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Bounded solutions for quasilinear modified Schrödinger equations*

Anna Maria Candela, Addolorata Salvatore and Caterina Sportelli
Dipartimento di Matematica
Università degli Studi di Bari Aldo Moro
Via E. Orabona 4, 70125 Bari, Italy
annamaria.candela@uniba.it
addolorata.salvatore@uniba.it
caterina.sportelli@uniba.it

Abstract

In this paper we establish a new existence result for the quasilinear elliptic problem

$$-\mathrm{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}A_t(x,u)|\nabla u|^p + V(x)|u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^N,$$

with $N \geq 2$, $p > 1$ and $V : \mathbb{R}^N \to \mathbb{R}$ suitable measurable positive function, which generalizes the modified Schrödinger equation. Here, we suppose that $A : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$–Caratheodory function such that $A_t(x,t) = \frac{\partial A}{\partial t}(x,t)$ and a given Carathéodory function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ has a subcritical growth and satisfies the Ambrosetti–Rabinowitz condition.

Since the coefficient of the principal part depends also on the solution itself, we study the interaction of two different norms in a suitable Banach space so to obtain a "good" variational approach. Thus, by means of approximation arguments on bounded sets we can state the existence of a nontrivial weak bounded solution.

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1 Introduction

In this paper we investigate the existence of weak bounded solutions for the quasilinear elliptic problem

$$-\mathrm{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}A_t(x,u)|\nabla u|^p + V(x)|u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

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with $N \geq 2$, $p > 1$, where $A : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$–Carathéodory function with partial derivative $A_t(x,t) = \frac{\partial A}{\partial t}(x,t)$, potential $V : \mathbb{R}^N \to \mathbb{R}$ is a suitable measurable function and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a given Carathéodory function.

Special examples of problem (1.1) are related to the existence of solitary waves for the quasilinear Schrödinger equation

$$iz_t = -\Delta z + V(x)z - h(|z|^2)z - \Delta(l(|z|^2))l'(|z|^2)z \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a given potential, l, $h : \mathbb{R} \to \mathbb{R}$ are real functions and the solution $z = z(x,t)$ is a complex function in $\mathbb{R}^N \times \mathbb{R}$. In fact, by using the ansatz $z(x,t) = \mathrm{e}^{-iEt}u(x)$ in (1.2) with $E \in \mathbb{R}$, the unknown strictly positive real function $u(x)$ is a solution of the corresponding elliptic problem

$$-\Delta u - \Delta(l(u^2))l'(u^2)u + V(x)u = Eu + h(u^2)u \quad \text{in } \mathbb{R}^N,$$

often referred as modified Schrödinger equation, which matches with (1.1), but taking $p = 2$, for suitable choices of function $l(s)$. And, again, the structure of the real term $l(s)$ allows one to use quasilinear equation (1.2) for describing several physical phenomena such as the self–channeling of a high–power ultrashort laser, or also some problems which arise in plasma physics, fluid mechanics, mechanics and in the condensed matter theory (see [33] and references therein or also [16] for some model problems).

On the other hand, if $A(x,t)$ is a constant, equation (1.1) turns into the p–Laplacian equation

$$-\Delta_p u + V(x)|u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^N \tag{1.3}$$

which has been widely investigated starting from the pioneer papers [8, 29]. Indeed, it is well known that (1.3) has a variational structure but there is a lack of compactness as such a problem is settled in the whole space $\mathbb{R}^N$ and classical variational tools cannot work easily; thus, suitable assumptions on potential $V(x)$ are required (see, e.g., [27] and references therein).

Clearly, the same difficulties still arise when we deal with the more general problem (1.1), but the presence of a coefficient which depends on the solution itself makes the variational approach more complicated. In fact, the "natural" functional associated to (1.1) is

$$\mathcal{J}(u) = \frac{1}{p}\int_{\mathbb{R}^N} A(x,u)|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} G(x,u)dx,$$

with $G(x,t) = \int_0^t g(x,s)ds$, but, even if $A(x,t)$ is a smooth strictly positive bounded function, if $A_t(x,t) \not\equiv 0$ functional $\mathcal{J}$ is well defined in $W^{1,p}(\mathbb{R}^N)$ while it is Gâteaux differentiable only along directions in $W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Such a problem has been overcome in different ways, for example by introducing suitable definitions of critical point for non–differentiable functionals (see, e.g., [3, 17, 19]), and in the whole Euclidean space $\mathbb{R}^N$ some existence results have been proved by using nonsmooth techniques (see, e.g., [4]), by means of constrained minimization arguments (see [23, 26, 28]), by introducing a suitable change of variable but only if $A(x,t)$ has a very special form, in particular it is independent of x (see, e.g., [18, 25, 31] and also [32] and references therein) or by making use of an approximation scheme via q–Laplacian regularization (see [22]). It is worth noting that the last mentioned technique turns out to be attractive also for studying the existence of multiple solutions of quasilinear equations of general forms to which the idea of changing variables does not apply (see [24]) and we think it should be interesting for possible future investigations to find out the right perturbation which may allow us to apply such a regularization approach to our more general problem (1.1).

Along with the aforementioned ideas, such a problem has been addressed also with a different approach which has been developed in [11, 12, 13]. More precisely, under some quite natural conditions, we are able to prove that functional $\mathcal{J}$ is C^1 in the Banach space $X = W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ equipped with the intersection norm $\|\cdot\|_X$ (see Proposition 3.10), then some abstract results can be applied in this setting as long as a weaker "compactness" condition is provided (see Definition 2.1).

Recently, if $V(x) \equiv 1$ this approach has been used for proving the existence of radial bounded solutions of (1.1) if $A(x,t)$ is quite general but all the involved functions are radially symmetric (see [15]) or they are 1–periodic with respect to x (see [14]).

Here, we get rid of both the symmetric and the periodic assumption and, as in the pioneer paper [9] which deals with a linear Schrödinger equation with $A(x,t) \equiv 1$ and $p = 2$ (see also [30] for the nonlinear case), we assume that the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a Lebesgue measurable function such that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0 \qquad \text{and} \qquad \lim_{|x|\to+\infty} \int_{B_1(x)} \frac{1}{V(y)}\, dy \;=\; 0, \tag{1.4}$$

where $B_1(x)$ denotes the unitary sphere of $\mathbb{R}^N$ centered in x.

Such assumptions (1.4) have already been used in [5, 6] for investigating the quasilinear equation (1.3) as they ensure a compactness embedding in suitable weighted Lebesgue spaces on $\mathbb{R}^N$ (see [9, Theorem 3.1]). A similar compact embedding has been stated in [7, 8] under suitable conditions on the level sets of V (for a comparison with assumption (1.4) see [30]).

Anyway, in the more general setting of problem (1.1) this is not enough and also an approximation argument is considered, i.e., for each $k \in \mathbb{N}$ we introduce the approximating quasilinear problem

$$\begin{cases} -\mathrm{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}A_t(x,u)|\nabla u|^p = g(x,u) - V(x)|u|^{p-2}u & \text{in } B_k,\\ u = 0 & \text{on } \partial B_k, \end{cases} \tag{1.5}$$

with $B_k = \{x \in \mathbb{R}^N : |x| < k\}$, and the related functional $\mathcal{J}_{B_k}$ on the "right" Banach space X_{B_k} (see definition (5.4), respectively (5.1)). Then, if "good" assumptions are satisfied (see Section 4) for all $k \in \mathbb{N}$ a weak bounded solution u_k of problem (1.5) exists and a nontrivial solution for equation (1.1) is constructed as a suitable limit of sequence $(u_k)_k$.

Thus, the first step is solving the given equation in bounded domains by following the ideas developed in [11, 12], then we pass to the limit in a "weak sense" so to find a weak bounded solution of (1.1) which has to be nontrivial (a similar approach is used in [14]).

We note that, since our variational principle requires that X is contained in $L^\infty(\mathbb{R}^N)$ (see definition (3.11)), in some sense the "transition" through bounded domain is mandatory as every "limit point" has to be a bounded function and the technical lemma, which allows it, holds only on bounded domains (see Lemma 5.6).

We note that our main theorem requires not only (1.4) but also $V(x)$ bounded on bounded sets, some hypotheses which describe suitable interaction properties between $A(x,t)$ and its derivative $A_t(x,t)$ and that $g(x,t)$ has a subcritical growth and satisfies an Ambrosetti–Rabinowitz type condition and a suitable assumption while approaching the origin. Thus, in order to not weigh this introduction down with too many details, we prefer to specify each hypothesis when required and to state our main result at the beginning of Section 4 (see Theorem 4.4).

This paper is organized as follows.

In Section 2 we introduce the abstract framework stating the weak Cerami–Palais–Smale condition and a generalized version of the classical Mountain Pass Theorem. In Section 3 we introduce some preliminary assumptions on the functions $A(x,t)$, $G(x,t)$ and on the potential $V(x)$, which allow us to give a variational principle for problem (1.1). In Section 4 we provide some further hypotheses on the involved functions, our main result is stated and some geometric properties are pointed out. Then, in Section 5 we investigate the existence of weak bounded solutions of problem (1.5) and, finally, in Section 6 we prove our main theorem.

2 Abstract setting

Throughout this section, we assume that:

- $(X, \|\cdot\|_X)$ is a Banach space with dual $(X', \|\cdot\|_{X'})$;
- $(W, \|\cdot\|_W)$ is a Banach space such that $X \hookrightarrow W$ continuously, i.e. $X \subset W$ and a constant $\sigma_0 > 0$ exists such that
$$\|\xi\|_W \ \leq\ \sigma_0\ \|\xi\|_X \qquad \text{for all } \xi \in X;$$
- $J : \mathcal{D} \subset W \to \mathbb{R}$ and $J \in C^1(X, \mathbb{R})$ with $X \subset \mathcal{D}$.

In order to avoid any ambiguity and simplify, when possible, the notation, from now on by X we denote the space equipped with its given norm $\|\cdot\|_X$ while, if the norm $\|\cdot\|_W$ is involved, we write it explicitly.

For simplicity, taking $\beta \in \mathbb{R}$, we say that a sequence $(\xi_n)_n \subset X$ is a *Cerami–Palais–Smale sequence at level* β, briefly $(CPS)_\beta$*–sequence*, if

$$\lim_{n\to+\infty} J(\xi_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ\,(\xi_n)\,\|_{X'}(1 + \|\xi_n\|_X) = 0.$$

Moreover, β is a *Cerami–Palais–Smale level*, briefly (CPS)*–level*, if there exists a $(CPS)_\beta$ – sequence.

As $(CPS)_\beta$ – sequences may exist which are unbounded in $\|\cdot\|_X$ but converge with respect to $\|\cdot\|_W$, we have to weaken the classical Cerami–Palais–Smale condition in a suitable way according to the ideas already developed in previous papers (see, e.g., [13]).

Definition 2.1. The functional J satisfies the *weak Cerami–Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ *condition*, if for every $(CPS)_\beta$–sequence $(\xi_n)_n$, a point $\xi \in X$ exists, such that

(i) $\lim_{n\to+\infty} \|\xi_n - \xi\|_W = 0$ (up to subsequences),

(ii) $J(\xi) = \beta,\ dJ(\xi) = 0$.

If J satisfies the $(wCPS)_\beta$ condition at each level $\beta \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I.

Since Definition 2.1 allows one to prove a Deformation Lemma (see [13, Lemma 2.3]), thus the following generalization of the Mountain Pass Theorem [2, Theorem 2.1] can be stated (for the proof, see [13, Theorem 1.7] with remarks in [15, Theorem 2.2]).

Theorem 2.2 (Mountain Pass Theorem). *Let $J \in C^1(X, \mathbb{R})$ be such that $J(0) = 0$ and the $(wCPS)$ condition holds in $\mathbb{R}$. Moreover, assume that two constants ϱ, $\alpha^* > 0$ and a point $e \in X$ exist such that*

$$u \in X,\ \|u\|_W = \varrho \quad \Longrightarrow \quad J(u) \geq \alpha^*,$$

$$\|e\|_W > \varrho \qquad \textit{and} \qquad J(e) < \alpha^*.$$

Then, J has a critical point $u_X \in X$ such that

$$J(u_X) = \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} J(\gamma(s)) \geq \alpha^*$$

with $\Gamma = \{\gamma \in C([0,1], X) :\ \gamma(0) = 0,\ \gamma(1) = e\}$.

3 The variational principle

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of the strictly positive integers and, taking any Ω open subset of $\mathbb{R}^N$, $N \geq 2$, we denote by:

- $B_R(x) = \{y \in \mathbb{R}^N : |y - x| < R\}$ the open ball with center in $x \in \mathbb{R}^N$ and radius $R > 0$;
- $|D|$ the usual N–dimensional Lebesgue measure of a measurable set D in $\mathbb{R}^N$;
- $(L^r(\Omega), |\cdot|_{\Omega,r})$ the classical Lebesgue space with norm $|u|_{\Omega,r} = \left(\int_\Omega |u|^r dx\right)^{1/r}$ if $1 \leq r < +\infty$;
- $(L^\infty(\Omega), |\cdot|_{\Omega,\infty})$ the space of Lebesgue–measurable essentially bounded functions endowed with norm $|u|_{\Omega,\infty} = \operatorname*{ess\,sup}_{\Omega} |u|$;
- $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ the classical Sobolev spaces both equipped with the standard norm $\|u\|_\Omega = (|\nabla u|^p_{\Omega,p} + |u|^p_{\Omega,p})^{\frac{1}{p}}$ if $1 \leq p < +\infty$.

Moreover, if $V : \mathbb{R}^N \to \mathbb{R}$ is a measurable function such that

$$(V_1) \qquad \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0,$$

we denote by

- $(L^r_V(\Omega), |\cdot|_{\Omega,V,r})$, if $1 \leq r < +\infty$, the weighted Lebesgue space with

$$L^r_V(\Omega) = \left\{u \in L^r(\Omega) : \int_\Omega V(x)|u|^r dx < +\infty\right\}, \quad |u|_{\Omega,V,r} = \left(\int_\Omega V(x)|u|^r dx\right)^{\frac{1}{r}}; \tag{3.1}$$

- $W_V^{1,p}(\Omega)$ and $W_{0,V}^{1,p}(\Omega)$, if $1 \le p < +\infty$, the weighted Sobolev spaces

$$W_V^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) : \int_\Omega V(x)|u|^p dx < +\infty \right\},$$
$$W_{0,V}^{1,p}(\Omega) = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega V(x)|u|^p dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_{\Omega,V} = (|\nabla u|_{\Omega,p}^p + |u|_{\Omega,V,p}^p)^{\frac{1}{p}}. \tag{3.2}$$

For simplicity, we put $B_R = B_R(0)$ for the open ball with center in the origin and radius $R > 0$ and, if $\Omega = \mathbb{R}^N$, we avoid to write the set in the norms, i.e., we put

- $|\cdot|_r = |\cdot|_{\mathbb{R}^N,r}$ for the norm in $L^r(\mathbb{R}^N)$, for all $1 \le r \le +\infty$;
- $|\cdot|_{V,r} = |\cdot|_{\mathbb{R}^N,V,r}$ for the norm in $L_V^r(\mathbb{R}^N)$, for all $1 \le r < +\infty$;
- $\|\cdot\| = \|\cdot\|_{\mathbb{R}^N}$ for the norm in $W^{1,p}(\mathbb{R}^N) = W_0^{1,p}(\mathbb{R}^N)$;
- $\|\cdot\|_V = \|\cdot\|_{\mathbb{R}^N,V}$ for the norm in $W_V^{1,p}(\mathbb{R}^N) = W_{0,V}^{1,p}(\mathbb{R}^N)$.

From now on, let $V : \mathbb{R}^N \to \mathbb{R}$ be a measurable function which satisfies not only condition (V_1) but also

(V_2) $$\lim_{|x|\to+\infty} \int_{B_1(x)} \frac{1}{V(y)}\, dy \;=\; 0.$$

Remark 3.1. If potential $V(x)$ satisfies assumption (V_1), then the following continuous embeddings hold:

$$L_V^r(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \quad \text{for all } 1 \le r < +\infty, \tag{3.3}$$
$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow W^{1,p}(\mathbb{R}^N) \quad \text{for all } 1 \le p < +\infty. \tag{3.4}$$

From Remark 3.1 and Sobolev Embedding Theorems, we deduce the following result (for the compact embeddings, see [9, Theorem 3.1]).

Theorem 3.2. *Let $V : \mathbb{R}^N \to \mathbb{R}$ be a Lebesgue measurable function satisfying assumption (V_1). Then, the following continuous embeddings hold:*

- *if $p < N$ then*
$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \quad \textit{for any } p \le r \le \frac{Np}{N-p}; \tag{3.5}$$
- *if $p = N$ then*
$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \quad \textit{for any } p \le r < +\infty; \tag{3.6}$$
- *if $p > N$ then*
$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \quad \textit{for any } p \le r \le +\infty. \tag{3.7}$$

Furthermore, if assumption (V_2) *also occurs, the compact embedding*

$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^r(\mathbb{R}^N) \quad \text{for any } p \le r < p^* \tag{3.8}$$

holds, with

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N. \end{cases}$$

From Theorem 3.2 it follows that, for any $r \ge p$ so that (3.5), respectively (3.6) or (3.7), holds then a constant $\tau_r > 0$ exists such that

$$|u|_r \le \tau_r \|u\|_V \quad \text{for all } u \in W_V^{1,p}(\mathbb{R}^N). \tag{3.9}$$

On the other hand, taking Ω open bounded domain in $\mathbb{R}^N$ and $p < N$, a classical embedding theorem implies that a constant $\sigma_* > 0$, independent of Ω and depending only on p and N, exists such that

$$|u|_{\Omega,p^*} \ \le \ \sigma_* \|u\|_\Omega \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{3.10}$$

From now on, we assume that potential $V(x)$ is a measurable function which satisfies condition (V_1) and we set

$$X := W_V^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \quad \text{and} \quad \|u\|_X = \|u\|_V + |u|_\infty \quad \text{for any } u \in X. \tag{3.11}$$

In the following, we assume $p \le N$ (otherwise, embedding (3.7) implies that $X = W_V^{1,p}(\mathbb{R}^N)$ and all the arguments and proofs can be simplified).

Lemma 3.3. *Taking* $p \le r < +\infty$*, we have that*

$$X \hookrightarrow L_V^r(\mathbb{R}^N) \qquad \text{with} \quad |u|_{V,r} \le \|u\|_X \quad \text{for all } u \in X. \tag{3.12}$$

Proof. Taking any $u \in X$, definitions (3.2) and (3.11) imply that

$$\int_{\mathbb{R}^N} V(x)|u|^r dx \le |u|_\infty^{r-p} \int_{\mathbb{R}^N} V(x)|u|^p dx \le |u|_\infty^{r-p} \|u\|_V^p \le \|u\|_X^r$$

which gives the proof. □

Remark 3.4. From (3.3), definition (3.11) and Lemma 3.3 it follows that

$$X \hookrightarrow L^r(\mathbb{R}^N) \quad \text{for any } p \le r \le +\infty.$$

Lemma 3.5. *If* $(u_n)_n \subset X$*,* $u \in X$ *and* $M > 0$ *are such that*

$$\|u_n - u\|_V \to 0 \quad \text{as } n \to +\infty \tag{3.13}$$

and

$$|u_n|_\infty \le M \quad \text{for all } n \in \mathbb{N}, \tag{3.14}$$

then $u_n \to u$ *strongly in* $L_V^r(\mathbb{R}^N)$ *for any* $p \le r < +\infty$*.*

Proof. From (3.14) we have that

$$\int_{\mathbb{R}^N} V(x)|u_n - u|^r dx \le |u_n - u|_\infty^{r-p} \int_{\mathbb{R}^N} V(x)|u_n - u|^p dx \le (M + |u|_\infty)^{r-p} \|u_n - u\|_V^p,$$

which, together with (V_1), (3.1) and (3.13), implies the desired result. □

We proceed recalling the following definition.

Definition 3.6. A function $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^k$–Carathéodory function, $k \in \mathbb{N} \cup \{0\}$, if

- $h(\cdot, t) : x \in \mathbb{R}^N \mapsto h(x, t) \in \mathbb{R}$ is measurable for all $t \in \mathbb{R}$,
- $h(x, \cdot) : t \in \mathbb{R} \mapsto h(x, t) \in \mathbb{R}$ is $\mathcal{C}^k$ for a.e. $x \in \mathbb{R}^N$.

Let $A : (x, t) \in \mathbb{R}^N \times \mathbb{R} \mapsto A(x, t) \in \mathbb{R}$ be a given function such that the following conditions hold:

(h_0) $A(x, t)$ is a $\mathcal{C}^1$–Carathéodory function with $A_t(x, t) = \frac{\partial}{\partial t} A(x, t)$;

(h_1) for any $\rho > 0$ we have that

$$\sup_{|t| \le \rho} |A(\cdot, t)| \in L^\infty(\mathbb{R}^N), \qquad \sup_{|t| \le \rho} |A_t(\cdot, t)| \in L^\infty(\mathbb{R}^N).$$

Furthermore, we assume that $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ exists such that:

(g_0) $g(x, t)$ is a $\mathcal{C}^0$–Carathéodory function;

(g_1) $a_1, a_2 > 0$ and $q \ge p$ exist such that

$$|g(x, t)| \le a_1 |t|^{p-1} + a_2 |t|^{q-1} \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R}.$$

Remark 3.7. Assumptions (g_0) and (g_1) imply that

$$G : (x, t) \in \mathbb{R}^N \times \mathbb{R} \mapsto \int_0^t g(x, s) ds \in \mathbb{R} \tag{3.15}$$

is a well defined $\mathcal{C}^1$–Carathéodory function and

$$|G(x, t)| \le \frac{a_1}{p} |t|^p + \frac{a_2}{q} |t|^q \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R}. \tag{3.16}$$

In particular, (g_1) and (3.15) imply that

$$G(x, 0) = g(x, 0) = 0 \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{3.17}$$

Taking any $u \in X$, from assumption (h_1) and definition (3.11) it follows that $A(\cdot, u)|\nabla u(\cdot)|^p \in L^1(\mathbb{R}^N)$, while hypotheses (g_0), (g_1) and Remark 3.4 provide that $G(\cdot, u) \in L^1(\mathbb{R}^N)$. Then, assumption (V_1) implies that functional

$$\mathcal{J}(u) = \frac{1}{p} \int_{\mathbb{R}^N} A(x, u) |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x) |u|^p dx - \int_{\mathbb{R}^N} G(x, u) dx \tag{3.18}$$

is well defined for all $u \in X$. Moreover, taking any u, $v \in X$, the same assumptions imply that $A_t(\cdot, u)v|\nabla u(\cdot)|^p \in L^1(\mathbb{R}^N)$ and $g(\cdot, u)v \in L^1(\mathbb{R}^N)$, then the Gâteaux differential of functional $\mathcal{J}$ in u along the direction v is well defined and is given by

$$\begin{aligned}\langle d\mathcal{J}(u), v\rangle = &\int_{\mathbb{R}^N} A(x,u)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx + \frac{1}{p}\int_{\mathbb{R}^N} A_t(x,u)v|\nabla u|^p dx \\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv dx - \int_{\mathbb{R}^N} g(x,u)v dx.\end{aligned} \tag{3.19}$$

As useful in the following, we recall same classical inequalities (for the proof, see, e.g., [20]).

Lemma 3.8. *A constant $C_0 > 0$ exists such that for any $\xi, \eta \in \mathbb{R}^N, N \geq 1$, it results*

$$||\xi|^{r-2}\xi - |\eta|^{r-2}\eta| \leq C_0|\xi - \eta| \left(|\xi| + |\eta|\right)^{r-2} \qquad \text{if } r > 2, \tag{3.20}$$

$$||\xi|^{r-2}\xi - |\eta|^{r-2}\eta| \leq C_0|\xi - \eta|^{r-1} \qquad \text{if } 1 < r \leq 2. \tag{3.21}$$

By reasoning as in [15, Lemma 3.4], from Lemma 3.8 we deduce the following result.

Lemma 3.9. *Taking $p > 1$, a constant $C_1 = C_1(p) > 0$ exists such that for any open domain $\Omega \subset \mathbb{R}^N$ it results*

$$\int_\Omega ||\nabla w|^p - |\nabla z|^p| dx \leq C_1\|w - z\|_\Omega \left(\|w\|_\Omega^{p-1} + \|z\|_\Omega^{p-1}\right) \quad \forall w, z \in W_0^{1,p}(\Omega).$$

Now, we are ready to state a "good" variational principle.

Proposition 3.10. *Let $p > 1$ and suppose that hypotheses (V_1), (h_0)–(h_1) and (g_0)–(g_1) hold. If $(u_n)_n \subset X$ and $u \in X$ are such that*

$$u_n \to u \qquad a.e. \text{ in } \mathbb{R}^N \tag{3.22}$$

and (3.13), (3.14) *hold for a constant $M > 0$, then*

$$\mathcal{J}(u_n) \to \mathcal{J}(u) \quad \text{and} \quad \|d\mathcal{J}(u_n) - d\mathcal{J}(u)\|_{X'} \to 0 \quad \text{as } n \to +\infty.$$

Hence, $\mathcal{J}$ is a $\mathcal{C}^1$ functional in X with Fréchet differential defined as in (3.19).

Proof. For the sake of convenience, we set $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 - \mathcal{J}_3$, where

$$\mathcal{J}_1 : u \in X \mapsto \mathcal{J}_1(u) = \frac{1}{p}\int_{\mathbb{R}^N} A(x,u)|\nabla u|^p dx \in \mathbb{R},$$

$$\mathcal{J}_2 : u \in X \mapsto \mathcal{J}_2(u) = \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx \in \mathbb{R},$$

$$\mathcal{J}_3 : u \in X \mapsto \mathcal{J}_3(u) = \int_{\mathbb{R}^N} G(x,u) dx \in \mathbb{R},$$

with related Gâteaux differentials

$$\langle d\mathcal{J}_1(u), v\rangle = \int_{\mathbb{R}^N} A(x,u)|\nabla u|^{p-2}\nabla u \cdot \nabla v\ dx + \frac{1}{p}\int_{\mathbb{R}^N} A_t(x,u)v|\nabla u|^p dx,$$

$$\langle d\mathcal{J}_2(u), v\rangle = \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\ dx,$$

$$\langle d\mathcal{J}_3(u), v\rangle = \int_{\mathbb{R}^N} g(x,u)v\ dx,$$

for any $u,\, v \in X$.
Now, let $(u_n)_n \subset X$, $u \in X$ and $M > 0$ be such that (3.13), (3.14) and (3.22) hold.
Firstly, we note that (3.4) and (3.13) imply $\|u_n - u\| \to 0$, then from the proof of [15, Proposition 3.6] it follows that

$$\begin{aligned} \mathcal{J}_1(u_n) \to \mathcal{J}_1(u) \quad &\text{and} \quad \|d\mathcal{J}_1(u_n) - d\mathcal{J}_1(u)\|_{X'} \to 0,\\ \mathcal{J}_3(u_n) \to \mathcal{J}_3(u) \quad &\text{and} \quad \|d\mathcal{J}_3(u_n) - d\mathcal{J}_3(u)\|_{X'} \to 0. \end{aligned}$$

Moreover, from definitions (3.1) and (3.2), limit (3.13) implies also that

$$|\mathcal{J}_2(u_n) - \mathcal{J}_2(u)| = \frac{1}{p}||u_n|^p_{V,p} - |u|^p_{V,p}| \to 0 \quad \text{as } n \to +\infty.$$

At last, taking $v \in X$ such that $\|v\|_X = 1$, so

$$|v|_\infty \le 1, \quad |v|_{V,p} \le 1, \tag{3.23}$$

from (V_1) and by definition we have that

$$|\langle d\mathcal{J}_2(u_n) - d\mathcal{J}_2(u), v\rangle| \;\le\; \int_{\mathbb{R}^N} V(x)||u_n|^{p-2}u_n - |u|^{p-2}u||v|dx. \tag{3.24}$$

Thus, if $1 < p \le 2$, from (3.21), Hölder inequality with $V(x) = V^{\frac{1}{p}}(x)V^{\frac{p-1}{p}}(x)$ and (3.23) we have that

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)||u_n|^{p-2}u_n - |u|^{p-2}u||v|dx &\le C_0 \int_{\mathbb{R}^N} V(x)|u_n - u|^{p-1}|v|dx\\ &\le C_0|u_n - u|^{p-1}_{V,p}|v|_{V,p} \le C_0|u_n - u|^{p-1}_{V,p}. \end{aligned} \tag{3.25}$$

On the other hand, if $p > 2$, again from Hölder inequality with $V(x) = V^{\frac{1}{p}}(x)V^{\frac{p-1}{p}}(x)$, and (3.20), (3.23) and direct computations we have that

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)||u_n|^{p-2}u_n - |u|^{p-2}u||v|dx &\le \left(\int_{\mathbb{R}^N} V(x)||u_n|^{p-2}u_n - |u|^{p-2}u|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}} |v|_{V,p}\\ &\le C_0\left(\int_{\mathbb{R}^N} V(x)|u_n - u|^{\frac{p}{p-1}}(|u_n| + |u|)^{\frac{p(p-2)}{p-1}}dx\right)^{\frac{p-1}{p}}, \end{aligned} \tag{3.26}$$

where once again from Hölder inequality but with $V(x) = V^{\frac{1}{p-1}}(x)V^{\frac{p-2}{p-1}}(x)$ it results

$$\left(\int_{\mathbb{R}^N} V(x)|u_n - u|^{\frac{p}{p-1}}(|u_n| + |u|)^{\frac{p(p-2)}{p-1}}dx\right)^{\frac{p-1}{p}} \le |u_n - u|_{V,p}\left(\int_{\mathbb{R}^N} V(x)(|u_n| + |u|)^p dx\right)^{\frac{p-2}{p}}.$$

Hence, from (3.26), direct computations imply that

$$\int_{\mathbb{R}^N} V(x)||u_n|^{p-2}u_n - |u|^{p-2}u||v|dx \le C_0^*|u_n - u|_{V,p}\big(|u_n|^{p-2}_{V,p} + |u|^{p-2}_{V,p}\big) \tag{3.27}$$

for a suitable constant $C_0^* > 0$.
Thus, summing up, from (3.24) and (3.25), respectively (3.27), it follows that

$$|\langle d\mathcal{J}_2(u_n) - d\mathcal{J}_2(u), v\rangle| \to 0 \quad \text{uniformly with respect to } v \in X,\ \|v\|_X = 1,$$

and the desired result holds. □

4 The main theorem and some first properties

From now on, besides hypotheses (V_1), (h_0)–(h_1), (g_0)–(g_1), which imply that $\mathcal{J}$ as in (3.18) is a $\mathcal{C}^1$ functional on X as in (3.11) (see Proposition 3.10), we consider also assumption (V_2) and the following conditions on functions $A(x,t)$, $g(x,t)$ and $V(x)$:

(h_2) a constant $\alpha_0 > 0$ exists such that

$$A(x,t) \geq \alpha_0 \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R};$$

(h_3) some constants $\mu > p$ and $\alpha_1 > 0$ exist so that

$$(\mu - p)A(x,t) - A_t(x,t)t \geq \alpha_1 A(x,t) \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R};$$

(h_4) a constant $\alpha_2 > 0$ exists such that

$$pA(x,t) + A_t(x,t)t \geq \alpha_2 A(x,t) \quad \text{a.e. in } \mathbb{R}^N \text{ for all } t \in \mathbb{R};$$

(g_2) we have that

$$\lim_{t \to 0} \frac{g(x,t)}{|t|^{p-2}t} = \bar{\alpha} \; < \; \frac{\alpha_0}{\tau_p^p} \quad \text{uniformly a.e. in } \mathbb{R}^N,$$

where α_0 is as in assumption (h_2) and τ_p is as in (3.9) with $r = p$;

(g_3) having μ as in hypothesis (h_3), then

$$0 < \mu G(x,t) \leq g(x,t)t \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R} \setminus \{0\};$$

(V_3) for any $\varrho > 0$, a constant $C_\varrho > 0$ exists such that

$$\operatorname*{ess\,sup}_{|x| \leq \varrho} V(x) \leq C_\varrho.$$

Remark 4.1. If (h_2) holds, then, without loss of generality, we can assume $\alpha_0 \leq 1$. Moreover, taking $t = 0$ in (h_3), we have also $\mu - p \geq \alpha_1$.

Remark 4.2. By means of (3.15), assumptions (g_0) and (g_2) imply that the existence of

$$\lim_{t \to 0} \frac{G(x,t)}{|t|^p} = \frac{\bar{\alpha}}{p} \quad \text{uniformly a.e. in } \mathbb{R}^N$$

is provided, too. In particular, from hypotheses (g_1)–(g_2) and direct computations it follows that for any $\varepsilon > 0$ a constant $c_\varepsilon > 0$ exists so that not only

$$|g(x,t)| \leq (\bar{\alpha} + \varepsilon)|t|^{p-1} + c_\varepsilon |t|^{q-1} \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R},$$

but also

$$|G(x,t)| \leq \frac{\bar{\alpha} + \varepsilon}{p}|t|^p + \frac{c_\varepsilon}{q}|t|^q \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R}, \tag{4.1}$$

if assumption (g_0) holds, too.

Remark 4.3. We note that (3.15), assumptions (g_0)–(g_1), (g_3) and straightforward computations imply that for any $\varepsilon > 0$ a function $\eta_\varepsilon \in L^\infty(\mathbb{R}^N)$ exists so that $\eta_\varepsilon > 0$ for a.e. $x \in \mathbb{R}^N$ and

$$G(x,t) \geq \eta_\varepsilon(x)|t|^\mu \quad \text{for a.e. } x \in \mathbb{R}^N \text{ if } |t| \geq \varepsilon. \tag{4.2}$$

Hence, from (3.16) and (4.2) it follows that

$$p < \mu \leq q. \tag{4.3}$$

Now, we are ready to state our main result.

Theorem 4.4. *Under assumptions* (V_1)–(V_3), (h_0)–(h_4) *and* (g_0)–(g_3), *with the growth exponent* q *in* (g_1) *such that*

$$q < p^*, \tag{4.4}$$

then problem (1.1) *admits at least one weak nontrivial bounded solution.*

Remark 4.5. In our set of hypotheses, summing up (4.3) and (4.4) we have that

$$1 < p < \mu \leq q < p^*,$$

with μ as in (h_3) and (g_3).

As useful in the following, firstly we give a convergence result.

Proposition 4.6. *Suppose that hypotheses* (V_1)–(V_2), (h_0)–(h_3), (g_0)–(g_1) *and* (g_3) *hold. Thus, taking any* $\beta \in \mathbb{R}$, *we have that any* $(CPS)_\beta$*–sequence* $(u_n)_n \subset X$ *is bounded in* $W^{1,p}_V(\mathbb{R}^N)$. *Furthermore,* $u \in W^{1,p}_V(\mathbb{R}^N)$ *exists such that, up to subsequences, as* $n \to +\infty$ *the following limits hold:*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}_V(\mathbb{R}^N), \tag{4.5}$$

$$u_n \to u \text{ strongly in } L^r(\mathbb{R}^N) \text{ for each } r \in [p, p^*[, \tag{4.6}$$

$$u_n \to u \text{ a.e. in } \mathbb{R}^N. \tag{4.7}$$

Proof. Taking $\beta \in \mathbb{R}$, let $(u_n)_n \subset X$ be a $(CPS)_\beta$–sequence of $\mathcal{J}$, i.e.,

$$\mathcal{J}(u_n) \to \beta \quad \text{and} \quad \|d\mathcal{J}(u_n)\|_{X'}(1 + \|u_n\|_X) \to 0 \qquad \text{as } n \to +\infty. \tag{4.8}$$

Thus, from (3.18), (3.19), (4.8), assumptions (h_2)–(h_3), (g_3) and also (3.2), direct computations give

$$\begin{aligned}\mu\beta + \varepsilon_n =& \mu\mathcal{J}(u_n) - \langle d\mathcal{J}(u_n), u_n\rangle \\ \geq& \frac{\alpha_0\alpha_1}{p}\int_{\mathbb{R}^N} |\nabla u_n|^p dx + \frac{\mu - p}{p}\int_{\mathbb{R}^N} V(x)|u_n|^p dx \geq c_1\|u_n\|_V^p,\end{aligned}$$

for a suitable constant $c_1 > 0$. So, $(u_n)_n$ is bounded in $W^{1,p}_V(\mathbb{R}^N)$ and (4.5)–(4.7) follows from (3.8). ☐

Now, we prove that the geometrical assumptions needed to apply Theorem 2.2 hold. To this aim, we start giving the following lemma.

Lemma 4.7. *Under assumptions* (h_0) *and* (h_2)*–*(h_3)*, we have that*

$$A(x, st) \le s^{\mu-p-\alpha_1} A(x,t) \quad \text{a.e. in } \mathbb{R}^N, \ \text{ for all } t \in \mathbb{R} \text{ and } s \ge 1. \tag{4.9}$$

Proof. Taking $t \in \mathbb{R}$, for a.e. $x \in \mathbb{R}^N$ and for all $s > 0$, assumptions (h_0) and (h_3) imply that

$$\frac{d}{ds} A(x, st) = A_t(x, st)t \le \frac{\mu - p - \alpha_1}{s} A(x, st).$$

Hence, from hypothesis (h_2) we obtain that

$$\frac{\frac{d}{ds} A(x, st)}{A(x, st)} \le \frac{\mu - p - \alpha_1}{s},$$

where Remark 4.1 ensures that $\mu - p - \alpha_1 \ge 0$. Thus, if $s \ge 1$ by integrating we obtain the desired result. □

Proposition 4.8. *Under assumptions* (V_1)*,* (h_0)*–*(h_2)*,* (g_0)*–*(g_2)*, if* $p < q < p^*$ *then two positive constants* ϱ*,* $\alpha^* > 0$ *exist so that*

$$u \in X, \ \|u\|_V = \varrho \quad \Longrightarrow \quad \mathcal{J}(u) \ge \alpha^*. \tag{4.10}$$

Proof. Let $u \in X$ and, from (g_2), take $\varepsilon > 0$ such that $\bar{\alpha} + \varepsilon < \frac{\alpha_0}{\tau_p^p}$, i.e.,

$$\alpha_0 - (\bar{\alpha} + \varepsilon)\tau_p^p > 0. \tag{4.11}$$

Then, from (h_2) with $\alpha_0 \le 1$ (see Remark 4.1), (4.1) and (3.9), definitions (3.2) and (3.18) imply that

$$\begin{aligned}
\mathcal{J}(u) &\ge \frac{\alpha_0}{p} \left(\int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{\mathbb{R}^N} V(x)|u|^p dx \right) - \frac{\bar{\alpha} + \varepsilon}{p} \int_{\mathbb{R}} |u|^p dx - \frac{c_\varepsilon}{\mu} \int_{\mathbb{R}^N} |u|^q dx \\
&\ge \frac{1}{p} \left(\alpha_0 - (\bar{\alpha} + \varepsilon)\tau_p^p \right) \|u\|_V^p - \frac{c_\varepsilon}{\mu} \tau_q^q \|u\|_V^q.
\end{aligned}$$

Then, (4.11) and $p < q$ allow us to find two positive constants ϱ and α^* so that (4.10) holds. □

Proposition 4.9. *Assume that* (V_1)*,* (h_0)*,* (h_2)*–*(h_3)*,* (g_0)*–*(g_1) *and* (g_3) *hold. Then, fixing* $\bar{u} \in X \setminus \{0\}$*, it follows that*

$$\mathcal{J}(s\bar{u}) \to -\infty \quad \text{as } s \to +\infty.$$

Proof. Since $\bar{u} \ne 0$, there exists $\varepsilon > 0$ such that

$$|\Omega_\varepsilon^{\bar{u}}| > 0 \qquad \text{with} \qquad \Omega_\varepsilon^{\bar{u}} = \{x \in \mathbb{R}^N : \ |\bar{u}(x)| > \varepsilon\}.$$

From (3.18), (4.9), inequality (4.2) and direct computations, for any $s \ge 1$ we have that

$$\begin{aligned}
\mathcal{J}(s\bar{u}) &= \frac{s^p}{p} \int_{\mathbb{R}^N} A(x, s\bar{u}) |\nabla \bar{u}|^p dx + \frac{s^p}{p} \int_{\mathbb{R}^N} V(x)|\bar{u}|^p dx - \int_{\mathbb{R}^N} G(x, s\bar{u}) dx \\
&\le \frac{s^{\mu-\alpha_1}}{p} \int_{\mathbb{R}^N} A(x, \bar{u}) |\nabla \bar{u}|^p dx + \frac{s^p}{p} \int_{\mathbb{R}^N} V(x)|\bar{u}|^p dx - s^\mu \int_{\Omega_\varepsilon^{\bar{u}}} \eta_1(x)|\bar{u}|^\mu dx.
\end{aligned}$$

Thus, being $\int_{\Omega_\varepsilon^{\bar{u}}} \eta_1(x)|\bar{u}|^\mu dx > 0$, inequality (4.3), together with assumption $\alpha_1 > 0$, gives the desired result. □

5 An existence result on bounded domains

From now on, let Ω denote an open bounded domain in $\mathbb{R}^N$. Thus, we define

$$X_\Omega = W^{1,p}_{0,V}(\Omega) \cap L^\infty(\Omega) \tag{5.1}$$

endowed with the norm

$$\|u\|_{X_\Omega} = \|u\|_{\Omega,V} + |u|_{\Omega,\infty} \quad \text{for any } u \in X_\Omega \tag{5.2}$$

and with dual space X'_Ω.
Actually, since any function $u \in X_\Omega$ can be trivially extended to a function $\tilde{u} \in X$ just assuming $\tilde{u}(x) = 0$ for all $x \in \mathbb{R}^N \setminus \Omega$, then

$$\|\tilde{u}\| = \|u\|_\Omega, \quad \|\tilde{u}\|_V = \|u\|_{\Omega,V}, \quad |\tilde{u}|_\infty = |u|_{\Omega,\infty}, \quad \|\tilde{u}\|_X = \|u\|_{X_\Omega}. \tag{5.3}$$

Remark 5.1. As Ω is a bounded domain, not only $\|u\|_\Omega$ and $|\nabla u|_{\Omega,p}$ are equivalent norms but also, if assumption (V_3) holds, a constant $c_\Omega \geq 1$ exists such that

$$\|u\|^p_{\Omega,V} = \int_\Omega |\nabla u|^p dx + \int_\Omega V(x)|u|^p dx \leq \int_\Omega |\nabla u|^p dx + c_\Omega \int_\Omega |u|^p dx \leq c_\Omega \|u\|^p_\Omega,$$

which, together with (3.4) and (5.3), implies that the norms $\|\cdot\|_{\Omega,V}$ and $\|\cdot\|_\Omega$ are equivalent, too.

From (3.17) and (3.18) it follows that the restriction of the functional $\mathcal{J}$ to X_Ω, namely

$$\mathcal{J}_\Omega = \mathcal{J}|_{X_\Omega},$$

is such that

$$\mathcal{J}_\Omega(u) = \frac{1}{p}\int_\Omega A(x,u)|\nabla u|^p dx + \frac{1}{p}\int_\Omega V(x)|u|^p dx - \int_\Omega G(x,u)dx, \quad u \in X_\Omega. \tag{5.4}$$

We note that, setting

$$\tilde{g}(x,t) = g(x,t) - V(x)|t|^{p-2}t \quad \text{for a.e. } x \in \mathbb{R}^N, \text{ all } t \in \mathbb{R}, \tag{5.5}$$

from (g_0) we have that $\tilde{g}$ is a $\mathcal{C}^0$–Caratheodory function such that

$$\tilde{G}(x,t) = \int_0^t \tilde{g}(x,s)ds = G(x,t) - \frac{1}{p}V(x)|t|^p \quad \text{for a.e. } x \in \mathbb{R}^N, \text{ all } t \in \mathbb{R}. \tag{5.6}$$

Then, by means of assumptions (g_1), (V_1) and (V_3), we have that

$$|\tilde{g}(x,t)| \leq (a_1 + |V|_{\Omega,\infty})|t|^{p-1} + a_2|t|^{q-1} \quad \text{for a.e. } x \in \mathbb{R}^N, \text{ all } t \in \mathbb{R}, \tag{5.7}$$

and from (5.4) and (5.6) we have that

$$\mathcal{J}_\Omega(u) = \frac{1}{p}\int_\Omega A(x,u)|\nabla u|^p dx - \int_\Omega \tilde{G}(x,u)dx, \quad u \in X_\Omega. \tag{5.8}$$

Hence, arguing as in [12, Proposition 3.1], it follows that $\mathcal{J}_\Omega : X_\Omega \to \mathbb{R}$ is a $\mathcal{C}^1$ functional such that, for any u, $v \in X_\Omega$, its Fréchet differential in u along the direction v is given by

$$\begin{aligned}\langle d\mathcal{J}_\Omega(u), v\rangle = &\int_\Omega A(x,u)|\nabla u|^{p-2}\nabla u\cdot\nabla v dx + \frac{1}{p}\int_\Omega A_t(x,u)v|\nabla u|^p dx\\ &+\int_\Omega V(x)|u|^{p-2}uv dx - \int_\Omega g(x,u)v dx.\end{aligned} \tag{5.9}$$

In order to apply Theorem 2.2 to functional $\mathcal{J}_\Omega$ as in (5.4), we start proving that it satisfies the $(wCPS)$ condition in $\mathbb{R}$ (see Definition 2.1).

Proposition 5.2. *Suppose that hypotheses* (V_1), (V_3), (h_0)–(h_4), (g_0)–(g_1) *and* (g_3) *hold. Then, if also* (4.4) *is verified, functional* $\mathcal{J}_\Omega$ *satisfies the* $(wCPS)$ *condition in* $\mathbb{R}$.

Proof. Firstly, we note that from Remark 5.1 the norm in (5.2) can be replaced with the equivalent one, still denoted by $\|\cdot\|_{X_\Omega}$, given by

$$\|u\|_{X_\Omega} = |\nabla u|_{\Omega,p} + |u|_{\Omega,\infty} \quad \text{for any } u \in X_\Omega.$$

Then, being $\mu > p$, from (g_3), (5.5)–(5.6) and direct computations we obtain that

$$\mu\tilde{G}(x,t) \le \tilde{g}(x,t)t \quad \text{for a.e. } x \in \mathbb{R}^N, \ \text{all } t \in \mathbb{R}.$$

Thus, from conditions (h_0)–(h_4) and (5.7), together with (4.4), we have that [12, Proposition 4.6] applies to functional $\mathcal{J}_\Omega$ written as in (5.8). ☐

Remark 5.3. In order to apply [12, Proposition 4.6] to functional $\mathcal{J}_\Omega$ in X_Ω condition $\tilde{G}(x,t) > 0$ for a.e. $x \in \Omega$ if $|t| \ge R$, for a suitable $R > 0$, is not necessary (such an assumption is required in [12] for proving one of the geometric conditions of the Mountain Pass Theorem).

For simplicity, let us assume that $0 \in \Omega$ so $\varepsilon^* > 0$ exists such that $B_{\varepsilon^*} \subset \Omega$.

Remark 5.4. Assume that the hypotheses of Propositions 4.8 and 4.9 are satisfied and fix $\bar{u} \in X \setminus \{0\}$ with $\operatorname{supp}\bar{u} \subset B_{\varepsilon^*}$. Then, $\bar{u} \in X_\Omega$ and, from Proposition 4.9, $\bar{s} > 0$ exists so that

$$\|u^*\|_{\Omega,V} > \varrho \quad \text{and} \quad \mathcal{J}_\Omega(u^*) < \alpha^* \tag{5.10}$$

with $u^* = \bar{s}\bar{u}$ and ϱ, α^* as in Proposition 4.8.
Clearly, we have that $\operatorname{supp} u^* \subset B_{\varepsilon^*}$, so $u^* \in X_\Omega$, and if we consider the segment joining 0 to u^*, namely

$$\gamma^* : s \in [0,1] \mapsto su^* \in X, \tag{5.11}$$

we obtain that not only $\gamma^*([0,1]) \subset X_\Omega$ but also $\operatorname{supp}\gamma^*(s) \subset B_{\varepsilon^*}$ for all $s \in [0,1]$.

Proposition 5.5. *Under the assumptions in Theorem 4.4, functional* $\mathcal{J}_\Omega$ *has at least a critical point* $u_\Omega \in X_\Omega$ *such that*

$$\alpha^* \ \le\ \mathcal{J}_\Omega(u_\Omega) \ \le\ \sup_{s\in[0,1]} \mathcal{J}(su^*), \tag{5.12}$$

with α^* *as in Proposition 4.8 and* u^* *as in Remark 5.4.*

Proof. From (3.17) we have that $\mathcal{J}_\Omega(0) = 0$. Moreover, from (5.3), (5.4) and Proposition 4.8 we have that

$$u \in X_\Omega, \quad \|u\|_{\Omega,V} = \varrho \qquad \Longrightarrow \qquad \mathcal{J}_\Omega(u) \geq \alpha^*.$$

Then, taking $u^* \in X_\Omega$ as in Remark 5.4, from (5.10) and Proposition 5.2 we have that Theorem 2.2 applies to $\mathcal{J}_\Omega$ in the Banach space X_Ω and a critical point $u_\Omega \in X_\Omega$ exists such that

$$\mathcal{J}_\Omega(u_\Omega) = \inf_{\gamma\in\Gamma_\Omega} \sup_{s\in[0,1]} \mathcal{J}_\Omega(\gamma(s)) \geq \alpha^*$$

with $\Gamma_\Omega = \{\gamma \in C([0,1], X_\Omega) : \gamma(0) = 0, \ \gamma(1) = u^*\}$.
On the other hand, also the second inequality in (5.12) is true as, from Remark 5.4, segment γ^* in (5.11) is such that $\gamma^* \in \Gamma_\Omega$. □

At last, let us point out that, in the following, we need to prove a uniform boundedness for a sequence which is bounded in $W_0^{1,p}(\Omega)$. Clearly, since Ω is a bounded subset of $\mathbb{R}^N$, if $p > N$ from the Sobolev Embedding Theorem such a boundedness is trivially satisfied but, on the contrary, if $p \leq N$ the following sufficient conditions are required.

Lemma 5.6. *Let Ω be an open bounded domain in $\mathbb{R}^N$ and consider p, q so that $1 < p \leq N$ and $p \leq q < p^*$ (if $N = p$ we just require that p^* is any number larger than q) and take $u \in W_0^{1,p}(\Omega)$. If $a^* > 0$ and $m_0 \in \mathbb{N}$ exist such that*

$$\int_{\Omega_m^+} |\nabla u|^p dx \leq a^* \left(m^q \ |\Omega_m^+| + \int_{\Omega_m^+} |u|^q dx \right) \quad \text{for all } m \geq m_0, \tag{5.13}$$

with $\Omega_m^+ = \{x \in \Omega : \ u(x) > m\}$, then $\operatorname{ess\,sup}_\Omega u$ is bounded from above by a positive constant which can be chosen so that it depends only on $|\Omega_{m_0}^+|$, N, p, q, a^*, m_0, $\|u\|_\Omega$, or better by a positive constant which can be chosen so that it depends only on N, p, q, a^*, m_0 and a_0^* for any $a_0^* > 0$ such that

$$\max\{|\Omega_{m_0}^+|, \ \|u\|_\Omega\} \ \leq \ a_0^*.$$

Vice versa, if inequality

$$\int_{\Omega_m^-} |\nabla u|^p dx \leq a^* \left(m^q \ |\Omega_m^-| + \int_{\Omega_m^-} |u|^q dx \right) \quad \text{for all } m \geq m_0,$$

holds, with $\Omega_m^- = \{x \in \Omega : \ u(x) < -m\}$, then $\operatorname{ess\,sup}_\Omega(-u)$ is bounded from above by a positive constant which can be chosen so that it depends only on N, p, q, a^*, m_0 and any constant which is greater than both $|\Omega_{m_0}^-|$ and $\|u\|_\Omega$.

Proof. The proof is essentially as in [21, Theorem II.5.1] but, in order to point out some uniform estimates on the L^∞–norms of a sequence which is bounded in $W_0^{1,p}(\Omega)$, here we prefer to give some more details.
Firstly, taking any $m \in \mathbb{N}$ we note that direct computations, the Hölder inequality and (3.10) imply that

$$\begin{aligned} \int_{\Omega_m^+} (u-m)^q dx \ &\leq |\Omega_m^+|^{1-\frac{q}{p^*}} \left(\int_{\Omega_m^+} (u-m)^{p^*} dx \right)^{\frac{q}{p^*}} \\ &\leq \sigma_*^q |\Omega_m^+|^{1-\frac{q}{p^*}} \left(\int_{\Omega_m^+} |\nabla u|^p dx \right)^{\frac{q}{p}}, \end{aligned} \tag{5.14}$$

with $\sigma_* > 0$ as in (3.10), since $\max\{u-m, 0\} \in W_0^{1,p}(\Omega)$. Thus, from (5.14) and direct computations it follows that

$$\begin{aligned}\int_{\Omega_m^+} |u|^q dx \ &\le\ 2^{q-1}\left(m^q|\Omega_m^+| + \int_{\Omega_m^+}(u-m)^q dx\right)\\ &\le 2^{q-1}\left(m^q|\Omega_m^+| + \sigma_*^q|\Omega_m^+|^{1-\frac{q}{p^*}}\left(\int_{\Omega_m^+}|\nabla u|^p dx\right)^{\frac{q}{p}}\right)\\ &\le\ 2^{q-1}\left(m^q\ |\Omega_m^+| + \sigma_*^q|\Omega_m^+|^{1-\frac{q}{p^*}}\|u\|_\Omega^{q-p}\int_{\Omega_m^+}|\nabla u|^p dx\right).\end{aligned}$$

Now, if $m \ge m_0$, from one hand (3.10) gives

$$m^{p^*}\ |\Omega_m^+|\ \le\ \int_{\Omega_m^+}|u|^{p^*}dx\ \le\ \int_\Omega |u|^{p^*}dx \le L, \tag{5.15}$$

with

$$L^{\frac{1}{p^*}} = \sigma_*\|u\|_\Omega, \tag{5.16}$$

while, from the other hand, (5.13) and (5.15), (5.16) imply that

$$\begin{aligned}\int_{\Omega_m^+}|\nabla u|^p dx\ &\le\ a^*(1+2^{q-1})m^q|\Omega_m^+|\\ &\quad + 2^{q-1}a^*\sigma_*^q|\Omega_m^+|^{1-\frac{q}{p^*}}\|u\|_\Omega^{q-p}\int_{\Omega_m^+}|\nabla u|^p dx\\ &\le\ a^*(1+2^{q-1})m^q|\Omega_m^+| + 2^{q-1}a^*\sigma_*^q\frac{L^{1-\frac{q}{p^*}}}{m^{p^*-q}}\|u\|_\Omega^{q-p}\int_{\Omega_m^+}|\nabla u|^p dx\\ &\le\ a^*(1+2^{q-1})m^q|\Omega_m^+| + 2^{q-1}a^*\sigma_*^{p^*}\frac{\|u\|_\Omega^{p^*-p}}{m^{p^*-q}}\ \int_{\Omega_m^+}|\nabla u|^p dx.\end{aligned} \tag{5.17}$$

Hence, taking $m_1 \in \mathbb{N}$ such that

$$m_1 \ge \max\left\{m_0, \left(2^q a^*\sigma_*^{p^*}\|u\|_\Omega^{p^*-p}\right)^{\frac{1}{p^*-q}}\right\}, \tag{5.18}$$

from (5.15), (5.17) and direct computations it follows that

$$\int_{\Omega_m^+}|\nabla u|^p dx\ \le\ a_1^* m^q|\Omega_m^+|\ \le\ a_1^* m^p\ |\Omega_m^+|\left(\frac{L}{|\Omega_m^+|}\right)^{\frac{q-p}{p^*}} \quad \text{for all } m \ge m_1,$$

with $a_1^* = 2a^*(1+2^{q-1})$, that is,

$$\int_{\Omega_m^+}|\nabla u|^p dx\ \le\ a_1^* L^{\frac{q-p}{p^*}}\ m^p\ |\Omega_m^+|^{1-\frac{p}{N}+\varepsilon} \quad \text{for all } m \ge m_1,$$

where $\varepsilon = \frac{p}{N} - \frac{q-p}{p^*} > 0$ as $q < p^*$. Then, by using the same arguments required by estimate (5.14) but with q replaced by 1, for all $m \ge m_1$ this last inequality implies that

$$\int_{\Omega_m^+}(u-m)dx \le \sigma_*|\Omega_m^+|^{1-\frac{1}{p^*}}\left(\int_{\Omega_m^+}|\nabla u|^p dx\right)^{\frac{1}{p}} \le a_2^* L^{\frac{q-p}{pp^*}}\ m\ |\Omega_m^+|^{1+\frac{\varepsilon}{p}}$$

with $a_2^* = \sigma_*(a_1^*)^{\frac{1}{p}}$. Thus, reasoning as in the proof of [21, Lemma II.5.1] but with our setting, from direct computations we have that

$$\operatorname*{ess\,sup}_{\Omega} u \ \le\ 2^{\frac{p}{\varepsilon}} \left(m_1 + a_3^* L^{\frac{q-p}{\varepsilon p^*}} \int_{\Omega^+_{m_1}} |u| dx \right) \tag{5.19}$$

with $a_3^* = (a_2^*)^{1+\frac{p}{\varepsilon}}$. So, since from (5.18) and Hölder inequality we have that

$$\int_{\Omega^+_{m_1}} |u| dx \le \int_{\Omega^+_{m_0}} |u| dx \le |\Omega^+_{m_0}|^{1-\frac{1}{p}} \left(\int_{\Omega^+_{m_0}} |u|^p dx \right)^{\frac{1}{p}} \ \le\ |\Omega^+_{m_0}|^{1-\frac{1}{p}} \|u\|_{\Omega},$$

from (5.16) estimate (5.19) becomes

$$\operatorname*{ess\,sup}_{\Omega} u \ \le\ 2^{\frac{p}{\varepsilon}} \left(m_1 + a_3^* \sigma_*^{\frac{q-p}{\varepsilon}} |\Omega^+_{m_0}|^{1-\frac{1}{p}} \|u\|_{\Omega}^{1+\frac{q-p}{\varepsilon}} \right). \tag{5.20}$$

At last, if we take $a_0^* \ge \max\{\|u\|_{\Omega},\ |\Omega^+_{m_0}|\}$, estimate (5.20) implies

$$\operatorname*{ess\,sup}_{\Omega} u \ \le\ 2^{\frac{p}{\varepsilon}} \left(m_1^* + a_3^* \sigma_*^{\frac{q-p}{\varepsilon}} (a_0^*)^{2+\frac{q-p}{\varepsilon}-\frac{1}{p}} \right)$$

with m_1^* defined as in (5.18) but replacing $\|u\|_{\Omega}$ with a_0^*.
Finally, the proof of the second statement of this lemma follows from the first part but applied to function $-u$. □

6 Proof of the main theorem

Now, we are ready to prove our main result. To this aim, we follow an approach which is similar to that one in [14] but, since our problem has a different setting and requires to rehash the proofs, for completeness here we provide all the details.

Throughout this section, we suppose that the assumptions in Theorem 4.4 are satisfied and for each $k \in \mathbb{N}$ we consider as bounded set the open ball B_k, its related Banach space X_{B_k} as in (5.1) and the functional

$$\mathcal{J}_k : u \in X_{B_k} \mapsto \mathcal{J}_k(u) = \mathcal{J}_{B_k}(u) \in \mathbb{R}$$

with $\mathcal{J}_{B_k}(u)$ defined as in (5.4).

Remark 6.1. For the sake of convenience, if $u \in X_{B_k}$ we always consider its trivial extension as 0 in $\mathbb{R}^N \setminus B_k$. Then, if we still denote such an extension by u, we have that $u \in X$, too. Moreover, from (3.17), definitions (3.18) and (5.4), respectively (3.19) and (5.9), imply that $\mathcal{J}_k(u) = \mathcal{J}(u)$, respectively

$$\langle d\mathcal{J}_k(u), v\rangle = \langle d\mathcal{J}(u), v\rangle \quad \text{for all } v \in X_{B_k}.$$

Remark 6.2. If in Remark 5.4 we take $\varepsilon^* \le 1$, then $u^* \in X_{B_1}$ and segment γ^* in (5.11) is such that $\operatorname{supp}\gamma^*(s) \subset B_1$ for all $s \in [0,1]$. Hence, for all $k \in \mathbb{N}$ we have that $\gamma^* \in \Gamma_{B_k}$, with

$$\Gamma_{B_k} = \{\gamma \in C([0,1], X_{B_k}) :\ \gamma(0) = 0,\ \gamma(1) = u^*\}.$$

Moreover, for the continuity of $\mathcal{J} \circ \gamma^* : s \in [0,1] \mapsto \mathcal{J}(su^*) \in \mathbb{R}$, $\alpha^{**} \in \mathbb{R}$ exists, independent of k, such that

$$\alpha^{**} = \max_{s\in[0,1]} \mathcal{J}(su^*).$$

Since for all $k \in \mathbb{N}$ Proposition 5.5 applies to $\mathcal{J}_k$ in X_{B_k}, from Remarks 6.1 and 6.2, a sequence $(u_k)_k \subset X$ exists such that for every $k \in \mathbb{N}$ it results:

(i) $u_k \in X_{B_k}$ with $u_k = 0$ in $\mathbb{R}^N \setminus B_k$,

(ii) $\alpha^* \leq \mathcal{J}(u_k) \leq \alpha^{**}$,

(iii) $\langle d\mathcal{J}(u_k), v\rangle = 0$ for all $v \in X_{B_k}$,

with α^* as in Proposition 4.8 and α^{**} as in Remark 6.2.

The proof of Theorem 4.4 requires different steps.

Firstly, we prove that sequence $(u_k)_k$ is bounded in X. If $p > N$, from the Sobolev Embedding Theorem it is enough to verify that $(\|u_k\|)_k$ is bounded while, if $p \leq N$, such a boundedness is not enough and Lemma 5.6 needs.

From now on, we denote any strictly positive constant independent of k by d_i.

Proposition 6.3. *A constant $M_0 > 0$ exists such that*

$$\|u_k\|_X \leq M_0 \quad \textit{for all } k \in \mathbb{N}. \tag{6.1}$$

Moreover, for every $r \geq p$ we have also that

$$|u_k|_{V,r} \leq M_0 \quad \textit{for all } k \in \mathbb{N}. \tag{6.2}$$

Proof. From conditions (i) and (iii) we infer that

$$\langle d\mathcal{J}(u_k), u_k\rangle = 0 \quad \text{for all } k \in \mathbb{N}, \tag{6.3}$$

then, taking μ as in assumption (h_3), from (6.3), definitions (3.18), (3.19), assumptions (h_2)–(h_3), (V_1), (g_3), (3.2) and direct computations we have that

$$\begin{aligned}\mu\mathcal{J}(u_k) &= \mu\mathcal{J}(u_k) - \langle d\mathcal{J}(u_k), u_k\rangle \\ &\geq \frac{\alpha_0\alpha_1}{p}\int_{\mathbb{R}^N} |\nabla u_k|^p dx + \left(\frac{\mu}{p} - 1\right)\int_{\mathbb{R}^N} V(x)|u_k|^p dx \;\geq\; d_1\|u_k\|_V^p\end{aligned}$$

for a suitable constant $d_1 > 0$. Whence, condition (ii) implies that

$$\|u_k\|_V \leq d_2 \quad \text{for all } k \in \mathbb{N} \tag{6.4}$$

with $d_2 = \left(\frac{\mu\alpha^{**}}{d_1}\right)^{\frac{1}{p}}$.

Now, in order to obtain estimate (6.1) if $p \leq N$, from definition (3.11) we need to prove that $(u_k)_k$ is a bounded sequence in $L^\infty(\mathbb{R}^N)$, too. To this aim, we note that for a fixed $k \in \mathbb{N}$ either $|u_k|_\infty \leq 1$ or $|u_k|_\infty > 1$.

If $|u_k|_\infty > 1$, then

$$\operatorname*{ess\,sup}_{\mathbb{R}^N} u_k > 1 \quad \text{and/or} \quad \operatorname*{ess\,sup}_{\mathbb{R}^N} (-u_k) > 1.$$

Assume that $\operatorname{ess\,sup}_{\mathbb{R}^N} u_k > 1$ and consider the set

$$B^+_{k,1} = \{x \in \mathbb{R}^N : \ u_k(x) > 1\}.$$

From condition (i) we have that

$$B^+_{k,1} \subset \ B_k,$$

moreover, $B^+_{k,1}$ is an open bounded domain such that not only $|B^+_{k,1}| > 0$ but also, by means of (3.4), it is

$$|B^+_{k,1}| \leq \int_{B^+_{k,1}} |u_k|^p dx \leq \int_{\mathbb{R}^N} |u_k|^p dx \leq \|u_k\|^p \leq d_3 \|u_k\|^p_V$$

for a suitable $d_3 > 0$. Hence, estimate (6.4) gives

$$|B^+_{k,1}| \leq d_4 \tag{6.5}$$

with $d_4 = d_2^p d_3$.
In order to prove that Lemma 5.6 applies, for any $m \in \mathbb{N}$ we consider the new function $R^+_m : \mathbb{R} \to \mathbb{R}$ defined as

$$t \mapsto R^+_m t = \begin{cases} 0 & \text{if } t \leq m \\ t - m & \text{if } t > m \end{cases}.$$

Since taking $t = 0$ in (h_4) we obtain $\alpha_2 \leq p$, for any $m \geq 1$, we have that $R^+_m u_k \in X_k$ and from condition (iii), definition (5.9) and hypotheses (V_1), (h_2), (h_4) and direct computations it follows that

$$\begin{aligned} 0 \ &= \langle d\mathcal{J}(u_k), R^+_m u_k \rangle = \int_{B^+_{k,m}} \left(1 - \frac{m}{u_k}\right) \left(A(x, u_k) + \frac{1}{p} A_t(x, u_k) u_k\right) |\nabla u_k|^p dx \\ &+ \int_{B^+_{k,m}} \frac{m}{u_k} A(x, u_k) |\nabla u_k|^p dx + \int_{B^+_{k,m}} \left(1 - \frac{m}{u_k}\right) V(x) |u_k|^p dx - \int_{\mathbb{R}^N} g(x, u_k) R^+_m u_k dx \\ &\geq \frac{\alpha_0 \alpha_2}{p} \int_{B^+_{k,m}} |\nabla u_k|^p dx - \int_{\mathbb{R}^N} g(x, u_k) R^+_m u_k dx, \end{aligned}$$

i.e.,

$$\frac{\alpha_0 \alpha_2}{p} \int_{B^+_{k,m}} |\nabla u_k|^p dx \leq \int_{\mathbb{R}^N} g(x, u_k) R^+_m u_k dx, \tag{6.6}$$

with $B^+_{k,m} = \{x \in \mathbb{R}^N : u_k(x) > m\}$. Clearly, it is $B^+_{k,m} \subseteq B^+_{k,1}$ so from (6.5) we have that

$$|B^+_{k,m}| \leq d_4. \tag{6.7}$$

Since (g_3) implies $g(x, t) > 0$ if $t > 0$ for a.e. $x \in \mathbb{R}^N$, then $g(x, u_k(x)) > 0$ for a.e. $x \in B^+_{k,m}$; so, from (g_1), (4.3) and the Young inequality with conjugate exponents $\frac{q}{p} > 1$ and $\frac{q}{q-p}$, it follows that

$$\begin{aligned} \int_{\mathbb{R}^N} g(x, u_k) R^+_m u_k dx &\leq \int_{B^+_{k,m}} g(x, u_k) u_k dx \leq \int_{B^+_{k,m}} (a_1 u_k^p + a_2 u_k^q) dx \\ &\leq \int_{B^+_{k,m}} \left(\frac{q-p}{q}\ a_1^{\frac{q}{q-p}} + \frac{p}{q}\ u_k^q\right) dx + a_2 \int_{B^+_{k,m}} u_k^q dx \\ &= \frac{q-p}{q}\ a_1^{\frac{q}{q-p}} |B^+_{k,m}| + \left(\frac{p}{q} + a_2\right) \int_{B^+_{k,m}} u_k^q dx. \end{aligned}$$

Hence, from (6.6) we obtain that

$$\int_{B^+_{k,m}} |\nabla u_k|^p dx \le a^* \left(|B^+_{k,m}| + \int_{B^+_{k,m}} u_k^q dx \right) \quad \text{for all } m \ge 1,$$

with $a^* > 0$ independent of m and k. Then, Lemma 5.6 with $\Omega = B_k$ applies and $d_5 > 1$ exists such that

$$\operatorname*{ess\,sup}_{B_k} u_k \le d_5$$

where, from Lemma 5.6, (3.4), (6.4) and (6.7) imply that constant d_5 can be choosen independent of $k \in \mathbb{N}$.
Similar arguments apply if $\operatorname*{ess\,sup}_{\mathbb{R}^N}(-u_k) > 1$, then, summing up, $d_6 \ge 1$ exists such that

$$|u_k|_\infty \le d_6 \qquad \text{for all } k \in \mathbb{N}.$$

So, (6.1) holds and from estimate (3.12) also (6.2) is satisfied. □

From estimate (6.1) and (3.8) it follows that $u_\infty \in W^{1,p}_V(\mathbb{R}^N)$ exists such that

$$u_k \rightharpoonup u_\infty \quad \text{weakly in } W^{1,p}_V(\mathbb{R}^N), \tag{6.8}$$

$$u_k \to u_\infty \quad \text{strongly in } L^r(\mathbb{R}^N) \text{ for any } p \le r < p^*, \tag{6.9}$$

$$u_k \to u_\infty \quad \text{a.e. in } \mathbb{R}^N. \tag{6.10}$$

Proposition 6.4. *$u_\infty \in L^\infty(\mathbb{R}^N)$; hence, $u_\infty \in X$.*

Proof. Taking

$$A = \{x \in \mathbb{R}^N : \lim_{k\to+\infty} u_k(x) = u_\infty(x)\},$$

for all $x \in A$ an integer $\nu_{x,1} \in \mathbb{N}$ exists such that

$$|u_k(x) - u_\infty(x)| < 1 \qquad \text{for all } k \ge \nu_{x,1},$$

then from (6.1) we have that

$$|u_\infty(x)| \le |u_\infty(x) - u_{\nu_{x,1}}(x)| + |u_{\nu_{x,1}}(x)| \le 1 + M_0.$$

On the other hand, from (6.10) it follows that $|\mathbb{R}^N \setminus A| = 0$, which completes the proof. □

Remark 6.5. Let $R \ge 1$ be fixed. Actually, since B_R is a bounded domain, from Remark 5.1 and (6.9) it follows that

$$u_k \to u_\infty \quad \text{strongly in } L^p_V(B_R). \tag{6.11}$$

Corollary 6.6. *For all $1 \le r < +\infty$ we have that*

$$u_k \to u_\infty \quad \textit{strongly in } L^r_V(B_R) \quad \textit{for all } R \ge 1. \tag{6.12}$$

Proof. Taking $1 \le r \le p$, the boundedness of B_R gives $L^p(B_R) \hookrightarrow L^r(B_R)$ which, together with (3.1), assumption (V_3) and (6.9), ensures that (6.12) holds.
On the other hand, if we take $r > p$, then Propositions 6.3 and 6.4, together with (6.11), allow us to apply Lemma 3.5, so (6.12) holds, too. □

Proposition 6.7. *We have that*

$$u_k \to u_\infty \quad \textit{strongly in } W^{1,p}_V(B_R) \quad \textit{for all } R \geq 1. \tag{6.13}$$

Proof. For simplicity, throughout this proof we denote any infinitesimal sequence by $(\varepsilon_k)_k$. Fixing any $R \geq 1$, from (6.11) it is enough to prove that

$$|\nabla u_k - \nabla u_\infty|_{B_R,p} \to 0 \quad \text{as } k \to +\infty. \tag{6.14}$$

To this aim, following an idea introduced in [10], let us consider the real map $\psi(t) = te^{\eta t^2}$, where $\eta > (\frac{\beta_2}{2\beta_1})^2$ will be fixed once β_1, $\beta_2 > 0$ are chosen in a suitable way later on. By definition, we have that

$$\beta_1 \psi'(t) - \beta_2 |\psi(t)| > \frac{\beta_1}{2} \qquad \text{for all } t \in \mathbb{R}. \tag{6.15}$$

Defining $v_k = u_k - u_\infty$, limit (6.8) implies that

$$v_k \rightharpoonup 0 \, \text{weakly in } W^{1,p}_V(\mathbb{R}^N), \tag{6.16}$$

while from (6.9), respectively (6.10), it follows that

$$v_k \to 0 \quad \text{strongly in } L^p(\mathbb{R}^N), \tag{6.17}$$

respectively

$$v_k \to 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{6.18}$$

Moreover, from Proposition 6.4 and (6.1) we obtain that

$$v_k \in X \quad \text{and} \quad |v_k|_\infty \leq \bar{M}_0 \quad \text{for all } k \in \mathbb{N}, \tag{6.19}$$

with $\bar{M}_0 = M_0 + |u_\infty|_\infty$. Now, let $\chi_R \in C^\infty(\mathbb{R}^N)$ be a cut–off function such that

$$\chi_R(x) = \begin{cases} 1 & \text{if } |x| \leq R \\ 0 & \text{if } |x| \geq R+1 \end{cases}, \qquad \text{with} \quad 0 \leq \chi_R(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^N, \tag{6.20}$$

and

$$|\nabla \chi_R(x)| \leq 2 \quad \text{for all } x \in \mathbb{R}^N. \tag{6.21}$$

Thus, for every $k \in \mathbb{N}$ we consider the new function

$$w_{R,k} : x \in \mathbb{R}^N \mapsto w_{R,k}(x) = \chi_R(x)\psi(v_k(x)) \in \mathbb{R}.$$

We note that (6.19) gives

$$|\psi(v_k)| \leq \psi(\bar{M}_0), \quad 0 < \psi'(v_k) \leq \psi'(\bar{M}_0) \qquad \text{a.e. in } \mathbb{R}^N, \tag{6.22}$$

while (6.18) implies

$$\psi(v_k) \to 0, \quad \psi'(v_k) \to 1 \qquad \text{a.e. in } \mathbb{R}^N. \tag{6.23}$$

Then, from (6.20) and (6.22) we have that $\operatorname{supp} w_{R,k} \subset \operatorname{supp} \chi_R \subset B_{R+1}$ and also

$$|w_{R,k}| \leq \psi(\bar{M}_0) \qquad \text{a.e. in } \mathbb{R}^N. \tag{6.24}$$

Moreover (6.23) implies that

$$w_{R,k} \to 0 \qquad \text{a.e. in } \mathbb{R}^N, \tag{6.25}$$

while from

$$\nabla w_{R,k} = \psi(v_k)\nabla\chi_R + \chi_R\psi'(v_k)\nabla v_k \qquad \text{a.e. in } \mathbb{R}^N \tag{6.26}$$

and (6.19)–(6.22) it follows that $w_{R,k} \in X_{B_{R+1}}$. Hence, for all $k \geq R+1$ we have that

$$w_{R,k} \in X_{B_k}$$

so (iii), (3.19) and (6.26) imply that

$$\begin{aligned} 0 \ &= \langle d\mathcal{J}(u_k), w_{R,k}\rangle \ = \int_{B_{R+1}} \psi(v_k)A(x,u_k)|\nabla u_k|^{p-2}\nabla u_k \cdot \nabla\chi_R dx \\ &+ \int_{B_{R+1}} \chi_R\psi'(v_k)A(x,u_k)|\nabla u_k|^{p-2}\nabla u_k \cdot \nabla v_k dx + \frac{1}{p}\int_{B_{R+1}} A_t(x,u_k)w_{R,k}|\nabla u_k|^p dx \\ &+ \int_{B_{R+1}} V(x)|u_k|^{p-2}u_k w_{R,k} dx - \int_{B_{R+1}} g(x,u_k)w_{R,k} dx. \end{aligned} \tag{6.27}$$

We note that (g_0)–(g_1) together with (6.1), (6.10), (6.24), (6.25) and Dominated Convergence Theorem on the bounded set B_{R+1} imply that

$$\int_{B_{R+1}} g(x,u_k)w_{R,k} dx \to 0. \tag{6.28}$$

Moreover, from assumption (h_1), (6.1), (6.21) and Hölder inequality we have that

$$\begin{aligned} \int_{B_{R+1}} |\psi(v_k)A(x,u_k)|\nabla u_k|^{p-2}\nabla u_k \cdot \nabla\chi_R| dx &\leq d_1 \int_{B_{R+1}} |\psi(v_k)||\nabla u_k|^{p-1} dx \\ &\leq d_1\|u_k\|_{B_{R+1}}^{p-1} \left(\int_{B_{R+1}} |\psi(v_k)|^p dx\right)^{\frac{1}{p}}; \end{aligned} \tag{6.29}$$

now, from (6.22), (6.23) and the boundedness of the set B_{R+1}, by means of Dominated Convergence Theorem, we obtain

$$\int_{B_{R+1}} |\psi(v_k)|^p dx \to 0. \tag{6.30}$$

Hence, estimate (6.1), together with (6.29) and (6.30), guarantees that

$$\int_{B_{R+1}} \psi(v_k)A(x,u_k)|\nabla u_k|^{p-2}\nabla u_k \cdot \nabla\chi_R dx \to 0. \tag{6.31}$$

Furthermore, from (V_3), (6.20), Hölder inequality, (6.1) and (6.30) it follows that

$$\begin{aligned} \left|\int_{B_{R+1}} V(x)|u_k|^{p-2}u_k w_{R,k} dx\right| &\leq C_{R+1}\int_{B_{R+1}} |u_k|^{p-1}|\psi(v_k)| dx \\ &\leq C_{R+1}|u_k|_{B_{R+1},p}^{p-1} \left(\int_{B_{R+1}} |\psi(v_k)|^p dx\right)^{\frac{1}{p}} \to 0. \end{aligned} \tag{6.32}$$

On the other hand, (6.1), (6.20), assumptions (h_1)–(h_2) and direct computations ensure the existence of a constant $d_2 > 0$, independent of k, such that

$$\begin{aligned}\left|\int_{B_{R+1}} \chi_R A_t(x,u_k)\psi(v_k)|\nabla u_k|^p dx\right| &\leq \frac{d_2}{\alpha_0}\int_{B_{R+1}} \chi_R A(x,u_k)|\psi(v_k)||\nabla u_k|^p dx \\ &= \frac{d_2}{\alpha_0}\int_{B_{R+1}} \chi_R A(x,u_k)|\psi(v_k)||\nabla u_k|^{p-2}\nabla u_k\cdot\nabla v_k dx \\ &\quad + \frac{d_2}{\alpha_0}\int_{B_{R+1}} \chi_R A(x,u_k)|\psi(v_k)||\nabla u_k|^{p-2}\nabla u_k\cdot\nabla u_\infty dx.\end{aligned}$$

We note that hypothesis (h_1), (6.1), (6.20) and Hölder inequality give

$$\begin{aligned}&\left|\int_{B_{R+1}} \chi_R A(x,u_n)|\psi(v_k)||\nabla u_k|^{p-2}\nabla u_k\cdot\nabla u_\infty dx\right| \\ &\leq d_3\left(\int_{B_{R+1}}|\psi(v_k)|^p|\nabla u_\infty|^p dx\right)^{\frac{1}{p}}\left(\int_{B_{R+1}}|\nabla u_k|^p dx\right)^{\frac{p-1}{p}} \qquad (6.33)\\ &\leq d_4\left(\int_{B_{R+1}}|\psi(v_k)|^p|\nabla u_\infty|^p dx\right)^{\frac{1}{p}} \ \to\ 0\end{aligned}$$

as (6.22), (6.23) and, again, Dominated Convergence Theorem imply that

$$\int_{B_{R+1}}|\psi(v_k)|^p|\nabla u_\infty|^p dx \to 0.$$

Then, from estimate (6.27) and (6.28), (6.31)–(6.33), we obtain that

$$\varepsilon_k \geq \int_{B_{R+1}} \chi_R A(x,u_k)h_k|\nabla u_k|^{p-2}\nabla u_k\cdot\nabla v_k dx \qquad (6.34)$$

if we set

$$h_k(x) = \psi'(v_k(x)) - \frac{d_2}{p\alpha_0}|\psi(v_k(x))| \quad \text{for a.e. } x\in\mathbb{R}^N.$$

Now, back to (6.15) and taking $\beta_1 = 1, \beta_2 = \frac{d_2}{p\alpha_0}$, from (6.22), (6.23) and direct computations not only we have that

$$h_k(x)\to 1 \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad |h_k(x)| \leq \psi'(\bar{M}_0) + d_5|\psi(\bar{M}_0)| \quad \text{a.e. in } \mathbb{R}^N, \qquad (6.35)$$

but also

$$h_k(x) > \frac{1}{2} \quad \text{a.e. in } \mathbb{R}^N. \qquad (6.36)$$

At last, back to (6.34), direct computations give

$$\begin{aligned}\varepsilon_k \ &\geq \int_{B_{R+1}} \chi_R A(x,u_k)h_k|\nabla u_k|^{p-2}\nabla u_k \cdot \nabla v_k dx \\ &= \int_{B_{R+1}} \chi_R A(x,u_\infty)|\nabla u_\infty|^{p-2}\nabla u_\infty \cdot \nabla v_k dx \\ &\quad + \int_{B_{R+1}} \chi_R (h_k A(x,u_k) - A(x,u_\infty))|\nabla u_\infty|^{p-2}\nabla u_\infty \cdot \nabla v_k dx \\ &\quad + \int_{B_{R+1}} \chi_R A(x,u_k)h_k(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2}\nabla u_\infty)\cdot \nabla v_k dx,\end{aligned} \tag{6.37}$$

where from (6.16) it follows that

$$\int_{B_{R+1}} \chi_R A(x,u_\infty)|\nabla u_\infty|^{p-2}\nabla u_\infty \cdot \nabla v_k dx \ \to \ 0.$$

Now, from Hölder inequality, we have that

$$\begin{aligned}&\int_{B_{R+1}} \chi_R |(h_k A(x,u_k) - A(x,u_\infty))|\nabla u_\infty|^{p-2}\nabla u_\infty \cdot \nabla v_k| dx \\ &\leq \left(\int_{B_{R+1}} |h_k A(x,u_k) - A(x,u_\infty)|^{\frac{p}{p-1}} |\nabla u_\infty|^p dx\right)^{\frac{p-1}{p}} \|v_k\|_{B_{R+1}} \ \to \ 0,\end{aligned} \tag{6.38}$$

since assumption (h_0), (6.10), (6.20) and (6.35) imply that

$$h_k A(x,u_k) - A(x,u_\infty) \to 0 \quad \text{a.e. in } \mathbb{R}^N,$$

while (6.1), (6.35), Proposition 6.4 and (h_1) give

$$|h_k A(x,u_k) - A(x,u_\infty)|^{\frac{p}{p-1}} |\nabla u_\infty|^p \leq d_6 |\nabla u_\infty|^p \quad \text{a.e. in } \mathbb{R}^N.$$

Hence, Dominated Convergence Theorem applies which, together with (6.17), guarantees that the limit in (6.38) holds.
Moreover, using the previous estimates in (6.37), from (6.20), (6.36), hypothesis (h_2) and the strong convexity of the power function with exponent $p > 1$, we have that

$$\varepsilon_k \geq \frac{\alpha_0}{2} \int_{B_{R+1}} (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2}\nabla u_\infty)\cdot \nabla v_k dx \geq 0,$$

which implies that

$$\int_{B_{R+1}} (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2}\nabla u_\infty)\cdot \nabla v_k dx \ \to \ 0;$$

hence, (6.14) follows from (6.8). □

Proposition 6.8. *We have that*

$$\langle d\mathcal{J}(u_\infty), \varphi\rangle = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N) \tag{6.39}$$

with $C_c^\infty(\mathbb{R}^N) = \{\varphi \in C^\infty(\mathbb{R}^N) : \operatorname{supp}\varphi \subset\subset \mathbb{R}^N\}$*. Hence,* $d\mathcal{J}(u_\infty) = 0$ *in* X*.*

Proof. Taking $\varphi \in C_c^\infty(\Omega)$, a radius $R \geq 1$ exists such that $\operatorname{supp}\varphi \subset B_R$. Thus, for all $k \geq R$ we have that $\varphi \in X_k$ so (iii) applies and $\langle d\mathcal{J}(u_k), \varphi\rangle = 0$. Furthermore, direct computations give

$$\begin{aligned}
0 \ &\leq |\langle d\mathcal{J}(u_\infty), \varphi\rangle| = |\langle d\mathcal{J}(u_k), \varphi\rangle - \langle d\mathcal{J}(u_\infty), \varphi\rangle| \\
&\leq \int_{B_R} |A(x,u_k)| \big| |\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2}\nabla u_\infty \big| |\nabla\varphi| dx \\
&\quad + \int_{B_R} |A(x,u_k) - A(x,u_\infty)| |\nabla u_\infty|^{p-1} |\nabla\varphi| dx \\
&\quad + \frac{|\varphi|_\infty}{p} \int_{B_R} |A_t(x,u_k)| \big| |\nabla u_k|^p - |\nabla u_\infty|^p \big| dx \\
&\quad + \frac{|\varphi|_\infty}{p} \int_{B_R} |A_t(x,u_k) - A_t(x,u_\infty)| |\nabla u_\infty|^p dx \\
&\quad + |\varphi|_\infty \int_{B_R} V(x) \big| |u_k|^{p-2}u_k - |u_\infty|^{p-2}u_\infty \big| dx + |\varphi|_\infty \int_{B_R} \big| g(x,u_k) - g(x,u_\infty) \big| dx.
\end{aligned} \tag{6.40}$$

We note that the boundedness of B_R together with (g_0)–(g_1), (6.1), (6.10) and Proposition 6.4 allow us to apply the Dominated Convergence Theorem and

$$\int_{B_R} \big| g(x,u_k) - g(x,u_\infty) \big| dx \to 0.$$

Moreover, by using some arguments similar to those ones in the proof of Proposition 3.10 we have that assumption (V_3) and (6.9) imply that

$$\int_{B_R} V(x) \big| |u_k|^{p-2}u_k - |u_\infty|^{p-2}u_\infty \big| dx \ \to \ 0.$$

On the other hand, by means of (h_1), Hölder inequality and direct computations, we obtain

$$\begin{aligned}
&\int_{B_R} |A(x,u_k)| \big| |\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2}\nabla u_\infty \big| |\nabla\varphi| dx \\
&\qquad \leq d_1 \left(\int_{B_R} \big| |\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2} u_\infty \big|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}},
\end{aligned} \tag{6.41}$$

for a suitable constant $d_1 = d_1(|\varphi|_\infty) > 0$. Now, if $p > 2$, then Lemma 3.8, again Hölder inequality with conjugate exponents $p-1$ and $\frac{p-1}{p-2}$, (6.1) and direct computations imply that

$$\left(\int_{B_R} \big| |\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2} u_\infty \big|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \leq d_2 \|u_k - u_\infty\|_{B_R}. \tag{6.42}$$

Conversely, if $1 < p \le 2$ again Lemma 3.8 applies so that

$$\left(\int_{B_R} \big||\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2}u_\infty\big|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \le \|u_k - u_\infty\|_{B_R}^{p-1}. \tag{6.43}$$

Thus, from (6.41)–(6.43) and (6.13) it follows that

$$\int_{B_R} |A(x,u_k)|\big||\nabla u_k|^{p-2}\nabla u_k - |\nabla u_\infty|^{p-2}\nabla u_\infty\big||\nabla\varphi| dx \ \to\ 0. \tag{6.44}$$

Furthermore, from assumption (h_1), (6.1), (6.13) and Lemma 3.9 we have that

$$\int_{B_R} |A_t(x,u_k)|\big||\nabla u_k|^p - |\nabla u_\infty|^p\big| dx \ \to\ 0,$$

while, from Dominated Convergence Theorem it results

$$\int_{B_R} |A_t(x,u_k) - A_t(x,u_\infty)||\nabla u_\infty|^p dx \ \to\ 0$$

as (h_0) and (6.13) imply that

$$|A_t(x,u_k) - A_t(x,u_\infty)||\nabla u_\infty|^p \to 0 \quad \text{ a.e. in } \mathbb{R}^N$$

and (h_1), (6.1) give

$$|A_t(x,u_k) - A_t(x,u_\infty)||\nabla u_\infty|^p \le d_3|\nabla u_\infty|^p \in L^1(B_R).$$

So, summing up, (6.40) guarantees that (6.39) holds and the proof follows from the density of $C_c^\infty(\mathbb{R}^N)$ in X. □

Proof of Theorem 4.4. From Proposition 6.8 we have that the statement of Theorem 4.4 is true if we prove $u_\infty \not\equiv 0$.
Arguing by contradiction, assume that $u_\infty = 0$. Hence, from assumption (g_1) with $q < p^*$ and (6.9) we have that

$$\int_{\mathbb{R}^N} g(x,u_k)u_k dx \ \to\ 0, \tag{6.45}$$

which, together with assumption (g_3), ensures that

$$\int_{\mathbb{R}^N} G(x,u_k)dx \to 0. \tag{6.46}$$

Furthermore, from (3.19), (iii), (h_4) and also (h_2) it results

$$\begin{aligned} 0 \ &=\ \langle d\mathcal{J}(u_k), u_k\rangle = \int_{\mathbb{R}^N} A(x,u_k)|\nabla u_k|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} A_t(x,u_k)u_k|\nabla u_k|^p dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|u_k|^p dx - \int_{\mathbb{R}^N} g(x,u_k)u_k dx \\ &\ge \frac{\alpha_2\alpha_0}{p}\int_{\mathbb{R}^N} |\nabla u_k|^p dx + \int_{\mathbb{R}^N} V(x)|u_k|^p dx - \int_{\mathbb{R}^N} g(x,u_k)u_k dx, \end{aligned}$$

which implies that

$$\|u_k\|_V \to 0 \quad \text{as } k \to +\infty \tag{6.47}$$

by means of (3.2) and (6.45). Hence, from (3.18), (h_1), (6.1), (6.46) and (6.47) we infer

$$\mathcal{J}(u_k) \leq d_1 \|u_k\|_V^p - \int_{\mathbb{R}^N} G(x, u_k) dx \to 0$$

in contradiction with estimate (ii). □

Remark 6.9. Unlike the results in [5, 6, 30] and other similar works, here managing our problem directly on $\mathbb{R}^N$ seems quite hard. In fact, the choice of the working space in (3.11) requires that the $(wCPS)$ condition holds if $u \in L^\infty(\mathbb{R}^N)$ too, but we have no knowledge of a result similar to Lemma 5.6 but settled in the whole Euclidean space $\mathbb{R}^N$.

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