This is a pre-print version. The final version is published in

DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S

doi:10.3934/dcdss.2020247

LARGE DATA SOLUTIONS FOR SEMILINEAR HIGHER ORDER EQUATIONS

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Abstract. In this paper we study local and global in time existence for a class of nonlinear evolution equations having order eventually greater than 2 and not integer. The linear operator has an homogeneous damping term; the nonlinearity is of polynomial type without derivatives:

$$u_{tt} + (-\Delta)^{2\theta} u + 2\mu(-\Delta)^{\theta} u_t + |u|^{p-1} u = 0, \quad t \geq 0,\ x \in \mathbb{R}^n,$$

with $\mu > 0$, $\theta > 0$. Since we are treating an absorbing nonlinear term, large data solutions can be considered.

1. Introduction.

In this paper, we analyze the evolution equation

$$\begin{cases} u_{tt} + (-\Delta)^{2\theta} u + 2\mu(-\Delta)^{\theta} u_t + |u|^{p-1} u = 0, & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

where $\mu > 0$, $\theta > 0$, and $p > 1$. In particular the order of the equation can be higher than 2 and not-integer.

This kind of operators have been considered in [7] and [2] with small data or with different kind of nonlinearities. See also [1] and the reference therein for other results on the linear case.
Here we prove the global existence of some energy solutions to (1) without any assumption on the size of initial data. In literature, to underline such lack of assumption, it is customary to speak about "large data solution". Hence, the sign of the nonlinear term is crucial that is we need an absorbing structure for the equation

The quasilinear version of (1) with large data has been treated in [4]: assuming $\theta > n/4$, θ integer, the equation

$$u_{tt} + (-\Delta)^{2\theta} u + 2\mu(-\Delta)^{\theta} u_t + |u_t|^{p-1} u_t = 0 \tag{2}$$

admits global existence in

$$u \in \mathcal{C}([0,\infty), H^{4\theta}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), H^{2\theta}(\mathbb{R}^n)) \cap \mathcal{C}^2([0,\infty), L^2(\mathbb{R}^n))\,.$$

For $p > 1 + 4/n$, and $\theta \in (n/4, n/2)$, such solutions satisfy optimal decay estimates, in the sense that the decay rate of Sobolev solution is the same as of the corresponding linear problem with vanishing right hand side, in particular the energy dissipates for $t \to \infty$. Moreover, with the same assumption on the p and θ exponents, the

2010 *Mathematics Subject Classification.* Primary: 35L71, 35L82, 35B40.
Key words and phrases. Large data, damping wave, energy estimate.

asymptotic profile of the solutions of (2) can be described by using a combination of solutions of the diffusion equations of type

$$v_t + a(-\Delta)^\theta v = 0, \qquad t \geq 0,\ x \in \mathbb{R}^n$$

In many papers on small data solutions, in order to improve decay estimates, L^1 regularity for initial data is required (see [2]). Here we do not consider this aspect, indeed we want to analyze the well posedness in energy spaces and we will pay in the decay estimates.

In particular our result applies to the semilinear plate equation with strong damping

$$u_{tt} + \Delta^2 u - \Delta u_t + |u|^{p-1}u = 0\,.$$

The same operator has been also considered in [3] for small data solutions and nonlinear memory term.

We shall prove the following global existence result.

Theorem 1.1. *Let $n \geq 1$, $\theta > n/4$, $p > 1$. For any $(u_0, u_1) \in H^{2\theta}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, there exists a unique global in time solution to the Cauchy problem* (1)*. More precisely*

$$u \in \mathcal{C}([0,\infty), H^{2\theta}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), L^2(\mathbb{R}^n))\,.$$

In the next theorem we emphasize other regularity results that comes from the energy estimates.

Theorem 1.2. *Let $n \geq 1$, $\theta > n/4$, $p > 1$. The global solution of* (1) *given by Theorem 1.1 satisfies*

$$u_t \in L^m([0,\infty), \dot{H}^{\frac{2\theta}{m}}(\mathbb{R}^n))\,, \quad m \in [2, +\infty)\,, \tag{3}$$

and

$$u \in L^\infty([0,\infty), \dot{H}^{\gamma, \frac{4\theta(p+1)}{p\gamma+2\theta}}(\mathbb{R}^n)) \quad \gamma \in (0, 2\theta]\,. \tag{4}$$

Assuming in addition that $n/2 < \theta < n$ then

$$u_t \in L^m([0,\infty), L^{\frac{2m}{m-2}}(\mathbb{R}^n)) \tag{5}$$

for any $m \in [2, +\infty)$.

Remark 1. Let us compare our result with the small data case analyzed in [2]. We have $u \in \mathcal{C}([0,\infty), H^{2\theta}(\mathbb{R}^n))$, but we do not know information on the global boundedness of u in time, apart of (4). Conversely in [2] a decay of $\|u(t,\cdot)\|_{L^\infty}$ and hence $u \in L^\infty([0,+\infty), L^\infty(\mathbb{R}^n))$ is given provided $p > 1 + 2n/(n-2\theta)$, $\theta < n/2$ and small initial data in $W^{2\theta,1} \cap W^{2\theta,\infty}$. This difference is due to our space choice: we consider large data in energy space. Interesting open problems are the study of the time decay of the solution and the validity of (4) in non-homogeneous spaces.

Remark 2. The assumption $\theta > n/4$ is necessary for our approach. Indeed, in the proof of local in-time existence theorem we use the embedding $H^{2\theta}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. On the other hand we do not have assumptions on p from above. In the last section of this paper we will give some information on the case $\theta < n/4$ and a critical exponent p will appear.

Remark 3. Finally, we suggest the interested reader to consider the global existence problem for the same operator with two kinds of nonlinear terms of focusing type, one dependent on u and the other on u_t. Another possibility is to deal with $\mu = \mu(t)$.

1.1. **Notation.** Let $f, g : \Omega \to \mathbb{R}$ be two functions. If there exists $C > 0$ such that $f(y) \leq Cg(y)$ for all $y \in \Omega$, then we write $f \lesssim g$. Similarly $f \approx g$ means that there exist two constants $C_1, C_2 > 0$ such that $C_1 g(y) \leq f(y) \leq C_2 g(y)$ for all $y \in \Omega$.

In all the paper, $*$ is the convolution with respect to the x variable.

We denote $\hat{f} = \mathfrak{F}f$, the Fourier transform of a function f with respect to the x variable.

For any $s \in \mathbb{R}$, we define the operator $|D|^s$, acting on suitable functions and distributions, as follows:

$$|D|^s f := \mathfrak{F}^{-1}(|\xi|^s \mathfrak{F}f).$$

For any $s > 0$, we denote by $H^s(\mathbb{R}^n)$ the usual Sobolev space $\{f \in L^2(\mathbb{R}^n) : \, |\xi|^s \hat{f} \in L^2(\mathbb{R}^n)\}$. We also denote by $\dot{H}^s(\mathbb{R}^n)$ the homogeneous space obtained only requiring $|\xi|^s \hat{f} \in L^2(\mathbb{R}^n)$, being f in the tempered distributions spaces modulo polynomials. In Section 3.1, given $m \geq 1$ and A a space of function depending on $x \in \mathbb{R}^n$, we put for brevity

$$L^m_t A_x := L^m([0, +\infty), A(\mathbb{R}^n))$$

to denote the space of $u(t,x)$ such that the function $t \to \|u(t,\cdot)\|_{A(\mathbb{R}^n)}$ is in $L^m([0, +\infty))$.

2. The analysis of Problem (1).

2.1. **The fundamental solution.** Let us consider the linear Cauchy problem with vanishing right hand side associated to (1):

$$\begin{cases} \varphi_{tt} + (-\Delta)^{2\theta}\varphi + 2\mu(-\Delta)^{\theta}\varphi_t = 0, & t \geq 0, \; x \in \mathbb{R}^n, \\ \varphi(0,x) = u_0(x), \\ \varphi_t(0,x) = u_1(x). \end{cases} \tag{6}$$

After applying the Fourier transform, we get

$$\begin{cases} \hat{\varphi}_{tt} + |\xi|^{4\theta}\hat{\varphi} + 2\mu|\xi|^{2\theta}\hat{\varphi}_t = 0\,, \\ \hat{\varphi}(0,\xi) = \hat{u}_0(\xi), \\ \hat{\varphi}_t(0,\xi) = \hat{u}_1(\xi). \end{cases}$$

We denote by $K := K(t,x)$ the fundamental solution; its Fourier transform solves

$$\begin{cases} \hat{K}_{tt} + |\xi|^{4\theta}\hat{K} + 2\mu|\xi|^{2\theta}\hat{K}_t = 0\,, \\ \hat{K}(0,\xi) = 0, \\ \hat{K}_t(0,\xi) = 1. \end{cases}$$

The solution of (6) is given by

$$\varphi(t,x) = (K_t(t,\cdot) + 2\mu(-\Delta)^{\theta}K(t,\cdot)) * u_0(x) + K(t,\cdot) * u_1(x). \tag{7}$$

Let us describe $\hat{K}(t,\xi)$. We shall distinguish three cases.

- If $\mu > 1$, then
$$\hat{K}(t,\xi) = \frac{e^{-bt|\xi|^{2\theta}} - e^{-at|\xi|^{2\theta}}}{2|\xi|^{2\theta}\sqrt{\mu^2-1}} = \frac{e^{-at|\xi|^{2\theta}}}{2\sqrt{\mu^2-1}} \frac{e^{-(b-a)t|\xi|^{2\theta}} - 1}{|\xi|^{2\theta}},$$
where
$$a = \mu + \sqrt{\mu^2-1}, \quad b = \mu - \sqrt{\mu^2-1};$$
- If $\mu = 1$, then $\hat{K}(t,\xi) = te^{-t|\xi|^{2\theta}}$;

- If $\mu \in (0,1)$, then

$$\hat{K}(t,\xi) = \frac{e^{-\mu t|\xi|^{2\theta}}}{\sqrt{1-\mu^2}} \frac{\sin(t\sqrt{1-\mu^2}|\xi|^{2\theta})}{|\xi|^{2\theta}} .$$

Remark 4. In the case $\mu > 1$ we can write

$$K(t,x) = \frac{1}{2\sqrt{\mu^2-1}} |D|^{-2\theta}(G_b(t,x) - G_a(t,x)),$$

where G_a is the fundamental solution to the diffusion equation

$$v_t + a(-\Delta)^\theta v = 0.$$

If $\mu = 1$, then $K = tG_1$.

Let us state some basic estimates for the fundamental solution in $L^2(\mathbb{R}^n)$.

Lemma 2.1. *Let $\theta \geq 0$ and $s \geq 0$. It holds*

$$\|K(t,\cdot) * g\|_{H^{2\theta}(\mathbb{R}^n)} \lesssim (1+t)\|g\|_{L^2(\mathbb{R}^n)} , \tag{8}$$

$$\|K_t(t,\cdot) * g\|_{H^s(\mathbb{R}^n)} \lesssim \|g\|_{H^s(\mathbb{R}^n)} , \tag{9}$$

$$\|K_{tt}(t,\cdot) * g\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{\dot{H}^{2\theta}(\mathbb{R}^n)} , \tag{10}$$

for any g such that the right side is finite.

The loss in time-decay in the right side of (8) is evident for $\mu = 1$ and it appears for $\mu \neq 1$ while considering $|\xi|$ close to zero. For other Sobolev estimates of (6) see [1], [2] and [4].

2.2. **Local existence.** Let us recall the classical contraction mapping principle in the version of [8].

Lemma 2.2. *Let X_1, X_2 be Banach spaces, $S : X_1 \to X_2$ a linear operator and $N : X_2 \to X_1$ a map such that $N0 = 0$. Given $\varphi \in X_2$, one considers the equation*

$$u = \varphi + SNu. \tag{11}$$

Assume that there exist $C_0 > 0$ and $R > 0$, such that

$$\|SG\|_{X_2} \leq C_0\|G\|_{X_1}, \quad \textit{for any } G \in X_1; \tag{12}$$

$$\|Nv - Nw\|_{X_1} \leq \frac{1}{2C_0}\|v-w\|_{X_2} \tag{13}$$

for any $v, w \in X_2$ with $\|v\|_{X_2} \leq R$, $\|w\|_{X_2} \leq R$. If $\|\varphi\|_{X_2} \leq R/2$, then there exists $u \in X_2$ the unique solution to (11). Moreover $\|u\|_{X_2} \leq 2\|\varphi\|_{X_2}$.

We take

$$X_1 = L^\infty([0,T], L^2(\mathbb{R}^n)), \quad \|v\|_{X_1} = \|v\|_{L^\infty([0,T],L^2(\mathbb{R}^n))}$$

and

$$\begin{aligned} X_2 &= \{u \in L^\infty([0,T], H^{2\theta}(\mathbb{R}^n)) \text{ such that } u_t \in L^\infty([0,T], L^2(\mathbb{R}^n))\} \\ \|u\|_{X_2} &= \|u\|_{L^\infty([0,T],H^{2\theta}(\mathbb{R}^n))} + \|u_t\|_{L^\infty([0,T],L^2(\mathbb{R}^n))} . \end{aligned}$$

We put

$$Sv(t,x) = \int_0^t K(t-\tau) * v(\tau,x) d\tau .$$

From (8) and (9) with $s = 0$, we deduce that $S : X_1 \to X_2$. More precisely, we can conclude that there exists $C_1 > 0$, independent of $T > 0$, such that

$$\|Sv\|_{X_2} \leq C_1(T^2 + T)\|v\|_{X_1}\,.$$

In particular S satisfies (12). Now we introduce φ as the solution of (6) written in the form (7). From (8), (9) with $s = 2\theta$ and (10), we see that if $u_0, u_1 \in H^{2\theta}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, then $\varphi \in X_2$. More precisely there exists $C_2 > 0$, independent of $T > 0$, such that

$$\|\varphi\|_{L^\infty([0,T],H^{2\theta}(\mathbb{R}^n))} + \|\varphi_t\|_{L^\infty([0,T],L^2(\mathbb{R}^n))} \leq C_2 T(\|u_0\|_{H^{2\theta}(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)})\,.$$

Finally, we put

$$Nv = |v|^{p-1}v\,.$$

The solution of (1) satisfies $u = \varphi + SNu$. In order to prove (13), we use

$$\|Nv - Nw\|_{L^2(\mathbb{R}^n)} \lesssim \|u - w\|_{L^2(\mathbb{R}^n)}(\|v\|^{p-1}_{L^\infty(\mathbb{R}^n)} + \|w\|^{p-1}_{L^\infty(\mathbb{R}^n)})\,.$$

This implies the existence of $C_3 > 0$, independent of $T > 0$, such that

$$\|Nv - Nw\|_{L^2(\mathbb{R}^n)} \leq C_3\|u - w\|_{X_2}(\|v\|_{X_2} + \|w\|_{X_2})^{p-1}$$

if

$$\theta > n/4.$$

For $w = 0$, being $N0 = 0$, we get $N : X_2 \to X_1$.

Let $R > 0$. Renaming

$$C_0 = 1/2^p C_3 R^{p-1}\,, \tag{14}$$

we get (13) for any $\|v\|_{X_2} \leq R$, $\|w\|_{X_2} \leq R$. For small $T > 0$, we get

$$T^2 + T < C_0/C_1 \tag{15}$$

so that Lemma 2.2 implies the local existence result once

$$\|u_0\|_{H^{2\theta}(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)} \leq R/(4C_2T)\,. \tag{16}$$

This proves that for suitable $T > 0$, one has

$$u \in L^\infty([0,T], H^{2\theta}(\mathbb{R}^n)) \text{ and } u_t \in L^\infty([0,T], L^2(\mathbb{R}^n)).$$

If we recall (7) and the continuity of K and K_t in time-variable, we can restrict X_1 and X_2 to

$$\begin{aligned} \tilde{X}_1 &= \mathcal{C}([0,T], L^2(\mathbb{R}^n)) \\ \tilde{X}_2 &= \mathcal{C}([0,T], H^{2\theta}(\mathbb{R}^n) \cap \mathcal{C}^1([0,T], L^2(\mathbb{R}^n))\}\,. \end{aligned}$$

These arguments give the following statement.

Theorem 2.3. *Let $n \geq 1$, $p > 1$ and $\theta > n/4$. Assume that $u_0 \in H^{2\theta}(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. There exists a suitable $T > 0$ and a unique solution of (1) in energy space:*

$$u \in \mathcal{C}([0,T], H^{2\theta}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T], L^2(\mathbb{R}^n)).$$

Comparing this result with Section 9 in [2], we see that for $\theta > n/4$, the upper bound on p that appears in such paper is not necessary.

2.3. **Energy estimates.** Let us introduce the energy

$$E(u)(t) = \frac{1}{2}\|u_t(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\|(-\Delta)^\theta u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{p+1}\|u(t,\cdot)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}. \quad (17)$$

It is crucial that any term of the energy is non negative.

Multiplying by $u_t(t,\cdot)$ the equation and integrating by parts, formally we get

$$\frac{d}{dt}E(u)(t) + 2\mu\||D|^\theta u_t(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} = 0\,.$$

Being $\mu > 0$, we see that $\frac{d}{dt}E(u)(t) \le 0$, that is the energy is decreasing and

$$E(u)(t) + 2\mu \int_0^t \||D|^\theta u_t(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} ds = E(u)(0)\,. \quad (18)$$

This formal computation is justified for solutions such that $u(t,\cdot) \in H^{4\theta}(\mathbb{R}^n)$ and $u_t(t,\cdot) \in H^{2\theta}(\mathbb{R}^n)$. We can assert that (18) holds for $u(t,\cdot) \in H^{2\theta}(\mathbb{R}^n)$ and $u_t(t,\cdot) \in L^2(\mathbb{R}^n)$ after an approximation procedure. Indeed proceeding as in Section 2.2, we see that a continuous dependence of solutions on the Cauchy data can be proved in $H^{2\theta}(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$ for functions having time derivative in $L^2(\mathbb{R}^n)$.

We can conclude, that the following lemma holds.

Lemma 2.4. *Let $n \ge 1$, $\theta > n/4$. Let $p > 1$.*

Let $u_0 \in H^{2\theta}(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. Let $T_{max} \in (0, +\infty]$ be the maximal existence time of the solution to (1). For any $t \in [0, T_{max})$, we have

$$\|u_t(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \le 2E(u)(0), \quad (19)$$

$$\|(-\Delta)^\theta u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \le 2E(u)(0), \quad (20)$$

$$\|u(t,\cdot)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} \le (p+1)E(u)(0)\,. \quad (21)$$

Moreover, it holds

$$\int_0^t \||D|^\theta u_t(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} ds \le \frac{1}{2\mu}E(u)(0). \quad (22)$$

2.4. **Blow-up dynamic.**

Corollary 1. *Let $n \ge 1$, $p > 1$ and $\theta > n/4$. Assume that $u_0 \in H^{2\theta}(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. Let $T_{\max} \in (0, \infty]$ be the maximal existence time of the solution to the Cauchy problem (1), in particular*

$$u \in \mathcal{C}([0, T_{\max}), H^{2\theta}(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T_{\max}), L^2(\mathbb{R}^n))\,.$$

It holds $T_{\max} < \infty$ if and only if,

$$\lim_{t \to T_{\max}} \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} = +\infty. \quad (23)$$

Proof. Having in mind (14), (15), (16), we see that $T_{max} > 0$ only depends on $\|u_0\|_{H^{2\theta}(\mathbb{R}^n)}$ and $\|u_1\|_{L^2(\mathbb{R}^n)}$. This means that if

$$\lim_{t \to T_{\max}} \left(\|u(t,\cdot)\|_{H^{2\theta}(\mathbb{R}^n)} + \|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)}\right) = +\infty$$

does not hold, then the solution to (1) may be prolonged. Since $\theta > n/4$, then $u_0 \in H^{2\theta}(\mathbb{R}^n)$ implies also $u_0 \in L^{p+1}(\mathbb{R}^n)$ and $E(u)(0)$ is finite. The energy estimate (19) gives L^2-boundedness of u_t. Hence, blow up will be determined by $H^{2\theta}(\mathbb{R}^n)$ norm. Recalling that

$$\|u(t,\cdot)\|_{H^{2\theta}(\mathbb{R}^n)} \simeq \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^\theta u(t,\cdot)\|_{L^2(\mathbb{R}^n)}\,,$$

we see that the uniform estimate (20) control the second term. We can conclude that the blow-up appears if and only if $L^2(\mathbb{R}^n)$ norm tends to infinity. □

3. Proof of main theorems.

3.1. Proof of Theorem 1.1.

Let $T_{\max}$ be the maximal existence time of the solution to (1). We assume, by contradiction, that $T_{max} < \infty$. We can use energy estimates, in particular (19), concluding that

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|u_0\|_{L^2(\mathbb{R}^n)} + \int_0^t \|u_t(s,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+T_{max}). \tag{24}$$

Hence, (23) does not hold. This conclude the proof of the global existence, since uniqueness is guaranteed by the local existence result.

3.2. Proof of Theorem 1.2.

Energy estimates can be summarized as

$$u_t \in L_t^\infty L_x^2\,, \tag{25}$$

$$u_t \in L_t^2 \dot{H}_x^\theta\,, \tag{26}$$

After interpolation we get (3). The energy estimate gives also

$$u \in L_t^\infty \dot{H}_x^{2\theta}\,,$$

$$u \in L_t^\infty L_x^{p+1}\,. \tag{27}$$

By interpolation we find (4).

Now we suppose that $n/2 < \theta < n$. We can take $\gamma_1 = \theta/2$ and $\gamma_2 = \theta$ such that $0 < 2\gamma_1 < n < 2\gamma_2$ and use the following relation that holds in fractional Sobolev spaces:

$$\|u_t(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u_t(t,\cdot)\|_{\dot{H}^{\gamma_1}(\mathbb{R}^n)} + \|u_t(t,\cdot)\|_{\dot{H}^{\gamma_2}(\mathbb{R}^n)}.$$

A proof of the previous inequality is given in [4]. On the other hand,

$$\begin{aligned}\|u_t(t,\cdot)\|^2_{\dot{H}^{\theta/2}(\mathbb{R}^n)} &= \int ||D|^{\theta/2}u_t(t,x)|^2\,dx = \int (|D|^\theta u_t(t,x))u_t(t,x)\,dx \\ &\leq \|u_t(t,\cdot)\|_{\dot{H}^\theta(\mathbb{R}^n)}\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq 2E[u](0)\|u_t(t,\cdot)\|_{\dot{H}^\theta(\mathbb{R}^n)}\,.\end{aligned}$$

Combining this with (26) we arrive to

$$u_t \in L_t^2 L_x^\infty\,.$$

By interpolation with (25), we conclude (5).

Remark 5. Having in mind (27), we see that an influence of the nonlinear term appears. Indeed in [4], for $\theta > n/4$, with nonlinear term dependent on u_t the solution does not belong to $u \in L_t^\infty L^r$ for any $r \geq 2$.

Remark 6. In Theorem 1.2, something better can be said on the continuity in time variable for u. For example from $u_t \in L^\infty([0,T],\dot{H}^{\gamma'})$ we can deduce $u \in \mathcal{C}([0,+\infty),H^{\gamma'})$ and $\gamma' \in (0,\theta]$. In general, given $t > s > 0$, we can compute

$$\begin{aligned}&\||D|^{\gamma'}u(t,\cdot) - |D|^{\gamma'}u(s,\cdot)\|_{L^2(\mathbb{R}^n)} \\ = &\int_s^t \frac{d}{d\tau}\||D|^{\gamma'}u(\tau,\cdot) - |D|^{\gamma'}u(s,\cdot)\|_{L^2(\mathbb{R}^n)}d\tau \\ = &\int_s^t \frac{1}{2\||D|^{\gamma'}u(\tau,\cdot) - |D|^{\gamma'}u(s,\cdot)\|_{L^2(\mathbb{R}^n)}}\frac{d}{d\tau}\||D|^{\gamma'}u(\tau,\cdot) - |D|^{\gamma'}u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}d\tau\end{aligned}$$

$$\begin{aligned}
&= \int_s^t \frac{\int_{\mathbb{R}^n} \frac{d}{d\tau}||D|^{\gamma'}u(\tau,x) - |D|^{\gamma'}u(s,x)|^2 dx}{2\||D|^{\gamma'}u(\tau,\cdot) - |D|^{\gamma'}u(s,\cdot)\|_{L^2(\mathbb{R}^n)}}\, d\tau \\
&\le \int_s^t \frac{\int_{\mathbb{R}^n} \left(|D|^{\gamma'}u(\tau,x) - |D|^{\gamma'}u(s,x)\right)\left(|D|^{\gamma'}u_t(\tau,x)\right) dx}{\||D|^{\gamma'}u(\tau,\cdot) - |D|^{\gamma'}u(s,\cdot)\|_{L^2(\mathbb{R}^n)}} d\tau \\
&\le \int_s^t \||D|^{\gamma'}u_t(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} d\tau \le |t-s|\||D|^{\gamma'}u_t\|_{L^\infty([0,T],L^2(\mathbb{R}^n))} \\
&= |t-s|\|u_t\|_{L^\infty([0,T],\dot H^{\gamma'}(\mathbb{R}^n))}.
\end{aligned}$$

4. **Some information on the case $\theta < n/4$.** The restriction $\theta > n/4$ appears as an assumption on the local existence Theorem 2.3. Indeed in the proof it was necessary to establish the next inequality

$$\||v|^{p-1}v - |w|^{p-1}w\|_{L^2(\mathbb{R}^n)} \lesssim \|u-w\|_{H^{2\theta}(\mathbb{R}^n)}(\|v\|^{p-1}_{H^{2\theta}(\mathbb{R}^n)} + \|w\|^{p-1}_{H^{2\theta}(\mathbb{R}^n)}).$$

For $\theta < n/4$ we can still prove this inequality provided

$$p < 1 + \frac{4\theta}{n-4\theta}.$$

Indeed we can use Hölder's inequality

$$\||v|^{p-1}v - |w|^{p-1}w\|_{L^2(\mathbb{R}^n)} \lesssim \|u-w\|_{L^r}(\|v\|^{p-1}_{L^{q(p-1)}(\mathbb{R}^n)} + \|w\|^{p-1}_{L^{q(p-1)}(\mathbb{R}^n)})$$

provided

$$\frac{1}{2} = \frac{1}{r} + \frac{1}{q}.$$

Hence, we need the embedding $H^{2\theta}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ and $H^{2\theta}(\mathbb{R}^n) \hookrightarrow L^{q(p-1)}(\mathbb{R}^n)$ given by

$$\begin{cases} (n-4\theta) < 2n/r\,, \\ (n-4\theta)(p-1) < n - 2n/r\,. \end{cases}$$

For $\theta > n/4$ these relations are trivially satisfied for any $r \ge 2$ and $p > 1$. Let $\theta < n/4$ then we are looking for

$$\begin{cases} 2 \le r < \frac{2n}{n-4\theta}\,, \\ (n-4\theta)(p-1) < n - 2n/r\,. \end{cases}$$

The first condition is optimized by taking $r = \frac{2n}{n-4\theta}$ hence

$$\theta < \frac{n}{4}, \quad 1 < p < 1 + \frac{4\theta}{n-4\theta}. \tag{28}$$

Also the energy estimates (19), (20), (21) and Corollary 1 holds true if we take $u_0 \in H^{2\theta}(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$. If we want to prove global in time existence in $H^{2\theta}(\mathbb{R}^n)$, then we need that $H^{2\theta}(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$ and then we assume

$$1 < p < \frac{4\theta + n}{n-4\theta}. \tag{29}$$

If the range for p in (28) is smaller than this, then Theorem 1.1 holds once (28) holds.

Still we have Theorem 1.2, but we can add something better. Indeed combining the energy estimates with the sharp fractional Sobolev inequality

$$\|u(t,\cdot)\|_{L^{\frac{2n}{n-4\theta}}(\mathbb{R}^n)} \lesssim \|(-\Delta)^\theta u(t,\cdot)\|_{L^2(\mathbb{R}^n)}$$

we get

$$u \in L_t^\infty L_x^{\frac{2n}{n-4\theta}}.$$

For the proof of the last sharp fractional Sobolev inequalities and its variant one can see [5]. As a conclusion, Theorem 1.2 holds when (28) is satisfied, moreover

$$u \in L_t^\infty L_x^m, \quad m \in \left[p+1, \frac{2n}{n-4\theta}\right].$$

Remark 7. Let us briefly discuss the condition (29), that is, the upper bound

$$p_c(n,\theta) = \frac{4\theta+n}{n-4\theta}. \tag{30}$$

Indeed the symbol of the equation is given by

$$p(\tau,\xi) = \tau^2 + |\xi|^{4\theta} - 2\mathrm{i}\mu|\xi|^{2\theta}\tau,$$

with $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. When we scale this symbol by using $\lambda > 0$, we see that

$$p(\lambda\tau, \lambda^{\frac{1}{2\theta}}\xi) = \lambda^2 p(\tau,\xi).$$

According to the notation of [6], the operator has quasi-homogeneous dimension $Q = 1 + \frac{n}{2\theta}$ and the corresponding Gagliardo-Nirenberg type exponent is $2^*(Q-1) = p_c$. This exponent also realizes the comparison level of the different terms of the energy (17), since it is the critical exponent for the embedding $H^{2\theta}(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$.

Acknowledgments. The Author is grateful to the organizers of Evolution Equation and PDEs on the occasion of the 60th birthday of Gisèle Ruiz Goldestein for the kind invitation to give a talk and write about these arguments. The Author thanks the referee for the valuable remarks.

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Received March 2019; revised June 2019.

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