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The critical exponent for semilinear σ-evolution equations with a strong non-effective damping

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Abstract

In this paper, we find the critical exponent for the existence of global small data solutions to:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^{\frac{\theta}{2}} u_t = f(u, u_t), & t \geq 0, \ x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (0, u_1(x)), \end{cases}$$

in the case of so-called *non-effective damping*, $\theta \in (\sigma, 2\sigma]$, where $\sigma \neq 1$ and $f = |u|^\alpha$ or $f = |u_t|^\alpha$, in low space dimension. By critical exponent we mean that global small data solution exists for supercritical powers $\alpha > \tilde{\alpha}$ and do not exist, in general, for subcritical powers $1 < \alpha < \tilde{\alpha}$. Assuming initial data to be small in L^1 or in some other L^p space, $p \in (1, 2)$, in addition to the energy space, the critical exponent only depends on the ratio $n/(\sigma p)$. We also prove the global existence of small data solutions in high space dimension for $\alpha > \bar{\alpha}$, but we leave open to determine if a counterpart nonexistence result for $\alpha < \bar{\alpha}$ holds or not.

Keywords: semilinear evolution equations, noneffective damping, $L^p - L^q$ estimates, critical exponent, global existence, small data solutions
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1. Introduction

In this paper we study the critical exponent of small data global-in-time solutions for the forward Cauchy problem for a σ-evolution equation with a so-called structural damping and with a power nonlinearity $f(u)$,

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^{\frac{\theta}{2}} u_t = f(u), & t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = 0, \\ u_t(0, x) = u_1(x), \end{cases} \tag{1}$$

or with a power nonlinearity $f(u_t)$,

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^{\frac{\theta}{2}} u_t = f(u_t), & t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = 0, \\ u_t(0, x) = u_1(x). \end{cases} \tag{2}$$

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We assume that $f(u) = |u|^\alpha$, for some $\alpha > 1$, or, more in general, that f verifies the following local Lipschitz-type condition

$$|f(u) - f(v)| \leq C\,|u-v|\,(|u|^{\alpha-1} + |v|^{\alpha-1}). \tag{3}$$

The important information is that the nonlinearity in (3) has no sign, i.e., it is not of prescribed type $\pm u|u|^{\alpha-1}$, so it is in general a perturbation which may create blow-up in finite time.

The term $(-\Delta)^\sigma$ in (1) and (2) denotes the σ-th power of the Laplace operator, and may possibly be non-integer. In the non-integer case, $(-\Delta)^\sigma\varphi = \mathscr{F}^{-1}(|\xi|^{2\sigma}\hat{\varphi})$, when $\varphi \in H^{2\sigma}$, where $\mathscr{F}$ denotes the Fourier transform with respect to the x variable.

In recent years there has been a growing attention to find the so-called *critical exponent* $\bar{\alpha}$ for problems like (1) and (2). By critical exponent, we mean that global solutions to (1) or (2) exist for sufficiently small data in some space, when $\alpha > \bar{\alpha}$, whereas solutions cannot exist globally, in general, when $\alpha \in (1, \bar{\alpha})$, assuming a suitable sign condition on the data. The critical case $\alpha = \bar{\alpha}$ sometimes belongs to the nonexistence interval (as in Theorems 1 and 2), and sometimes to the existence interval (as in Theorems 3 and 4).

The study of these problems goes back to the work of H. Fujita [21] about the semilinear heat equation. In general, nonlinear phenomena may break the boot-strap argument which allows to prolong local-in-time solutions. H. Fujita investigated how this occurrence is prevented for sufficiently small initial data if, and only if, the power nonlinearity is larger than a given threshold exponent.

It is clear that the action of the damping term $(-\Delta)^{\frac{\theta}{2}}u_t$ dissipates the energy

$$E[u](t) = \frac{1}{2}\,\|u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\,\|(-\Delta)^{\frac{\sigma}{2}}u(t,\cdot)\|_{L^2}^2, \tag{4}$$

of the corresponding linear problem for the σ-evolution equation

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^{\frac{\theta}{2}}u_t = 0, & t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = 0, \\ u_t(0,x) = u_1(x). \end{cases} \tag{5}$$

However, the presence of the fractional power $(-\Delta)^{\frac{\theta}{2}}$ also deeply modifies the asymptotic profile, as $t \to \infty$, of the solution to (5). According to a classification introduced for more general problems in [10] (see also [57, 58] for the original definition with classical damping and time-dependent coefficients), we say that the damping is *effective* when $\theta \in [0, \sigma)$ and *non-effective* when $\theta \in (\sigma, 2\sigma]$. In the limit case $\theta = \sigma$ the equation in (5) is scale-invariant.

In the effective case $\theta \in [0, \sigma)$, the asymptotic profile of the solution to (5) is determined by the solution to a diffusive problem (see [8, 11, 33]). This phenomenon allowed in recent years to determinate that the critical exponents in the effective case for (1) and (2) (see [9, 11, 16]), respectively, are

$$\alpha_0^{\text{eff}}(n, \sigma, \theta) = 1 + \frac{2\sigma}{n-\theta}, \quad \text{for } n > \theta, \text{ and} \qquad \alpha_1^{\text{eff}}(n/\theta) = 1 + \frac{\theta}{n}.$$

These are the same exponents appearing in Proposition 1.1 in the effective case and in the limit scale-invariant case $\theta = \sigma$. When $n \leq \theta$, there is no global-in-time solution to (1), in general (see Proposition 1.1). The limit scale-invariant case $\theta = \sigma$ is in general easier, due to the fact that the oscillations have the same scaling of the diffusive part of the solution.

The case of wave equation with classical damping, i.e. $\sigma = 1$ and $\theta = 0$, has been first investigated by A. Matsumura [39], who determined the existence of small data global-in-time solutions to (1) in the supercritical case $\alpha > 1 + 2/n$ in space dimension $n = 1, 2$. Only later on, the result has been extended to any space dimension $n \geq 3$ by G. Todorova and B. Yordanov [56] (see also [29]), with the nonexistence counterpart proved in [60] and the diffusion phenomenon showed in [27, 28, 38, 43]. The critical case for more general nonlinearities has been recently discussed in [18].

The situation is completely different in the non-effective case, since oscillations appear in the asymptotic profile of the solution, as in the case of the undamped model. This in general deeply modifies the critical exponent for the semilinear problem, as one may see comparing the Fujita exponent $1+2/n$ of the heat equation [21] and of the damped wave equation [39, 56], with the Strauss [52] exponent $\bar{\alpha}$ of the wave equation (see [32], see also [1, 23, 24, 25, 26, 31, 35, 37, 49, 51, 55, 59, 61]), obtained as the solution to

$$(\alpha-1)\left(\frac{n-1}{2}+\frac{1}{\alpha}\right)=2. \tag{6}$$

The transition from Fujita exponent $1+2/n$ to a shifted Strauss exponent, for a special time-dependent damping, has been recently shown in space dimension $n=1$ in [7].

In model (5), the interplay between the diffusive part and the oscillating part of the solution to (5) leads to a delicate equilibrium for the asymptotic profile of the solution. Until now, it was not clear how to find the critical exponent of problems (1) and (2). In this paper, we give a final and someway apparently surprising answer to this question. The critical exponents for (1) and (2) in the noneffective case $\theta\in(\sigma,2\sigma]$ are, respectively, $\alpha_0(n/\sigma)$ and $\alpha_1(n/\sigma)$, where

$$\alpha_0(n/\sigma)=1+\frac{2}{\frac{n}{\sigma}-1}\,,\quad \text{for } n>\sigma, \text{ and} \qquad \alpha_1(n/\sigma)=1+\frac{\sigma}{n}\,, \tag{7}$$

at least in low space dimension (see Theorems 1 and 2). When $n\leq\sigma$, there is no global solution to (1), in general (see Proposition 1.1). In particular, α_0 and α_1 do not depend at all on the power θ of the damping, and they are the ones obtained by scaling arguments and test function method in [11]. In particular, they are the same obtained in the limit case $\theta=\sigma$ (but in this latter, easier, case, they are valid in any space dimension $n\geq 1$). The critical exponent $\alpha_0(n/\sigma)$ is also the same obtained for the undamped $\sigma-$ evolution equation with power nonlinearity $|u|^\alpha$ in space dimension $\sigma<n\leq 2\sigma$, see [20, Theorem 2.2]. We stress that, on the contrary, the decay rate of the energy for (1) and (2) only depends on the power θ, and is independent of the power σ (see later, (11) and (14)). This phenomenon highlights how energy methods were not sufficient to find the critical exponents for (1) and (2), and the use of L^p-L^q estimates is crucial in the case of non-effective damping.

In order to obtain our global existence result we apply some optimal L^p-L^q decay estimates, $1\leq p\leq q\leq\infty$, for the solution to the linear Cauchy problem (5) recently obtained in [12], which we collect, for the ease of reading, in Section 2. Our existence results are also optimal, that is, global-in-time solutions do not exists in the subcritical and critical cases, in general (see Proposition 1.1).

In Section 5, we collect some existence results in high space dimension for $\sigma>1$, but the existence exponent for (1) becomes larger than the one in (7), as a consequence of the oscillations in the asymptotic profile of the solution. So far, we do not know whether these existence results in high space dimension are optimal or not, since we do not know if a counterpart nonexistence result hold for this larger range of powers. For $\sigma>1$ and high space dimension, we are able to prove the global existence of small data solutions for $\alpha\geq\bar{\alpha}(n,\sigma)$, where $\bar{\alpha}(n,\sigma)$ is the solution to

$$(\alpha-1)\left(\frac{n}{2}+\frac{\sigma-1}{\alpha}\right)=2\sigma. \tag{8}$$

We mention that $\bar{\alpha}(n,\sigma)$ is defined as the solution to a quadratic equation, as the Strauss exponent for the wave equation is, whereas $\alpha_0(n/\sigma)$ in (7) and the Fujita exponent for the heat equation are defined as the solution to a linear equation.

The proof of this result is based on the fact that the critical exponents to (1) and (2) change if initial data are assumed to be small in the energy space and in L^p, $p\in(1,2)$, instead of L^1, namely, the quantity n/σ in (7) is replaced by $n/(\sigma p)$, see Theorems 3 and 4 for the existence results, and Proposition 4.1 for the nonexistence results. From this point of view, we may say that the existence exponent in (8), valid in high space dimension, is critical if initial data are assumed in L^p, where p solves $\bar{\alpha}(n,\sigma)=\alpha_0(n/(p\sigma))$. That is, assuming a suitable sign condition for L^p initial data (see (46)), no global-in-time solution may exist for $\alpha\in(1,\bar{\alpha})$. It is not clear if this exponent may be lowered assuming better integrability of the initial data.

The possibility that the critical exponent for a dissipative equation changes expression when the space dimension is larger than some threshold, is not new. Keel and Tao [34] conjectured for the semilinear Klein-Gordon equation that the critical exponent changes from $1+2/n$ in space dimension $n\leq 3$ to some larger exponent in space dimension $n\geq 4$; indeed, the existence of global-in-time small data solutions holds for powers larger than Strauss exponent [52]. For a semilinear wave equation with time-dependent damping $2t^{-1}u_t$ the critical exponent changes from $1+2/n$ in space dimension $n = 1, 2$ to a shifted Strauss exponent [14, 15, 44] (namely, n is replaced by $n + 2$ in (8)) in space dimension $n \geq 3$.

The case $\sigma = 1$ is not included in this paper, since the $L^p - L^q$ estimate obtained in [12], see (20), fails at $\sigma = 1$. This singularity at $\sigma = 1$ is a well-known difference between the case $\sigma = 1$ and $\sigma \neq 1$ (for instance, the Hessian matrix of $|\xi|^\sigma$ is singular if, and only if, $\sigma = 1$), see [45, 53]. So far, only a partial result is provided for $\sigma = 1$ and $\theta \in (1, 2]$ in [16], but the existence exponent therein is likely far from being optimal.

However, the general case $\sigma \neq 1$ is of interest. Equations whose "principal part" is $u_{tt} + (-\Delta)^\sigma u = 0$, like the plate equation which is attained for $\sigma = 2$, are called σ-evolution equations in the sense of Petrowsky (see [19]), since their symbols $\tau^2 + |\xi|^{2\sigma}$ have only pure imaginary, distinct, roots $\tau = \pm i|\xi|^\sigma$ for all $\xi \neq 0$. The set of 1-evolution operators coincides with the set of strictly hyperbolic operators.

The case $\sigma = 2$ in (5) is an important model in the literature, known as beam operator and plate operator in the case of space dimension $n = 1$ and $n = 2$, respectively. Models to study the vibrations of thin plates ($n = 2$) given by the full von Kármán system have been studied by several authors, in particular, see [6, 36, 46]. We mention that some plate models include also a term $-\Delta u_{tt}$ called rotational inertia. Energy estimates for solutions, for which a regularity-loss type decay appears, have been investigated in [2, 3, 4, 54].
Models in (5) with $\theta = 0$ and $\theta = 2\sigma$ have been studied in the abstract setting:

$$u''(t) + u'(t) + Au(t) = 0, \quad \text{and} \qquad u''(t) + Au'(t) + Au(t) = 0,$$

where A is a nonnegative self-adjoint operator in a real Hilbert space H (see [5] and [30] and reference therein). For a study of the dissipative effects for thermoelastic plate equations, where the heat conduction is modeled by Cattaneo's law or by Fourier's law, we address the reader to [47] for the linear case and to [48] for the global existence of small data solutions to a quasilinear model.

We are now ready to state our two main results, in which initial data are assumed to be small in the energy space and in the L^1 space. This class of initial data includes, in particular, compactly supported smooth initial data.

Theorem 1. *Assume that* $\sigma \in (2/3, 1)$ *and* $n = 1$*, or that* $1 < \sigma < n \leq \bar{n}(\sigma)$*, where*

$$\bar{n}(\sigma) = (3\sigma - 2)\left[1 + \frac{1}{2}\left(\sqrt{1 + 8\sigma(3\sigma - 2)^{-2}} - 1\right)\right]. \tag{9}$$

Also assume that the damping is noneffective*, i.e.,* $\theta \in (\sigma, 2\sigma]$*. Fix* $\alpha > \alpha_0(n/\sigma)$*, where* $\alpha_0(n/\sigma)$ *is as in* (7)*, and let* $\eta = 2$ *if* $n < 2\theta$*, or* $\eta > n/\theta$ *if* $n \geq 2\theta$*. Then there is a constant* $\epsilon > 0$ *such that for any*

$$u_1 \in L^1 \cap L^\eta \quad \textit{with} \qquad \|u_1\|_{L^1} + \|u_1\|_{L^\eta} < \epsilon, \tag{10}$$

there exists a uniquely determined energy solution

$$u \in C([0, \infty), H^\sigma \cap L^\alpha \cap L^\infty) \cap C^1([0, \infty), L^2),$$

to (1)*. Moreover, the solution satisfies the energy estimate*

$$E[u](t) \leq C\,(1 + t)^{-\frac{n}{\theta}}\,(\|u_1\|^2_{L^1} + \|u_1\|^2_{L^\eta}), \tag{11}$$

where $E[u](t)$ *is as in* (4)*, and the decay estimate*

$$\|u(t, \cdot)\|_{L^q} \leq C\,(1 + t)^{1 - \frac{n}{\sigma}\left(1 - \frac{1}{q}\right)}\,(\|u_1\|_{L^1} + \|u_1\|_{L^\eta}), \qquad \forall q \in [\alpha, \infty],$$

where the constant $C > 0$ *does not depend on the initial data.*

In particular, Theorem 1 applies to the case of plate equation, i.e., $\sigma = 2$, for $\alpha > 5$ if $n = 3$ and for $\alpha > 3$ if $n = 4$.

Remark 1.1. We stress that $\bar{n}(\sigma)$ in (9) is the solution to the second order equation

$$n^2 - (3\sigma - 2)n - 2\sigma = 0, \tag{12}$$

and it satisfies the following properties: $\bar{n}(\sigma) \sim 3\sigma - 2$ as $\sigma \to \infty$, and $\bar{n}(\sigma) - (3\sigma - 2)$ is a decreasing function with respect to σ, with its infimum given by $\bar{n}(\sigma) - (3\sigma - 2) \to 1$ as $\sigma \to 1$. In particular, $\bar{n}(\sigma) \in (3\sigma - 2, 3\sigma - 1)$ for any $\sigma > 1$. We postpone the discussion of global solutions to (1) in high space dimension to Section 5.

Theorem 2. *Assume that $\sigma \geq 3$ and that $n \leq \sigma - 2$. Also assume that the damping is* noneffective*, i.e., $\theta \in (\sigma, 2\sigma]$. Fix $\alpha > \alpha_1(n/\sigma)$, where $\alpha_1(n/\sigma)$ is as in* (7)*. Then there is a constant $\epsilon > 0$ such that for any*

$$u_1 \in L^1 \cap L^\alpha \quad \text{with} \qquad \|u_1\|_{L^1} + \|u_1\|_{L^\alpha} < \epsilon, \tag{13}$$

there exists a uniquely determined energy solution $u \in C([0,\infty), H^\sigma) \cap C^1([0,\infty), L^2 \cap L^\alpha)$ to (2)*.*

Moreover, the solution satisfies the energy estimate

$$E[u](t) \leq C\,(1+t)^{-\frac{n}{\theta}}\,(\|u_1\|_{L^1}^2 + \|u_1\|_{L^\alpha}^2), \tag{14}$$

where $E[u](t)$ is as in (4)*, and the decay estimate*

$$\|u_t(t,\cdot)\|_{L^\alpha} \leq C\,(1+t)^{-\frac{n}{\sigma}(1-\frac{1}{\alpha})}\,(\|u_1\|_{L^1} + \|u_1\|_{L^\alpha}),$$

where the constant $C > 0$ does not depend on the initial data.

We mention that a local existence result to (1) and (2) with initial data in an appropriate space may be obtained by standard methods (with no need of assuming supercritical power nonlinearities and small data). In this paper, we are interested in the case in which the local solution may be globally-in-time prolonged. Having this in mind, we will provide a proof for Theorems 1 and 2, based on a contraction argument, which at a glance guarantees the existence of both local and global in time solutions.

The nonexistence counterpart of Theorems 1 and 2 has been given in [11, Theorem 1, Theorem 2] for integer powers $\theta/2$ and σ, in both the effective and noneffective cases, using a test function method which goes back to [40] and some strategies introduced in [17, 41]. More in general, by using a novel test function recently developed in [22], this result has been extended to fractional powers $\theta/2$ and σ. The nonexistence result holds for weak solutions to (1) and (2), in an appropriate sense (see later, (62) and (63)), in particular it applies to energy solutions.

Proposition 1.1 (Examples 6.3 and 6.4 in [13]). *Let $0 \leq \theta \leq 2\sigma$, and assume that $u_1 \in L^1$ verifies*

$$\int_{\mathbb{R}^n} u_1(x)\,dx > 0. \tag{15}$$

Then there exists no global weak solution to (1) *with $f(u) = |u|^\alpha$,*

- *for any $\alpha > 1$ if $n \leq \min\{\theta, \sigma\}$;*
- *for any*
$$\alpha \in \left(1, 1 + \frac{2\sigma}{n - \min\{\theta,\sigma\}}\right],$$
if $n > \min\{\theta, \sigma\}$.

Moreover, there exists no global weak solution to (2) *with $f(u_t) = |u_t|^\alpha$, for any*

$$\alpha \in \left(1, 1 + \frac{\min\{\theta,\sigma\}}{n}\right].$$

Proposition 1.1 proves that the exponents $\alpha_0(n/\sigma)$ and $\alpha_1(n/\sigma)$ for (1) and (2), in the noneffective case $\theta \in (\sigma, 2\sigma]$ are really critical, in the sense that global existence of small data solutions do not hold in the subcritical and critical cases, in general.

In Section 4 we discuss how the critical exponents for Cauchy problems (1) and (2) change if the initial data are small in L^p, for some $p > 1$, and not in L^1. In Section 5, we discuss how the use of the results in Section 4, together with some other results, may be used to obtain a global existence result for (1) in high space dimension. However, in high space dimension, we leave open the problem to investigate whether the proposed exponent is critical or not. That is, we cannot prove a nonexistence result for powers smaller than the proposed exponent, neither we have basis to conjecture that this can be done, in some way, or not.

Remark 1.2. In this paper, we assumed $u(0, x) = 0$ in (1) and (2), for the sake of brevity. However, it is not difficult to extend the smallness assumption on the initial data to include a non-zero first initial data in the statements.

2. Estimates for the linear Cauchy problem (5)

In this section, we collect the sharp estimates for the solution to (5), when $\sigma \neq 1$, $\sigma < \theta \leq 2\sigma$, which will be used to prove Theorems 1 and 2, via the application of Duhamel's principle. The following estimates and their sharpness are proved in the recent paper [12]. The sharpness of the estimates employed is connected to the optimality of the critical exponent for the nonlinear problems (1) and (2), which is guaranteed by Proposition 1.1.

Assuming $u_1 \in L^p \cap L^2$ for some $p \in [1,2]$, the following energy estimate for the solution to (5) may be easily derived (see also [12, Example 3.1]):

$$E[u](t) \leq C\,(1+t)^{-\frac{2n}{\theta}\left(\frac{1}{p}-\frac{1}{2}\right)}\,(\|u_1\|_{L^p}^2 + \|u_1\|_{L^2}^2), \tag{16}$$

where $E[u](t)$ is as in (4), for some $C > 0$ and $c > 0$. Assuming initial data only in L^p for $p \in [1,2)$ and $n(1/p - 1/2) \leq \theta$, we may replace (16) with a singular (at $t = 0$) energy estimate:

$$E[u](t) \leq C\,(1+t)^{-\frac{2n}{\theta}\left(\frac{1}{p}-\frac{1}{2}\right)}\,\|u_1\|_{L^p}^2 + t^{-2\delta}\,e^{-ct}\,\|u_1\|_{L^p}^2, \tag{17}$$

where $\delta > n/(2\theta)$ if $p = 1$ or

$$\delta = \frac{n}{\theta}\left(\frac{1}{p} - \frac{1}{2}\right),$$

if $p \in (1,2)$.

When dealing with more general $L^p - L^q$ estimates, with $1 \leq p \leq q \leq \infty$, due to the nature of the σ-evolution equation with a noneffective damping, the decay rate is influenced by the quantity:

$$d(p,q) = n\left(\frac{1}{p} - \frac{1}{q}\right) + n\sigma \max\left\{\frac{1}{2} - \frac{1}{p}, \frac{1}{q} - \frac{1}{2}\right\}. \tag{18}$$

The quantity $d(p,q)$ is related to the following property (see [42, Theorem 4.2]): assuming $u_1 \in L^p$, the solution $u(t,\cdot)$ to the Cauchy problem for the undamped σ-evolution equation

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u = 0, & t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = 0, \\ u_t(0,x) = u_1(x), \end{cases} \tag{19}$$

is in L^q for any $t > 0$ if, and only if, $d(p,q) \leq \sigma$ if $1 < p \leq q < \infty$, or $d(p,q) < \sigma$ if $1 = p \leq q \leq \infty$ or $1 < p \leq q = \infty$.

For a given $p_2 \in [p,q]$, we also define

$$a = a(p_2,q) = n\left(\frac{1}{p_2} - \frac{1}{q}\right).$$

In [12, Theorems 2.1 and 9.1], it is proved that

$$\|\partial_t^j u(t,\cdot)\|_{L^q} \leq C\,(1+t)^{1-j-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)}\,\|u_1\|_{L^p} + C\,t^{-\delta}\,e^{-ct}\,\|u_1\|_{L^{p_2}}, \tag{20}$$

for $j = 0, 1, \ldots$, provided that $d(p,q) < \sigma(1-j)$ and where $\delta \geq 0$ is defined as follows:

- if $\theta < 2\sigma$ and $a \geq \theta$, then
$$\delta = j + \frac{a - \theta}{2\sigma - \theta},$$
if $p_2, q \in (1,\infty)$, whereas δ may be any positive number verifying
$$\delta > j + \frac{a - \theta}{2\sigma - \theta},$$
if $p_2 = 1$ or $q = \infty$;

- if $\theta \leq 2\sigma$ and $a \leq \theta$, then
$$\delta = \left(j - 1 + \frac{a}{\theta}\right)_+ ,$$
if $p_2, q \in (1, \infty)$, whereas δ may be any nonnegative number verifying
$$\delta > j - 1 + \frac{a}{\theta},$$
if $p_2 = 1$ or $q = \infty$, provided that we assume $a < \theta$ and $1/p_2 - 1/q \geq 1/2$ if $\theta = 2\sigma$.

In particular, when $\delta > 0$, estimate (20) is singular at $t = 0$.

We stress a crucial difference between (16)-(17), and (20): the decay rate in (16) and (17) depends on n/θ, whereas the decay rates in (20) depend on n/σ. This latter property allows us to use estimate (20) to find the critical exponents $\alpha_0(n/\sigma)$ and $\alpha_1(n/\sigma)$, which are independent of θ, for the Cauchy problems (1) and (2).

When the condition $d(p, q) < \sigma(1 - j)$ is violated, it is still possible to obtain $L^p - L^q$ decay estimates, but with a worse decay rate. Since the decay rate is crucial to determine the critical exponents $\alpha_0(n/\sigma)$ and $\alpha_1(n/\sigma)$, we may obtain our global existence results for supercritical powers $\alpha > \alpha_0(n/\sigma)$ and $\alpha > \alpha_1(n/\sigma)$ only in low space dimension. In Section 5, we discuss the case of estimates with a different decay rate and how this influences the condition on α for the global existence of small data solutions.

3. Proof of Theorems 1 and 2

Let K denote the fundamental solution to (5), that is,
$$u^{\mathrm{lin}}(t, x) = K(t, x) *_{(x)} u_1(x) ,$$
is the solution to the linear Cauchy problem (5).

A function $u \in X$, where X is a suitable space, is a solution to (1) or (2) in X if, and only if, it satisfies the equality
$$u(t, x) = u^{\mathrm{lin}}(t, x) + Fu(t, x) , \qquad \text{in } X, \tag{21}$$
where F is the operator such that, for any $u \in X$,
$$Fu(t, x) = \int_0^t K(t - s, x) *_{(x)} f(u(s, x))\, ds , \tag{22}$$
if we consider Cauchy problem (1) or, respectively,
$$Fu(t, x) = \int_0^t K(t - s, x) *_{(x)} f(u_t(s, x))\, ds , \tag{23}$$
if we consider Cauchy problem (2).

The proof of our global existence results is based on the following scheme. We define an appropriate initial data function space $\mathcal{A}$ and a space for solutions $X(T)$, equipped with a norm induced by some of the decay estimates we obtained for u^{lin}, assuming initial data in $\mathcal{A}$. In particular, we look for a norm such that the estimate
$$\|u^{\mathrm{lin}}\|_{X(T)} \leq C_1 \|u_1\|_{\mathcal{A}}, \tag{24}$$
holds with C_1 independent of T. Then we prove that the estimate
$$\|Fu - Fv\|_{X(T)} \leq C_2 \|u - v\|_X \big(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\big) , \tag{25}$$
with C_2 independent of T. In particular, for $v = 0$, estimate (25) reduces to
$$\|Fu\|_{X(T)} \leq C \|u\|_{X(T)}^{\alpha} . \tag{26}$$

We now define

$$R = 2C_1\,\|u_1\|_{\mathcal{A}}.$$

For sufficiently small data, $2C_2R^{\alpha-1} \leq 1/2$. Then, by (24) and (26) it follows that the operator $u^{\rm lin}(t,x) + F$ maps the ball $B_R = \{u : \|u\|_X \leq R\}$ in itself. Due to (25), it is a contraction. Therefore, there is a unique fixed point for $u^{\rm lin}(t,x) + F$ in B_R, that is, a unique solution to (21). Moreover, $\|u\|_X \leq 2C_1\,\|u_1\|_{\mathcal{A}}$.

The information that $u \in X$ plays a fundamental role to estimate $f(u(s,\cdot))$ or $f(u_t(s,\cdot))$ in suitable norms. We will employ the following well-known result.

Lemma 3.1. *Let $\nu > -1$, $\mu \in \mathbb{R}$, and $c > 0$. Then it holds*

$$\int_0^t (t-s)^\nu\,(1+s)^\mu\,ds \leq \begin{cases} C\,(1+t)^\nu & \text{if } \mu < -1,\\ C\,(1+t)^\nu\,\log(e+t) & \text{if } \mu = -1,\\ C\,(1+t)^{1+\nu+\mu} & \text{if } \mu > -1,\end{cases}$$

and

$$\int_0^t (t-s)^\nu\,e^{-c(t-s)}\,(1+s)^\mu\,ds \leq C\,(1+t)^\mu,$$

for some $C = C(\nu,\mu,c) > 0$. The estimate is also valid if $(t-s)^\nu$ is replaced by $(1+t-s)^\nu$ in the integral.

Lemma 3.1 has been proved in many different versions by many authors. One earlier version of this lemma goes back to [50].

3.1. *Proof of Theorem 1*

In order to prove Theorem 1, for any $T > 0$, we fix the initial data space to be $\mathcal{A} = L^1 \cap L^\eta$, where $\eta = 2$ if $n < 2\theta$ and $\eta > n/\theta$ if $n \geq 2\theta$, and we introduce the solution space

$$X(T) = C([0,T], H^\sigma \cap L^\alpha \cap L^\infty) \cap C^1([0,T], L^2),$$

equipped with norm

$$\|u\|_{X(T)} = \max_{t\in[0,T]} \Big((1+t)^{\frac{n}{2\theta}}\,\sqrt{E[u](t)} + \sum_{q=\alpha,\infty} (1+t)^{-1+\frac{n}{\sigma}\left(1-\frac{1}{q}\right)} \|u(t,\cdot)\|_{L^q} \Big), \tag{27}$$

where $E[u](t)$ is as in (4).

In particular, any function u in $X(T)$ verifies the decay estimates

$$\begin{aligned} \|u(t,\cdot)\|_{L^\alpha} &\leq (1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{\alpha}\right)}\,\|u\|_{X(T)}, \quad \forall t \in [0,T],\\ \|u(t,\cdot)\|_{L^\infty} &\leq (1+t)^{1-\frac{n}{\sigma}}\,\|u\|_{X(T)}, \quad \forall t \in [0,T], \end{aligned}$$

and, by interpolation, the estimate

$$\|u(t,\cdot)\|_{L^{\beta\alpha}} \leq (1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{\beta\alpha}\right)}\,\|u\|_{X(T)}, \quad \forall t \in [0,T], \tag{28}$$

for any $u \in X(T)$ and $\beta \in [1,\infty]$. By Hölder inequality, using (3) and (28), we find that

$$\begin{aligned} &\|(f(u)-f(v))(s,\cdot)\|_{L^\beta}\\ &\quad \leq C_1\,\|(u-v)(|u|^{\alpha-1}+|v|^{\alpha-1})(s,\cdot)\|_{L^\beta}\\ &\quad \leq C_1\,\|(u-v)(s,\cdot)\|_{L^{\beta\alpha}}\,\big(\|u(s,\cdot)\|_{L^{\beta\alpha}}^{\alpha-1} + \|v(s,\cdot)\|_{L^{\beta\alpha}}^{\alpha-1}\big)\\ &\quad \leq C_2\,(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)-\frac{n}{\sigma}\left(1-\frac{1}{\beta}\right)}\,\|u-v\|_{X(T)}\,\big(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\big), \end{aligned} \tag{29}$$

for any $\beta \geq 1$.

We divide the proof of (24) and (25) in two lemmas.

Lemma 3.2. *Let $n = 1$ and $\sigma \in (2/3, 1)$ or $1 < \sigma < n \leq \bar{n}(\sigma)$, and let $\alpha > \alpha_0(n/\sigma)$. If $u_1 \in L^1 \cap L^\eta$, where $\eta > n/\theta$, then*

$$\|u^{\mathrm{lin}}(t,\cdot)\|_{L^q} \leq C_1\,(1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\,\|u_1\|_{L^1\cap L^\eta}, \quad q = \alpha, \infty, \tag{30}$$

with $C_1 > 0$ independent of $T > 0$. If $u, v \in X(T)$, then

$$\|(Fu - Fv)(t,\cdot)\|_{L^q} \leq C_2\,(1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\,\|u - v\|_X(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}), \quad q = \alpha, \infty, \tag{31}$$

with $C_2 > 0$ independent of $T > 0$.

We mention that the restriction $\eta \geq 2$ is not necessary in Lemma 3.2, but it will come into play in the statement of Theorem 1, since we also use Lemma 3.3 to prove it.

Proof. We assume in a first moment that $\theta \in (\sigma, 2\sigma)$.

We notice that $d(p,q) < \sigma$, where d is defined in (18) holds with $p = 1$ if, and only if,

$$n(\sigma - 1)\,\frac{1}{q} + n\left(1 - \frac{\sigma}{2}\right) < \sigma. \tag{32}$$

If $\sigma < 1$, the previous inequality holds for any q if $n(1 - \sigma/2) < \sigma$, that is, $n = 1$ and $\sigma > 2/3$. If $\sigma > 1$, (32) holds for any $q > \alpha_0(n/\sigma)$ if, and only if,

$$n(\sigma - 1)\,\frac{n - \sigma}{n + \sigma} + n\left(1 - \frac{\sigma}{2}\right) \leq \sigma,$$

that is, $n \leq \bar{n}(\sigma)$. On the other hand, setting $p_2 = \min\{q, \eta\}$, we may take $\delta = 0$ in (20), due to $\eta > n/\theta$.

Therefore, by (20) with $j = 0$, $p = 1$ and $p_2 = \min\{q, \eta\}$, we immediately obtain (30). To deal with $Fu - Fv$, we distinguish two cases.

If $n < 2\sigma$, by using (20) with $j = 0$, $p = 1$, $q = \alpha, \infty$, $p_2 = q$ and $\delta = 0$ (here there is no need to take $p_2 = \min\{q, \eta\}$, as we did for u^{lin}), we obtain

$$\begin{aligned}\|(Fu - Fv)(t,\cdot)\|_{L^q} \leq C &\int_0^t (1+t-s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\,\|(f(u) - f(v))(s,\cdot)\|_{L^1}\,ds\\ &+ C \int_0^t e^{-c(t-s)}\,\|(f(u) - f(v))(s,\cdot)\|_{L^q}\,ds.\end{aligned}$$

Thanks to (29) with $\beta = 1, q$, we may estimate

$$\|(Fu - Fv)(t,\cdot)\|_{L^q} \leq C\,\|u - v\|_{X(T)}\,(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1})\,(I_1(t) + I_2(t)),$$

where

$$\begin{aligned}I_1(t) &= \int_0^t (1+t-s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\,(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)}\,ds,\\ I_2(t) &= \int_0^t e^{-c(t-s)}\,(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)}\,ds.\end{aligned}$$

We notice that $1 - n/\sigma > -1$, due to $n < 2\sigma$, whereas

$$\alpha - \frac{n}{\sigma}(\alpha - 1) < -1,$$

if, and only if, $\alpha > \alpha_0(n/\sigma)$. Thanks to Lemma 3.1, we find that

$$\begin{aligned}I_1(t) &\leq C\,(1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)},\\ I_2(t) &\leq C\,(1+t)^{\alpha-\frac{n}{\sigma}(\alpha-1)-\frac{n}{\sigma}\left(1-\frac{1}{q}\right)},\end{aligned}$$

the latter decay rate being faster than the previous one; hence, we obtain (31).

If $n \geq 2\sigma$ (which, in turn, implies that $n > 2$, as a consequence of the assumptions on σ) and $q = \infty$, we cannot use Lemma 3.1. Then we split the integral in (22) in $[0, t/2]$ and $[t/2, t]$, and we use (20) with $j = 0$, $p_2 = \infty$, $\delta = 0$, and $p = 1$ for $s \in [0, t/2]$ or with $p = n/(2\sigma)$ for $s \in [t/2, t]$. We stress that the condition $d(p, q) < \sigma$ with $p = n/(2\sigma)$ and $q = \infty$ reads as

$$\frac{n}{\sigma}\left(\frac{2\sigma}{n} - \frac{1}{\infty}\right) + n\left(\frac{1}{2} - \frac{2\sigma}{n}\right) < 1, \quad \text{i.e.,} \qquad n < 4\sigma - 2.$$

This latter inequality holds as a consequence of $n \leq \bar{n}(\sigma)$ (see Remark 1.1). Using (29) with $\beta = 1, n/(2\sigma), \infty$, we obtain:

$$\|(Fu - Fv)(t, \cdot)\|_{L^\infty} \leq C\, \|u - v\|_{X(T)} \left(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\right) (I_3(t) + I_4(t)),$$

where

$$I_3(t) = \int_0^{t/2} (1 + t - s)^{1-\frac{n}{\sigma}} (1 + s)^{\alpha - \frac{n}{\sigma}(\alpha-1)}\, ds + \int_{t/2}^{t} (1 + t - s)^{-1} (1 + s)^{\alpha - \frac{n}{\sigma}(\alpha-1) - \frac{n}{\sigma} + 2}\, ds,$$

$$I_4(t) = \int_0^t e^{-c(t-s)} (1 + s)^{\alpha - \frac{n}{\sigma}\alpha}\, ds.$$

By using that $t - s \approx t$ for $s \in [0, t/2]$ and $s \approx t$ for $s \in [t/2, t]$, we get

$$I_3(t) = (1 + t)^{1-\frac{n}{\sigma}} \int_0^{t/2} (1 + s)^{\alpha - \frac{n}{\sigma}(\alpha-1)}\, ds + (1 + t)^{\alpha - \frac{n}{\sigma}(\alpha-1) - \frac{n}{\sigma} + 2} \int_{t/2}^{t} (1 + t - s)^{-1}\, ds.$$

The last integral is uniformly bounded with respect to t if, and only if, $\alpha > \alpha_0(n/\sigma)$. On the other hand, under this assumption, we also get

$$(1 + t)^{\alpha - \frac{n}{\sigma}(\alpha-1) - \frac{n}{\sigma} + 2} \int_{t/2}^{t} (1 + t - s)^{-1}\, ds \leq C\, (1 + t)^{1-\frac{n}{\sigma}}.$$

By Lemma 3.1,

$$I_4(t) \leq C\, (1 + t)^{\alpha - \frac{n}{\sigma}\alpha},$$

the latter decay being faster than $(1 + t)^{1-\frac{n}{\sigma}}$.

We proceed similarly when $q = \alpha$. We use (20) with $j = 0$, $p_2 = \alpha$, and $p = 1$ for $s \in [0, t/2]$ and with

$$p = \begin{cases} 2 & \text{if } \alpha < \frac{2n}{(n-2\sigma)_+}, \\ \frac{n\alpha}{n+2\alpha\sigma} & \text{if } \alpha \geq \frac{2n}{(n-2\sigma)_+}, \end{cases} \tag{33}$$

for $s \in [t/2, t]$. We stress that $\alpha > 2$, as a consequence of $n \leq \bar{n}(\sigma)$. With p as in (33), the condition $d(p, \alpha) < \sigma$ holds, as well as

$$1 - \frac{n}{\sigma}\left(\frac{1}{p} - \frac{1}{\alpha}\right) \geq -1.$$

Then we obtain

$$\|(Fu - Fv)(t, \cdot)\|_{L^\alpha} \leq C\, \|u - v\|_{X(T)} \left(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\right) (I_5(t) + I_6(t) + I_7(t))$$

where

$$I_5(t) = \int_0^{t/2} (1 + t - s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{\alpha}\right)} (1 + s)^{\alpha - \frac{n}{\sigma}(\alpha-1)}\, ds,$$

$$I_6(t) = \int_{t/2}^{t} (1 + t - s)^{1-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{\alpha}\right)} (1 + s)^{\alpha - \frac{n}{\sigma}(\alpha-1) - \frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\, ds,$$

$$I_7(t) = \int_0^t e^{-c(t-s)} (1 + s)^{\alpha - \frac{n}{\sigma}\alpha + \frac{n}{\sigma\alpha}}\, ds \leq C(1 + t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{\alpha}\right)}.$$

By using that $t-s\approx t$ for $s\in[0,t/2]$, we get

$$I_5(t)\leq C\,(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{\alpha})}\int_0^{t/2}(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)}\,ds.$$

The last integral is uniformly bounded with respect to t if, and only if, $\alpha>\alpha_0(n/\sigma)$. On the other hand, under this assumption, applying Lemma 3.1, we obtain a faster decay rate for $I_6(t)$ and $I_7(t)$. This concludes the proof for $\theta<2\sigma$.

In the case $\theta=2\sigma$, we proceed as before, but when we deal with $\|(Fu-Fv)(t,\cdot)\|_{L^\infty}$, we fix $p_2=2$, instead of $p_2=\infty$, in (20). Indeed, the condition $n<2\theta$ holds, due to $n\leq\bar{n}(\sigma)\leq 3\sigma-1$ (see Remark 1.1) and $\theta=2\sigma$. Then, using (29) with $\beta=2$, we shall replace $I_2(t)$ and $I_4(t)$ by

$$I_2^\dagger(t)=\int_0^t e^{-c(t-s)}\,(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)-\frac{n}{2\sigma}}\,ds\,.$$

The proof follows again by Lemma 3.1, since the decay rate $(1+t)^{\alpha-\frac{n}{\sigma}(\alpha-1)-\frac{n}{2\sigma}}$ is faster than $(1+t)^{1-\frac{n}{\sigma}}$, due to $n\leq\bar{n}(\sigma)$.

Lemma 3.2 is sufficient to construct a solution to (1) in $C([0,\infty),L^\alpha\cap L^\infty)$. However, to construct an energy solution, we supplement it with the following.

Lemma 3.3. *Let $n>\sigma$ and $\alpha>\alpha_0(n/\sigma)$. If $u_1\in L^1\cap L^2$, then*

$$\sqrt{E[u^{\mathrm{lin}}](t)}\leq C_1\,(1+t)^{-\frac{n}{2\theta}}\,\|u_1\|_{L^1\cap L^2},\tag{34}$$

with $C_1>0$ independent of $T>0$. If $u,v\in X(T)$, then

$$\sqrt{E[Fu-Fv](t)}\leq C_2\,(1+t)^{-\frac{n}{2\theta}}\,\|u-v\|_{X(T)}\,(\|u\|_{X(T)}^{\alpha-1}+\|v\|_{X(T)}^{\alpha-1}),\tag{35}$$

with $C_2>0$ independent of $T>0$.

We stress that Lemma 3.3 holds with no bound from above on the space dimension n.

Proof. The proof of (34) follows applying (16) with $p=1$.

Let $u,v\in X(T)$. We split the integral in (22) in the intervals $[0,t/2]$ and $[t/2,t]$ and we apply (16) in the integral, with $p=1$ for $s\in[0,t/2]$ and with $p=2$ for $s\in[t/2,t]$. Using (29) with $\beta=1,2$, we obtain

$$\sqrt{E[Fu-Fv](t)}\leq C\,\|u-v\|_{X(T)}\,(\|u\|_{X(T)}^{\alpha-1}+\|v\|_{X(T)}^{\alpha-1})\,I(t)$$

where

$$I(t)=\int_0^{t/2}(1+t-s)^{-\frac{n}{2\theta}}\,(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)}\,ds+\int_{t/2}^t(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)-\frac{n}{2\sigma}}\,ds\,.$$

By using that $t-s\approx t$ for $s\in[0,t/2]$ and $s\approx t$ for $s\in[t/2,t]$, we then obtain

$$I(t)\approx(1+t)^{-\frac{n}{2\theta}}\int_0^{t/2}(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)}\,ds+\int_{t/2}^t(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)-\frac{n}{2\sigma}}\,ds.$$

The first integral is uniformly bounded with respect to t if, and only if, $\alpha>\alpha_0(n/\sigma)$. On the other hand, under this assumption, we also get

$$\int_{t/2}^t(1+s)^{\alpha-\frac{n}{\sigma}(\alpha-1)-\frac{n}{2\sigma}}\,ds\leq C\,(1+t)^{-\frac{n}{2\sigma}}\leq C\,(1+t)^{-\frac{n}{2\theta}},$$

due to $\sigma\leq\theta$. Therefore, $I(t)\leq C\,(1+t)^{-\frac{n}{2\theta}}$, that is, we proved (35).

Thanks to Lemmas 3.2 and 3.3, we conclude the proof of Theorem 1. Indeed, (24) follows from (30) and (34), whereas (25) follows from (31) and (35).

3.2. Proof of Theorem 2

In order to prove Theorem 2, for any $T > 0$, we fix the initial data space to be $\mathcal{A} = L^1 \cap L^\alpha$, and we introduce the solution space

$$X(T) = C([0,T], H^\sigma) \cap C^1([0,T], L^2 \cap L^\alpha),$$

equipped with norm

$$\|u\|_{X(T)} = \max_{t\in[0,T]} \big((1+t)^{\frac{n}{2\theta}} \sqrt{E[u](t)} + (1+t)^{\frac{n}{\sigma}(1-\frac{1}{\alpha})}\|u_t(t,\cdot)\|_{L^\alpha}\big),$$

where $E[u](t)$ is as in (4).

In particular, any function u in $X(T)$ verifies the decay estimate

$$\|u_t(t,\cdot)\|_{L^\alpha} \leq (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{\alpha})} \|u\|_{X(T)}, \quad \forall t \in [0,T], \tag{36}$$

for any $u \in X(T)$. By Hölder inequality, using (3) and (36), we find that

$$\begin{aligned} \|(f(u_t) - f(v_t))(s,\cdot)\|_{L^1} &\leq C_1 \|(u_t - v_t)(|u_t|^{\alpha-1} + |v_t|^{\alpha-1})(s,\cdot)\|_{L^1} \\ &\leq C_1 \|(u_t - v_t)(s,\cdot)\|_{L^\alpha} \big(\|u_t(s,\cdot)\|_{L^\alpha}^{\alpha-1} + \|v_t(s,\cdot)\|_{L^\alpha}^{\alpha-1}\big) \\ &\leq C_2 (1+s)^{-\frac{n}{\sigma}(\alpha-1)} \|u-v\|_{X(T)} \big(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\big). \end{aligned} \tag{37}$$

We divide the proof of (24) and (25) in two lemmas.

Lemma 3.4. *Let $n \leq \sigma - 2$ and $\alpha > \alpha_1(n/\sigma)$. If $u_1 \in L^1 \cap L^\alpha$, then*

$$\|\partial_t u^{\text{lin}}(t,\cdot)\|_{L^\alpha} \leq C_1 (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{\alpha})} \|u_1\|_{L^1\cap L^\alpha}, \tag{38}$$

with $C_1 > 0$ independent of $T > 0$. If $u, v \in X(T)$, then

$$\|\partial_t(Fu - Fv)(t,\cdot)\|_{L^\alpha} \leq C_2 (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{\alpha})} \|u-v\|_X \big(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\big), \tag{39}$$

with $C_2 > 0$ independent of $T > 0$.

Proof. We notice that the condition $d(1,q) < 0$ holds for any $q > \alpha_1(n/\sigma)$ if, and only if, $n \leq \sigma - 2$. Therefore, by (20) with $j = 1$, $p = 1$ and $p_2 = q = \alpha$, we obtain (38).

On the other hand, let $p_2 = 1$ and fix $\delta = n/\theta$; we stress that $\delta < 1$. Then, by (20) with $j = 1$, $p = p_2 = 1$ and $q = \alpha$, we obtain

$$\begin{aligned} \|\partial_t(Fu - Fv)(t,\cdot)\|_{L^\alpha} &\leq C \int_0^t (1+t-s)^{-\frac{n}{\sigma}(1-\frac{1}{\alpha})} \|(f(u_t) - f(v_t))(s,\cdot)\|_{L^1}\, ds \\ &\quad + C \int_0^t (t-s)^{-\delta} e^{-c(t-s)} \|(f(u_t) - f(v_t))(s,\cdot)\|_{L^1}\, ds \\ &\leq C \|u-v\|_{X(T)} \big(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\big) I(t), \end{aligned}$$

with

$$I(t) = \int_0^t (1+t-s)^{-\frac{n}{\sigma}(1-\frac{1}{\alpha})} (1+s)^{-\frac{n}{\sigma}(\alpha-1)}\, ds + \int_0^t (t-s)^{-\delta} e^{-c(t-s)} (1+s)^{-\frac{n}{\sigma}(\alpha-1)}\, ds\,,$$

where we used (37). Due to $n < \sigma$ and $\alpha > \alpha_1(n/\sigma)$, by Lemma 3.1, the first integral in $I(t)$ may be estimated by $C(1+t)^{-\frac{n}{\sigma}(1-\frac{1}{\alpha})}$. For the second integral in $I(t)$, we find the faster decay rate $(1+t)^{-\frac{n}{\sigma}(\alpha-1)}$. Therefore, we proved (39).

Lemma 3.4 is sufficient to construct a solution to (2) in $C^1([0,\infty), L^\alpha)$. However, to construct an energy solution, we supplement it with the following.

Lemma 3.5. *Let $n < 2\theta$ and $\alpha > \alpha_1(n/\sigma)$. If $u_1 \in L^1 \cap L^2$, then* (34) *holds with $C_1 > 0$ independent of $T > 0$. If $u, v \in X(T)$, then* (35) *holds with $C_2 > 0$ independent of $T > 0$.*

Proof. The proof of (34) follows applying (16) with $p = 1$, as in the proof of Lemma 3.3.

Let $u, v \in X(T)$. Due to $n < 2\theta$, we may apply (17) with $p = 1$ and choosing $n/(2\theta) < \delta < 1$, obtaining

$$\begin{aligned}\sqrt{E[Fu - Fv](t)} &\leq C \int_0^t (1 + t - s)^{-\frac{n}{2\theta}} \|f(u_t) - f(v_t)(s, \cdot)\|_{L^1}\, ds \\ &\quad + C \int_0^t (t - s)^{-\delta} e^{-c(t-s)} \|f(u_t) - f(v_t)(s, \cdot)\|_{L^1}\, ds \\ &\leq C \|u - v\|_{X(T)} \left(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\right) I(t),\end{aligned}$$

with

$$I(t) = \int_0^t \left((1 + t - s)^{-\frac{n}{2\theta}} + (t - s)^{-\delta} e^{-c(t-s)}\right) (1 + s)^{-\frac{n}{\sigma}(\alpha-1)}\, ds,$$

where we used (37). Due to $n < 2\theta$, as in the proof of Lemma 3.4, $I(t) \leq C(1 + t)^{-\frac{n}{2\theta}}$, therefore, we proved (35).

Thanks to Lemmas 3.4 and 3.5, we conclude the proof of Theorem 2. Indeed, (24) follows from (38) and (34), whereas (25) follows from (39) and (35).

4. Global-in-time solutions with small data in L^p and critical powers

Dropping the assumption of initial data in L^1, and replacing it with the assumption that u_1 is small in L^p, for some $p > 1$, the critical exponent for Cauchy problem (1) changes to $\alpha_0(n/(\sigma p))$ where α_0 is as in (7), i.e.,

$$\alpha_0(n/(\sigma p)) = 1 + \frac{2}{\frac{n}{\sigma p} - 1}.$$

Moreover, the global existence of small data solutions to (1) holds for supercritical and critical powers $\alpha \geq \alpha_0(n/(\sigma p))$.

In the following, for simplicity we only prove a global existence result of small data solutions for critical powers $\alpha = \alpha_0(n/(\sigma p))$, since the case of supercritical powers is easier and it may be treated with a simpler approach. We only consider solutions in $C([0, \infty), L^{\alpha p})$, and not in the energy space. This choice simplifies the proof.

This result may be proved in any space dimension n such that $d(p, \alpha p) < \sigma$, where d is defined in (18), that is,

$$\frac{n}{p}\left(1 - \frac{1}{\alpha}\right) + n\sigma \max\left\{\frac{1}{2} - \frac{1}{p}, \frac{1}{\alpha p} - \frac{1}{2}\right\} < \sigma. \tag{40}$$

Explicitly, (40) is

$$\begin{cases} \frac{n}{p}\left(1 - \frac{1}{\alpha}\right) + n\sigma\left(\frac{1}{2} - \frac{1}{p}\right) < \sigma & \text{if } p \geq 1 + \frac{1}{\alpha}, \\ \frac{n}{p}\left(1 - \frac{1}{\alpha}\right) + n\sigma\left(\frac{1}{\alpha p} - \frac{1}{2}\right) < \sigma & \text{if } p \leq 1 + \frac{1}{\alpha}, \end{cases} \tag{41}$$

with $\alpha = \alpha_0(n/(\sigma p))$.

Theorem 3. *Fix $p \in (1, 2)$, $n > \sigma p$, $\alpha = \alpha_0(n/(\sigma p))$, and assume that* (40) *holds. Also assume that the damping is* noneffective, *i.e., $\theta \in (\sigma, 2\sigma]$. Then there is a constant $\epsilon > 0$ such that for any*

$$u_1 \in L^p \cap L^\eta \quad \textit{with} \qquad \|u_1\|_{L^p} + \|u_1\|_{L^\eta} < \epsilon, \tag{42}$$

where

$$\eta = \begin{cases} p & \textit{if } \frac{n}{p}\left(1 - \frac{1}{\alpha}\right) \leq \theta, \\ \left(\frac{\theta}{n} + \frac{1}{\alpha p}\right)^{-1} & \textit{otherwise,} \end{cases} \tag{43}$$

there exists a uniquely determined weak solution $u \in C([0, \infty), L^{\alpha p})$ to (1). *Moreover, the solution satisfies the decay estimate*

$$\|u(t, \cdot)\|_{L^{p\alpha}} \leq C (1 + t)^{-\frac{1}{\alpha}} (\|u_1\|_{L^1} + \|u_1\|_{L^\eta}),$$

where the constant $C > 0$ does not depend on the initial data.

Example 4.1. Let $\sigma = 2$. Then $\alpha_0(n/(2p)) = 1 + \frac{4p}{n-2p}$. We first consider the case of small p, that is, $p \leq 1 + 1/\alpha = 2n/(n+2p)$, or, equivalently, high space dimension $n \geq 2p^2/(2-p)$ (we will see in Section 5 that this is the case of interest). Condition (41) gives us the interval in which Theorem 3 is applicable for a given p when $p \leq 1 + 1/\alpha_0(n/(2p))$:

$$2p\,\frac{p}{2-p} \leq n < \frac{p}{2-p}\left(p + 1 + \sqrt{p(p-2)+9}\right).$$

In particular, we stress that the upper bound is larger than the lower bound for any $p \in (1,2)$, that the both lower and upper bounds are increasing with respect to p and diverge to ∞ as $p \to 2$.

On the other hand, for large $p \geq 2n/(n+2p)$, or, equivalently, low space dimension $n \leq 2p^2/(2-p)$, condition (41) takes the form $(2-p)n^2 - 2p(p-1)n + 4p^2 > 0$. We notice that the previous inequality is always verified for $p < 2\sqrt{2} - 1$, that is, for $p < 2\sqrt{2} - 1$ Theorem 3 is applicable for

$$2p < n \leq 2p\,\frac{p}{2-p}\,.$$

On the other hand, when $2\sqrt{2} - 1 \leq p < 2$, Theorem 3 is applicable for

$$2p < n < n_-(p), \quad \text{or} \qquad n_+(p) < n \leq 2p\,\frac{p}{2-p},$$

where

$$n_\pm = \frac{p}{2-p}\,(p-1)\left(1 \pm \sqrt{1 - \frac{4(2-p)}{(p-1)^2}}\right).$$

For instance, if $p = 3/2$, then $\alpha_0(n/3) = 1 + \frac{6}{n-3}$ and Theorem 3 is applicable in space dimension $4 \leq n \leq 16$. We stress that Theorem 1 was not applicable in space dimension $n \geq 5$. We postpone to Section 5 the discussion about the existence exponents in higher space dimension.

Dropping the assumption of initial data in L^1, and replacing it with the assumption that u_1 is small in L^p, for some $p \in (1,2)$, the critical exponent for Cauchy problem (2) changes to $\alpha_1(n/(\sigma p))$ where α_1 is as in (7), i.e.,

$$\alpha_1(n/(\sigma p)) = 1 + \frac{\sigma p}{n}.$$

Moreover, the global existence of small data solutions to (2) holds for supercritical and critical powers $\alpha \geq \alpha_1(n/(\sigma p))$.

As in Theorem 3, we only prove the result for critical powers $\alpha = \alpha_1(n/(\sigma p))$.

In order to do this, we assume that the space dimension n is such that

$$\frac{1}{p}\left(1 - \frac{1}{\alpha}\right) + \sigma \max\left\{\frac{1}{2} - \frac{1}{p}, \frac{1}{\alpha p} - \frac{1}{2}\right\} < 0. \tag{44}$$

Remark 4.1. If $p \leq 1 + \frac{1}{\alpha}$ i.e, $n \geq \frac{p\sigma(p-1)}{2-p}$, condition (44) is satisfied for

$$n < \frac{p}{2-p}\,(p\sigma - 2)\,.$$

On the other hand, if $p \geq 1 + \frac{1}{\alpha}$, condition (44) is satisfied for

$$n > p\left(\frac{2}{2-p} - \sigma\right) = \frac{p}{2-p}(p\sigma + 2 - 2\sigma).$$

Then we may prove the following.

Theorem 4. *Fix $p \in (1,2)$, $\sigma > 2$ and $\alpha = \alpha_1(n/(\sigma p))$ be such that (44) holds. Also assume that the damping is* noneffective*, i.e., $\theta \in (\sigma, 2\sigma]$. Then there is a constant $\epsilon > 0$ such that for any*

$$u_1 \in L^p \cap L^{\alpha p} \quad \textit{with} \qquad \|u_1\|_{L^p} + \|u_1\|_{L^{\alpha p}} < \epsilon, \tag{45}$$

there exists a uniquely determined solution $u \in C^1([0,\infty), L^{\alpha p})$ to (2).

Moreover, the solution satisfies the decay estimate

$$\|u_t(t,\cdot)\|_{L^{\alpha p}} \leq C\,(1+t)^{-\frac{1}{\alpha}}\,(\|u_1\|_{L^p} + \|u_1\|_{L^{\alpha p}}),$$

where the constant $C > 0$ does not depend on the initial data.

Example 4.2. Let $p = 4/3$ and $\sigma = 3$. Then $\alpha_1(n/4) = 1 + \frac{4}{n}$ and Theorem 4 is applicable in space dimension $n \leq 3$. We stress that Theorem 2 was not applicable in space dimension $n \geq 2$.

Example 4.3. Let $p = 3/2$ and $\sigma = 4$. Then $\alpha_1(n/6) = 1 + \frac{6}{n}$ and Theorem 4 is applicable in space dimension $n \leq 11$. We stress that Theorem 2 was not applicable in space dimension $n \geq 3$ for $\sigma = 4$.

The exponents $\alpha_0(n/(\sigma p))$ in Theorem 3 and $\alpha_1(n/(\sigma p))$ in Theorem 4 are critical, in the sense that they cannot be improved in general, assuming initial data in L^p. In the case of integer powers σ and $\theta/2$, a nonexistence result for (1) and (2) with initial data in L^p, for any subcritical power $\alpha \in (1, \alpha_1(n/(\sigma p)))$, has been proved in [11, Theorems 3 and 4]. In the general case of possibly fractional powers, with minor modifications to the proof of Theorem 1 in [13], we may prove the following.

Proposition 4.1. *Let $0 \leq \theta \leq 2\sigma$, and assume that $u_1 \in L^p$ verifies*

$$\liminf_{|x|\to\infty} (\log|x|)\,|x|^{\frac{n}{p}}\,u_1(x) > 0. \tag{46}$$

Then there exists no global (weak) solution $u \in C([0,\infty), L^q)$ to (1) with $f(u) = |u|^\alpha$, for any $q \in [1,\infty]$ and for any

- *for any $\alpha > 1$ if $n \leq p\min\{\theta,\sigma\}$;*
- *for any*

$$\alpha \in \left(1, 1 + \frac{2\sigma}{\frac{n}{p} - \min\{\theta,\sigma\}}\right),$$

  if $n > p\min\{\theta,\sigma\}$.

Moreover, there exists no global (weak) solution $u \in C^1([0,\infty), L^q)$ to (2) with $f(u_t) = |u_t|^\alpha$, for any $q \in [1,\infty]$ and for any

$$\alpha \in \left(1, 1 + \frac{p\min\{\theta,\sigma\}}{n}\right).$$

For the definition of weak solution to (1) and (2) see later, (62) and (63).

We mention that Proposition 4.1 also provides the counterpart of the existence results with L^p data in the case of effective damping $\theta \in [0,\sigma]$, discussed in [11, Section 2.5].

4.1. Proof of Theorem 3

To prove Theorem 3, we partially follow the proof of Theorem 1, but thanks to the assumption (40), we may fix $p_1 \in [1,p)$ such that

$$\frac{n}{p_1}\left(1 - \frac{1}{\alpha}\right) + n\sigma \max\left\{\frac{1}{2} - \frac{1}{p_1}, \frac{1}{\alpha p_1} - \frac{1}{2}\right\} < \sigma. \tag{47}$$

Moreover, due to $\alpha > 1$ and

$$\frac{n}{\sigma}\left(\frac{1}{p} - \frac{1}{p\alpha}\right) = 1 + \frac{1}{\alpha} < 2,$$

we may also assume that p_1 verifies the following:

$$p_1\alpha > p, \qquad n\left(\frac{1}{p_1} - \frac{1}{p\alpha}\right) < 2\sigma. \tag{48}$$

We equip the solution space

$$X(T) = C([0,T], L^{\alpha p_1} \cap L^{\alpha p}),$$

with norm

$$\|u\|_{X(T)} = \max_{t\in[0,T]} (1+t)^{\frac{1}{\alpha}} \Big((1+t)^{-\frac{n}{\sigma\alpha}\left(\frac{1}{p_1}-\frac{1}{p}\right)} \|u(t,\cdot)\|_{L^{\alpha p_1}} + \|u(t,\cdot)\|_{L^{\alpha p}} \Big). \tag{49}$$

In particular, if $u, v \in X(T)$, (29) is replaced by

$$\|(f(u)-f(v))(s,\cdot)\|_{L^{p_1}} \leq C_2\,(1+s)^{\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{p}\right)-1}\,\|u-v\|_{X(T)}\,(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}). \tag{50}$$

Then we may prove a result similar to Lemma 3.2.

Lemma 4.1. *Let $p \in (1,2]$ and $\alpha = \alpha_0(n/(\sigma p))$, and assume that* (40) *holds. Let $p_1 \in [1,p)$ be such that* (47) *and* (48) *hold. If $u_1 \in L^p \cap L^\eta$, where η is as in* (43)*, then*

$$\|u^{\mathrm{lin}}(t,\cdot)\|_{L^{\alpha p}} \leq C_1\,(1+t)^{-\frac{1}{\alpha}}\,\|u_1\|_{L^p}, \tag{51}$$

$$\|u^{\mathrm{lin}}(t,\cdot)\|_{L^{\alpha p_1}} \leq C_1\,(1+t)^{\frac{n}{\sigma\alpha}\left(\frac{1}{p_1}-\frac{1}{p}\right)-\frac{1}{\alpha}}\,\|u_1\|_{L^p}, \tag{52}$$

with $C_1 > 0$ independent of $T > 0$. If $u, v \in X(T)$, then

$$\|(Fu-Fv)(t,\cdot)\|_{L^{\alpha p}} \leq C_2\,(1+t)^{-\frac{1}{\alpha}}\,\|u-v\|_X(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}), \tag{53}$$

$$\|(Fu-Fv)(t,\cdot)\|_{L^{\alpha p_1}} \leq C_2\,(1+t)^{\frac{n}{\sigma\alpha}\left(\frac{1}{p_1}-\frac{1}{p}\right)-\frac{1}{\alpha}}\,\|u-v\|_X(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}), \tag{54}$$

with $C_2 > 0$ independent of $T > 0$.

We stress that the fact that $p_1 < p$ appears in the decay rate in (52) and (54) is the key to deal with the critical case $\alpha = \alpha_0(n/(\sigma p))$.

Proof. Recalling that α is such that

$$1 - \frac{n}{\sigma}\left(\frac{1}{p} - \frac{1}{\alpha p}\right) = -\frac{1}{\alpha},$$

by (20) with $j = 0$, $p_2 = \eta$, and $q = \alpha p, \alpha p_1$, we immediately obtain (51) and (52). To deal with $Fu - Fv$, we use (20) with p_1 in the place of p, so that we obtain

$$\begin{aligned}\|(Fu-Fv)(t,\cdot)\|_{L^q} \leq{}& C\int_0^t (1+t-s)^{1-\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{q}\right)}\,\|(f(u)-f(v))(s,\cdot)\|_{L^{p_1}}\,ds\\ &+ C\int_0^t (t-s)^{-\delta}\,e^{-c(t-s)}\,\|(f(u)-f(v))(s,\cdot)\|_{L^{p_1}}\,ds,\end{aligned}$$

where $q = \alpha p_1, \alpha p$, and

$$\delta = \frac{1}{2\sigma-\theta}\left(n\left(\frac{1}{p_1} - \frac{1}{\alpha p}\right) - \theta\right)_+.$$

Due to (48), $\delta < 1$.

Thanks to (50), we may estimate

$$\|(Fu-Fv)(t,\cdot)\|_{L^q} \leq C\,\|u-v\|_{X(T)}\,(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1})\,(I_8(t) + I_9(t)),$$

where

$$I_8(t) = \int_0^t (1+t-s)^{1-\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{q}\right)} (1+s)^{\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{p}\right)-1}\, ds,$$
$$I_9(t) = \int_0^t (t-s)^{-\delta}\, e^{-c(t-s)}\, (1+s)^{\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{p}\right)-1}\, ds.$$

Due to

$$1-\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{q}\right) > -1, \qquad \frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{p}\right)-1 > -1,$$

thanks to Lemma 3.1, we find that $I_8(t) \leq C\,(1+t)^{1-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)}$. Recalling that $\delta < 1$ due to (48), by Lemma 3.1 we also obtain

$$I_9(t) \leq C\,(1+t)^{\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{p}\right)-1} \leq C\,(1+t)^{1-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{\alpha p}\right)},$$

where in the last inequality we used the second part of (48). Therefore, we proved (53) and (54).

This concludes the proof of Theorem 3.

4.2. *Proof of Theorem 4*

Assumption (44) corresponds to $d(p, \alpha p) < 0$. As a consequence, we may also fix $p_1 \in [1, p)$ such that

$$\frac{n}{p_1}\left(1-\frac{1}{\alpha}\right) + n\sigma \max\left\{\frac{1}{2}-\frac{1}{p_1}, \frac{1}{\alpha p_1}-\frac{1}{2}\right\} < 0. \tag{55}$$

Moreover, due to $\alpha > 1$ and due to

$$\frac{n}{p\sigma}\left(1-\frac{1}{\alpha}\right) = \frac{1}{\alpha} < 1,$$

we may also assume that p_1 is sufficiently close to p to verify

$$\alpha p_1 > p, \qquad n\left(\frac{1}{p_1}-\frac{1}{\alpha p}\right) < \sigma. \tag{56}$$

To prove Theorem 4, we follow the proof of Theorem 2, but we equip the solution space

$$X(T) = C^1([0,T], L^{\alpha p_1} \cap L^{\alpha p}),$$

with norm

$$\|u\|_{X(T)} = \max_{t\in[0,T]} (1+t)^{\frac{1}{\alpha}} \left((1+t)^{-\frac{n}{\sigma\alpha}\left(\frac{1}{p_1}-\frac{1}{p}\right)}\|u_t(t,\cdot)\|_{L^{\alpha p_1}} + \|u_t(t,\cdot)\|_{L^{\alpha p}}\right).$$

In particular, if $u, v \in X(T)$, (37) is replaced by

$$\|(f(u_t)-f(v_t))(s,\cdot)\|_{L^{p_1}} \leq C_2\,(1+s)^{\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{p}\right)-1}\, \|u-v\|_{X(T)}\,\big(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\big). \tag{57}$$

Then we may prove a result similar to Lemma 3.4.

Lemma 4.2. *Let $p \in (1,2)$ and $\alpha = \alpha_1(n/(\sigma p))$. Let p_1 be such that* (55) *and* (56) *hold. If $u_1 \in L^p \cap L^{\alpha p}$, then*

$$\|\partial_t u^{\mathrm{lin}}(t,\cdot)\|_{L^{\alpha p}} \leq C_1\,(1+t)^{-\frac{1}{\alpha}}\, \|u_1\|_{L^p\cap L^{\alpha p}}, \tag{58}$$

$$\|\partial_t u^{\mathrm{lin}}(t,\cdot)\|_{L^{\alpha p_1}} \leq C_1\,(1+t)^{\frac{n}{\sigma\alpha}\left(\frac{1}{p_1}-\frac{1}{p}\right)-\frac{1}{\alpha}}\, \|u_1\|_{L^p\cap L^{\alpha p}}, \tag{59}$$

with $C_1 > 0$ independent of $T > 0$. If $u, v \in X(T)$, then

$$\|\partial_t(Fu-Fv)(t,\cdot)\|_{L^{\alpha p}} \leq C_2\,(1+t)^{-\frac{1}{\alpha}}\, \|u-v\|_X\big(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\big), \tag{60}$$

$$\|\partial_t(Fu-Fv)(t,\cdot)\|_{L^{\alpha p_1}} \leq C_2\,(1+t)^{\frac{n}{\sigma\alpha}\left(\frac{1}{p_1}-\frac{1}{p}\right)-\frac{1}{\alpha}}\, \|u-v\|_X\big(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1}\big), \tag{61}$$

with $C_2 > 0$ independent of $T > 0$.

We stress that the fact that p_1 appears in the decay rate in (59) and (61), is the key to deal with the critical case $\alpha = \alpha_1(n/(\sigma p))$.

Proof. By (20) with $j = 1$ and $p_2 = q = \alpha p, \alpha p_1$, we immediately obtain (58) and (59).

On the other hand, we may apply (20) with p_1 in place of p, and $\delta < 1$, obtaining

$$\begin{aligned}\|\partial_t(Fu - Fv)(t,\cdot)\|_{L^q} &\leq C \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{q}\right)} \|(f(u_t) - f(v_t))(s,\cdot)\|_{L^{p_1}}\,ds \\ &\quad + C \int_0^t (t-s)^{-\delta}\, e^{-c(t-s)}\, \|(f(u_t) - f(v_t))(s,\cdot)\|_{L^{p_1}}\,ds \\ &\leq C\,\|u-v\|_{X(T)}\,(\|u\|_{X(T)}^{\alpha-1} + \|v\|_{X(T)}^{\alpha-1})\, I(t),\end{aligned}$$

with

$$I(t) = \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{q}\right)} (1+s)^{\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{p}\right)-1}\,ds + \int_0^t (t-s)^{-\delta}\, e^{-c(t-s)}\,(1+s)^{\frac{n}{\sigma}\left(\frac{1}{p_1}-\frac{1}{p}\right)-1}\,ds\,,$$

where we used (57). We now notice that $n(1/p_1 - 1/q) < \sigma$, as a consequence of (56), so that, by Lemma 3.1, we get $I(t) \leq C(1+t)^{-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)}$. Therefore, we proved (60) and (61).

This concludes the proof of Theorem 4.

4.3. Proof of Proposition 4.1

We say that $u \in C([0,\infty), L^q)$, for some $q \in [1,\infty]$, is a weak solution to (1) with $f(u) = |u|^\alpha$ if, for any $\psi \in C_c^\infty([0,\infty)$, and for any $\varphi \in C^\infty(\mathbb{R}^n)$ with all derivatives in $L^1 \cap L^\infty$, the following equality holds true:

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^\alpha\, \psi(t)\,\varphi(x)\,dx\,dt + \psi(0)\int_{\mathbb{R}^n} u_1(x)\,\varphi(x)\,dx \\ &\qquad = \int_0^\infty \int_{\mathbb{R}^n} u(t,x)\,(\partial_t^2 - \partial_t(-\Delta)^{\frac{\theta}{2}} + (-\Delta)^\sigma)\psi(t)\varphi(x)\,dx\,dt.\end{aligned} \tag{62}$$

Indeed, if u is a smooth solution to (1), then (62) follows integrating by parts.

We say that $u \in C^1([0,\infty), L^q)$, for some $q \in [1,\infty]$, is a weak solution to (2) with $f(u_t) = |u_t|^\alpha$ if, for any $\psi \in C_c^\infty([0,\infty)$, and for any $\varphi \in C^\infty(\mathbb{R}^n)$ with all derivatives in $L^1 \cap L^\infty$, the following equality holds true:

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^n} |u_t(t,x)|^\alpha\, \psi(t)\,\varphi(x)\,dx\,dt + \psi(0)\int_{\mathbb{R}^n} u_1(x)\,\varphi(x)\,dx \\ &\qquad = \int_0^\infty \int_{\mathbb{R}^n} u_t(t,x)\,(-\partial_t + (-\Delta)^{\frac{\theta}{2}})\psi(t)\varphi(x)\,dx\,dt + \int_0^\infty \Psi(t)\int_{\mathbb{R}^n} u_t(t,x)\,(-\Delta)^\sigma\varphi(x)\,dx\,dt,\end{aligned} \tag{63}$$

where

$$\Psi(t) = \int_t^\infty \psi(s)\,ds.$$

Proof. We assume that at least one among σ or $\theta/2$ is not integer, otherwise Theorems 3 and 4 in [11] apply. We follow the proof of Theorem 1 in [13] for general higher order equations of order m. We sketch it here, with special reference to Cauchy problems (1) and (2), to show where condition (46) comes into play.

Let $s = \min\{\sigma - [\sigma], \theta/2 - [\theta/2]\}$, where $[x] = \max\{n \in \mathbb{N} : n \leq x\}$ denotes the floor function of x. We fix $\varphi = \langle x\rangle^{-n-2s}$, where $\langle x\rangle^2 = 1 + |x|^2$, and $\psi \in C_c^\infty([0,\infty))$, nonnegative, with $\psi(0) > 0$.

If u is a weak solution to (1) with $f(u) = |u|^\alpha$, as in (62), then

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^\alpha\, \psi(R^{-\eta}\,t)\,\varphi(R^{-1}x)\,dx\,dt + \psi(0)\int_{\mathbb{R}^n} u_1(x)\,\varphi(R^{-1}x)\,dx \\ &\qquad = \int_0^\infty \int_{\mathbb{R}^n} u(t,x)\,(\partial_t^2 - \partial_t(-\Delta)^{\frac{\theta}{2}} + (-\Delta)^\sigma)\psi(R^{-\eta}\,t)\varphi(R^{-1}\,x)\,dx\,dt\end{aligned}$$

for any $R \geq 1$ and $\eta > 0$. Following as in the proof of Theorem 1 in [13], we may use Young inequality to estimate

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)| \left|(\partial_t^2 - \partial_t(-\Delta)^{\frac{\theta}{2}} + (-\Delta)^\sigma)\psi(R^{-\eta}\, t)\varphi(R^{-1}\, x)\right| dx\, dt \\
&\quad \leq \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^\alpha\, \psi(R^{-\eta}\, t)\, \varphi(R^{-1}x)\, dx\, dt \\
&\qquad + C \int_0^\infty \int_{\mathbb{R}^n} \left|(\partial_t^2 - \partial_t(-\Delta)^{\frac{\theta}{2}} + (-\Delta)^\sigma)\psi(R^{-\eta}\, t)\varphi(R^{-1}\, x)\right|^{\alpha'} (\psi(R^{-\eta}\, t)\, \varphi(R^{-1}x))^{-\frac{\alpha'}{\alpha}}\, dx\, dt,
\end{aligned}$$

where $\alpha' = \alpha/(\alpha-1)$. By [13, Lemma 3.1], we may estimate

$$|(-\Delta)^{\frac{\theta}{2}}\varphi| \leq C\varphi, \quad |(-\Delta)^\sigma\varphi| \leq C\varphi,$$

so that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \left|(\partial_t^2 - \partial_t(-\Delta)^{\frac{\theta}{2}} + (-\Delta)^\sigma)\psi(R^{-\eta}\, t)\varphi(R^{-1}\, x)\right|^{\alpha'} (\psi(R^{-\eta}\, t)\, \varphi(R^{-1}x))^{-\frac{\alpha'}{\alpha}}\, dx\, dt \\
&\quad \leq C\, (R^{-2\eta} + R^{-(\eta+\theta)} + R^{-2\sigma})^{\alpha'} \int_0^\infty \int_{\mathbb{R}^n} \psi(R^{-\eta}\, t)\, \varphi(R^{-1}x)\, dx\, dt.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)| \left|(\partial_t^2 - \partial_t(-\Delta)^{\frac{\theta}{2}} + (-\Delta)^\sigma)\psi(R^{-\eta}\, t)\varphi(R^{-1}\, x)\right| dx\, dt \\
&\quad \leq \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^\alpha\, \psi(R^{-\eta}\, t)\, \varphi(R^{-1}x)\, dx\, dt + C\, R^{n+\eta}\, (R^{-2\eta} + R^{-(\eta+\theta)} + R^{-2\sigma})^{\alpha'} \\
&\quad \leq \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^\alpha\, \psi(R^{-\eta}\, t)\, \varphi(R^{-1}x)\, dx\, dt + C_1\, R^{n+\eta-2\sigma\,\alpha'},
\end{aligned}$$

for any $R \geq 1$, where we chose $\eta = 2\sigma - \min\{\theta, \sigma\}$. Using (46) we find

$$\psi(0) \int_{\mathbb{R}^n} u_1(x)\, \varphi(R^{-1}x)\, dx \geq c\, \langle 1\rangle^{-n-2s} \int_{R/2\leq|x|\leq R} (\log|x|)^{-1}|x|^{-\frac{n}{p}}\, dx \geq c_1 R^{n\left(1-\frac{1}{p}\right)}\, (\log R)^{-1},$$

for some $K > 0$ and $c > 0$, and for any $R \geq 2K$. The contradiction follows from the inequality

$$c_1 R^{n\left(1-\frac{1}{p}\right)}\, (\log R)^{-1} \leq C_1\, R^{n+\eta-2\sigma\,\alpha'},$$

which does not hold for sufficiently large R, when

$$\alpha' > \frac{\frac{n}{p}+\eta}{2\sigma}, \qquad \text{i.e., } n \leq p\min\{\theta,\sigma\}, \text{ or } \quad \alpha < 1 + \frac{2\sigma}{\frac{n}{p} - \min\{\theta,\sigma\}}\,.$$

If u is a global weak solution to (2) with $f(u_t) = |u_t|^\alpha$, as in (63), then

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} |u_t(t,x)|^\alpha\, \psi(R^{-\eta}\, t)\, \varphi(R^{-1}x)\, dx\, dt + \psi(0) \int_{\mathbb{R}^n} u_1(x)\, \varphi(R^{-1}x)\, dx \\
&\quad = \int_0^\infty \int_{\mathbb{R}^n} u_t(t,x)\, (-\partial_t + (-\Delta)^{\frac{\theta}{2}})\psi(R^{-\eta}\, t)\varphi(R^{-1}\, x)\, dx\, dt + \int_0^\infty \Psi(R^{-\eta}\, t) \int_{\mathbb{R}^n} u_t(t,x)\, (-\Delta)^\sigma\varphi(R^{-1}\, x)\, dx\, dt,
\end{aligned}$$

for any $R \geq 1$ and $\eta > 1$. Choosing $\eta = \min\{\theta,\sigma\}$ and proceeding as in the previous step, we find

$$c_1 R^{n\left(1-\frac{1}{p}\right)}\, (\log R)^{-1} \leq C\, R^{n+\eta}(R^{-\eta} + R^{-\theta} + R^{\eta-2\sigma})^{\alpha'} \leq C_1\, R^{n+\eta-\eta\alpha'},$$

which does not hold for sufficiently large R, when

$$\alpha' > \frac{\frac{n}{p}+\eta}{\eta}, \qquad \text{i.e.} \qquad \alpha < 1 + \frac{p\eta}{n}\,.$$

This concludes the proof.

5. Global-in-time solutions in high space dimension

In this section, we investigate Cauchy problem (1) in the case of high space dimension, that is, $n > \bar{n}(\sigma)$ if $\sigma > 1$. For brevity, we omit the discussion of the case $\sigma < 1$.

As we noticed in the proof of Lemma 3.2, $n \leq \bar{n}(\sigma)$ is equivalent to ask that $d(1, \alpha_0(n/\sigma)) < \sigma$. Similarly, (40) corresponds to $d(p, \alpha_0(n/(\sigma p))) < \sigma$. When $d(p, q) \geq \sigma$, that is,

$$n\left(\frac{1}{p} - \frac{1}{q}\right) + n\sigma \max\left\{\frac{1}{2} - \frac{1}{p}, \frac{1}{q} - \frac{1}{2}\right\} \geq \sigma, \tag{64}$$

then (20) is modified into

$$\|u(t,\cdot)\|_{L^q} \leq C\,(1+t)^{\frac{\sigma}{\theta} - \frac{n}{\theta}\left(\frac{1}{p}-\frac{1}{q}\right) + n\left(1-\frac{\sigma}{\theta}\right)\,\max\left\{\frac{1}{2}-\frac{1}{p}, \frac{1}{q}-\frac{1}{2}\right\}}\,\log(e+t)\,\|u_1\|_{L^p} + C\,t^{-\delta}\,e^{-ct}\,\|u_1\|_{L^{p_2}}, \tag{65}$$

for some $C > 0$ and $c > 0$ (see [12, Theorem 2.1]). We notice that the estimate in (65) depends on both θ and σ. As a consequence of (64), the exponent of $1 + t$ in (65) is equal to or larger than the exponent of $1 + t$ in (20).

Having in mind that the global existence of solutions to (1) with small data in L^p, $p \in [1, 2]$, is found when the decay rate of $L^p - L^{p\alpha}$ estimate is strictly smaller than or equal to $-1/\alpha$ (see the proof of Theorems 1 and 3), we look for the best decay rate of $L^p - L^{p\alpha}$ estimates, according to the cases when either $d(p, \alpha p) - \sigma$ is negative or not.

If we put $q = \alpha p$ in (18), we have

$$d(p, \alpha p) = \frac{n}{p}\left(1 - \frac{1}{\alpha}\right) + n\sigma \max\left\{\frac{1}{2} - \frac{1}{p}, \frac{1}{\alpha p} - \frac{1}{2}\right\}$$

where we see that

$$\max\left\{\frac{1}{2} - \frac{1}{p}, \frac{1}{\alpha p} - \frac{1}{2}\right\} = \begin{cases} \frac{1}{\alpha p} - \frac{1}{2} & \text{if } 1 \leq p \leq 1 + 1/\alpha \\ \frac{1}{2} - \frac{1}{p} & \text{if } 1 + 1/\alpha \leq p \leq 2. \end{cases}$$

In particular,

$$\frac{\partial}{\partial p} d(p, \alpha p) = \begin{cases} -\frac{n}{p^2}\left(1 + \frac{\sigma - 1}{\alpha}\right) < 0 & \text{if } 1 < p < 1 + \frac{1}{\alpha}, \\ \frac{n}{p^2}\left(\sigma - 1 + \frac{1}{\alpha}\right) > 0 & \text{if } p > 1 + \frac{1}{\alpha}, \end{cases}$$

that is, the condition $d(p, \alpha p) < \sigma$ is less restrictive, as p increases, up to $1 + 1/\alpha$. However, enlarging p, the decay rate of the $L^p - L^{\alpha p}$ estimate in (20) becomes worse, so that it leads to a larger existence exponent. This effect leads to the following problem. When $n > \bar{n}(\sigma)$ and we fix $\alpha > \alpha_0(n/\sigma)$, is it possible to find p such that using $L^p - L^{\alpha p}$ estimates, a global existence of small data solutions follow for the fixed α?

Ignoring the logarithmic term in (65), the function $\zeta : [1, 1 + 1/\alpha] \to \mathbb{R}$, which describes the power of $1 + t$ when we put $q = \alpha p$ in (20) or in (65), is:

$$\zeta(p) = \begin{cases} 1 - \frac{n}{\sigma p}\left(1 - \frac{1}{\alpha}\right) & \text{if } d(p, \alpha p) < \sigma, \\ \frac{\sigma}{\theta} - \frac{n}{\theta p}\left(1 - \frac{1}{\alpha}\right) + n\left(\frac{1}{\alpha p} - \frac{1}{2}\right)\left(1 - \frac{\sigma}{\theta}\right) & \text{if } d(p, \alpha p) \geq \sigma. \end{cases}$$

With the exception of the threshold values of p when $p = 1 + 1/\alpha$ or when $d(p, \alpha p) = \sigma$, we may differentiate $\zeta(p)$, obtaining

$$\zeta'(p) = \begin{cases} \frac{n}{\sigma p^2}\left(1 - \frac{1}{\alpha}\right) & \text{if } d(p, \alpha p) < \sigma, \\ \frac{n}{\theta \alpha p^2}(\alpha + \sigma - \theta - 1) & \text{if } d(p, \alpha p) > \sigma. \end{cases}$$

First of all, we notice that if $\alpha \geq \theta + 1 - \sigma$, then $\zeta'(p)$ is nonnegative, so that $\min \zeta = \zeta(1)$. Therefore, we assume in the following that $\alpha < \theta + 1 - \sigma$, and we distinguish three scenarios.

If $d(1, \alpha) \leq \sigma$ then $\zeta'(p)$ is nonnegative for $p \in [1, 1 + 1/\alpha]$, so that $\min \zeta(p) = \zeta(1)$. This scenario corresponds to the case treated in Theorem 1, that is, the critical exponent is $\alpha_0(n/\sigma)$ when $n \leq \bar{n}(\sigma)$.

If $d(1 + 1/\alpha, 1 + \alpha) < \sigma < d(1, \alpha)$, that is,

$$\frac{n}{\alpha + 1}(\alpha + \sigma - 1) < \frac{(n+2)\sigma}{2} < \frac{n}{\alpha}(\alpha + \sigma - 1).$$

then $\zeta'(p)$ is negative in $[1,\bar p)$ and positive in $(\bar p, 1+1/\alpha]$, so that $\min\zeta(p)=\zeta(\bar p)$, where $\bar p$ is the solution to

$$\sigma = d(\bar p, \alpha\bar p) = \frac{n}{\bar p}\left(1-\frac{1}{\alpha}\right) + n\sigma\left(\frac{1}{\alpha\bar p}-\frac{1}{2}\right) = \frac{n}{\bar p}\,\frac{\alpha+\sigma-1}{\alpha} - \frac{n\sigma}{2},$$

Finally, if $\sigma\leq d(1+1/\alpha, 1+\alpha)$, that is,

$$\frac{2\sigma}{\alpha-1} \leq n\,\frac{2-\sigma}{\alpha+1}, \tag{66}$$

then ζ' is nonpositive, so that

$$\min\zeta(p) = \zeta(1+1/\alpha) = \frac{\sigma}{\theta} - \frac{n}{\theta}\frac{\alpha-1}{\alpha+1}\,\frac{2+\theta-\sigma}{2}.$$

We stress that condition (66) fails for any $\alpha>1$, if $\sigma\geq 2$.

Now that we have determined $\min\zeta(p)$, according to α, we may compute the three candidate exponents for the global existence, obtained by solving $\alpha\zeta(1)=-1$, $\alpha\zeta(\bar p)=-1$ and $\alpha\zeta(1+1/\alpha)=-1$.

If we work with L^1-L^α estimates when $\alpha\geq\theta+1-\sigma$, we may find global existence of small data solutions for $\alpha>\alpha^*$, where $\alpha^*=\alpha^*(n,\theta,\sigma)$ is the solution to $0=\alpha\zeta(1)+1$, that is,

$$\begin{aligned} 0 &= 2\theta\alpha\zeta(1)+2\theta = 2\sigma\alpha - 2n(\alpha-1) + n(\theta-\sigma)(2-\alpha) + 2\theta \\ &= -(\alpha-1)(n(2+\theta-\sigma)-2\sigma) + 2(\theta+\sigma) + n(\theta-\sigma). \end{aligned}$$

Therefore, we find the exponent

$$\alpha^* = 1 + \frac{n(\theta-\sigma)+2(\theta+\sigma)}{n(2+\theta-\sigma)-2\sigma}.$$

This existence exponent is the best possible working with $L^p-L^{\alpha p}$ estimates when $\alpha^*\geq\theta+1-\sigma$, that is,

$$n \leq \tilde n(\sigma,\theta) = 2\frac{\theta+\sigma(1+\theta-\sigma)}{(1+\theta-\sigma)(\theta-\sigma)}. \tag{67}$$

We stress that $\tilde n(\sigma,\theta)$ is decreasing w.r.t θ and that

$$\tilde n(\sigma, 2\sigma) = 2\,\frac{3+\sigma}{1+\sigma}$$

Working with $L^{\bar p}-L^{\alpha\bar p}$ estimates, we may compute the global existence exponent replacing $p=\bar p$ in $\alpha_0(n/(\sigma\bar p))$ in Theorem 3, that is,

$$\bar p = \frac{n(\alpha-1)}{\sigma(\alpha+1)}.$$

Replacing this expression in the equality

$$\sigma = \frac{n}{\bar p}\,\frac{\alpha+\sigma-1}{\alpha} - \frac{n\sigma}{2} = \frac{\sigma(\alpha+1)}{\alpha-1}\,\frac{\alpha+\sigma-1}{\alpha} - \frac{n\sigma}{2},$$

we find the quadratic equation

$$\frac{n}{2}(\alpha-1) = (\alpha+1)\left(1+\frac{\sigma-1}{\alpha}\right) - (\alpha-1) = \sigma+1+\frac{\sigma-1}{\alpha}\,.$$

Therefore, global existence of small data solutions hold for $\alpha>\bar\alpha$, where $\bar\alpha=\bar\alpha(n,\sigma)$ is the exponent which solves (8).

Finally, we may exclude the case in which the critical exponent is obtained when $\min\zeta(p)=\zeta(1+1/\alpha)$. To show this, it is sufficient to notice that condition (66) is not verified for $\alpha=\bar\alpha(n,\sigma)$. Indeed, replacing the equality in (8), we find that (66) holds with $\alpha=\bar\alpha$ if, and only if,

$$n(1-\sigma) \geq 2\sigma + 2\frac{\sigma-1}{\alpha}.$$

However, the inequality above is verified for no n, due to our assumption $\sigma > 1$.

Therefore, we get the existence exponent $\bar{\alpha}$, obtained working with $L^p - L^{\alpha p}$ estimates, if $n > \max\{\bar{n}(\sigma), \tilde{n}(\sigma, \theta)\}$. Namely, in this latter case, $d(\bar{p}, \bar{p}\bar{\alpha}) < \sigma$, so that Theorem 3 is applicable with $p = \bar{p}$ and $\alpha = \bar{\alpha}$.

We notice that $\bar{\alpha} \to 1$ as $n \to \infty$. More precisely,

$$\bar{\alpha} \approx 1 + \frac{4\sigma}{n}, \qquad \text{as } n \to \infty.$$

We stress that the same limit is obtained as $\sigma \to 1$, that is,

$$\lim_{\sigma \to 1} \bar{\alpha}(n, \sigma) = 1 + \frac{4}{n}.$$

Example 5.1. Let $\sigma = 2$, $\theta \in (2, 4]$, $n \geq 3$. The critical exponent is

$$\alpha_0(n/2) = 1 + \frac{4}{n-2},$$

as long as $n \leq \bar{n}(\sigma)$, that is, $n \leq 4$. Computing

$$\tilde{n}(2, \theta) = 2\frac{3\theta - 2}{(\theta - 1)(\theta - 2)},$$

we find that $\tilde{n}(2, \theta) \geq 5$ if, and only if,

$$2 < \theta \leq \bar{\theta} = 2 + \frac{1 + \sqrt{161}}{10}.$$

We may now compute

$$\alpha^*(n, \theta, 2) = 1 + \frac{n(\theta - 2) + 2(\theta + 2)}{n\theta - 4}$$

$$\bar{\alpha}(n, 2) = 1 + \frac{6 - n + \sqrt{(n+6)^2 + 8n}}{2n}.$$

Summarizing, we find global small data solutions for:

$$\alpha > \begin{cases} 1 + \frac{4}{n-2} & \text{if } n = 3, 4, \\ \bar{\alpha}(n, 2) & \text{if } n \geq 5, \end{cases}$$

when $\bar{\theta} < \theta \leq 4$, and for

$$\alpha > \begin{cases} 1 + \frac{4}{n-2} & \text{if } n = 3, 4, \\ \alpha^*(n, \theta, 2) & \text{if } 5 \leq n \leq \tilde{n}(2, \theta), \\ \bar{\alpha}(n, 2) & \text{if } n \geq \tilde{n}(2, \theta), \end{cases}$$

when $2 < \theta \leq \bar{\theta}$. For instance, if $\theta = 3$, we find global small data solutions for:

$$\alpha > \begin{cases} 1 + \frac{4}{n-2} & \text{if } n = 3, 4, \\ \alpha^*(5) = 1 + 15/11 & \text{if } n = 5, \\ \alpha^*(6) = 1 + 8/7 & \text{if } n = 6, \\ \alpha^*(7) = \bar{\alpha}(7) = 2 & \text{if } n = 7, \\ \bar{\alpha}(n) & \text{if } n \geq 8. \end{cases}$$

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