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QUANTIZATION OF THE FREE POISSON CENTRAL LIMIT THEOREM

YUNGANG LU*

Abstract. This paper is devoted to quantization of the free Poisson central limit theorem. We construct a sequence of free independent binomial random variables on an interacting Fock space in terms of creation–annihilation operators. By using these random variables, one studies a quantization of the free Poisson central limit theorem with respect to the convergence both in mixed–moments and in law.

1. Introduction

In the present paper, as a continuation of [6] and [7], we set up a **quantization** of the free Poisson central limit theorem (CLT in short). Its monotone analogue will be dealt with in [8].

The free Poisson CLT (see [12] and references within) can be formulated practically as follows: *Let* $\{p_n\}_{n=1}^{\infty} \subset [0,1]$ *be such sequence that* $np_n \to \lambda$*, then, in the weak convergence*

$$\lim_{n\to\infty} \left(\left(1-p_n\right)\delta_0 + p_n\delta_1\right)^{\star n} = P_f\left(\lambda\right) \tag{1.1}$$

where, "$\star$" *is the free convolution and* $P_f(\lambda)$ *is the free Poisson distribution with the parameter* λ *(see* [9]*,* [10]*,* [11] *and* [12] *for the detail), in particular,* $P_f(0) := \delta_0$. Throughout, for any $x \in \mathbb{R}$, δ_x :=the Dirac measure centred on x.

One recalls that the free convolution is usually defined by means of the Cauchy transform: for any $\mu, \nu \in \mathcal{P} := \{\text{probability measure on } (\mathbb{R}, \mathcal{B})\}$ with the Cauchy transforms G_μ and G_ν respectively, the free convolution $\mu \star \nu$ is the element of $\mathcal{P}$ with the Cauchy transform $G_\mu + G_\nu$. Moreover, the properties of the Cauchy transform and the characterization of the weak convergence of a sequence of $\mathcal{P}$ in terms of the Cauchy transform have already been studied systematically (see, e.g. [3], [5], [10] and references within).

Clearly, the free Poisson CLT (1.1) can be reformulated in terms of algebraic random variables and the free independence as follows: *Let* $(\mathcal{X}, \psi)$ *be an algebraic probability space and* $\{\xi_{n,k} : n \in \mathbb{N}^* \text{ and } k \le n\}$ *be a family of algebraic random variables, let* $\{p_n\}_{n=1}^{\infty} \subset [0,1]$*. If*

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- $\psi\left(\xi_{n,k}^m\right) = \psi\left(\xi_{n,k}\right) = p_n$ *for any* $m, n \in \mathbb{N}^*$ *and* $k \le n$ *(therefore, the* $\psi-$*distribution of* $\xi_{n,k}$ *is* $(1-p_n)\,\delta_0 + p_n\delta_1$*, i.e. the binomial distribution with the parameter* $(1, p_n)$*, one writes this fact simply as* $\xi_{n,k} \overset{\psi}{\sim} b\,(1, p_n)$*);*
- *for any* $n \ge 2$*,* $\{\xi_{n,1}, \dots, \xi_{n,n}\}$ *is a free independent family with respect to* ψ*,*

the $\psi-$ *distribution of* $\sum_{k=1}^n \xi_{n,k}$ *goes to* $P_f\,(\lambda)$*, whenever* $np_n \longrightarrow \lambda$*.*

For any $n \in \mathbb{N}^*$, since $\xi_{n,k} \overset{\psi}{\sim} b(1, p_n)$ for any $k \le n$ and since $\{\xi_{n,1}, \dots, \xi_{n,n}\}$ is a free independent family with respect to ψ, $\sum_{k=1}^n \xi_{n,k}$ is indeed a sum of n (free) independent random variables and each of them has the binomial distribution with the parameter $(1, p_n)$. So, one calls naturally its distribution the *free binomial distribution with the parameter* (n, p_n).

Notice that this reformulation of the free Poisson CLT does **not** depend on the specific construction of the algebraic probability space and random variables. In other words, if we take such a **particular** $(\mathcal{A}, \phi)$ and $\{X_{n,k} : n \in \mathbb{N}^* \text{ and } k \le n\}$ that

- $X_{n,k} \overset{\phi}{\sim} b\,(1, p_n)$ for any $n \in \mathbb{N}^*$ and $k \le n$,
- $\{X_{n,1}, \dots, X_{n,n}\}$ is a free independent family with respect to ϕ for any $n \ge 2$,

then the $\phi-$distribution of $\sum_{k=1}^n X_{n,k}$ is $((1-p_n)\,\delta_0 + p_n\delta_1)^{\star n}$ and the usual free Poisson CLT tells us that it goes to $P_f\,(\lambda)$ whenever $np_n \to \lambda$.

In this paper, we start from a given Hilbert space $\mathcal{H}$ with an orthogonal normal base (onb in short) $\{e_k\}_{k=1}^\infty$ and introduce a particular interacting Fock space (IFS in short) $\Gamma_{sf}\,(\mathcal{H})$ as a modification the usual full (free) Fock space over $\mathcal{H}$ (for the detail, see Section 2). Then we defined, on the IFS $\Gamma_{sf}\,(\mathcal{H})$, the creation (resp. annihilation) operator a_f^+ (resp. a_f) with the test function $f \in \mathcal{H}$. By denoting, for any $k \in \mathbb{N}^*$, $a_k^+ := a_{e_k}^+$ (resp. $a_k := a_{e_k}$) and

$$a_k^{(\varepsilon)} := \begin{cases} a_k, & \text{if } \varepsilon = -1 \\ a_k^+, & \text{if } \varepsilon = 1 \\ a_k a_k^+, & \text{if } \varepsilon = 0 \\ a_k^+ a_k, & \text{if } \varepsilon = 2 \end{cases} \tag{1.2}$$

we will take the following particular algebraic probability space $(\mathcal{A}, \phi)$ and algebraic random variables $\{X_{n,k} : n \in \mathbb{N}^* \text{ and } k \le n\}$: Let

- $\mathcal{A}$ be the algebra generated by $\mathcal{A}_k$'s and
$$\mathcal{A}_k := \left\{\text{polynomial in } a_k \text{ and } a_k^+ \text{ with degree} \ge 1\right\}, \quad \forall k \tag{1.3}$$
and any monomial in a_k and a_k^+ with degree ≥ 1 will be called a **word** of the algebra $\mathcal{A}_k$;
- ϕ :=the vacuum state, i.e., $\phi\,(\cdot) := \langle \Psi, \cdot\Psi \rangle$ with Ψ :=the vacuum vector of $\Gamma_{sf}\,(\mathcal{H})$;
- for any $n \in \mathbb{N}^*$, $k \le n$ and for any given $\{p_n\}_{n=1}^\infty \subset [0,1]$,

$$X_{n,k} := \sqrt{p_n\,(1-p_n)}\,a_k^{(-1)} + \sqrt{p_n\,(1-p_n)}\,a_k^{(+1)} + p_n a_k^{(0)} + (1-p_n)\,a_k^{(2)} \tag{1.4}$$

Remark 1.1. We will say that, $\sqrt{p_n(1-p_n)}a_k^{(-1)}$ and $\sqrt{p_n(1-p_n)}a_k^{(+1)}$ are **off–diagonal components** of $X_{n,k}$ since they are conjugate each other; $p_n a_k^{(0)}$ and $(1-p_n)\,a_k^{(2)}$ are **diagonal components** of $X_{n,k}$ since they are self–adjoint.

In Section 2, we give concrete construction of the IFS $\Gamma_{sf}(\mathcal{H})$ and show

- the algebras $\mathcal{A}_k$'s defined in (1.3) are free independent;
- any $X_{n,k}$ is a projector and $\phi(X_{n,k})=p_n$; consequently $X_{n,k}\overset{\phi}{\sim} b(1,p_n)$.

Thus, we introduce, for any $n\in\mathbb{N}^*$

$$B_n^{(\pm1)}:=\sqrt{p_n(1-p_n)}\sum_{k=1}^n a_k^{(\pm1)};\ B_n^{(0)}:=p_n\sum_{k=1}^n a_k^{(0)};\ B_n^{(2)}:=(1-p_n)\sum_{k=1}^n a_k^{(2)}$$

$$B_n:=B_n^{(-1)}+B_n^{(+1)}+B_n^{(0)}+B_n^{(2)}=\sum_{k=1}^n X_{n,k} \tag{1.5}$$

and moreover, for any $\{c_0,c_1,c_2\}\subset\mathbb{R}$,

$$\begin{aligned}&D_F(n,p_n;c_0,c_1,c_2)\\ :=&\phi-\text{distribution of } c_1\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0B_n^{(0)}+c_2B_n^{(2)}\end{aligned} \tag{1.6}$$

As said above, the usual free Poisson CLT confirms that the ϕ−distribution of B_n (i.e., $D_F(n,p_n;1,1,1)$) goes to $P_f(\lambda)$ whenever $np_n\to\lambda$. Contrast with this, the *quantized* Poisson type CLT is:

1) to calculate, for any $\{c_0,c_1,c_2\}\subset\mathbb{R}$, the limit

$$\lim_{N\to\infty}\phi\left(\exp\left(it\left(c_1\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0B_n^{(0)}+c_2B_n^{(2)}\right)\right)\right),\qquad\forall t\in\mathbb{R} \tag{1.7}$$

i.e. to know the weak limit of $D_F(n,p_n;c_0,c_1,c_2)$ by means of the characteristic function;

2) to study the moment version of above limit, i.e., to see the limit

$$\lim_{n\to\infty}\phi\left(B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m))}\right) \tag{1.8}$$

for any $m\in\mathbb{N}$ and $\varepsilon\in\{-1,0,1,2\}^m$;

3) to give a suitable representation to the above limits.

As a consequence of the above 1) and 2), the quantized Poisson CLT makes in evidence the *individual* contributions of each $B_n^{(\varepsilon)}$ in the CLT's procedure, not only their sum. In particular, by noticing that

$$B_n^{(\pm1)}:=\sqrt{p_n(1-p_n)}\sum_{k=1}^n a_k^{(\pm1)}\overset{np_n\to\lambda}{\approx}\sqrt{\lambda}\frac{1}{\sqrt{n}}\sum_{k=1}^n a_k^{(\pm1)}$$

and by taking $c_0=c_2=0$ and $c_1=1$ in the above 1), one gets the free Laplace–de Moivre CLT. Therefore, the quantization of the free Poisson CLT gives a view to understand the relationship between the free Poisson CLT and corresponding Laplace–de Moivre CLT: *the free Laplace–de Moivre CLT is the off–diagonal part of the free Poisson CLT*. Moreover, by using the *representation* mentioned in the above 3), one can also understand the relationship between the free Poisson distribution and the free Gaussian distribution, namely, the Wigner (e.g. semi–circle) distribution.

In Section 3, we

- show that the limit (1.7) equals to
$$\left\langle \Phi, \exp\left(it\left(c_1\sqrt{\lambda}\left(b^{(-1)}+b^{(+1)}\right)+\lambda c_0 b^{(0)}+c_2 b^{(2)}\right)\right)\Phi\right\rangle \tag{1.9}$$
in other words, the vacuum distribution of $\left\{c_1\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0B_n^{(0)}+c_2B_n^{(2)}\right\}_{n=1}^{\infty}$ (i.e. $D_n\left(n,p_n;c_0,c_1,c_2\right)$) goes to the vacuum distribution of $c_1\sqrt{\lambda}\left(b^{(-1)}+b^{(+1)}\right)+\lambda c_0 b^{(0)}+c_2 b^{(2)}$ in the weak convergence, where and throughout the present paper,
$$b^{(\varepsilon)} := \begin{cases} b, & \text{if } \varepsilon=-1 \\ b^+, & \text{if } \varepsilon=1 \\ \mathbf{1}, & \text{if } \varepsilon=0 \\ \mathbf{1}-P_\Phi, & \text{if } \varepsilon=2 \end{cases} \tag{1.10}$$
b^+ and b are the creation–annihilation operators on the 1–mode–interacting Fock space (1M–IFS in short) $\Gamma_{fr}(\mathbb{C}) := \Gamma\left(\mathbb{C},\{\omega_n\}_n\right)$ with $\omega_n=1$ for any $n\in\mathbb{N}^*$, Φ is the vacuum vector and P_Φ is the projector from $\Gamma_{fr}(\mathbb{C})$ to its vacuum subspace;
- prove the existence of the limit (1.8) and show that it has the form
$$\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(n+1))}\Phi\right\rangle \lambda^{\sum_{k=1}^{n+1}(1-|\varepsilon(k)|/2)} \tag{1.11}$$

Remark 1.2. On the 1M–IFS $\Gamma_{fr}(\mathbb{C})$, one has $bb^+=\mathbf{1}$ and $b^+b=\mathbf{1}-P_\Phi$.

2. Definition and Some Elementary Properties of $\Gamma_{sf}(\mathcal{H})$

The following notations will be used frequently
$$\begin{aligned} \overline{\mathbb{F}_n} &:= \{\text{functions from } \{1,\dots,n\} \text{ to } \mathbb{N}\} \\ \mathbb{F}_n &:= \left\{\mathbf{k}\in\overline{\mathbb{F}_n} : \mathbf{k}(i)\neq\mathbf{k}(i+1) \text{ for any } 1\le i<n\right\} \\ \mathbb{F}_n^0 &:= \left\{\mathbf{k}\in\mathbb{F}_n : \mathbf{k}(i)\neq\mathbf{k}(j) \text{ for any } 1\le i\neq j\le n\right\} \end{aligned} \tag{2.1}$$

Let $\mathcal{H}$ be a Hilbert space with an onb $\{e_k\}_{k=1}^{\infty}$, one defines $\Gamma_{sf}(\mathcal{H})$ as $\bigoplus_{n=0}^{\infty}\mathcal{H}_n$, where $\mathcal{H}_0:=\mathbb{C}$, $\mathcal{H}_1:=\mathcal{H}$ and for any $n\ge 2$, $\mathcal{H}_n$ is the (pre–)Hilbert space obtained by equipping the usual the tensor scalar product $\langle\cdot,\cdot\rangle_n$ on the vector space
$$lin.sp.\left\{e_{\mathbf{k}(n)}\otimes\dots\otimes e_{\mathbf{k}(1)} : \mathbf{k}\in\mathbb{F}_n^0\right\} \tag{2.2}$$
i.e.,
$$\left\langle e_{\mathbf{h}(n)}\otimes\dots\otimes e_{\mathbf{h}(1)}, e_{\mathbf{k}(n)}\otimes\dots\otimes e_{\mathbf{k}(1)}\right\rangle_n := \prod_{1\le j\le n}\delta_{\mathbf{h}(j),\mathbf{k}(j)},\quad \forall \mathbf{k},\mathbf{h}\in\mathbb{F}_n^0 \tag{2.3}$$

As usual, one calls $\Psi := 1\oplus 0\oplus 0\oplus ...$ and $\mathcal{H}_n$ the vacuum vector and the $n-$particle space respectively.

$\Gamma_{sf}(\mathcal{H})$ is obviously a subspace of the usual full (free) Fock space over $\mathcal{H}$ which is $\bigoplus_{n=0}^{\infty}\mathcal{H}^{\otimes n}$. Contrast with $\mathcal{H}_n$, $\mathcal{H}^{\otimes n}$ is obtained by introducing the same scalar product (2.3) on the vector space

$$lin.sp.\left\{e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)} : \mathbf{k}\in\overline{\mathbb{F}_n}\right\}$$

On the other hand, $\Gamma_{sf}(\mathcal{H})$ can be viewed as a particular IFS over $\mathcal{H}$: for any $n\geq 2$, one defines $\Omega_n:\mathcal{H}^{\otimes n}\longmapsto\mathcal{H}^{\otimes n}$ by the linearity and

$$\begin{aligned}&\Omega_n\left(e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)}\right)\\ :=&\prod_{1\leq j<r\leq n}\left(1-\delta_{\mathbf{k}(j),\mathbf{k}(r)}\right)\left(e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)}\right),\ \forall\mathbf{k}\in\overline{\mathbb{F}_n}\end{aligned}\tag{2.4}$$

It is easy to see that Ω_n is a projector and moreover, the following $(\cdot,\cdot)_n$

$$(x,y)_n := \langle x,\Omega_n y\rangle\,,\qquad\forall x,y\in\mathcal{H}^{\otimes n}$$

defines a positive quadratic form and our $\mathcal{H}_n$ (more precisely, $(\mathcal{H}_n,\langle\cdot,\cdot\rangle_n)$) is isomorphic to $(\mathcal{H}^{\otimes n},(\cdot,\cdot)_n)/\ker((\cdot,\cdot)_n)$. Clearly,

- for any $n\in\mathbb{N}^*$, $\left\{e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)} : \mathbf{k}\in\mathbb{F}_n^0\right\}$ is an onb of $\mathcal{H}_n$;
- a general element of $\Gamma_{sf}(\mathcal{H})$ has the form $\sum_{n\in I}x_n$ with $I\subset\mathbb{N}$ finite and $x_n\in\mathcal{H}_n$ for any $n\in I$.

For any $k\in\mathbb{N}^*$, we introduce the linear operator (called the creation operator with the test function e_k) a_k^+ on $\Gamma_{sf}(\mathcal{H})$ by linearity and

$$\begin{aligned}&a_k^+\Psi := e_k\\ &a_k^+ e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)} := \begin{cases}e_k\otimes e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)}, & \text{if } k\notin range(\mathbf{k})\\ 0, & \text{if } k\in range(\mathbf{k})\end{cases}\\ =&\prod_{j=1}^{n}\left(1-\delta_{k,\mathbf{k}(j)}\right)e_k\otimes e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)},\qquad\forall n\geq 1,\mathbf{k}\in\mathbb{F}_n^0\end{aligned}\tag{2.5}$$

Proposition 2.1. *For any $n\in\mathbb{N}$ and $k\in\mathbb{N}^*$, one has the following affirmations.*

1) $\left\|a_k^+|_{\mathcal{H}_n}\right\| = 1$, $\left\|a_k^+\right\| = 1$ *and* $\left(a_k^+\right)^2 = 0$.

2) a_k *:=the conjugate of* a_k^+*, is called the annihilation operator with the test function* e_k *and verifies*

$$\begin{aligned}&a_k\Psi = 0,\\ &a_k\left(e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)}\right) = \delta_{k,\mathbf{k}(n)}e_{\mathbf{k}(n-1)}\otimes\ldots\otimes e_{\mathbf{k}(1)},\quad\forall\,\mathbf{k}\in\mathbb{F}_n^0\end{aligned}\tag{2.6}$$

Moreover, $\left\|a_k|_{\mathcal{H}_n}\right\| = 1$ *for any* $n\in\mathbb{N}^*$, $\|a_k\| = 1$ *and* $(a_k)^2 = 0$.

3) By denoting,

$$\begin{aligned}&\mathcal{H}_{n,k}^{(s)} := lin.sp.\left\{e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)} : \mathbf{k}\in\mathbb{F}_n^0 \text{ with } \mathbf{k}(n) = k\right\}\\ &\mathcal{H}_{n,k}^{(d)} := lin.sp.\left\{e_{\mathbf{k}(n)}\otimes\ldots\otimes e_{\mathbf{k}(1)} : \mathbf{k}\in\mathbb{F}_n^0 \text{ with } k\notin range(\mathbf{k})\right\}\end{aligned}\tag{2.7}$$

then $\mathcal{H}_{n,k}^{(s)}$ *and* $\mathcal{H}_{n,k}^{(d)}$ *are closed,*

$$\left(\mathcal{H}_{n,k}^{(s)}\right)^{\perp} = lin.sp.\left\{e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \mathbb{F}_n^0 \text{ with } \mathbf{k}(n) \neq k\right\}$$
$$\left(\mathcal{H}_{n,k}^{(d)}\right)^{\perp} = lin.sp.\left\{e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \mathbb{F}_n^0 \text{ with } k \in range(\mathbf{k})\right\} \tag{2.8}$$

and

$$\mathcal{H}_n = \mathcal{H}_{n,k}^{(s)} \oplus \left(\mathcal{H}_{n,k}^{(s)}\right)^{\perp} = \mathcal{H}_{n,k}^{(d)} \oplus \left(\mathcal{H}_{n,k}^{(d)}\right)^{\perp}$$

Proof. All affirmations can be checked easily and we omit their detail. □

Proposition 2.2. *1) For any $k \in \mathbb{N}^*$,*

$$a_k^+ a_k = P_k^{(s)} := \bigoplus_{n=1}^{\infty} (\text{the projector to } \mathcal{H}_{n,k}^{(s)})$$
$$a_k a_k^+ = P_k^{(d)} := (\text{the projector to } \mathcal{H}_0) \oplus \bigoplus_{n=1}^{\infty} (\text{the projector to } \mathcal{H}_{n,k}^{(d)}) \tag{2.9}$$

and

$$a_k^+ P_k^{(d)} = P_k^{(s)} a_k^+; \quad a_k P_k^{(d)} = P_k^{(s)} a_k = 0 \tag{2.10}$$

For any different $h, k \in \mathbb{N}^*$,

$$a_k a_h^+ = 0, \quad P_k^{(s)} P_h^{(s)} = P_h^{(s)} P_k^{(s)} = 0 \tag{2.11}$$

and

$$\begin{aligned} & a_h^+ a_k \left(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) \\ =& \delta_{k,\mathbf{k}(n)} \prod_{j=1}^{n-1} \left(1 - \delta_{h,\mathbf{k}(j)}\right) \left(e_h \otimes e_{\mathbf{k}(n-1)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right), \ \forall n \geq 1, \ \mathbf{k} \in \mathbb{F}_n^0 \end{aligned} \tag{2.12}$$

2) For any $n \in \mathbb{N}^*$,

$$\Big\| \sum_{k=1}^{n} a_k^+ a_k \Big\| = 1, \quad \Big\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k^{(\pm 1)} \Big\| \leq 1, \quad \Big\| \frac{1}{n} \sum_{k=1}^{n} a_k a_k^+ \Big\| \leq 1$$

and so $\left\{B_n^{(-1)}, B_n^{(+1)}, B_n^{(0)}, B_n^{(2)} : n \in \mathbb{N}^*\right\}$ is a uniform bounded family whenever there exists the limit $\lim_{n\to\infty} np_n$.

Proof. Let $J \subset \mathbb{N}$ be finite, let $n_r \in \mathbb{N}^*$, $\alpha_r \in \mathbb{C}$ and $\mathbf{k}_r \in \mathbb{F}_{n_r}^0$ for any $r \in J$, by writing $\sum_{r\in J} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)}$ to

$$\Big(\sum_{r\in J:\mathbf{k}_r(n_r)=k} + \sum_{r\in J:\mathbf{k}_r(n_r)\neq k} \Big) \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)} \tag{2.13}$$

(2.5) and (2.6) tell us that

$$a_k^+ a_k \sum_{r\in J:\mathbf{k}_r(n_r)=k} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)} = \sum_{r\in J:\mathbf{k}_r(n_r)=k} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)}$$

and

$$a_k^+ a_k \Psi = 0 = a_k^+ a_k \sum_{r\in J:\mathbf{k}_r(n_r)\neq k} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)}$$

therefore, $a_k^+ a_k = P_k^{(s)}$.

Similarly, by writing $\sum_{r\in J} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)}$ to

$$\Big(\sum_{r\in J:k\in range(\mathbf{k}_r)} + \sum_{r\in J:k\notin range(\mathbf{k}_r)}\Big)\alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)} \tag{2.14}$$

(2.5) and (2.6) say that

$$a_k a_k^+ \sum_{r\in J:k\in range(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)} = 0$$

and

$$\begin{aligned} &a_k a_k^+ \Psi = \Psi \\ &a_k a_k^+ \sum_{r\in J:k\notin range(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)} \\ =& \sum_{r\in J:k\notin range(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)} \end{aligned}$$

therefore, $a_k a_k^+ = P_k^{(d)}$.

Clearly, in order to obtain (2.10), one needs only to prove the first equality and others are trivial: in fact, $a_k P_k^{(d)} = a_k a_k a_k^+ = 0$, $P_k^{(s)} a_k = a_k^+ a_k a_k = 0$.

For any $k \in \mathbb{N}^*$, (2.9) and the definition of creation operator tell us that

$$\begin{aligned} &P_k^{(d)} \sum_{r\in J:k\in range(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)} \\ =&0 = a_k^+ \sum_{r\in J:k\in range(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)} \end{aligned}$$

So both $a_k^+ P_k^{(d)}$ and $P_k^{(s)} a_k^+$ bring the vector in the form of

$$\sum_{r\in J:k\in range(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)}$$

to zero. Moreover, it follows just from the definitions of a_k^+, $P_k^{(d)}$ and $P_k^{(s)}$ that

$$\begin{aligned} &a_k^+ P_k^{(d)} \Big(\alpha\Psi + \sum_{r:k\notin range(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)}\Big) \\ =&\alpha e_k + \sum_{r:k\notin range(\mathbf{k}_r)} \alpha_r e_k \otimes e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)} \end{aligned}$$

and

$$\begin{aligned}
&P_k^{(s)} a_k^+ \Big(\alpha\Psi + \sum_{r:k\notin range(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)}\Big) \\
=&P_k^{(s)} \Big(\alpha e_k + \sum_{r:k\notin range(\mathbf{k}_r)} \alpha_r e_k \otimes e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)}\Big) \\
=&\alpha e_k + \sum_{r:k\notin range(\mathbf{k}_r)} \alpha_r e_k \otimes e_{\mathbf{k}_r(n_r)} \otimes \ldots \otimes e_{\mathbf{k}_r(1)}
\end{aligned}$$

So one gets the first equality of (2.10).

In the case of $h \neq k$, the equalities in (2.11) are obtained just from (2.5), (2.6) and the fact $\mathcal{H}_{n,k}^{(s)} \bigcap \mathcal{H}_{n,h}^{(s)} = \emptyset$. Moreover, (2.12) is proved simply as follows:

$$\begin{aligned}
a_h^+ a_k \left(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) &= \delta_{k,\mathbf{k}(n)} a_h^+ \left(e_{\mathbf{k}(n-1)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) \\
&= \delta_{k,\mathbf{k}(n)} \prod_{j=1}^{n-1} \left(1 - \delta_{h,\mathbf{k}(j)}\right) \left(e_h \otimes e_{\mathbf{k}(n-1)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right)
\end{aligned}$$

Summing up, the affirmation 1) is proved.

The first equality of (2.9) says that $a_k^+ a_k$ is a non–zero projector for any k; then (2.11) tells us that $\sum_{k=1}^n a_k^+ a_k$ is a non–zero projector and so $\left\|\sum_{k=1}^n a_k^+ a_k\right\| = 1$. The second equality of (2.9) guarantees that $\left\|a_k a_k^+\right\| = 1$ for any k. Consequently

$$\Big\|\frac{1}{n}\sum_{k=1}^n a_k a_k^+\Big\| \leq \frac{1}{n}\sum_{k=1}^n \left\|a_k a_k^+\right\| = 1$$

Therefore, the first equality of (2.11) makes sure that

$$\Big(\sum_{k=1}^n a_k^+\Big)^* \Big(\sum_{k=1}^n a_k^+\Big) = \sum_{k,h=1}^n a_k a_h^+ = \sum_{k=1}^n a_k a_k^+$$

and so

$$\begin{aligned}
&\Big\|\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^n a_k^+\Big)^*\Big\|^2 = \Big\|\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^n a_k^+\Big)\Big\|^2 \\
=&\Big\|\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^n a_k^+\Big)^* \Big(\frac{1}{\sqrt{n}}\sum_{k=1}^n a_k^+\Big)\Big\| = \Big\|\frac{1}{n}\sum_{k=1}^n a_k a_k^+\Big\|^2 \leq 1
\end{aligned}$$ □

Remark 2.3. One has clearly the conjugate formulation of (2.10):

$$P_k^{(d)} a_k = a_k P_k^{(s)}, \qquad P_k^{(d)} a_k^+ = a_k^+ P_k^{(s)} = 0$$

Proposition 2.4. *1) For any* k,

$$\mathcal{A}_k = lin.sp.\left\{a_k^+, a_k, P_k^{(d)}, a_k^+ P_k^{(d)}, P_k^{(s)}, a_k P_k^{(s)}\right\} \tag{2.15}$$

i.e., any word of $\mathcal{A}_k$ *is either the following:* a_k^+, $a_k, P_k^{(d)}$, $a_k^+ P_k^{(d)}$, $P_k^{(s)}$, $a_k P_k^{(s)}$.

2) For any k *and* x_k *being a word of* $\mathcal{A}_k$,

$$x_k\Psi = \begin{cases} e_k, & \text{if } x_k \in \left\{a_k^+,\ a_k^+ P_k^{(d)}\right\} \\ \Psi, & \text{if } x_k = P_k^{(d)} \\ 0, & \text{if } x_k \in \left\{a_k, P_k^{(s)}, a_k P_k^{(s)}\right\} \end{cases} \tag{2.16}$$

and for any $n \in \mathbb{N}^*$ *and* $\mathbf{k} \in \mathbb{F}_n^0$

$$\begin{aligned} & x_k\left(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) \\ &= \begin{cases} \prod_{h=1}^n \left(1-\delta_{k,\mathbf{k}(h)}\right) e_k \otimes e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}, & \text{if } x_k \in \left\{a_k^+,\ a_k^+ P_k^{(d)}\right\} \\ \prod_{h=1}^n \left(1-\delta_{k,\mathbf{k}(h)}\right) e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}, & \text{if } x_k = P_k^{(d)} \\ \delta_{k,\mathbf{k}(n)} e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}, & \text{if } x_k = P_k^{(s)} \\ \delta_{k,\mathbf{k}(n)} e_{\mathbf{k}(n-1)} \otimes \ldots \otimes e_{\mathbf{k}(1)}, & \text{if } x_k \in \left\{a_k, a_k P_k^{(s)}\right\} \end{cases} \end{aligned} \tag{2.17}$$

hereinafter,

$$e_{\mathbf{k}(r)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\Big|_{r=0} := \Psi \tag{2.18}$$

3) For any $k,\, n \in \mathbb{N}^*$, $\mathbf{k} \in \mathbb{F}_n$ *(not necessarily belongs to* $\mathbb{F}_n^0$*) and* $j \in \{1, \ldots, n\}$, *for any* $x_j \in \left\{a_{\mathbf{k}(j)}^+, a_{\mathbf{k}(j)}, a_{\mathbf{k}(j)}^+ P_{\mathbf{k}(j)}^{(d)}, P_{\mathbf{k}(j)}^{(s)}, a_{\mathbf{k}(j)} P_{\mathbf{k}(j)}^{(s)}\right\}$ *(notice that* x_j *is NOT be taken as* $P_{\mathbf{k}(j)}^{(d)}$*),*

$$\begin{aligned} & x_n \ldots x_1 \Psi \\ &= \begin{cases} e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}, & \text{if } \mathbf{k} \in \mathbb{F}_n^0 \text{ and } x_j \in \left\{a_{\mathbf{k}(j)}^+, a_{\mathbf{k}(j)}^+ P_{\mathbf{k}(j)}^{(d)}\right\},\ \forall j \in \{1, \ldots, n\} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2.19}$$

Proof. Just by applying definitions of operators $a_k^+, a_k,\ P_k^{(d)}$ and $P_k^{(s)}$, one gets easily (2.16) and (2.17) for $\mathbf{k} \in \mathbb{F}_n^0$.

The affirmation 3) is a consequence of the affirmation 2), more precisely, is the consequence of the fact:

$$x_k\left(e_{\mathbf{k}(m)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) = \prod_{h=1}^m \left(1-\delta_{k,\mathbf{k}(h)}\right) e_k \otimes e_{\mathbf{k}(m)} \otimes \ldots \otimes e_{\mathbf{k}(1)}$$

for any $m \in \mathbb{N}$, $\mathbf{k} \in \mathbb{F}_m^0$ and $x_k \in \left\{a_k^+, a_k^+ P_k^{(d)}\right\}$. Now we turn to show the affirmation 1).

By the definition,

$$\mathcal{A}_k = lin.sp.\left\{a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(n)} : n \in \mathbb{N}^*, \varepsilon \in \{-1,1\}^n\right\}$$

where, one prefers to write $\left\{a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(n)} : n \in \mathbb{N}^*, \varepsilon \in \{-1,1\}^n\right\}$ to

$$\begin{aligned} & \left\{a_k, a_k^+\right\} \cup \left\{a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(2m)} : m \in \mathbb{N}^*, \varepsilon \in \{-1,1\}^{2m}\right\} \\ & \cup \left\{a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(2m+1)} : m \in \mathbb{N}^*, \varepsilon \in \{-1,1\}^{2m+1}\right\} \end{aligned}$$

It follows from the fact $a_k^{+2} = a_k^2 = 0$ (see Proposition 2.1) that a general word of $\mathcal{A}_k$, i.e., $a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(n)}$ with $n \in \mathbb{N}^*$ and $\varepsilon \in \{-1,1\}^n$, differs from zero only if ε is *consecutively different*: $\varepsilon(1) \neq \varepsilon(2) \neq \ldots \neq \varepsilon(n-1) \neq \varepsilon(n)$. Moreover

- if $n = 2m$ and $1 = \varepsilon(1) \neq \varepsilon(2) \neq \ldots \neq \varepsilon(n-1) \neq \varepsilon(n)$, one has
$$a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(n)} = a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(2m)} = \left(a_k^+ a_k\right)^m = P_k^{(s)}$$
- if $n = 2m$ and $-1 = \varepsilon(1) \neq \varepsilon(2) \neq \ldots \neq \varepsilon(n-1) \neq \varepsilon(n)$, one has
$$a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(n)} = a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(2m)} = \left(a_k a_k^+\right)^m = P_k^{(d)}$$
- if $n = 2m+1$ and $1 = \varepsilon(1) \neq \varepsilon(2) \neq \ldots \neq \varepsilon(n-1) \neq \varepsilon(n)$, one has
$$a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(n)} = a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(2m+1)} = a_k^+ \left(a_k a_k^+\right)^m = a_k^+ P_k^{(d)}$$
- if $n = 2m+1$ and $-1 = \varepsilon(1) \neq \varepsilon(2) \neq \ldots \neq \varepsilon(n-1) \neq \varepsilon(n)$, one has
$$a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(n)} = a_k^{\varepsilon(1)} \ldots a_k^{\varepsilon(2m+1)} = a_k \left(a_k^+ a_k\right)^m = a_k P_k^{(s)}$$

Summing up, the thesis is obtained. □

Remark 2.5. Among all words of $\mathcal{A}_k$, $P_k^{(d)}$ has a particular feature: its vacuum expectation equals to 1 while anyone of other words has zero vacuum expectation.

For any $k, n \in \mathbb{N}^*$ and $\mathbf{k} \in \mathbb{F}_n^0$, we have the following easily proved result:
$$1 - \prod_{h=1}^{n} \left(1 - \delta_{k,\mathbf{k}(h)}\right) = \sum_{j=1}^{n} \delta_{k,\mathbf{k}(j)} = \chi_{range(\mathbf{k})}(k) \tag{2.20}$$
where, $\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$ for any set E.

In fact, for any $\mathbf{k} \in \mathbb{F}_n^0$, $\{\mathbf{k}^{-1}(\{j\})\}$'s are 2–2 disjoint and so one gets the second equality of (2.20):
$$\chi_{range(\mathbf{k})}(k) = \chi_{\cup_{j=1}^n \mathbf{k}^{-1}(\{j\})}(k) = \sum_{j=1}^{n} \chi_{\mathbf{k}^{-1}(\{j\})}(k) = \sum_{j=1}^{n} \delta_{k,\mathbf{k}(j)}$$
Moreover, the first equality of (2.20) is obtained by noticing that
- both $1 - \prod_{h=1}^n \left(1 - \delta_{k,\mathbf{k}(h)}\right)$ and $\sum_{j=1}^n \delta_{k,\mathbf{k}(j)}$ take values in $\{0, 1\}$;
- $1 - \prod_{h=1}^n \left(1 - \delta_{k,\mathbf{k}(h)}\right) = 0$ iff $\prod_{h=1}^n \left(1 - \delta_{k,\mathbf{k}(h)}\right) = 1$ iff $\delta_{k,\mathbf{k}(h)} = 0$ for all $h \in \{1, \ldots, n\}$ iff $\sum_{h=1}^n \delta_{k,\mathbf{k}(h)} = 0$.

Proposition 2.6. *For any $k, n \in \mathbb{N}^*$ and $\mathbf{k} \in \mathbb{F}_n^0$, one has the following affirmations:*

1) as a particular case of (2.16) *and* (2.17),
$$(1 - P_k^{(d)})\Psi = 0$$
$$(1 - P_k^{(d)}) \left(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) = \chi_{range(\mathbf{k})}(k) e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} \tag{2.21}$$
2) let
$$\widetilde{\mathcal{A}}_k := \left\{a_k^+, a_k, a_k^+ P_k^{(d)}, P_k^{(s)}, a_k P_k^{(s)}, 1 - P_k^{(d)}\right\} \tag{2.22}$$
and
$$\mathcal{A}_k^{(+1)} := \left\{a_k^+, a_k^+ P_k^{(d)}\right\},\ \mathcal{A}_k^{(-1)} := \left\{a_k, a_k P_k^{(s)}\right\},\ \mathcal{A}_k^{(0)} := \left\{P_k^{(s)}, 1 - P_k^{(d)}\right\} \tag{2.23}$$
Then

$$\mathcal{A}_k^{(+1)},\ \mathcal{A}_k^{(-1)} \text{ and } \mathcal{A}_k^{(0)} \text{ are 2–2 disjoint;} \quad \mathcal{A}_k^{(+1)} \cup \mathcal{A}_k^{(-1)} \cup \mathcal{A}_k^{(0)} = \widetilde{\mathcal{A}}_k \tag{2.24}$$

and

$$x_k(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}) \in \begin{cases} \{ce_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} : c \in \mathbb{C}\}, & \text{if } x_k \in \mathcal{A}_k^{(0)} \\ \{ce_k \otimes e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} : c \in \mathbb{C}\}, & \text{if } x_k \in \mathcal{A}_k^{(+1)} \\ \{ce_{\mathbf{k}(n-1)} \otimes \ldots \otimes e_{\mathbf{k}(1)} : c \in \mathbb{C}\}, & \text{if } x_k \in \mathcal{A}_k^{(-1)} \end{cases} \tag{2.25}$$

Proof. The proof is completed easily as follows:

- the first equality of (2.21) is trivial in virtue of (2.16);
- the second equality of (2.21) is obtained just by combining (2.17) and (2.20);
- (2.24) is just a consequence of (2.22) and (2.23);
- finally, one gets (2.25) thanks to (2.16), (2.17) and (2.23). □

Proposition 2.7. *For any $m \in \mathbb{N}^*$ and $\mathbf{k} \in \mathbb{F}_m^0$, for any $x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)} \cup \mathcal{A}_{\mathbf{k}(j)}^{(0)}$ with $j \in \{1, \ldots, m\}$, by denoting $r := \big|\{j \in \{1, \ldots, m\} : x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\}\big|$, there exist a constant C (depending on $\mathbf{k}$ in general) and $1 \le i_1 < \ldots < i_r \le m$ such that*

$$x_m \ldots x_1 \Psi = C e_{\mathbf{k}(i_r)} \otimes \ldots \otimes e_{\mathbf{k}(i_1)} \tag{2.26}$$

hereinafter, we have adopted the conventions (2.18) *and* $\{i_1, \ldots, i_r\}\big|_{r=0} := \emptyset$.

Proof. (2.16) and the first equality of (2.21) say that

$$x_1 \Psi = \begin{cases} e_{\mathbf{k}(1)}, & \text{if } x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)} \\ 0, & \text{if } x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(0)} \cup \mathcal{A}_{\mathbf{k}(1)}^{(-1)} \end{cases} \tag{2.27}$$

So we obtain (2.26) by taking (recall that we have assumed $x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)} \cup \mathcal{A}_{\mathbf{k}(1)}^{(0)}$)

- $C = 0$ if $x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(0)}$;
- $C = 1$, $r = 1$ and $i_1 = 1$ if $x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)}$.

By assuming the validity of our thesis for $m = n$, one has $n + 1 > n \ge i_r$ and

$$\begin{aligned} r' :=& \big|\{j \in \{1, \ldots, n+1\} : x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\}\big| \\ =& \begin{cases} \big|\{j \in \{1, \ldots, n\} : x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\}\big| + 1, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(+1)} \\ \big|\{j \in \{1, \ldots, n\} : x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\}\big|, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(0)} \end{cases} \\ =& \begin{cases} r + 1, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(+1)} \\ r, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(0)} \end{cases} \end{aligned}$$

So, by applying (2.25) and noticing $x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(+1)} \cup \mathcal{A}_{\mathbf{k}(n+1)}^{(0)}$,

$$\begin{aligned} & x_{n+1} x_n \ldots x_1 \Psi = C x_{n+1} e_{\mathbf{k}(i_r)} \otimes \ldots \otimes e_{\mathbf{k}(i_1)} \\ =& C \begin{cases} c_1\, e_{\mathbf{k}(n+1)} \otimes e_{\mathbf{k}(i_r)} \otimes \ldots \otimes e_{\mathbf{k}(i_1)}, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(+1)} \\ c_2\, e_{\mathbf{k}(i_r)} \otimes \ldots \otimes e_{\mathbf{k}(i_1)}, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(0)} \end{cases} \end{aligned} \tag{2.28}$$

with $c_1, c_2 \in \mathbb{C}$. The induction principle gives the thesis. □

In the following, we denote, for any $k \in \mathbb{N}^*$,

$$\overline{\mathcal{A}}_k := lin.sp. \left\{ \mathbf{1}, a_k^+, a_k, P_k^{(d)}, a_k^+ P_k^{(d)}, P_k^{(s)}, a_k P_k^{(s)} \right\} \tag{2.29}$$

Theorem 2.8. *With respect to the vacuum state,* $\left\{\overline{\mathcal{A}}_k\right\}_k$ *and* $\{\mathcal{A}_k\}_k$ *are free independent families.*

Proof. Clearly we need only to prove the free independence of $\left\{\overline{\mathcal{A}}_k\right\}_k$. That is, by the definition, to show

$$\langle \Psi, x_n \dots x_1 \Psi \rangle = 0 \tag{2.30}$$

for any $n \geq 2$, $\mathbf{k} \in \mathbb{F}_n$ and for any such $\{x_j\}_{j=1}^n$ that $x_j \in \overline{\mathcal{A}}_{\mathbf{k}(j)}$ and $\langle \Psi, x_j \Psi \rangle = 0$ for all $j \in \{1, \dots, n\}$.

(2.29) says that any $x \in \overline{\mathcal{A}}_k$ must have the form

$$\alpha_1 a_k^+ + \alpha_2 a_k + \alpha_3 a_k^+ P_k^{(d)} + \alpha_4 P_k^{(s)} + \alpha_5 a_k P_k^{(s)} + \alpha_6 P_k^{(d)} + \alpha_7 \mathbf{1}$$

By combining this with the following facts

- $\langle \Psi, \mathbf{1}\Psi \rangle = 1 = \left\langle \Psi, P_k^{(d)} \Psi \right\rangle$ for any $k \in \mathbb{N}^*$,
- $\langle \Psi, x\ \Psi \rangle = 0$ for any $x \in \{a_k^+,\ a_k,\ a_k^+ P_k^{(d)},\ P_k^{(s)},\ a_k P_k^{(s)}\}$ and $k \in \mathbb{N}^*$,

one knows that any such $x_j \in \overline{\mathcal{A}}_{\mathbf{k}(j)}$ that $\langle \Psi, x_j \Psi \rangle = 0$ must have the form:

$$\begin{aligned} x_j = {} & \alpha_{\mathbf{k}(j),1} a_{\mathbf{k}(j)}^+ + \alpha_{\mathbf{k}(j),2} a_{\mathbf{k}(j)} + \alpha_{\mathbf{k}(j),3} a_{\mathbf{k}(j)}^+ P_{\mathbf{k}(j)}^{(d)} \\ & + \alpha_{\mathbf{k}(j),4} P_{\mathbf{k}(j)}^{(s)} + \alpha_{\mathbf{k}(j),5} a_{\mathbf{k}(j)} P_{\mathbf{k}(j)}^{(s)} + \alpha_{\mathbf{k}(j),6} (\mathbf{1} - P_{\mathbf{k}(j)}^{(d)}) \end{aligned}$$

and so $x_n \dots x_1$ equals to a sum of finite products in the form $y_n \dots y_1$, where y_j belongs to $\widetilde{\mathcal{A}}_{\mathbf{k}(j)}$ introduced in (2.22) for any $j \in \{1, \dots, n\}$. So, (2.30) will be proved if one gets

$$\langle \Psi, y_n \dots y_1 \Psi \rangle = 0, \qquad \forall y_j \in \widetilde{\mathcal{A}}_{\mathbf{k}(j)},\ \forall j \in \{1, \dots, n\} \tag{2.31}$$

and now we turn to show it.

If $n = 1$, (2.27) tells us that $y_1 \Psi \in \{0, e_{\mathbf{k}(1)}\}$ and so $\langle \Psi, y_1 \Psi \rangle = 0$. Now we examine (2.31) for $n \geq 2$. Moreover, since $y_1 \Psi = 0$ if $y_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(-1)} \cup \mathcal{A}_{\mathbf{k}(1)}^{(0)}$, we need only to prove (2.31) with the additional condition $y_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)}$. Now we perform it in two cases.

<u>The 1^{st} case: $\left\{ j : y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(-1)} \right\} \neq \emptyset$.</u> In this case, let

$$m + 1 := \min \left\{ j : y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(-1)} \right\}$$

then,

- $m \leq n - 1$, $y_{m+1} \in \mathcal{A}_{\mathbf{k}(m+1)}^{(-1)}$ and $y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(0)} \cup \mathcal{A}_{\mathbf{k}(j)}^{(+1)}$ for all $j \leq m$;
- the additional condition $y_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)}$ mentioned in above tells us that $m \geq 1$.

For any $m \in \{1, \dots, n-1\}$, Proposition 2.7 says that $y_m \dots y_1 \Psi$ has the form $C e_{\mathbf{k}'(r)} \otimes \dots \otimes e_{\mathbf{k}'(1)}$ with $C \in \mathbb{C}$, $1 \leq i_1 < \dots < i_r \leq m$ and $\mathbf{k}'(j) := \mathbf{k}(i_j)$ for any $j \in \{1, \dots, r\}$. Moreover, the facts $\mathbf{k} \in \mathbb{F}_n^0$ and $1 \leq i_1 < \dots < i_r < m + 1$

make sure that $\mathbf{k}' \in \mathbb{F}_r^0$ and $\mathbf{k}(m+1) \neq \mathbf{k}'(r)$, where $r := \left|\left\{j \leq m : y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\right\}\right|$. Therefore, thank to the fact $y_{m+1} \in \mathcal{A}_{\mathbf{k}(m+1)}^{(-1)}$ and the formula (2.17), one gets

$$y_{m+1}y_m \dots y_1\Psi = C\, y_{m+1}e_{\mathbf{k}'(r)} \otimes \dots \otimes e_{\mathbf{k}'(1)} = 0 \tag{2.32}$$

The 2^{nd} case: $\left\{j : y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(-1)}\right\} = \emptyset$. In this case, Proposition 2.7 says that the vector $y_n \dots y_1\Psi$ has the form $C\, e_{\mathbf{k}(i_r)} \otimes \dots \otimes e_{\mathbf{k}(i_1)}$, where, it follows form the fact $y_1 \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}$ that $i_1 = 1$ and $r \geq 1$. Consequently one gets the equality in (2.31).

Summing up, the proof is completed. □

Remark 2.9. We do not take the standard free Fock space over $\mathcal{H}$ and the corresponding creation–annihilation operators, even if on any free Fock space, the family $\{X_{n,k} : k \leq n\}$ is free independent for any $n \geq 2$. The reason is, on the standard free Fock space, for any annihilation (resp. creation) operator a (resp. a^+) with uni–norm test function, for any $p \in (0,1)$, the vacuum distribution of $X := \sqrt{p\,(1-p)}\,(a + a^+) + paa^+ + (1-p)\,a^+a$ can not be $b\,(1,p)$.

3. Quantization of the Free Poisson CLT

Now we are ready to see the quantized free Poisson CLT, which says essentially that if $np_n \longrightarrow \lambda$, then for any $\varepsilon \in \{1,-1,0,2\}$, $B_n^{(\varepsilon)}$ introduced in (1.5) equals asymptotically to $\lambda^{1-|\varepsilon|/2}b^{(\varepsilon)}$, where $b^{(\varepsilon)}$ is defined in (1.10).

Proposition 3.1. *1) On the IFS* $\Gamma_{sf}(\mathcal{H})$, *one has, for any* $\{p_N\}_{N=1}^{\infty} \subset [0,1]$ *and* $N \in \mathbb{N}^*$,

$$\begin{aligned}
&B_N^{(-1)}\Psi = B_N^{(2)}\Psi = 0; \quad B_N^{(0)}\Psi = Np_N\Psi; \quad \left\langle \Psi, \left(B_N^{(+1)}\right)^n \Psi \right\rangle = 0 \\
&B_N^{(-1)}\left(B_N^{(+1)}\right)^n \Psi = \chi_{N]}(n)\ (N-n+1)\,p_N\,(1-p_N)\left(B_N^{(+1)}\right)^{n-1}\Psi \\
&B_N^{(0)}\left(B_N^{(+1)}\right)^n \Psi = \chi_{N]}(n)\ (N-n)\,p_N\left(B_N^{(+1)}\right)^n \Psi \\
&B_N^{(2)}\left(B_N^{(+1)}\right)^n \Psi = \chi_{N]}(n)\ (1-p_N)\left(B_N^{(+1)}\right)^n \Psi, \quad \forall n \in \mathbb{N}^*
\end{aligned} \tag{3.1}$$

hereinafter, $\chi_{N]}(n) := \begin{cases} 1, & \text{if } n \leq N \\ 0, & \text{if } n > N \end{cases}$.

2) On the 1M–IFS $\Gamma_{fr}(\mathbb{C})$, *one has*

$$\begin{aligned}
&b^{(-1)}\Phi = b^{(2)}\Phi = 0; \quad \left\langle \Phi, \left(b^{(+1)}\right)^n \Phi \right\rangle = 0 \\
&b^{(2)}\left(b^{(+1)}\right)^n \Phi = \left(b^{(+1)}\right)^n \Phi; \quad b^{(-1)}\left(b^{(+1)}\right)^n \Phi = \left(b^{(+1)}\right)^{n-1}\Phi, \ \forall n \in \mathbb{N}^*
\end{aligned} \tag{3.2}$$

Proof. Clearly, we need only to show the last three equalities of (3.1) and others are trivial. Moreover, one uses simply p instead of p_N.

It follows from the definitions of $a^{(\varepsilon)}$'s and $B_N^{(\varepsilon)}$'s that

$$\begin{aligned}
\left(B_N^{(+1)}\right)^n \Psi &= \left(\sqrt{p\,(1-p)}\right)^n \sum_{\mathbf{k}\in\overline{\mathbb{F}_n},\ \mathbf{k}(j)\le N \text{ for any } j} a^+_{\mathbf{k}(n)} \cdots a^+_{\mathbf{k}(1)} \Psi \\
&= \chi_{N]}(n) \left(\sqrt{p\,(1-p)}\right)^n \sum_{\mathbf{k}\in\mathbb{F}^0_n,\ \mathbf{k}(j)\le N \text{ for any } j} e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} \qquad (3.3)
\end{aligned}$$

So the last three equalities of (3.1) are trivial for $n > N$. For $n \le N$, one gets

$$\begin{aligned}
& B_N^{(0)} \left(B_N^{(+1)}\right)^n \Psi \\
\overset{(3.3)}{=} & p \left(\sqrt{p\,(1-p)}\right)^n \sum_{k=1}^N \sum_{\mathbf{k}\in\mathbb{F}^0_n,\ \mathbf{k}(j)\le N \text{ for any } j} a_k a_k^+ \left(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) \\
\overset{(2.9)}{=} & p \left(\sqrt{p\,(1-p)}\right)^n \sum_{\mathbf{k}\in\mathbb{F}^0_n,\ \mathbf{k}(j)\le N \text{ for any } j} \ \sum_{1\le k\le N,\, k\notin range(\mathbf{k})} e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} \\
= & (N-n)\, p \left(\sqrt{p\,(1-p)}\right)^n \sum_{\mathbf{k}\in\mathbb{F}^0_n,\ \mathbf{k}(j)\le N \text{ for any } j} e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} \\
\overset{(3.3)}{=} & (N-n)\, p \left(B_N^{(+1)}\right)^n \Psi \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
& B_N^{(2)} \left(B_N^{(+1)}\right)^n \Psi \\
\overset{(3.3)}{=} & p \left(\sqrt{p\,(1-p)}\right)^n \sum_{k=1}^N \sum_{\mathbf{k}\in\mathbb{F}^0_n,\ \mathbf{k}(j)\le N \text{ for any } j} a_k^+ a_k \left(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) \\
= & (1-p) \left(\sqrt{p\,(1-p)}\right)^n \sum_{k=1}^N \sum_{\mathbf{k}\in\mathbb{F}^0_n,\ \mathbf{k}(j)\le N \text{ for any } j} \delta_{k,\mathbf{k}(n)} \left(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) \\
= & (1-p) \left(\sqrt{p\,(1-p)}\right)^n \sum_{\mathbf{k}\in\mathbb{F}^0_n,\ \mathbf{k}(j)\le N \text{ for any } j} e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} \\
\overset{(3.3)}{=} & (1-p) \left(B_N^{(+1)}\right)^n \Psi \qquad (3.5)
\end{aligned}$$

Finally, one has

$$\begin{aligned}
& B_N^{(-1)} \left(B_N^{(+1)}\right)^n \Psi \\
\overset{(3.3)}{=} & \left(\sqrt{p\,(1-p)}\right)^{n+1} \sum_{k=1}^N \sum_{\mathbf{k}\in\mathbb{F}^0_n,\ \mathbf{k}(j)\le N \text{ for any } j} a_k \left(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) \\
= & \left(\sqrt{p\,(1-p)}\right)^{n+1} \sum_{k=1}^N \sum_{\mathbf{k}\in\mathbb{F}^0_n,\ \mathbf{k}(j)\le N \text{ for any } j} \delta_{k,\mathbf{k}(n)} \left(e_{\mathbf{k}(n-1)} \otimes \ldots \otimes e_{\mathbf{k}(1)}\right) \qquad (3.6)
\end{aligned}$$

Since

$$\sum_{k=1}^{N}\sum_{\mathbf{k}\in\mathbb{F}_n^0,\ \mathbf{k}(j)\le N\ \forall j\le n}\delta_{k,\mathbf{k}(n)}e_{\mathbf{k}(n-1)}\otimes\ldots\otimes e_{\mathbf{k}(1)}$$

$$=\sum_{\mathbf{k}\in\mathbb{F}_{n-1}^0,\ \mathbf{k}(j)\le N\ \forall j\le n-1}e_{\mathbf{k}(n-1)}\otimes\ldots\otimes e_{\mathbf{k}(1)}\sum_{k=1}^{N}\sum_{1\le\mathbf{k}(n)\le N,\ \mathbf{k}(n)\ne\mathbf{k}(j)\ \forall j\le n-1}\delta_{k,\mathbf{k}(n)}$$

$$=(N-n+1)\sum_{\mathbf{k}\in\mathbb{F}_{n-1}^0,\ \mathbf{k}(j)\le N\ \forall j\le n-1}e_{\mathbf{k}(n-1)}\otimes\ldots\otimes e_{\mathbf{k}(1)}$$

one obtains, by applying this formula to (3.6),

$$B_N^{(-1)}\left(B_N^{(+1)}\right)^n\Psi$$

$$=(N-n+1)\left(\sqrt{p(1-p)}\right)^{n+1}\sum_{\mathbf{k}\in\mathbb{F}_{n-1}^0,\ \mathbf{k}(j)\le N\text{ for any }j}e_{\mathbf{k}(n-1)}\otimes\ldots\otimes e_{\mathbf{k}(1)}$$

$$\overset{(3.3)}{=}(N-n+1)\,p(1-p)\left(B_N^{(+1)}\right)^{n-1}\Psi \tag{3.7}$$

□

Theorem 3.2. *On the IFS* $\Gamma_{sf}(\mathcal{H})$, *if* $\{p_N\}_{N=1}^{\infty}\subset[0,1]$ *verifies* $\lim_{N\to\infty}Np_N=\lambda$, *then for any* $n\in\mathbb{N}$ *and* $\varepsilon\in\{-1,0,1,2\}^n$, *the limits, as* $N\to\infty$, *of* (1.7) *and (1.8) exist; moreover, they equal to the expressions* (1.9) *and (1.11) respectively. Consequently, the Jacobi coefficients of the free Poisson distribution with the parameter* λ *(more precisely,* $(\lambda,1)$*) is*

$$\alpha_0=\omega_n=\lambda,\ \alpha_n=\lambda+1,\quad\forall n\in\mathbb{N}^* \tag{3.8}$$

Proof. (3.1) and (3.2) tell us

$$\left\langle\Psi,B_N^{(\varepsilon)}\Psi\right\rangle=\begin{cases}Np_N, & \text{if }\varepsilon=0\\ 0, & \text{otherwise}\end{cases}\longrightarrow\begin{cases}\lambda, & \text{if }\varepsilon=0\\ 0, & \text{otherwise}\end{cases}$$
$$=\lambda\left\langle\Phi,b^{(\varepsilon)}\Phi\right\rangle=\left\langle\Phi,b^{(\varepsilon)}\Phi\right\rangle\lambda^{1-|\varepsilon|/2}$$

and

$$\left\langle\Psi,B_N^{(\varepsilon(1))}B_N^{(\varepsilon(2))}\Psi\right\rangle$$
$$=\begin{cases}Np_N(1-p_N), & \text{if }\varepsilon(1)=-1\text{ and }\varepsilon(2)=1\\ 0, & \text{otherwise}\end{cases}$$
$$\longrightarrow\begin{cases}\lambda, & \text{if }\varepsilon(1)=-1\text{ and }\varepsilon(2)=1\\ 0, & \text{otherwise}\end{cases}=\lambda\left\langle\Phi,b^{(\varepsilon(1))}b^{(\varepsilon(2))}\Phi\right\rangle$$
$$=\left\langle\Phi,b^{(\varepsilon(1))}b^{(\varepsilon(2))}\Phi\right\rangle\lambda^{\sum_{k=1}^{2}(1-|\varepsilon(k)|/2)}$$

i.e. the thesis is proved for $n=1$ and 2.

Suppose that the thesis is proved up to n, let's see it for $n+1$. Since we are considering the limits of (1.7) and (1.8) as $N\to\infty$, one can assume that $N\geq n+1$.

If $\varepsilon(n+1)\in\{-1,2\}$, (3.1) and (3.2) tell us that

$$\begin{aligned}\left\langle\Psi,B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n+1))}\Psi\right\rangle=0=&\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(n+1))}\Phi\right\rangle\\ =&\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(n+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n+1}(1-|\varepsilon(k)|/2)}\end{aligned}$$

If $\varepsilon(n+1)=0$, (3.1) gives

$$\begin{aligned}\left\langle\Psi,B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n+1))}\Psi\right\rangle&=\left\langle\Psi,B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n))}B_N^{(0)}\Psi\right\rangle\\&=Np_N\left\langle\Psi,B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n))}\Psi\right\rangle\end{aligned}$$

The assumption of induction and the fact $Np_N\longrightarrow\lambda$ make sure that

$$\left\langle\Psi,B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n+1))}\Psi\right\rangle\longrightarrow\lambda\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n}(1-|\varepsilon(k)|/2)}$$

On the other hand, (3.2) gives

$$\begin{aligned}&\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}b^{(\varepsilon(n+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n+1}(1-|\varepsilon(k)|/2)}\Big|_{\varepsilon(n+1)=0}\\=&\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}b^{(0)}\Phi\right\rangle\lambda^{1+\sum_{k=1}^{n}(1-|\varepsilon(k)|/2)}\\=&\lambda\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n}(1-|\varepsilon(k)|/2)}\end{aligned}$$

i.e.

$$\begin{aligned}&\left\langle\Psi,B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n))}B_N^{(0)}\Psi\right\rangle\\\longrightarrow&\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}b^{(\varepsilon(n+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n+1}(1-|\varepsilon(k)|/2)}\Big|_{\varepsilon(n+1)=0}\end{aligned}$$

Finally, one sees the case of $\varepsilon(n+1)=1$.

If $\varepsilon(j)=1$ for all $j\in\{1,\dots,n+1\}$, one gets, thanks to (3.1) and (3.2),

$$\begin{aligned}&\left\langle\Psi,B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n+1))}\Psi\right\rangle=\left\langle\Psi,\left(B_N^{(+1)}\right)^{n+1}\Psi\right\rangle\\=&0=\left\langle\Phi,\left(b^{+}\right)^{n+1}\Phi\right\rangle=\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}b^{(\varepsilon(n+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n+1}(1-|\varepsilon(k)|/2)}\end{aligned}$$

So one needs only to see the case of $\{k:\varepsilon(k)\neq1\}\neq\emptyset$. In this case, one denotes $m:=\max\{k:\varepsilon(k)\neq1\}$, then $\varepsilon(m)\in\{-1,0,2\}$, $\varepsilon(m+1)=\ldots=\varepsilon(n+1)=1$,

$$B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n+1))}=B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(m-1))}B_N^{(\varepsilon(m))}\left(B_N^{(+1)}\right)^{n+1-m}\tag{3.9}$$

In virtue of the formula (3.1), the formula (3.9), the assumption of induction and the fact $Np_N\longrightarrow\lambda$, one gets the following:

If $\varepsilon(m)=0$,

$$
\begin{aligned}
&\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n+1))}\Psi\right\rangle\Big|_{\varepsilon(m)=0,\varepsilon(j)=1\ \forall j>m}\\
=&\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(m-1))}B_N^{(0)}\left(B_N^{(+1)}\right)^{n+1-m}\Psi\right\rangle\\
=&\left(N-(n+1-m)\right)p_N\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(m-1))}\left(B_N^{(+1)}\right)^{n+1-m}\Psi\right\rangle\\
\longrightarrow&\lambda\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}\left(b^{(+1)}\right)^{n+1-m}\Phi\right\rangle\lambda^{\sum_{k=1}^{m-1}(1-|\varepsilon(k)|/2)+(n+1-m)/2}\\
=&\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(n+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n+1}(1-|\varepsilon(k)|/2)}\Big|_{\varepsilon(m)=0,\varepsilon(j)=1\ \forall j>m}
\end{aligned}
$$

If $\varepsilon(m)=-1$,

$$
\begin{aligned}
&\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n+1))}\Psi\right\rangle\Big|_{\varepsilon(m)=-1,\varepsilon(j)=1\ \forall j>m}\\
=&\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(m-1))}B_N^{(-1)}\left(B_N^{(+1)}\right)^{n+1-m}\Psi\right\rangle\\
=&Np_N\left(1-p_N\right)\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(m-1))}\left(B_N^{(+1)}\right)^{n-m}\Psi\right\rangle\\
\longrightarrow&\lambda\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}\left(b^{(+1)}\right)^{n-m}\Phi\right\rangle\lambda^{\sum_{k=1}^{m-1}(1-|\varepsilon(k)|/2)+(n-m)/2}
\end{aligned}
$$

On the other hand, (3.2) gives

$$
\begin{aligned}
&\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(n+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n+1}(1-|\varepsilon(k)|/2)}\Big|_{\varepsilon(m)=-1,\varepsilon(j)=1\ \forall j>m}\\
=&\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}b^{(-1)}\left(b^{(+1)}\right)^{n+1-m}\Phi\right\rangle\lambda^{\sum_{k=1}^{m-1}(1-|\varepsilon(k)|/2)+(n-m+2)/2}
\end{aligned}
$$

So

$$
\begin{aligned}
&\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n+1))}\Psi\right\rangle\Big|_{\varepsilon(m)=-1,\varepsilon(j)=1\ \forall j>m}\\
\longrightarrow&\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(n+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n+1}(1-|\varepsilon(k)|/2)}\Big|_{\varepsilon(m)=-1,\varepsilon(j)=1\ \forall j>m}
\end{aligned}
$$

If $\varepsilon(m)=2$,

$$
\begin{aligned}
&\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(n+1))}\Psi\right\rangle\Big|_{\varepsilon(m)=2,\varepsilon(j)=1\ \forall j>m}\\
=&\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(m-1))}B_N^{(2)}\left(B_N^{(+1)}\right)^{n+1-m}\Psi\right\rangle\\
=&\left(1-p_N\right)\left\langle \Psi, B_N^{(\varepsilon(1))}\dots B_N^{(\varepsilon(m-1))}\left(B_N^{(+1)}\right)^{n+1-m}\Psi\right\rangle\\
\longrightarrow&\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}\left(b^{(+1)}\right)^{n+1-m}\Phi\right\rangle\lambda^{\sum_{k=1}^{m-1}(1-|\varepsilon(k)|/2)+(n-m)/2}\\
=&\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}b^{(2)}\left(b^{(+1)}\right)^{n+1-m}\Phi\right\rangle\lambda^{\sum_{k=1}^{m-1}(1-|\varepsilon(k)|/2)+(n-m)/2}\\
=&\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(n+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{n+1}(1-|\varepsilon(k)|/2)}\Big|_{\varepsilon(m)=2,\varepsilon(j)=1\ \forall j>m}
\end{aligned}
$$

Summing up, we have proved that the limit of (1.8) equals to the expression (1.11). By combining together this and the boundedness of $\{B_n^{(\varepsilon)}\}_{n=1}^{\infty}$, we know that the limit of (1.7) equals to the expression (1.9). $\square$

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