SPECHT PROPERTY FOR THE ALGEBRA OF UPPER TRIANGULAR MATRICES OF SIZE TWO WITH A TAFT'S ALGEBRA ACTION

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Abstract. Let F be a field of characteristic zero and UT_2 be the algebra of 2×2 upper triangular matrices over F. In a previous paper by Centrone and Yasumura the authors give a description of the action of Taft's algebras H_m on UT_2 and its H_m-identities. In this paper we give a complete description of the space of multilinear H_m-identities in the language of Young diagrams through the representation theory of the hyperoctahedral group. We finally prove that the variety of H_m-module algebras generated by UT_2 has the Specht property, i.e., every T^{H_m}-ideal containing the H_m-identities of UT_2 is finitely based.

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1. Introduction

One of the most interesting problems in the theory of algebras with polynomial identities (PI-algebras) is the so called *Specht problem.* We outline briefly what the Specht problem is: given a variety of algebras (associative, Lie, Jordan, graded, etc.) one can ask whether or not any subvariety is finitely generated. In the languages of T-ideals (the ideals of polynomial identities of a given algebra), the Specht problem can be formulated as follows: given any algebra A is it true that any T-ideal containing the T-ideal of A is finitely generated (or based) as a T-ideal? If we restrict our attention to the associative environment, the Specht problem was solved positively in [30] and [31] by Kemer provided the ground field of the algebras therein is of characteristic 0. Further generalizations of Kemer's result are due to Sviridova [42] (PI-algebras graded by a finite abelian group), Aljadeff and Kanel-Belov [2] (PI-algebras graded by a finite group), Karasik [28] (PI-algebras that are module algebras under the action of a finite dimensional semisimple Hopf algebra), Centrone, Estrada and Ioppolo [15] (PI-algebras that are superalgebras with superinvolution).

In this paper we study the Specht property for the variety of H_m-module algebras generated by the algebra UT_2 of 2×2 upper triangular matrices over a field of characteristic 0 containing a primitive m-th root of unit and where H_m denotes a Taft's Hopf algebra of dimension m^2. We want to point out although H_m is finite dimensional, it is not semisimple. Hence we are not allowed to use Karasik's result in order to establish whether or not our variety satisfies the Specht property. Anyway, the main result of the paper is the next (see Theorem 16):

Theorem 1. *Let $m \geq 2$ an integer and let us consider the Taft's Hopf algebra H_m over a field of characteristic 0 containing a primitive m-th root of unit. Then the variety of H_m-module algebras generated by UT_2 satisfies the Specht property.*

As far as we know this is the first result in the literature toward Specht property of varieties of algebras under the action of a Taft's Hopf algebra.

Hereby we would like to highlight the role of UT_2 in the theory of PI-algebras. In [40] Regev proved the codimension sequence of any associative PI-algebra is exponentially bounded. Later Kemer in [32] showed such codimensions are either polynomially bounded or grow exponentially. Moreover, Giambruno and Zaicev in a famous couple of paper (see [22] and [23]) computed the exponential rate of growth of a PI-algebra and proved that it is a non-negative integer. By a well known Kemer's result [29] we get the variety of algebras generated by UT_2 is a variety of almost polynomial growth, i.e., it has exponential growth but every proper subvariety has polynomial growth. An analogous result was found by Valenti in [44] for varieties of algebras graded by a finite group and by Mishchenko and

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Valenti in [37] for varieties of algebras with involution. Notice that in the latter paper the authors constructed out of UT_2 a suitable algebra generating a variety of almost polynomial growth. We would also like to cite the paper [21] by Giambruno and Rizzo toward differential identities: here the authors prove that UT_2 under the action of its algebra of derivation does not generate a variety of almost polynomial growth and they construct a subvariety of almost polynomial growth. Notice that the variety of H_m-module algebras generated by UT_2 is not of almost polynomial growth too as showed by one of the authors and Yasumura in [14].

In order to prove Theorem 1 and as its consequences, in the present paper we also get:

the H_m-cocharacter sequence of UT_2 and its colength,

the relatively free H_m-algebra of UT_2 is automata,

The Gelfand-Kirillov dimension of the relatively free H_m-algebra of UT_2,

confirming, in the case of Taft's Hopf algebra, a result obtained by one of the authors in [12].

2. **Preliminaries**

2.1. **Gradings.**

We start off with the classical notion of *grading* by a group. Let $G = \{g_1, \dots, g_s\}$ be any group of finite order s and let F be a field. If A is an F-algebra, we say that A is a *G-graded algebra* if there are subspaces A_g for each $g \in G$ such that

$$A = \bigoplus_{g\in G} A_g \text{ and } A_gA_h \subseteq A_{gh}.$$

If $0 \neq a \in A_g$ we say that a is *homogeneous of G-degree g* or *G-graded homogeneous of G-degree g*, and we write $\deg(a) = g$.

Assume that the characteristic of F does not divide m and F contains primitive m-th roots of the unit. Let α be an automorphism of A of order m, (i.e., α has order m as an element of the group $\mathrm{Aut}(A)$ of the automorphisms of A). It is well known that α induces a $\mathbb{Z}_m$-grading on A, that is,

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{m-1},$$

where, for a fixed primitive m-root of unit γ,

$$A_k = \{a \in A \mid \alpha(a) = \gamma^k a\}.$$

Moreover, there is a one-to-one correspondence (*duality*) between gradings by a group and the action of a group of automorphisms under some restrictions (see [24, Theorem 3.2.1]).

2.2. **Taft's algebras.**

Let F be a field containing an m-th root of the unit γ for some positive integer m. Let $H_{m^2}(\gamma)$ be the bialgebra defined by generators c and d with relations

$$c^m = 1, \quad d^m = 0, \quad dc = \gamma cd$$

The coalgebra structure is given by

$$\Delta(c) = c \otimes c, \quad \Delta(d) = c \otimes d + d \otimes 1$$

$$\varepsilon(c) = 1, \quad \varepsilon(d) = 0.$$

As an F-vector space, $H_{m^2}(\gamma)$ has dimension m^2 with basis $\{c^ix^j \mid 0 \leq i, j, \leq m-1\}$. The bialgebra $H_{m^2}(\gamma)$ becomes a Hopf algebra if we define the antipode as $S(c) = c^{-1}$ and $S(d) = -c^{-1}d$. This Hopf algebra is known as the m-th *Taft's Hopf algebra*.

2.3. ***H*-module algebras.**

Let F be a field of characteristic zero and H a Hopf algebra over F. We remand to the books [17,38,39,43] for basic definitions, examples and further information about Hopf algebras. An algebra A is an *H-module algebra* if A is endowed with a left H-action $h \otimes a \mapsto ha$ or, equivalently, with a homomorphism $H \to \mathrm{End}_F(A)$ such that

1. $h(ab) = (h_{(1)}a)(h_{(2)}b)$,
2. $h(1_A) = \varepsilon(h)1_A$, for all $h \in H, a, b \in A$.

Here we use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$, where Δ and ε are the comultiplication and the counit in H, respectively.

Let A be a finite dimensional algebra over a field F; assume that the characteristic of F does not divide m. Since $c^m = 1$, then c acts as an automorphism of A of order m, d acts as a c-derivation, that is, it satisfies $d(ab) = d(a)b + c(a)d(b)$, for all $a, b \in A$, and the actions of c and d are related by $dc = \gamma cd$. Thus, an action of H_m on A is completely determined by a choice of:

(1) an automorphism α of A of order m,
(2) an α-derivation d of A such that $d^m = 0$ and $d\alpha = \gamma\alpha d$.

or equivalently (see [14] Proposition 4), a choice of:

(1) a $\mathbb{Z}_m$-grading $A = \bigoplus_{i\in\mathbb{Z}_m} A_i$,
(2) an α-derivation d (where α defines the $\mathbb{Z}_m$-grading) such that $d(A_i) \subseteq A_{i-1}$ and $d^m = 0$.

The proof of [24, Proposition 3.3.6] gives us a linear basis $\mathcal{B}_1 = \{\chi_1, \ldots, \chi_m\}$ of the subalgebra $\langle c\rangle$ of H_m generated by c, such that each χ_i corresponds to a projection $\phi_i \colon A \to A_i$ with respect to the decomposition $A = \bigoplus_{i\in\mathbb{Z}_m} A_i$. Then $\{d^j\chi_i \mid i, j = 0, 1, \ldots, m-1\}$ turns out to be a basis of H_m. Let $D_m = \mathrm{span}_F\{1, d, d^2, \ldots, d^{m-1}\}$, then by [4] we have that the variables $d^j\chi_i \cdot x$ correspond to graded variables under the action of d^j. Then H_m-polynomials correspond to $\mathbb{Z}_m$-graded polynomials with the action of D_m and their polynomial identities coincide, that is, $\mathrm{Id}^{H_m}(A) = \mathrm{Id}^{gr,D_m}(A)$ (see [14] Proposition 14).

2.4. **Free H-module algebras.** Let $F\langle X\rangle$ be the free F-algebra on the set of countable non-commutative variables $X = \{x_1, x_2, \ldots\}$ and consider the vector space $V = F\langle X\rangle \otimes_F H$. The free H-module algebra over X, denoted by $F^H\langle X\rangle$ is the tensor algebra over V. Any element of $F^H\langle X\rangle$ will be called *H-polynomial*. In what follows we shall use the notation:

$$x_{i_1}^{h_1} x_{i_2}^{h_2} \cdots x_{i_n}^{h_n} := (x_{i_1} \otimes h_1) \otimes (x_{i_2} \otimes h_2) \otimes \cdots \otimes (x_{i_n} \otimes h_n).$$

Now, let H be finite dimensional and $\{b_1, \ldots, b_m\}$ be a basis (as a vector space) of H. It follows that $F^H\langle X\rangle$ is isomorphic to the free algebra over F with free formal (non-commutative) generators x^{b_j}, $j \in \{1, \ldots, m\}$, $x \in X$. Notice that $F^H\langle X\rangle$ has a structure of left H-module algebra by defining the next H-action:

$$h(x_{i_1}^{h_1} x_{i_2}^{h_2} \cdots x_{i_n}^{h_n}) = x_{i_1}^{h_{(1)}h_1} x_{i_2}^{h_{(2)}h_2} \cdots x_{i_n}^{h_{(n)}h_n},$$

where $h_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(n)}$ is the image of $h \in H$ under the comultiplication Δ of H applied $(n-1)$ times. Thus $F^H\langle X\rangle$ is the free H-module algebra on X. This means that, for any H-module algebra W and for every function $\alpha\colon X \to W$, there exists a unique homomorphism of algebras and H-modules (we call this kind of homomorphisms simply H-homomorphisms) $\beta\colon F^H\langle X\rangle \to W$ extending α. In what follows, we shall identify X with the set $\{x^{1_H} | x \in X\} \subset F^H\langle X\rangle$.

Given any H-module algebra W, we say that an H-polynomial $f \in F^H\langle X\rangle$ is an *H-identity* for W if for every H-homomorphism $\varphi\colon F^H\langle X\rangle \to W$ the polynomial f is in the kernel of φ. In other words, $f(x_1, \ldots, x_n) \in F^H\langle X\rangle$ is an H-identity of W if and only if $f(w_1, \ldots, w_n) = 0$, for all $w_1 \ldots, w_n \in W$. The set $\mathrm{Id}^H(W)$ of all identities satisfied by W is an ideal of $F^H\langle X\rangle$ and it is invariant under all H-endomorphisms of $F^H\langle X\rangle$. The ideals having such a property are called *T^H-ideals*. Moreover, all T^H-ideals are of this form: in fact, it is not difficult to see that, given a T^H-ideal I of $F^H\langle X\rangle$, then $\mathrm{Id}^H(F^H\langle X\rangle/I) = I$.

Two H-module algebras W_1 and W_2 are said to be *T^H-equivalent*, and we write $W_1 \sim_{T^H} W_2$, if $\mathrm{Id}^H(W_1) = \mathrm{Id}^H(W_2)$.

Given a non-empty set $S \subseteq F^H\langle X\rangle$, the class $\mathrm{var}^H(S)$ of all H-module algebras W such that f is an H-identity for W for all $f \in S$ is called the *variety determined (or generated)* by S. Similarly, given an H-module algebra W, the variety of H-module algebras generated by W, denoted by $\mathrm{var}^H(W)$, is the class of all H-module algebras satisfying the H-identities of W. Hence we say that $A \in \mathrm{var}^H(W)$ if and only if $\mathrm{Id}^H(W) \subseteq \mathrm{Id}^H(A)$.

Let $f(x_1, \ldots, x_n, Y) \in F^H\langle X\rangle$ be a multilinear H-polynomial, where Y is a set of variables disjoint from $x_1, \ldots, x_n$. We say f is *alternating* on $\{x_1, \ldots, x_n\}$ if there exists a multilinear H-polynomial $h(x_1, \ldots, x_n, Y)$ such that

$$f(X) = \sum_{\sigma\in S_n} (-1)^\sigma h(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, Y).$$

Equivalently, f is alternating on $\{x_1, \ldots, x_n\}$ if substituting in f x_i by x_j (for every i and j) and viceversa, we get $-f$.

2.5. ***H*-cocharacters and *H*-PI-exponent.** From now on any Hopf algebra is supposed to be finite dimensional over F. Let A be an H-module algebra. Denote by P_n^H the space (of dimension $(\dim_F H)^n \cdot n!$) of all multilinear H-polynomials in $x_1, \ldots, x_n$, $n \in \mathbb{N}$, i.e.,

$$P_n^H := \langle x_{\sigma(1)}^{h_1} x_{\sigma(2)}^{h_2} \cdots x_{\sigma(n)}^{h_n} \mid h_i \in H, \sigma \in S_n \rangle \subset F^H\langle X \rangle$$

where S_n is the symmetric group on n elements. The space P_n^H has a natural structure of left S_n-module induced by $\sigma \cdot x_1^{h_1} x_2^{h_2} \cdots x_n^{h_n} = x_{\sigma(1)}^{h_1} x_{\sigma(2)}^{h_2} \cdots x_{\sigma(n)}^{h_n}$, if $\sigma \in S_n$. Since $P_n^H \cap \mathrm{Id}^H(A)$ is a subspace invariant under the above action, hence $P_n^H(A) := P_n^H/(P_n^H \cap \mathrm{Id}^H(A))$ is a left S_n-module. This leads us to consider the S_n-character of $P_n^H(A)$, namely $\chi_n^H(A)$, which is called *n-th H-cocharacter of A*. The non-negative integer

$$c_n^H(A) := \chi_n^H(A)(1) = \dim_F P_n^H(A)$$

is called *n-th H-codimension of A*. Moreover the sequences $\{\chi_n^H(A)\}_{n \geq 0}$ and $\{c_n^H(A)\}_{n \geq 0}$ are called the *H-cocharacter sequence of A* and the *H-codimension sequence of A* respectively.

Given an H-module algebra A, if the limit

$$\limsup_{n \to \infty} \sqrt[n]{c_n^H(A)}$$

exists we shall call it *H PI-exponent of A* and we shall denote it by $\exp^H(A)$ (see [15, Section 9]).

The existence of the exponent for H-module algebras was studied in [28] in the case H is finite dimensional and semisimple acting on an associative algebra over a field of characteristic 0. In particular, in [28] the author proved that the H-exponent exists and is an integer. It is easy to see Taft's algebras are not semisimple algebras. In [25] the author proved the existence of the exponent for finite dimensional algebras over an algebraically closed field of characteristic 0 that are simple under the action of a Taft's algebra. We recall Taft's algebras are non-commutative, non-cocommutative and not semisimple Hopf algebras.

Let $n \geq 1$ be an integer. A *partition* λ of n is a finite sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \cdots \geq \lambda_k > 0$ and $\sum_{i=1}^r \lambda_i = n$. In this case we write $\lambda \vdash n$ or $|\lambda| = n$. By representation theory of S_n in characteristic zero, there is a one-to-one correspondence between irreducible S_n-characters and partitions of n. If χ_λ denotes the irreducible S_n-character corresponding to the partition $\lambda \vdash n$, then we can write

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m_\lambda^H \chi_\lambda$$

where $m_\lambda^H \geq 0$ are the corresponding multiplicities. The irreducible S_n-submodules of $P_n^H(A)$ can be written as $FS_n e_{T_\lambda} \cdot f$, where f is some H-polynomial in $P_n^H(A)$, T_λ is some Young tableau of the partition $\lambda \vdash n$ and

$$e_{T_\lambda} = \sum_{\sigma \in \mathcal{R}_{T_\lambda}, \tau \in \mathcal{C}_{T_\lambda}} (-1)^\tau \sigma\tau,$$

where $\mathcal{R}_{T_\lambda}$ and $\mathcal{C}_{T_\lambda}$ are the rows and columns stabilizers respectively.

2.6. **Gradings on UT_2.** Let UT_2 be the algebra of 2×2 upper triangular matrices over the field F. A detailed description of the G-graded identities satisfied by the algebra UT_2 when the characteristic of F is 0 is given in [44]. In particular, in [44] the author shows that, up to isomorphism, there is only one non-trivial grading. So any G-grading on UT_2 is actually a $\mathbb{Z}_2$-grading. The algebras with $\mathbb{Z}_2$-grading are called *superalgebras*.

2.7. **H_m actions on UT_2.** Form now on, F is a field of characteristic zero containing a primitive m-th root of the unit; H_m is an m-th Taft's Hopf algebra over F.

Consider the H_m-action on UT_2. Then by a result of Centrone and Yasumura (see page 738 of [14]), there exist three structures of H_m-module algebra on UT_2:

i) the trivial grading and d acts trivially: in this case, $\mathrm{Id}^{H_m}(UT2)$ is merely the ideal of ordinary polynomial identities of UT_2;

ii) the canonical $\mathbb{Z}_2$-grading and d acts trivially: in this case, $\mathrm{Id}^{H_m}(UT2)$ coincides with the ideal of $\mathbb{Z}_2$-graded polynomial identities of UT_2 which was originally calculated by Valenti in [44];

iii) the canonical $\mathbb{Z}_2$-grading and d acts non-trivially. In this case, necessarily $d = \mathrm{ad}_\alpha(ae_{12})$, for some $0 \neq a \in F$, that is, if $A = \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} \in UT_2$, then

$$A^d = \mathrm{ad}_\alpha(ae_{12})\left(\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix}\right) = \begin{pmatrix} 0 & a(x_{22}-x_{11}) \\ 0 & 0 \end{pmatrix}. \tag{1}$$

The Specht properties for (i) and (ii) are particular cases of the Specht property for ordinary PI-algebras [31] and G-graded PI-algebras [2], respectively. Therefore, we will study the case (iii). Thus, from now on, an H_m-action on UT_2 means the canonical $\mathbb{Z}_2$-grading on UT_2 with a non-trivial action of d on UT_2. This forces us to see an action of H_m on the algebra UT_2 as an action of H_2 on UT_2. It is worth recalling in [26] the author gives an explicit description of the simple algebras that are module algebra under the action of a Sweedler's algebra that is a Taft's algebra of dimension 4.

2.8. **The H_m-identities of UT_2.** Let $F\langle X\rangle$ be the free associative algebra over the countable set $X = \{x_1, x_2, \dots\}$. If we write $X = Y \cup Z$ where $Y = \{y_1, y_2, \dots\}$ is the countable set of variables of degree zero and $Z = \{z_1, z_2, \dots\}$ is the countable set of variables of degree one, and $Y \cap Z = \emptyset$, then $F\langle Y \cup Z\rangle$ has a natural structure of free superalgebra on $Y \cup Z$. The elements of $F\langle Y \cup Z\rangle$ are called graded polynomials.

A graded polynomial $f(y_1, \dots, y_t, z_1, \dots, z_s) \in F\langle Y \cup Z\rangle$ is a graded identity of the superalgebra $A = A_0 \oplus A_1$, and we write $f \equiv 0$, if, for all $a_1, \dots, a_t \in A_0$, $b_1, \dots, b_s \in A_1$,, we have $f(a_1, \dots, a_t, b_1, \dots, b_s) = 0$. We denote by $\mathrm{Id}^{gr}(A) = \{f \in F\langle Y \cup Z\rangle \mid f \equiv 0 \text{ on } A\}$ the ideal of graded identities of A. Notice that $\mathrm{Id}^{gr}(A)$ is a T_2-ideal of $F\langle Y \cup Z\rangle$, i.e., an ideal that is invariant under all $\mathbb{Z}_2$-graded endomorphisms of the free superalgebra $F\langle Y \cup Z\rangle$. Since the characteristic of F is zero, it is well known that $\mathrm{Id}^{gr}(A)$ is completely determined by its multilinear graded polynomials.

Now, we construct $F\langle Y \cup Z \mid D_2\rangle$ the free superalgebra on $X = Y \cup Z$ with action of $D_2 = F\langle d \mid d^2 = 0\rangle$ as follows. The algebra $F\langle Y \cup Z \mid D_2\rangle$ is the algebra freely generated by the set $\{x^{d_1} = d_1(x) \mid x \in Y \text{ or } x \in Z, d_1 \in D_2\}$. We let D_2 act on $F\langle Y \cup Z \mid D_2\rangle$ by requiring that if $d_1, d_2 \in D_2$, then $(x^{d_1})^{d_2} = x^{d_1 d_2}$, and then by extending this action on all of $F\langle Y \cup Z \mid D_2\rangle$ as follows: if v, w are monomials, then define $(vw)^d = v^d w + (-1)^{\deg(v)} v w^d$ and then extend this action by linearity to all of $F\langle Y \cup Z \mid D_2\rangle$. The elements of $F\langle Y \cup Z \mid D_2\rangle$ are called *$\mathbb{Z}_2$-D_2-polynomials.*

The algebra $F\langle Y \cup Z \mid D_2\rangle$ has the following universal property: Given any superalgebra $A = A_0 \oplus A_1$ with D_2-action, any set theoretical map $\varphi\colon Y \cup Z \to A$ such that $\varphi(Y) \subseteq A_0$ and $\varphi(Z) \subseteq A_1$, extends uniquely to a homomorphism of superalgebras $\bar\varphi\colon F\langle Y \cup Z \mid D_2\rangle \to A$ such that $\bar\varphi(f_1^d) = \bar\varphi(f)_1^d$, for any $f \in F\langle Y \cup Z \mid D_2\rangle$, $d_1 \in D_2$.

If we let Φ be the set of all such homomorphisms, then $\mathrm{Id}^{\mathbb{Z}_2, D_2}(A) = \cap_{\bar\phi \in \Phi} \ker \bar\phi$ is the ideal of $\mathbb{Z}_2$-D_2-polynomials identities of A. This means that a $\mathbb{Z}_2$-D_2-polynomial $f(y_1, \dots, y_s, z_1, \dots, z_t) \in F\langle Y \cup Z \mid D_2\rangle$ is a $\mathbb{Z}_2$-D_2-identity for A if $f(a_1, \dots, a_s, b_1, \dots, b_t) = 0$ for all $a_1, \dots, a_s \in A_0$ and $b_1, \dots, b_t \in A_1$. We write $f \equiv 0$ on A, in this case.

Assume $\mathbb{Z}_2 = \{1, c\}$.

Definition 2. $\mathrm{Id}^{\mathbb{Z}_2, D_2}(A) = \{f \in F\langle Y \cup Z \mid D_2\rangle \mid f \equiv 0$ *on* $A\}$ *is the ideal of $\mathbb{Z}_2$-D_2-polynomial identities of A.*

Proposition 3 ([14] Proposition 14)**.**

$$\mathrm{Id}^{H_2}(A) = \mathrm{Id}^{\mathbb{Z}_2, D_2}(A),$$

and

$$F^{H_2}\langle X\rangle \cong F\langle Y \cup Z \mid D_2\rangle.$$

Theorem 4 (Theorem 17, [14]). *For each* $j = 0, 1, \ldots, m-1$, *let* $\beta_j = \sum_{l=0}^{m-1} \gamma^{jl} c^l$, $y_i = x_i^{\beta_0}$ *and* $z_i = x_i^{\beta_1}$. *Then the* T^{H_m}*-ideal of* UT_2 *is generated by the following polynomials*

$$[y_1, y_2],\ z_1 x^h z_2,\ z^d,\ x^{d^2},\ y_1^d x^h y_2^d,\ x^{\beta_j},$$

where $h \in H_m$ *and* $j = 2, \ldots, m-1$.

3. The space of multilinear H_m-polynomials

Let $P_n^{\mathbb{Z}_2, D_2}$ the vector space of multilinear $\mathbb{Z}_2$-D_2-polynomials of degree n in $x_1, \ldots, x_n$, i.e.,

$$P_n^{\mathbb{Z}_2,D_2} = \operatorname{span}_F\{x_{\sigma(1)}^{d_1} \cdots x_{\sigma(n)}^{d_n} \mid \sigma \in S_n, d_i \in D_2, x_i = y_i \text{ or } x_i = z_i, i = 1, \ldots, n\} \subset F\langle Y \cup Z \mid D_2\rangle.$$

Recall that the wreath product of $\mathbb{Z}_2$ and S_n (called the *hyperoctahedral* group) is the group defined by

$$\mathbb{Z}_2 \wr S_n = \{(g_1, \ldots, g_n; \sigma) \mid g_1, \ldots, g_n \in \mathbb{Z}_2, \sigma \in S_n\}$$

with multiplication given by

$$(g_1, \ldots, g_n; \sigma)(h_1, \ldots, h_n; \tau) = (g_1 h_{\sigma^{-1}(1)}, \ldots, g_n h_{\sigma^{-1}(n)}; \sigma\tau).$$

Let $\mathbb{Z}_2 = \{1, c\}$. Then the space $P_n^{\mathbb{Z}_2,D_2}$ has a structure of left $\mathbb{Z}_2 \wr S_n$-module induced by defining for $(g_1, \ldots, g_n; \sigma) \in \mathbb{Z}_2 \wr S_n$ and $f(x_1, \ldots, x_n) \in P_n^{\mathbb{Z}_2,D_2}$ (see [24, Lemma 10.1.5]),

$$(g_1, \ldots, g_n; \sigma) f(x_1, \ldots, x_n) = f(x_{\sigma(1)}^{g_{\sigma(1)}^{-1}}, \ldots, x_{\sigma(n)}^{g_{\sigma(n)}^{-1}}),$$

where $y_{\sigma(i)}^c = y_{\sigma(i)}$ and $z_{\sigma(i)}^c = -z_{\sigma(i)}$.

Notice that the vector space $P_n^{\mathbb{Z}_2,D_2} \cap \mathrm{Id}^{\mathbb{Z}_2,D_2}(A)$ is invariant under this action, hence $P_n^{\mathbb{Z}_2,D_2}(A) := P_n^{\mathbb{Z}_2,D_2}/(P_n^{\mathbb{Z}_2,D_2} \cap \mathrm{Id}^{\mathbb{Z}_2,D_2}(A))$ is a left $\mathbb{Z}_2 \wr S_n$-module. Let $\chi_n^{\mathbb{Z}_2,D_2}(A)$ be its character. It is known (see for instance Section 10.4 of [24]) that there is a one-to-one correspondence between irreducible $\mathbb{Z}_2 \wr S_n$-character and pairs of partitions (λ, μ), where $\lambda \vdash r$, $\mu \vdash n-r$, for all $r = 0, 1, \ldots, n$. If $\chi_{\lambda,\mu}$ denotes the irreducible $\mathbb{Z}_2 \wr S_n$-character corresponding to (λ, μ) then we can write

$$\chi_n^{\mathbb{Z}_2,D_2}(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} \chi_{\lambda,\mu},$$

where $m_{\lambda,\mu} \geq 0$ are the corresponding multiplicities.

For fixed $r \in \{0, \ldots, n\}$, let

$$\begin{aligned} P_{r,n-r} = \operatorname{span}_F\{x_{\sigma(1)}^{d_1} \cdots x_{\sigma(n)}^{d_n} \mid \sigma \in S_n, d_i \in D_2, x_i = y_i \text{ for } i = 1, \ldots, r, \\ \text{and } x_i = z_i \text{ for } i = r+1, \ldots, n\} \end{aligned}$$

be the subspace of multilinear $\mathbb{Z}_2$-D_2-polynomials in the variables $y_1, \ldots, y_r, z_{r+1}, \ldots, z_n$. In order to study $P_n^{\mathbb{Z}_2,D_2}(A)$ it is enough to study

$$P_{r,n-r}(A) = \frac{P_{r,n-r}}{P_{r,n-r} \cap \mathrm{Id}^{\mathbb{Z}_2,D_2}(A)}$$

for all $r = 0, \ldots, n$. If we let S_r acting on the variables $y_1, \ldots, y_r$ and S_{n-r} acting on the variables $z_{r+1}, \ldots, z_n$, we obtain an action of $S_r \times S_{n-r}$ on $P_{r,n-r}$ and $P_{r,n-r}(A)$ becomes a left $S_r \times S_{n-r}$-module. Let $\chi_{r,n-r}(A)$ be its character. It is well known that the irreducible $S_r \times S_{n-r}$-characters are obtained by taking the outer tensor product of S_r and S_{n-r} irreducible characters, respectively. Then, we can write

$$\chi_{r,n-r}(A) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash n-r}} m_{\lambda,\mu} (\chi_\lambda \otimes \chi_\mu),$$

where χ_λ (respectively, χ_μ) denotes the irreducible S_r-character (respectively S_{n-r}-character) and $m_{\lambda,\mu} \geq 0$ are the corresponding multiplicities.

The relation between the character $\chi_n^H(A)$ and the character $\chi_{r,n-r}(A)$ for any H_m-module algebra A is given by

$$\chi_n^{\mathbb{Z}_2,D_2}(A)=\sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu} \quad \text{and} \quad \chi_{r,n-r}(A)=\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}(\chi_\lambda\otimes\chi_\mu)$$

for all $r\leq n$. Moreover,

$$c_n^H(A)=\sum_{r=0}^{n}\binom{n}{r}\dim_F P_{r,n-r}(A).$$

Remark that since $[y_1,y_2]^d$ is an H_m-identity of UT_2, then we have the following equality modulo $\mathrm{Id}^{H_m}(UT_2)$

$$y_1^d y_2 - y_2^d y_1 = y_2 y_1^d - y_1 y_2^d. \tag{2}$$

Moreover, for every $n\geq 0$, a linear basis for the space $P_{n,0}(UT_2)$ is given by the following set of polynomials:

- $y_1\cdots y_n$,
- $w_S := y_{i_1}\cdots y_{i_{k-1}}y_{i_k}^d y_{i_{k+1}}\cdots y_{i_n}$,

where S denotes the ordered k-tuple $(i_1,\dots,i_k)$, $i_j\in\{1,\dots,n\}$ and all the other indexes are ordered. This implies that the space $P_{n,0}(UT_2)$ has dimension $\sum_{k=0}^{n}\binom{n}{k}=2^n$.

A linear basis for the space $P_{n-1,1}(UT_2)$ is given by the following set of polynomials:

- $u_S := y_{i_1}\cdots y_{i_{k-1}}zy_{i_{k+1}}\cdots y_{i_n}$,

where S denotes the ordered k-tuple $(i_1,\dots,i_k)$, $i_j\in\{1,\dots,n\}$ and all the other indexes are ordered. Since the number of polynomials u_S is given by $\sum_{k=0}^{n-1}\binom{n-1}{k}$, then the space $P_{n-1,1}(UT_2)$ has dimension 2^{n-1}. The spaces $P_{r,n-r}(UT_2)$ vanishes for $r=0,1,\dots,n-2$. Therefore we obtain the following.

Proposition 5. *The n-th H_m-codimension of UT_2 is*

$$c_n^{H_m}(UT_2)=\sum_{r=0}^{n}\binom{n}{r}\dim_F P_{r,n-r}(UT_2)=n2^{n-1}+2^n=(n+2)2^{n-1},$$

and the H_m PI-exponent of UT_2 is

$$\exp^{H_m}(UT_2)=\limsup_{n\to\infty}\sqrt[n]{c_n^{H_m}(UT_2)}=2.$$

4. H_m-Cocharacters of UT_2

The goal of this section is giving a complete description of the H_m-cocharacter sequence of UT_2, where H_m is an m-th Taft's Hopf algebra.

Let $\lambda\vdash r$, $\mu\vdash n-r$ and let $W_{\lambda,\mu}$ be a left irreducible $S_r\times S_{n-r}$-module. It is well known that if T_λ is a tableau of shape λ and T_μ is a tableau of shape μ, then $W_{\lambda,\mu}\cong F(S_r\times S_{n-r})e_{T_\lambda}e_{T_\mu}$ where S_r and S_{n-r} act on disjoint sets of integers.

For a partition $\lambda\vdash n$ we denote by $h(\lambda)$ the height of the diagram associated to λ, that is, if $\lambda=(\lambda_1,\dots,\lambda_k)$, then $h(\lambda)=k$.

We can now write the explicit decomposition of the n-th H_m-cocharacter of UT_2 into irreducibles.

Theorem 6. *Let*

$$\chi_n^{\mathbb{Z}_2,D_2}(UT_2)=\sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

be the n-th H_m-cocharacter of the H-module algebra UT_2. Then

i) $m_{\lambda,\emptyset}=l+1$ if $\lambda=(k+l,k)$;
ii) $m_{\lambda,\mu}=l+1$ if $\lambda=(k+l,k)$, $\mu=(1)$;
iii) $m_{\lambda,\mu}=0$ in all other cases.

Proof. Let $A=UT_2$ and consider the canonical grading $A=A_0\oplus A_1$, where $A_0=\mathrm{span}\{e_{11},e_{22}\}$ and $A_1=\mathrm{span}\{e_{12}\}$. Since $\dim A_0=2$ and $\dim A_1=1$, any H_m-polynomial alternating on three even variables or in two odd variables vanishes on A; it follows that $m_{\lambda,\mu}=0$ if either $h(\lambda)\geq 3$ or $h(\mu)\geq 2$

and this proves the case (*iii*). By Proposition 4, $z_1xz_2 \in \mathrm{Id}^{H_m}(A)$, then $m_{\lambda,\mu} = 0$ whenever $|\mu| \geq 2$. So we have two cases left to study, namely $\mu = \emptyset$ or $\mu = (1)$.

First we consider the case $\mu = \emptyset$. Let $\lambda = (k+l, k)$, with $k \geq 0$, $l \geq 0$ and $2k + l = n$. For each $i = 0, \dots, l$ let us consider the following tableau:

$$T_\lambda^{(i)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline i+1 & i+2 & \cdots & i+k & 1 & 2 & \cdots & i & i+2k+2 & \cdots & n \\ \hline i+k+2 & i+k+3 & \dots & i+2k+1 \\ \cline{1-4}\end{array}.$$

We associate to $T_\lambda^{(i)}$ the H_m-polynomial

$$b_{k,l}^{(i)}(y_1, y_2) = y_1^i \underbrace{\overline{y_1} \cdots \widehat{y_1} (\widetilde{y_1})^d}_{k} \underbrace{\overline{y_2} \cdots \widehat{y_2} \widetilde{y_2}}_{k} y_1^{l-i},$$

where $-, {}^\wedge, \sim$ mean alternation on the corresponding elements. It is not hard to see

$$b_{k,l}^{(i)}(y_1, y_2) = \sum_{\sigma_1, \dots, \sigma_k \in S_2} (-1)^{\sigma_1} \cdots (-1)^{\sigma_k} y_1^i y_{\sigma_1(1)} \cdots y_{\sigma_k(1)}^d y_{\sigma_1(2)} \cdots y_{\sigma_k(2)} y_1^{l-i}.$$

We shall prove the $l+1$ H_m-polynomials $b_{k,l}^{(i)}(y_1, y_2)$, $i = 0, \dots, l$, are linearly independent over F modulo $\mathrm{Id}^{H_m}(A)$. Suppose by absurd $\sum_{i=0}^l \beta_i b_{k,l}^{(i)}(y_1, y_2) = 0 \pmod{\mathrm{Id}^{H_m}(A)}$ and let $t = \max\{i \mid \beta_i \neq 0\}$. Then $\beta_t b_{k,l}^{(t)}(y_1, y_2) + \sum_{i<t} \beta_i b_{k,l}^{(i)}(y_1, y_2) = 0 \pmod{\mathrm{Id}^{H_m}(A)}$. If we consider the substitution $y_1 = y_1 + y_3$, we get

$$\begin{aligned}(3)\quad & \beta_t (y_1+y_3)^t \overline{(y_1+y_3)} \cdots \widehat{(y_1+y_3)} \widetilde{(y_1+y_3)}^d \, \overline{y_2} \cdots \widehat{y_2} \widetilde{y_2} (y_1+y_3)^{l-t} \\ & \qquad + \sum_{i<t} \beta_i (y_1+y_3)^i \overline{(y_1+y_3)} \cdots \widehat{(y_1+y_3)} \widetilde{(y_1+y_3)}^d \, \overline{y_2} \cdots \widehat{y_2} \widetilde{y_2} (y_1+y_3)^{l-i} \\ & \qquad\qquad = 0 (\mathrm{mod}\ \mathrm{Id}^{H_m}(A)).\end{aligned}$$

Let us consider the homogeneous component of degree $t+k$ in y_1 and of degree $l-t$ in y_3. Considering the substitution $y_1 = e_{11}$ and $y_2 = y_3 = e_{22}$, then, by Equation (1) we get $y_1^d = -ae_{12}$ and we obtain $(-\beta_t a)e_{12} = 0$, which implies $\beta_t = 0$, a contradiction. Hence the H_m-polynomials $b_{k,l}^{(i)}(y_1, y_2)$, $i = 0, \dots, l$, are linearly independent $(\mathrm{mod}\ \mathrm{Id}^{H_m}(A))$.

Notice that, for all i, $e_{T_\lambda^{(i)}}(y_1, \dots, y_n)$ is the complete linearization of the H_m-polynomial $b_{k,l}^{(i)}(y_1, y_2)$. It follows that the H_m-polynomials $e_{T_\lambda^{(i)}}$, $i = 0, \dots, l$, are linearly independent $(\mathrm{mod}\ \mathrm{Id}^{H_m}(A))$ and this implies that $m_{\lambda,\mu} \geq l+1$.

We want to prove the multiplicities are exactly $l+1$. For, let T_λ be any tableau and $e_{T_\lambda}(y_1, \dots, y_n)$ the corresponding H_m-polynomial. If $e_{T_\lambda} \notin \mathrm{Id}^{H_m}(A)$, then any two alternating variables in e_{T_λ} must lie on different sides of the elements of type y_i^d. Since e_{T_λ} is a linear combination $(\mathrm{mod}\ \mathrm{Id}^{H_m}(A))$ of H_m-polynomials, each alternating on k pairs of y_i's, we get e_{T_λ} is a linear combination of the H_m-polynomials $e_{T_\lambda^{(i)}}$, $i = 0, \dots, l$. Hence $m_{\lambda,\mu} = l+1$ and this proves item (*i*) of the sentence.

We only need to study the case $\mu = (1)$. Let $\lambda = (k+l, k)$, with $k \geq 0$, $l \geq 0$ and $2k + l = n - 1$. This case can be proved following word by word the last part of the proof of Theorem 3 of [44], where the H_m-polynomials

$$a_{k,l}^{(i)}(y_1, y_2, z) = y_1^i \underbrace{\overline{y_1} \cdots \widetilde{y_1}}_{k} z \underbrace{\overline{y_2} \cdots \widetilde{y_2}}_{k} y_1^{l-i}, \qquad i = 0, 1, \dots, l,$$

are the highest weight vectors corresponding to λ. As above,

$$a_{k,l}^{(i)}(y_1, y_2, z) = \sum_{\sigma_1, \dots, \sigma_k \in S_2} (-1)^{\sigma_1} \cdots (-1)^{\sigma_k} y_1^i y_{\sigma_1(1)} \cdots y_{\sigma_k(1)} z y_{\sigma_1(2)} \cdots y_{\sigma_k(2)} y_1^{l-i}.$$

This proves (*ii*) and the proof is complete. □

Recall that in characteristic zero, any result on multilinear polynomial identities obtained in the language of representations of the symmetric group is equivalent to a corresponding result on homogeneous polynomial identities obtained in the language of representations of the general linear group.

Notice that the H_m-polynomial $b_{k,l}^{(i)}$ is obtained from the essential idempotent corresponding to the tableau $T_\lambda^{(i)}$ by identifying all the elements in each row of λ. Therefore, the H_m-polynomial $b_{k,l}^{(i)}$ is a highest weight vector, according to the representation theory of GL_n (see [19, Chapter 12] for more details). We recall that the complete linearization of a highest weight vector associated to an irreducible GL_n-module generates an irreducible S_n-module.

Corollary 7. *The highest weight vectors whose characters appear with non-zero multiplicity in the decomposition of $\chi_n^{\mathbb{Z}_2,D_2}(UT_2)$ are linear combinations of H_m-polynomials of the form:*

(1)
$$b_{k,l}^{(i)}(y_1,y_2) = y_1^i\,\underbrace{\overline{y_1}\cdots\widehat{y_1}(\widetilde{y_1})^d}_{k}\underbrace{\overline{y_2}\cdots\widehat{y_2}\widetilde{y_2}}_{k}\,y_1^{l-i}, \qquad i=0,1,\dots,l,$$

where $2k+l=n$; and

(2)
$$a_{k,l}^{(i)}(y_1,y_2,z) = y_1^i\,\underbrace{\overline{y_1}\cdots\widetilde{y_1}}_{k}\,z\,\underbrace{\overline{y_2}\dots\widetilde{y_2}}_{k}\,y_1^{l-i}, \qquad i=0,1,\dots,l,$$

where $2k+l+1=n$.

If $\chi_n^{\mathbb{Z}_2,D_2}(A)=\sum_{r=0}^n\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu}$ is the decomposition of the $\mathbb{Z}_2\wr S_n$-character of A, then one defines the n-th $\mathbb{Z}_2\wr S_n$-*colength* of A as

$$l_n^{\mathbb{Z}_2,D_2}(A)=\sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}.$$

By Theorem 6 we immediately get the following.

Corollary 8. *For all $n\geq 1$,*

$$l_n^{\mathbb{Z}_2,D_2}(UT_2)=\sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}=\frac{n^2+3n+2}{2}.$$

5. Specht property for the H_m-module algebra UT_2

In this section we prove that the variety of H_m-module algebras generated by the H_m-module algebra UT_2 has the Specht property.

Definition 9. *Let W be an H-module algebra. We say that $\mathrm{Id}^H(W)$ has the Specht property if any T^H-ideal I such that $I\supseteq \mathrm{Id}^H(W)$, has a finite basis, that is, I is finitely generated as a T^H-ideal. We say that the variety $\mathcal{V}$ has the Specht property if the corresponding T^H-ideal has the Specht property.*

We recall that a binary relation $\leq$ on a set A is a *quasi-order* if $\leq$ is reflexive and transitive, i.e., (i) $a\leq a$ for all $a\in A$, and (ii) $a\leq b$ and $b\leq c$ imply $a\leq c$, with $a,b,c\in A$. If B is a subset of a quasi-ordered set A, the *closure* of B, written $\overline{B}$, is defined as

$$\overline{B}=\{a\in A \mid \text{ exists } b\in B \text{ such that } b\leq a\}.$$

We say that the quasi-ordered set A has the *finite basis property* (**f.b.p.**) if for any subset B of A, there exists a finite subset B_0 of A such that $B_0\subseteq B\subseteq\overline{B_0}$. Every well-ordered set has **f.b.p.**. In particular, the set $\mathbb{N}$ of natural numbers with standard ordering has **f.b.p.**. The following theorem gives a equivalent definition for **f.b.p.**.

Theorem 10. *[27, theorem 2.1] The following conditions on a quasi-ordered set A are equivalent.*

1. *If B is any subset of A, there is a finite set B_0 such that $B_0\subseteq B\subseteq\overline{B_0}$;*
2. *There exists neither an infinite strictly descending sequence in A nor an infinite one of mutually incomparable elements of A.*

Let $A_1,A_2,\dots,A_n$ be quasi-ordered sets. The cartesian product $A_1\times A_2\times\cdots\times A_n$ ordered by $(a_1,a_2,\dots,a_n)\leq(b_1,b_2,\dots,b_n)$ if and only if $a_i\leq b_i$ for all $i\in\{1,2,\dots,n\}$ is a quasi-ordered set.

The following theorems are useful.

Theorem 11. *[27, Theorem 2.3] Let $A_1, A_2, \ldots, A_n$ be quasi-ordered sets satisfying* **f.b.p.**, *so their cartesian product satisfies* **f.b.p.**.

Theorem 12. *Let $A_1, A_2, \ldots, A_n$ be quasi-ordered sets satisfying* **f.b.p.**, *so the disjoint union $A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ endowed with the quasi-order $a \leq b$ if and only if $a, b \in A_i$ and $a \leq_{A_i} b$ for some $i \in \{1, \ldots, n\}$ satisfies* **f.b.p.**.

The free H-module algebra $F^H\langle X\rangle$ is a quasi-ordered set if we define for $f, g \in F^H\langle X\rangle$,

$$f \leq g \text{ if and only if } g \in \langle f\rangle_{T^H},$$

where $\langle f\rangle_{T^H}$ denotes the T^H-ideal generated by f.

If I is a T^H-ideal of $F^H\langle X\rangle$, the quasi-order on $F^H\langle X\rangle$ is inherited by $\frac{F^H\langle X\rangle}{I}$. Here we shall consider $\frac{F^{H_m}\langle X\rangle}{\mathrm{Id}^{H_m}(UT_2)}$ as a quasi-ordered set. Hence, if $f, g \in F^{H_m}\langle X\rangle$,we define

$$f \leq g \text{ if and only if } g \in \langle\{f\} \cup \mathrm{Id}^{H_m}(UT_2)\rangle_{T^{H_m}}.$$

In this case we way that g is a consequence of f modulo $\mathrm{Id}^{H_m}(UT_2)$ or simply that g is a consequence of f.

Remark 13. *Let M be a subset of $F^H\langle X\rangle$. Then $\overline{M} \subseteq \langle M\rangle_{T^H}$ by definition. On the other hand, since $M \subseteq \overline{M}$ we have that $\langle\overline{M}\rangle_{T^H} = \langle M\rangle_{T^H}$*

Let M be the set of all the highest weight vectors corresponding to the cocharacters appearing with non-zero multiplicities in $\chi_n^{\mathbb{Z}_2, D_2}(UT_2)$. By Corollary 7, the highest weight vectors lying in M are a linear combination of H_m-polynomials of the form:

(1) $y_1^i \underbrace{\overline{y_1} \cdots \widetilde{y_1}}_{k} z \underbrace{\overline{y_2} \ldots \widetilde{y_2}}_{k} y_1^{l-i}$,

(2) $y_1^i \underbrace{\overline{y_1} \cdots \widehat{y_1} (\widetilde{y_1})^d}_{k} \underbrace{\overline{y_2} \cdots \widehat{y_2} \widetilde{y_2}}_{k} y_1^{l-i}$.

Let us denote by $\mathcal{B}_1$ the set of H_m-polynomials of the form (1) and $\mathcal{B}_2$ the set of H_m-polynomials of the form (2). For $i = 1, 2$, we define the quasi-order $\leq$ in $\mathcal{B}_i$ by $f \leq g$ if and only if g is a consequence of f, where $f, g \in \mathcal{B}_i$. We consider the following sets which are in one-to-one correspondence with the highest weight vectors of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively:

$$\begin{aligned} B_1 &= \{(i, l-i, k) \mid 0 \leq i \leq q\} \\ &= \{(i, j, k)\} = \mathbb{N}^3; \\ B_2 &= \{(i, l-i, k) \mid 0 \leq i \leq l\} \\ &= \{(i, j, k)\} = \mathbb{N}^3. \end{aligned}$$

By theorem 11, B_1 and B_2 have **f.b.p.** with the natural quasi-order of $\mathbb{N}^3$. We shall show that the quasi-order $\leq$ in B_1 and B_2 induces the quasi-order $\leq$ in $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively.

Lemma 14. $(i, j, k) \leq (i', j', k')$ *implies*

(1) $y_1^i \underbrace{\overline{y_1} \cdots \widetilde{y_1}}_{k} z \underbrace{\overline{y_2} \ldots \widetilde{y_2}}_{k} y_1^j \leq y_1^{i'} \underbrace{\overline{y_1} \cdots \widetilde{y_1}}_{k'} z \underbrace{\overline{y_2} \ldots \widetilde{y_2}}_{k'} y_1^{j'}$, *and*

(2) $y_1^i \underbrace{\overline{y_1} \cdots \widehat{y_1} \widetilde{y_1}^d}_{k} \underbrace{\overline{y_2} \cdots \widehat{y_2} \widetilde{y_2}}_{k} y_1^j \leq y_1^{i'} \underbrace{\overline{y_1} \cdots \widehat{y_1} \widetilde{y_1}^d}_{k'} \underbrace{\overline{y_2} \cdots \widehat{y_2} \widetilde{y_2}}_{k'} y_1^{j'}$.

Proof. By transitivity of the quasi-order, in order to prove (1) we prove that

(i) $(i, j, k) \leq (i', j, k)$ implies $a_{i,j,k} \leq a_{i',j,k}$;
(ii) $(i, j, k) \leq (i, j', k)$ implies $a_{i,j,k} \leq a_{i,j',k}$;
(iii) $(i, j, k) \leq (i, j, k')$ implies $a_{i,j,k} \leq a_{i,j,k'}$,

where i, i', j, j', k, k' are integers and

$$a_{i,j,k} = a_{i,j,k}(y_1, y_2, z) = y_1^i \underbrace{\overline{y_1} \cdots \widetilde{y_1}}_{k} z \underbrace{\overline{y_2} \ldots \widetilde{y_2}}_{k} y_1^j. \tag{4}$$

The statements (i) and (ii) follow from the fact that $a_{i,j,k} \equiv y_1^{i'-i} a_{i,j,k} \mod (\mathrm{Id}^H(UT_2))$ and $a_{i,j,k} \equiv a_{i,j,k} y_1^{j'-j} \mod (\mathrm{Id}^H(UT_2))$ respectively. In order to prove the statement (iii), without loss of generality, we may suppose $k' = k+1$. The general statement will follows by a standard induction argument.

Notice that $a_{i,j,k}$ is a linear combination (mod $(\mathrm{Id}^H(UT_2))$) of the polynomials:

$$y_1^{i+t} y_2^{k-t} z y_1^{k-t} y_2^t y_1^j, \qquad t = 0, 1, \dots, k.$$

Thus, if we multiply by appropriate variables y's to the right or to the left of these polynomials, we obtain that for all $t = 0, 1, \dots, k$,

$$y_1^{i+t} y_2^{k-t} z y_1^{k-t} y_2^t y_1^j \le y_1^{i+t} y_2^{(k+1)-t} z y_1^{(k+1)-t} y_2^t y_1^j,$$

and therefore $a_{i,j,k} \le a_{i,j,k'}$.

The proof of (2) is analogous. ☐

Lemma 15. *The sets $\mathcal{B}_1$ and $\mathcal{B}_2$ with the quasi-order given above satisfy the* **f.b.p.**.

Proof. Let $\mathcal{B}_1'$ be a subset of $\mathcal{B}_1$ and B_1' a subset of $B_1 = \{(i, l-i, k) \mid a_{k,l}^{(i)} \in \mathcal{B}_1\}$ corresponding to $\mathcal{B}_1'$, i.e., $B_1' = \{(i, l-i, k) \mid a_{k,l}^{(i)} \in \mathcal{B}_1'\}$. Since $B_1' \subseteq B_1 \subseteq \mathbb{N}^3$, and, by Theorem 11, $\mathbb{N}^3$ has **f.b.p.**, we have there is a finite set $B_1^0 \subseteq B_1'$ such that $B_1^0 \subseteq B_1' \subseteq \overline{B_1^0}$. Consider $\mathcal{B}_1^0 = \{a_{k,l}^{(i)} \mid (i, l-i, k) \in B_1^0\} \subseteq \mathcal{B}_1'$ and $a_{k,l}^{(i)} \in \mathcal{B}_1'$. This implies $(i, l-i, k) \in B_1' \subseteq \overline{B_1^0}$; therefore there is $(i_0, l_0 - i_0, k_0) \in B_1^0$, where $(i_0, l_0 - i_0, k_0) \le (i, l-i, k)$. By the previous lemma, $a_{k_0,l_0}^{(i_0)} \le a_{k,l}^{(i)}$, where $a_{k_0,l_0}^{(i_0)} \in \mathcal{B}_1^0$. Thus $a_{k,l}^{(i)} \in \overline{\mathcal{B}_1^0}$ and consequently $\mathcal{B}_1^0 \subseteq \mathcal{B}_1' \subseteq \overline{\mathcal{B}_1^0}$, where $\mathcal{B}_1^0$ is a finite set. This shows $(\mathcal{B}_1, \le)$ satisfies **f.b.p.**.

The proof for the set $(\mathcal{B}_2, \le)$ is analogous and we are done. ☐

We already have the key ingredients to prove the main result of this section. We want to highlight we are going to use the algorithm described in full details in the paper [13].

Theorem 16. *$var^{H_m}(UT_2)$ has the Specht property.*

Proof. If $I = \mathrm{Id}^{H_m}(UT_2)$, then Theorem 4 ensures us that I is finitely generated. So let us suppose $I \supsetneq \mathrm{Id}^{H_m}(UT_2)$. Let M be the set of highest weight vectors corresponding to cocharacters appearing with non-zero multiplicities in $\chi_n^{\mathbb{Z}_2, D_2}(UT_2)$, $n \ge 0$; hence, $F^{H_m}\langle X\rangle$ is generated by M modulo $\mathrm{Id}^{H_m}(UT_2)$. Since $F^{H_m}\langle X\rangle \supseteq I \supsetneq \mathrm{Id}^{H_m}(UT_2)$, there exists $M' \subseteq M$ such that I is generated by M' modulo $\mathrm{Id}^{H_m}(UT_2)$. We will show that $(M, \le)$ satisfies **f.b.p.**, where $\le$ is the quasi-order given by the consequence, i.e., $f \le g$ if and only if g is a consequence of f in $F^{H_m}\langle X\rangle / \mathrm{Id}^{H_m}(UT_2)$.

A highest weight vector of degree n in M is a linear combination of H_m-polynomials of the form $a_{k,l}^{(i)}$, $i = 0, \dots, l$ and k, l fixed such that $2k + l + 1 = n$, or H_m-polynomials of the form $b_{k,l}^{(i)}$, $i = 0, \dots, l$ and k, l fixed such that $2k + l = n$ because they correspond to different modules. Thus $M = \mathcal{S}_1 \sqcup \mathcal{S}_2$, where $\mathcal{S}_1$ is the set of highest weight vectors associated to $\mathcal{B}_1$ and $\mathcal{S}_2$ is the set of highest weight vectors associated to $\mathcal{B}_2$. Then, by Theorem 12, it suffices to show that the sets $\mathcal{S}_i$ satisfy **f.b.p.**, where $f \le g$ if and only if g is a consequence of f, where $f, g \in \mathcal{S}_i$ for $i = 1, 2$.

Consider the set $\mathcal{S}_1$. A highest weight vector of degree n in $\mathcal{S}_1$ is of the form $\sum_{i=0}^{l} \alpha_i a_{k,l}^{(i)}$. Define the leading term of this highest weight vector as the element $a_{k,l}^{(i_0)}$, where $i_0 = \min\{i \mid \alpha_i \ne 0\}$. Notice that $\mathcal{B}_1$ can be seen as the set of all the leading terms of the set $\mathcal{S}$ and, by Lemma 15, $(\mathcal{B}_1, \le)$ satisfies **f.b.p.**. Hence, $\mathcal{B}_1$ has a finite subset $\mathcal{B}_1^0$ such that every element in $\mathcal{B}_1$ is bigger than some element of $\mathcal{B}_1^0$. Let $\mathcal{S}_1^0 \subseteq \mathcal{S}_1$ be the finite subset with leading terms in $\mathcal{B}_1^0$.

Let

$$h_1 = \sum_{i=0}^{l} \alpha_i a_{k,l}^{(i)} \in \mathcal{S}_1^0 \quad \text{and} \quad h_2 = \sum_{j=0}^{l'} \beta_j a_{k',l'}^{(j)} \in \mathcal{S}_1$$

be two highest weight vectors with leading terms $a_{k,l}^{(i_0)}, a_{k',l'}^{(j_0)}$ respectively, and such that $a_{k,l}^{(i_0)} \le a_{k',l'}^{(j_0)}$. Then,

$$a_{k',l'}^{(j_0)} \equiv y_1^{j_0 - i_0} \underbrace{\overline{y_1} \cdots \widetilde{y_1}}_{k'-k} a_{k,l}^{(i_0)} \underbrace{\overline{y_2} \cdots \widetilde{y_2}}_{k'-k} y_1^{l' - j_0 - l + i_0} \quad (\mathrm{mod}\ \mathrm{Id}^{H_m}(UT_2)).$$

At light of this, we consider the highest weight vector

$$h := \sum_{i=0}^{l} \alpha_i y_1^{j_0-i_0} \underbrace{\overline{y_1}\cdots\widetilde{y_1}}_{k'-k} a_{k,l}^{(i)} \underbrace{\overline{y_2}\cdots\widetilde{y_2}}_{k'-k} y_1^{l'-j_0-l+i_0}$$

which is a consequence of h_1 and its leading term is exactly $a_{k',l'}^{(j_0)}$. Therefore the leading term of

$$h_2 - \frac{\beta_{j_0}}{\alpha_{i_0}} h$$

is smaller than the leading term of h_2 and by inductive arguments is a consequence of $\mathcal{S}_1^0$. This shows that $\mathcal{S}_1$ satisfies **f.b.p.**.

Similarly, $\mathcal{S}_2$ satisfies **f.b.p.** too.

Finally, since I is generated by M' modulo $\mathrm{Id}^{H_m}(UT_2)$ and $(M, \leq)$ satisfies **f.b.p.**, then there exists a finite set $M_0 \subseteq M' \subseteq M$ such that $M_0 \subseteq M' \subseteq \overline{M_0}$. By Remark 13,

$$I = \langle M' \rangle_{T^{H_m}} = \langle \overline{M_0} \rangle_{T^{H_m}} = \langle M_0 \rangle_{T^{H_m}}$$

and we are done. □

6. Final remarks: Gelfand-Kirillov dimension and automata structure

In this section we would like to add more details in the depicted description of the relatively free H_m-algebra of UT_2. We add some information regarding the asympthotic of its growth function and a note toward the regularity of the language it represents. For more details about the Gelfand-Kiriilov dimension (GK dimension) of an algebra we remand to the books [34] by Krause and Lenagan and [36] by McConnell and Robson.

6.1. **Gelfand-Kirillov dimension.** In what follow, we shall introduce an H-module algebra version of the GK dimension of the relatively free H-module algebra. Let A be an finitely generated H-module algebra over F, where H is a finite dimensional Hopf algebra over F with F-basis $\{b_1, \ldots, b_m\}$. We shall denote with the symbol $F_k^H(A)$ the *relatively free H-module algebra of A in k variables*, that is,

$$F_k^H(A) := F^H \langle x_1, \ldots, x_k \rangle / (F^H \langle x_1, \ldots, x_k \rangle \cap T^H(A)).$$

Recall that $F^H \langle x_1, \ldots, x_k \rangle$ is isomorphic to the free algebra over F with free formal generators $x_i^{b_j}$, where $i \in \{1, \cdots, k\}$ and $j \in \{1, \cdots, m\}$. Thus,

$$F_k^H(A) = \frac{F^H \left\langle x_1^{b_1}, \ldots, x_1^{b_m}, \ldots, x_k^{b_1}, \ldots, x_k^{b_m} \right\rangle}{F^H \left\langle x_1^{b_1}, \ldots, x_1^{b_m}, \ldots, x_k^{b_1}, \ldots, x_k^{b_m} \right\rangle \cap T^H(A)}.$$

Definition 17 (H-Gelfand-Kirillov dimension in k variables)**.** *Let H be a finite dimensional Hopf algebra over a field F with F-basis $\{b_1, \ldots, b_m\}$ and A a finitely generated H-module algebra over F. The H-Gelfand-Kirillov dimension of A in k variables is*

$$\mathrm{GKdim}_k^H(A) := \mathrm{GKdim}(F_k^H(A)).$$

In [5] the author studies several properties of $\mathrm{GKdim}_r(A)$ (that is when H is the trivial Hopf algebra). In particular, it can be proved $\mathrm{GKdim}_r(A)$ is defined by the complexity type of the algebra A or by a set of semidirect products of matrix algebras over the ring of polynomials from the variety generated by A. See also the paper [8] by Berele for explicit computations of the GK dimension of some remarkable PI-algebras, the papers [9] and [10] by Centrone or the surveys [18] by Drensky and [11] by Centrone.

Here we have the next result based on direct computations.

Theorem 18. *Let $m, k \geq 2$ integers and let us consider H_m. Then*

$$\mathrm{GKdim}_k^{H_m}(UT_2) = 2k.$$

Proof. From the results showed in Section 2.8, it turns out the monomials

$$y_1^{a_1} \cdots y_k^{a_k}, y_1^{b_1} \cdots y_k^{b_k} y^d y_1^{c_1} \cdots y_k^{c_k}, y_1^{b_1} \cdots y_k^{b_k} z y_1^{c_1} \cdots y_k^{c_k}$$

constitute a linear basis of $F_k^{H_m}(UT_2)$. Then the growth function of $F_k^{H_m}(UT_2)$, for sufficiently large n, is

$$g_V(n) := 2\binom{n-1+2k}{2k} + \binom{n+k}{k} + k + 2$$

which grows as a polynomial in n of degree $2k$ and we are done at light of the definition of GK dimension of an algebra. □

Because the H_m-exponent of UT_2 is 2, the previous result is an experimental confirmation of the following result obtained by one of the authors in [12] in the environment of graded algebras:

Theorem 19. *Let A be a finite dimensional G-graded algebra. Then*

$$\mathrm{GKdim}_k^G(A) = \exp^G(A)k + \alpha,$$

where α is integer and $\exp^G(A)$ denotes the G-graded exponent of A.

We can formulate the next conjecture.

Conjecture 20. *Let A be a finite dimensional H_m-module algebra. Then*

$$\mathrm{GKdim}_k^{H_m}(A) = \exp^{H_m}(A)k + \alpha,$$

where α is integer and $\exp^{H_m}(A)$ denotes the H_m-exponent of A.

6.2. **Automata algebras and regular languages.** Given a (possibly infinite) set of variables X, we consider the set of monomials $M(X)$ in the free algebra $F\langle X\rangle$. We define *formal language* any subset L of $M(X)$.

Definition 21. *Given two languages $L, L' \subseteq M(X)$, we consider the set-theoretic union $L \cup L'$ and the product $L \cdot L' = \{w \cdot w' | w \in L, w' \in L'\}$. Moreover, one defines the star operation $L^* = \bigcup_{d\geq 0} L_d$, where $L_0 = \{1\}$ and $L_d = LL_{d-1}$, for any $d \geq 1$. The union, the product and the star operation are called the regular operations over the languages. One also considers the set-theoretic intersection $L \cap L'$ and the complement $L^c = \{w \in M(X) | w \notin L\}$.*

A language is said to be *regular* if it can be generated by a regular grammar. We remand to the book of [20] for a general overview toward grammars, languages and formal computer science. Notice that regular languages can be obtained from finite languages by means of regular operations. We have the following, well known, result due to Kleene (see [33]).

Theorem 22. *A language $L \subseteq M(X)$ is regular if and only if it can be obtained from finite languages by applying a finite number of regular operations.*

We recall a *monomial* algebra over a field F is an algebra that is isomorphic to $F\langle X\rangle/I$, where I is generated by elements of $M(X)$. Moreover, a monomial algebra is said to be *automata* if I is a regular language. We advice the reader to read the book by [6] for a compendium about monomial algebras. With a slight modification of the original definition of automata algebra, we have the notion of automata algebra on relatively free algebra.

Definition 23. *Let A be a PI-algebra and let $R_A := F\langle X\rangle/I$ its relatively free algebra, where $I = T(A)$. We say R_A is automata if $\hat{I}$ is a regular language, where $\hat{I}$ denotes the ideal of leading terms of I.*

Of course, the previous definition can be adapted trivially to graded algebras, module algebras, etc. It is also convenient to recall a sort of duality between regularity of a monomial ideal and regularity of its quotient algebra.

Proposition 24. *Let I be a monomial ideal of $F\langle X\rangle$ and let $\mathcal{B}$ be a basis of $F\langle X\rangle/I$ as a vector space. Then I is regular if and only if $\mathcal{B}$ is regular.*

By the proof of Theorem 18 we get a basis of the relatively free H_m-algebra of UT_2 in k variables is monomial and can be written in terms of regular operation as

$$\{y_1\}^* \cdots \{y_k\}^* \cup \{y_1\}^* \cdots \{y_k\}^* \{y^d\} \{y_1\}^* \cdots \{y_k\}^* \cup \{y_1\}^* \cdots \{y_k\}^* \{z\} \{y_1\}^* \cdots \{y_k\}^*,$$

i.e., at light of Proposition 24 and Theorem 22, $\hat{I}$ is regular. This is the content of the following.

Theorem 25. *Let $k \geq 2$ an integer. Then the relatively free H_m-algebra of UT_2 in k variables is an automata algebra.*

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