\|_{L^m} \leq \|(u - v)(\tau, \cdot)\|_{L^{mp}} \left(\|u(\tau, \cdot)\|_{L^{mp}}^{p-1} + \|v(\tau, \cdot)\|_{L^{mp}}^{p-1}\right)$$

$$\lesssim (1 + \tau)^{-p \min\left\{\frac{n}{2m}\left(1 - \frac{1}{p}\right), \beta\right\}} \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right).$$

Similarly,

$$\|g(\tau, \cdot)\|_{L^q} \leq \|(u - v)(\tau, \cdot)\|_{L^{qp}} \left(\|u(\tau, \cdot)\|_{L^{qp}}^{p-1} + \|v(\tau, \cdot)\|_{L^{qp}}^{p-1}\right)$$

$$\lesssim (1 + \tau)^{-p \min\left\{\frac{n}{2}\left(\frac{1}{m} - \frac{1}{qp}\right), \beta\right\}} \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right),$$

$$= (1 + \tau)^{-\min\left\{\frac{n}{2m}(p-1) - \frac{a}{2}, p\beta\right\}} \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right),$$

provided that $a = n(1/q - 1/m)$ is sufficiently small, that is, q is sufficiently close to m, in particular, $qp \geq m$. We may now apply Theorem 1.5.

We distinguish two cases. We first assume that $p_c = 1 + 2m/n$ verifies $p_c \geq 1/\beta$. It is clear that

$$\min\left\{\frac{n}{2m}(p-1), p\beta\right\} \geq \min\left\{\frac{n}{2m}(p_c - 1), p_c\beta\right\} = \min\{1, p_c\beta\} = 1,$$

due to the assumption $p_c \geq 1/\beta$. Similarly,

$$\min\left\{\frac{n}{2m}(p-1) - \frac{a}{2}, p\beta\right\} \geq 1 - \frac{a}{2}.$$

Due to the assumption $\kappa \leq s < 2$, letting $b = s/2 \in (0, 1)$, we get

$$\|g(t, \cdot)\|_{H^{\kappa - 2b, m}} \leq \|g(t, \cdot)\|_{H^{s - 2b, m}} = \|g(t, \cdot)\|_{L^m} \leq (1 + t)^{-1} \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right),$$

and $(1 + t)^{-1} \leq (1 + t)^{-\min\left\{\frac{\kappa}{2}, \beta\right\}}$. Therefore, we may estimate the quantity A in (15) (see Example 1.1) by

$$A \leq \frac{4}{2 - s} \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right).$$

We also assume that $a > 0$ is sufficiently small to get $s + a < 2$ as well. Therefore,

$$\|g(t, \cdot)\|_{H^{\kappa - 2 - 2\beta + a, q}} \leq \|g(t, \cdot)\|_{H^{s - 2 - 2\beta + a, q}} \leq \|g(t, \cdot)\|_{L^q} \leq t^{-\left(1 - \frac{a}{2}\right)} \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right),$$

for any $\kappa \leq s$. Applying Theorem 1.5 (see Example 1.2), we get the desired estimate

$$\|Fu(t, \cdot) - Fv(t, \cdot)\|_{\dot{H}^{\kappa, m}} \leq C(1 + t)^{-\min\left\{\frac{\kappa}{2}, \beta\right\}} \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right),$$

for any $\kappa \leq s$.

Now let $p_c < 1/\beta$. In this case, we fix $a = 0$. Now for any $p > 1/\beta$,

$$\min\left\{\frac{n}{2m}(p-1), p\beta\right\} = d(p) > 1,$$

since $p > p_c$ as well. Therefore,

$$\|g(t,\cdot)\|_{H^{\kappa-2-2\beta,m}} \leq \|g(t,\cdot)\|_{H^{\kappa-2b,m}} \leq \|g(t,\cdot)\|_{L^m} \leq t^{-d(p)} \|u-v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right),$$

for any $\kappa \leq s$ and for $b = 1 - s/2$. Applying Theorem 1.5 (see Examples 1.1 and 1.3), we get again the desired decay estimate.

This concludes the proof.   □

4.1. **A variant of Theorem 4.2 in space dimension** $n = 1, 2$. It is easy to see that the range in which the existence exponent is $1 + 2m/n$ may be enlarged taking smaller q. In the following result, we discuss the case of low space dimension $n = 1, 2$, in which it is possible to choose q close to 1 as one desires.

Proposition 4.3. *Let* $n = 1, 2$, *and* $m \in (1, \infty)$. *Fix* $s = 2$. *Assume that*

$$\beta \, p_c > 1 - \frac{n}{2}\left(1 - \frac{1}{m}\right). \tag{49}$$

Then for all $p \geq p_c$ *there exists* $\varepsilon_0 > 0$ *such that if* (44) *holds and* $f(u)$ *satisfies* (5), *for any* $\delta \in (0, 2 - n/m)$ *there is a uniquely determined solution* u *to* (4) *which belongs to* $C([0, \infty), H^{2-\delta,m})$.

Proof of Proposition 4.3. We follow the proof of Theorem 4.2, but now we take q very close to 1. This is possible because, on the one hand for any $\delta \in (0, 2 - n/m)$ it holds

$$H^{\kappa,m} \hookrightarrow L^{qp}, \quad \text{for all} \quad 2 - \delta > \kappa = n\left(\frac{1}{m} - \frac{1}{qp}\right)$$

and

$$\kappa + a = \frac{n}{q}\left(1 - \frac{1}{p}\right) < 2$$

for any $q \in (1, m)$; on the other hand, since (49) holds and $p \geq p_c$ we can fix $q \in (1, m)$ sufficiently small such that

$$\beta p \geq 1 - \frac{a}{2}.$$

Now, noticing also that $qp \geq m$, since $2m/n \geq m$ we may apply (46) to $r = qp_c$. As a consequence we obtain

$$\|g(t,\cdot)\|_{L^q} \lesssim (1+t)^{-1+\frac{a}{2}} \|u-v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right).$$

Moreover, as in the proof of Theorem 4.2 we get

$$\|g(t,\cdot)\|_{L^m} \lesssim (1+t)^{-p\beta} \|u-v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right);$$

thus, since $p\beta > \min\{\kappa/2, \beta\}$, letting $b = 1 - \delta/2$ we may estimate the quantity A in (15) (see Example 1.1) by

$$A \leq \frac{4}{\delta} \|u-v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right).$$

Applying Theorem 1.5 (see Example 1.2 to estimate the quantity B in (16)), we conclude the proof.   □

4.2. **Application of the decay estimates to another nonlinear problem.** The smoothing effect of K_1 also allows us to investigate global-in-time existence results for semilinear problems with nonlinear terms which are different by the classical $|u|^p$, e.g. $f(u, \nabla u) = \nabla(u|u|^{p-1})$ with $p > 1$; for brevity, we discuss only the case of space dimension $n = 1$, fixing $\beta \geq 1/2$ in (4).

Theorem 4.4. *Let $m \in (1, \infty)$; fix the space dimension $n = 1$, $\beta \in [1/2, 1)$, and the regularity $s = 1$. Let $p \geq 1 + m$. Then, there exists $\varepsilon_0 > 0$ such that if*

$$u_0 \in H^{1,m}, \quad with \ \|u_0\|_{H^{1,m}} \leq \varepsilon_0, \tag{50}$$

there is a uniquely determined solution $u \in \mathcal{C}([0,\infty), H^{1,m})$ to (4) with $f = \nabla|u|^p$. Moreover, the solution u satisfies the following estimates:

$$\|u(t, \cdot)\|_{\dot{H}^{\kappa,m}} \leq C\,(1+t)^{-\frac{\kappa}{2}} \|u_0\|_{H^{1,m}}, \qquad \kappa \in [0,1],$$

where $C > 0$ is independent of $t > 0$.

Proof. To prove Theorem 4.4, we consider the evolution space

$$X = C([0, \infty), H^{1,m}),$$

and the norm

$$\|u\|_X = \sup_{t \geq 0}\ \sup_{\kappa \in [0,1]} (1+t)^{\frac{\kappa}{2}} \|u(t, \cdot)\|_{\dot{H}^{\kappa,m}}.$$

Applying the fractional Sobolev embedding or Gagliardo-Nirenberg inequality we conclude that

$$\|u(\tau, \cdot)\|_{L^r} \lesssim (1+\tau)^{-\frac{1}{2}\left(\frac{1}{m} - \frac{1}{r}\right)} \|u\|_X, \tag{51}$$

for any $u \in X$ and $r \geq m$, since

$$H^{\kappa,m} \hookrightarrow L^r, \qquad \|h\|_{L^r} \leq C\,\|h\|_{\dot{H}^{\kappa,m}}, \quad \kappa = \frac{1}{m} - \frac{1}{r} < 1.$$

Moreover, applying Proposition 3.1, being $\beta \geq 1/2$, we get that

$$\|K_0 *_{(x)} u_0\|_X \leq C_1\, \|u_0\|_{H^{1,m}},$$

for some $C_1 > 0$. As in the proof of Theorem 4.2 we consider the operator $N : X \to X$ defined as in (47) and we prove that for any $u, v \in X$ estimate (48) is satisfied for some $C_2 > 0$; as a consequence, we will get the existence of a unique weak solution u to (4) in X, provided that $\|u_0\|_{H^{1,m}}$ is sufficiently small. For any u and v in X we define

$$g(\tau, x) = \nabla(u(\tau, x)|u(\tau, x)|^{p-1}) - \nabla(v(\tau, x)|v(\tau, x)|^{p-1}),$$

with $p \geq \tilde{p}_c := 1 + m$. By (51), since it holds

$$|\nabla(u|u|^{p-1}) - \nabla(v|v|^{p-1})| \leq |\nabla(u-v)|(|u|^{p-1} + |v|^{p-1}) + |\nabla v||u-v|(|u|^{p-2} - |v|^{p-2}),$$

we may fix $b = \kappa/2$ and estimate

$$
\begin{aligned}
\|g(t, \cdot)\|_{H^{\kappa-2b,m}} = \|g(t, \cdot)\|_{L^m} &\leq \|u-v\|_{\dot{H}^{1,m}}\left(\|u\|_{L^\infty}^{p-1} + \|v\|_{L^\infty}^{p-1}\right) \\
&\quad + \|v\|_{\dot{H}^{1,m}}\|u-v\|_{L^\infty}\left(\|u\|_{L^\infty}^{p-2} + \|v\|_{L^\infty}^{p-2}\right) \\
&\leq (1+t)^{-\frac{1}{2} - \frac{p-1}{2m}}\|u-v\|_X\left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right),
\end{aligned}
\tag{52}
$$

for any $\kappa \in [0,1]$; moreover, for any $q \in (1, m)$ we have

$$
\begin{aligned}
\|g(t, \cdot)\|_{H^{\kappa-2\beta-2+a,q}} \leq \|g(t, \cdot)\|_{L^q} &\leq \|u-v\|_{\dot{H}^{1,m}}\left(\|u\|_{L^{r(p-1)}}^{p-1} + \|u\|_{L^{r(p-1)}}^{p-1}\right) \\
&\quad + \|v\|_{\dot{H}^{1,m}}\|u-v\|_{L^\infty}\left(\|u\|_{L^{r(p-1)}}^{p-2} + \|u\|_{L^{r(p-1)}}^{p-2}\right) \\
&\leq (1+t)^{-\frac{1}{2} - \frac{1}{2}\left(\frac{p-1}{m} - \frac{1}{r}\right)}\|u-v\|_X\left(\|u\|_X^{p-1} + \|v\|_X^{p-1}\right),
\end{aligned}
$$

where $r \in [q, \infty)$ satisfies $1/q = 1/m + 1/r$ and a, defined as in (13), is less than 1.

In particular, as a consequence of (52), we may estimate the quantity A in (15) (see Example 1.1) by

$$A \leq 4 \left\| u - v \right\|_X \left(\left\| u \right\|_X^{p-1} + \left\| v \right\|_X^{p-1} \right).$$

On the other hand, noticing that

$$\frac{1}{2} + \frac{1}{2} \Big(\frac{p-1}{m} - \frac{1}{r} \Big) \geq 1 - \frac{a}{2},$$

for any $p \geq \tilde{p}_c$, we may estimate the quantity B in (16) (see Example 1.2) and apply Theorem 1.5. The proof of the desired result follows if the initial datum u_0 satisfies condition (50). $\square$

5. Proof of Theorem 4.1

Let us consider the Cauchy-type problem

$$\begin{cases} \partial_t u + \partial_t^\beta (-\Delta)^{1-\beta} u - \Delta u = |u|^p \\ u(0, x) = u_0(x), \end{cases} \tag{53}$$

with $p > 1$ and u_0 satisfying condition (42) or (43). For any $\alpha > 0$ the left-sided and, respectively, right-sided Riemann-Liouville fractional integral of order α of a given function f defined on $[a, b]$ are given by

$$\left(J_{a+}^\alpha f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} \, ds,$$

and, respectively,

$$\left(J_{b-}^\alpha f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} \, ds,$$

for any $t \in [a, b]$. Moreover, for any $\alpha \in (0, 1)$ we define the left-sided and, respectively, right-sided Riemann-Liouville fractional derivatives of order α as

$$\left({}^{RL}D_{a+} f \right)(t) = \partial_t \left(J_{a+}^{1-\alpha} f \right)(t),$$

and

$$\left({}^{RL}D_{b-} f \right)(t) = -\partial_t \left(J_{b-}^{1-\alpha} f \right)(t).$$

In the following we employ a modified test function method to prove the desired results; in order to treat the nonlocal operators we replace compactly supported test functions by suitable test functions with polynomial decay. We first give a definition of global-in-time weak solution.

Definition 5.1. Let us fix $q = n + 2 - 2\beta$; we define the space $C_q^\infty(\mathbb{R}^n)$ as the subspace of infinitely differentiable functions φ such that $\langle x \rangle^q \varphi$ is bounded and the function $\langle x \rangle^q (-\Delta)^\sigma \varphi$ is bounded for $\sigma = 1 - \beta$ and $\sigma = 1$.

The following statement guarantees that the space $C_q^\infty(\mathbb{R}^n)$ is not empty (see Corollary 3.1 in [9]).

Proposition 5.2. *Let* $f(x) = \langle x \rangle^{-\omega}$, *for* $\omega > n$, *and let* $\sigma > 0$. *We set* $s = \sigma - \lfloor \sigma \rfloor$. *Then*

$$\forall x \in \mathbb{R}^n : \quad |(-\Delta)^\sigma f(x)| \leq C \langle x \rangle^{-\omega_\sigma},$$

where $\omega_\sigma = \omega + 2\sigma$ *if* σ *is an integer, or* $\omega_\sigma = n + 2s$ *otherwise, and the constant* C *verifies the following bound from below:*

$$C = C(n, \sigma, \omega) \geq (-\Delta)^\sigma f(0) = 2^{2\sigma} \frac{\Gamma(\sigma + n/2)}{\Gamma(n/2)} \frac{\Gamma(\sigma + \omega/2)}{\Gamma(\omega/2)}.$$

Remark 5.1. The space C_q^∞ is a vector space; as a consequence of Proposition 5.2 it is not empty, being the function $\varphi(x) = \langle x \rangle^{-\omega}$ in C_q^∞ for any $\omega \geq q$. Moreover, due to $q > n$, we get the inclusion

$$C_q^\infty \subset L^\infty(\mathbb{R}^n, \langle x \rangle^q \, dx) \subset L^1.$$

Definition 5.3. We say that $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+, L^p(\mathbb{R}^n, \langle x \rangle^{-q} dx))$ is a global-in-time weak solution if for any test function $\psi \in C^1_c([0,\infty))$ and $\varphi \in C^\infty_q(\mathbb{R}^n)$, it holds

$$\int_{\mathbb{R}^n} \int_0^\infty |u(t,x)|^p \psi(t) \varphi(x) \, dt \, dx$$

$$= \int_{\mathbb{R}^n} \int_0^\infty u(t,x) \big(- \psi'(t) \varphi(x) +^{RL} D^\beta_{\infty-} \psi(t) (-\Delta)^{1-\beta} \varphi(x) - \psi(t) \Delta \varphi(x) \big) \, dt \, dx$$

$$- \psi(0) \int_{\mathbb{R}^n} u_0(x) \varphi(x) \, dx - \big(J^{1-\beta}_{\infty-} \psi \big)(0) \int_{\mathbb{R}^n} u_0(x) (-\Delta)^{1-\beta} \varphi(x) \, dx.$$

This definition of global-in-time weak solution is motivated by the following result about fractional integration by parts which allows to prove that any classical solution of problem (53) is also a weak solution in the sense of Definition 5.3.

Lemma 5.4. *(Lemma 2.7 in [18]) Let $b > 0$, $f \in L^{p_1}([0,b])$, $f \in L^{p_2}([0,b])$, and either $p_1, p_2 \geq 1$ such that $1/p_1 + 1/p_2 < 1 + \gamma$, or $p_1, p_2 > 1$ and $1/p_1 + 1/p_2 = 1 + \gamma$. Then, we have the following:*

$$\int_0^b (J^\gamma_{0+} f)(t) g(t) \, dt = \int_0^b f(t) (J^\gamma_{b-} g)(t) \, dt.$$

Proof of Theorem 4.1. Let u be a global-in-time nontrivial weak solution to (53), in the sense of Definition 5.3. We introduce $\psi \in C^1([0,\infty))$, a non-increasing function, such that $\mathrm{supp}\,\psi \subset [0,1]$ and

$$\psi(t) = \begin{cases} 1 & \text{if } t \in [0, 1/2), \\ c_0 (1-t)^{\ell+1} & \text{if } t \in [1-\varepsilon, 1), \end{cases}$$

for some $c_0 > 0$, $\ell > 1/(p-1)$ and $\varepsilon > 0$ arbitrarily small. On the other hand, we fix $\varphi \in C^\infty_q$ defined as $\varphi(x) = \langle x \rangle^{-n-2(1-\beta)}$. For any $R \geq 1$ and $\eta > 0$ we define

$$\psi_R(t) = \psi(R^{-\eta} t), \qquad \varphi_R(x) = \varphi(R^{-1} x).$$

According to Definition 5.3 we have that

$$I_R := \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p \psi_R(t) \varphi_R(x) \, dx \, dt$$

$$= \int_{\mathbb{R}^n} \int_0^\infty u(t,x) \big(- \psi'_R(t) \varphi_R(x) +^{RL} D^\beta_{\infty-} \psi_R(t) (-\Delta)^{1-\beta} \varphi_R(x) - \psi_R(t) \Delta \varphi_R(x) \big) \, dt \, dx$$

$$- \psi_R(0) \int_{\mathbb{R}^n} u_0(x) \varphi_R(x) \, dx - \big(J^{1-\beta}_{\infty-} \psi_R \big)(0) \int_{\mathbb{R}^n} u_0(x) (-\Delta)^{1-\beta} \varphi_R(x) \, dx.$$

We preliminary notice that

$$\psi'_R(t) = R^{-\eta} \psi'(R^{-\eta} t), \qquad {}^{RL} D^\beta_{\infty-} \psi_R(t) = R^{-\beta\eta} \big({}^{RL} D^\beta_{\infty-} \psi \big)(R^{-\eta} t),$$

$$(J^{1-\beta}_{\infty-} \psi_R)(t) = R^{\eta(1-\beta)} (J^{1-\beta}_{\infty-} \psi)(R^{-\eta} t),$$

and

$$(-\Delta)^{1-\beta} \varphi_R(x) = R^{-2(1-\beta)} (-\Delta)^{1-\beta} \varphi(R^{-1} x), \quad (-\Delta) \varphi_R(x) = R^{-2} (-\Delta) \varphi(R^{-1} x).$$

We set

$$k_0 := (J^{1-\beta}_{\infty-} \psi)(0) \in \mathbb{R}_+;$$

thus, we conclude

$$I_R = -R^{-\eta} \int_{\mathbb{R}^n} \int_0^\infty u(t,x) \psi'(R^{-\eta} t) \varphi(R^{-1} x) \, dt \, dx - R^{-2} \int_{\mathbb{R}^n} \int_0^\infty u(t,x) \psi(R^{-\eta} t) \Delta \varphi(R^{-1} x) \, dt \, dx$$

$$+ R^{-\beta\eta - 2(1-\beta)} \int_{\mathbb{R}^n} \int_0^\infty u(t,x) \big({}^{RL} D^\beta_{\infty-} \psi \big)(R^{-\eta} t) (-\Delta)^{1-\beta} \varphi(R^{-1} x) \, dt \, dx$$

$$- \int_{\mathbb{R}^n} u_0(x) \varphi(R^{-1} x) \, dx - k_0 R^{(\eta-2)(1-\beta)} \int_{\mathbb{R}^n} u_0(x) (-\Delta)^{1-\beta} \varphi(R^{-1} x) \, dx. \tag{54}$$

By Hölder inequality, we derive

$$R^{-\eta} \int_{\mathbb{R}^n} \int_0^\infty |u(t,x)||\psi'(R^{-\eta}t)|\varphi(R^{-1}x)\,dt\,dx$$

$$\leq C I_R^{\frac{1}{p}} R^{-\eta} \left(\int_{\mathbb{R}^n} \int_0^\infty |\psi'(R^{-\eta}t)|^{p'}\psi(R^{-\eta}t)^{-\frac{p'}{p}}\varphi(R^{-1}x)\,dt\,dx \right)^{\frac{1}{p'}}$$

$$\leq C I_R^{\frac{1}{p}} R^{(n+\eta)/p'-\eta} \leq \frac{I_R}{3p} + \frac{C}{p'} R^{n+\eta-\eta p'}, \tag{55}$$

$$R^{-2} \int_{\mathbb{R}^n} \int_0^\infty |u(t,x)|\psi(R^{-\eta}t)|\Delta\varphi(R^{-1}x)|\,dt\,dx$$

$$\leq C I_R^{\frac{1}{p}} R^{-2} \left(\int_{\mathbb{R}^n} \int_0^\infty \psi(R^{-\eta}t)|\Delta\varphi(R^{-1}x)|^{p'}\varphi(R^{-1}x)^{-\frac{p'}{p}}\,dt\,dx \right)^{\frac{1}{p'}}$$

$$\leq C I_R^{\frac{1}{p}} R^{(n+\eta)/p'-2} \leq \frac{I_R}{3p} + \frac{C}{p'} R^{n+\eta-2p'}, \tag{56}$$

and

$$R^{-\beta\eta-2(1-\beta)} \int_{\mathbb{R}^n} \int_0^\infty |u(t,x)||(^{RL}D_{\infty-}^\beta\psi)(R^{-\eta}t)||(-\Delta)^{1-\beta}\varphi(R^{-1}x)|\,dt\,dx$$

$$\leq C I_R^{\frac{1}{p}} R^{-\beta\eta-2(1-\beta)} \left(\int_{\mathbb{R}^n} \int_0^\infty \frac{|(^{RL}D_{\infty-}^\beta\psi)(R^{-\eta}t)|^{p'}|(-\Delta)^{1-\beta}\varphi(R^{-1}x)|^{p'}}{\psi(R^{-\eta}t)^{\frac{p'}{p}}\varphi(R^{-1}x)^{\frac{p'}{p}}}\,dt\,dx \right)^{\frac{1}{p'}}$$

$$\leq C I_R^{\frac{1}{p}} R^{(n+\eta)/p'-\beta\eta-2(1-\beta)} \leq \frac{I_R}{3p} + \frac{C}{p'} R^{n+\eta-(\beta\eta+2(1-\beta))p'}, \tag{57}$$

provided that

$$(\psi')\psi^{-\frac{1}{p}} \leq C, \qquad |(^{RL}D_{\infty-}^\beta\psi)|\psi^{-\frac{1}{p}} \leq C, \tag{58}$$

for some constant $C > 0$, and

$$|(-\Delta)^{1-\beta}\varphi|\varphi^{-\frac{1}{p}} \in L^{p'}, \qquad |\Delta\varphi|\varphi^{-\frac{1}{p}} \in L^{p'}, \qquad \varphi \in L^1. \tag{59}$$

Indeed, being $\operatorname{supp}(\psi)$, $\operatorname{supp}(\psi')$, and $\operatorname{supp}(^{RL}D_{\infty-}^\beta\psi)$ included in $[0,1]$, estimates (58) and (59) are sufficient to guarantee the boundedness of the integral terms in (55), (56) and (57).

Being $\psi \in C^1([0,1])$ it follows that for any $\gamma \in [0,1]$ the test function ψ' belongs to the weighted space $C_\gamma([0,1])$, i.e. $t^\gamma\psi'$ belongs to $C([0,1])$; as a consequence $^{RL}D_{1-}^\beta\psi =^C D_{1-}^\beta\psi = J_{1-}^{1-\beta}\psi'$ is continuous in $[0,1]$ (see Lemma 28(a) in [18]); then, both ψ' and $^{RL}D_{1-}^\beta\psi =^{RL} D_{\infty-}^\beta\psi$ are uniformly bounded in $[0,1]$; moreover, there exists $c_\varepsilon > 0$ such that $\psi(t) > c_\varepsilon$ uniformly in $[0, 1-\varepsilon]$; thus, condition (58) trivially holds in $[0, 1-\varepsilon]$. Else, in the interval $(1-\varepsilon, 1)$ we have

$$|\psi'(t)\psi(t)^{-\frac{1}{p}}| \lesssim (1-t)^{\ell - \frac{\ell+1}{p}},$$

and, by property 2.1 in [18],

$$\left| (^{RL}D_{\infty-}^\beta\psi)(t)\psi(t)^{-\frac{1}{p}} \right| = C(1-t)^{\ell+1-\beta-\frac{\ell+1}{p}},$$

for some constant $C > 0$ independent of $t \in [0,1]$; thus, choosing $\ell \geq 1/(p-1)$ condition (58) is satisfied also for $t \in [1-\varepsilon, 1]$, for a suitable $C > 0$. Furthermore, by Proposition 5.2 we know

$$|(-\Delta)^{1-\beta}\varphi|\varphi^{-\frac{1}{p}} \leq \varphi^{\frac{1}{p'}} = \langle x \rangle^{-\frac{n+2-2\beta}{p'}} \in L^{p'},$$

and

$$|\Delta\varphi|\varphi^{-\frac{1}{p}} \lesssim \langle x \rangle^{-\frac{n+4-2\beta}{p'}} \in L^{p'};$$

finally, by Remark 5.1 we know $\varphi \in L^1$.

Morover, if $u_0 \in L^m$ with $m \in (1,\infty)$ and condition (42) holds, for any $R > 1$ we have

$$\int_{\mathbb{R}^n} u_0(x)\varphi_R(x)\,dx \geq \int_{|x|\leq R} u_0(x)\,dx$$
$$\geq C\varepsilon_0 \int_{|x|\leq R} (1+|x|)^{-\frac{n}{m}} (\ln(e+|x|))^{-1}\,dx \geq C\varepsilon_0 R^{n\left(1-\frac{1}{m}\right)} (\ln(e+R))^{-1}, \qquad (60)$$

where the constant $C > 0$ does not depend on R; moreover, applying the Hölder inequality we can estimate

$$-R^{(\eta-2)(1-\beta)} \int_{\mathbb{R}^n} u_0(x)(-\Delta)^{1-\beta}\varphi(R^{-1}x)\,dx \leq R^{(\eta-2)(1-\beta)}\|u_0\|_{L^m} \left(\int_{\mathbb{R}^n} \langle R^{-1}x\rangle^{-(n+2(1-\beta))m'}\,dx \right)^{\frac{1}{m'}}$$
$$\leq R^{n\left(1-\frac{1}{m}\right)+(\eta-2)(1-\beta)}\|u_0\|_{L^m}, \qquad (61)$$

being m' the the conjugate exponent of m, i.e. $1/m + 1/m' = 1$.

If $p < 1 + 2m/n$ there exists $\delta > 0$ sufficiently small such that $p < 1 + (2-\delta)m/n$; let us fix $\eta = 2 - \delta$. Collecting together estimates (55), (56), (57), (60) and (61), by (54) we obtain

$$I_R \lesssim R^{n\left(1-\frac{1}{m}\right)} \ln(e+R)^{-1}\left(\left(R^{\frac{n}{m}-(2-\delta)(p'-1)} + k_0 R^{-\delta(1-\beta)}\|u_0\|_{L^m} \right) \ln(e+R) - \varepsilon_0 \right);$$

the contradiction follows taking $R \to \infty$.

If $m = 1$, taking $\eta = 2$ we get

$$I_R + \int_{\mathbb{R}^n} u_0(x)\varphi(R^{-1}x)\,dx$$
$$+ k_0 \int_{\mathbb{R}^n} u_0(x)(-\Delta)^{1-\beta}\varphi(R^{-1}x)\,dx \lesssim R^{n+2-2p'};$$

since $u_0 \in L^1$ we have

$$\lim_{R\to\infty} \int_{\mathbb{R}^n} u_0(x)\big(\varphi(R^{-1}x) + k_0(-\Delta)^{1-\beta}\varphi(R^{-1}x)\big)\,dx$$
$$= \big(1 + k_0(-\Delta)^{1-\beta}\varphi(0)\big) \int_{\mathbb{R}^n} u_0(x)\,dx = \bar{K} > 0,$$

as a consequence of assumption (43); in particular, $(-\Delta)^{1-\beta}\varphi(0) > 0$ can be explicitly evaluated (see Proposition 5.2). Thus, on the one hand, applying the monotone convergence theorem for any $p < 1 + 2/n$ we get

$$0 = \lim_{R\to\infty} R^{n+2-2p'} \gtrsim \lim_{R\to\infty} I_R + \bar{K} = \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p\,dx\,dt + \bar{K},$$

which is impossible since $\bar{K} > 0$.

On the other hand, for $p = p_c$ we get

$$\lim_{R\to\infty} I_R = \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^{p_c}\,dx\,dt \leq C,$$

for some constant $C > 0$ independent of R, that is $u \in L^{p_c}([0,\infty)\times\mathbb{R}^n)$.

Now, we repeat the same reasoning used in subcritical case, fixing $\eta = 2$ and replacing the test function φ_R by the test function $\varphi_{RK} = \langle R^{-1}K^{-1}x\rangle^{-n-2(1-\beta)}$, for a given constant $K \gg 1$. Being supp$\,(\psi') \subset [1/2,1]$ we find

$$I_R + \int_{\mathbb{R}^n} u_0(x)\varphi(R^{-1}K^{-1}x)\,dx$$
$$+ k_0 \int_{\mathbb{R}^n} u_0(x)(-\Delta)^{1-\beta}\varphi(R^{-1}K^{-1}x)\,dx$$
$$\lesssim \tilde{I}_R^{\frac{1}{p_c}} K^{\frac{2n}{n+2}} + I_R^{\frac{1}{p_c}} K^{-\frac{4}{n+2}} + I_R^{\frac{1}{p_c}} K^{-\frac{4}{n+2}+2\beta},$$

where

$$\tilde{I}_R := \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p \tilde{\psi}_R(t) \varphi_R(x)\, dx\, dt, \quad \tilde{\psi}_R(t) = \begin{cases} 0 & \text{if } t \in [0, R/2), \\ \psi_R(t) & \text{otherwise.} \end{cases}$$

In particular, being $u \in L^p([0,\infty) \times \mathbb{R}^n)$ it holds $\tilde{I}_R \to 0$ as $R \to \infty$. If both R and K tend to infinity we get

$$0 < \|u\|_{L^p([0,\infty) \times \mathbb{R}^n)} + \left(1 + k_0(-\Delta)^{1-\beta}\varphi(0)\right) \int_{R^n} u_0(x)\, dx < \|u\|_{L^p([0,\infty) \times \mathbb{R}^n)} \lim_{K \to \infty} \left(K^{-\frac{4}{n+2}} + K^{-\frac{4}{n+2}+2\beta}\right).$$

If $\beta < 2/(n+2)$ we get a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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