

## GLOBAL EXISTENCE FOR EQUIVALENT NONLINEAR SPECIAL SCALE INVARIANT DAMPED WAVE EQUATIONS

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**ABSTRACT.** In this paper we give the notion of equivalent damped wave equations. As an application we study global in time existence for the solution of special scale invariant damped wave equation with small data. To gain such results, without radial assumption, we deal with Klainerman vector fields. In particular we can treat some potential behind the forcing term.

**1. Introduction.** Let us consider the Cauchy problem

$$\begin{cases} v_{tt} - \Delta v + \frac{\mu}{(1+t)}v_t + \frac{\nu}{(1+t)^2}v = (1+t)^s|v|^p, & t \geq 0, x \in \mathbb{R}^n, \\ v(0, x) = \varepsilon v_0(x) \\ v_t(0, x) = \varepsilon v_1(x) \end{cases} \quad (1)$$

with  $(\mu, \nu, s) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ ,  $p > 1$ ,  $n \geq 2$  and  $v_0, v_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ . The operator

$$\mathcal{L} = \partial_{tt} - \Delta + \frac{\mu}{1+t}\partial_t + \frac{\nu}{(1+t)^2}$$

is called *scale invariant damped wave operator* since for any  $\lambda \in \mathbb{R}$  it holds

$$\mathcal{L}T_\lambda = \lambda^2 T_\lambda \mathcal{L} \quad \text{with} \quad T_\lambda u(t, x) = u(\lambda(1+t) - 1, \lambda x).$$

Once we take the transformation

$$u(t, x) := (1+t)^r v(t, x) \quad r \in \mathbb{R}, \quad (2)$$

we get

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu-2r}{(1+t)}u_t + \frac{\nu-r(\mu-r-1)}{(1+t)^2}u = (1+t)^{s-r(p-1)}|u|^p. \\ u(0, x) = \varepsilon v_0(x) \\ u_t(0, x) = \varepsilon(rv_0(x) + v_1(x)) \end{cases} \quad (3)$$

This transformation rule motivates the following definition.

**Definition 1.1.** The equation

$$v_{tt} - \Delta v + \frac{\mu}{(1+t)}v_t + \frac{\nu}{(1+t)^2}v = (1+t)^s|v|^p, \quad t \geq 0, x \in \mathbb{R}^n, \quad (4)$$

is *equivalent* to the damped equation

$$w_{tt} - \Delta w + \frac{\tilde{\mu}}{1+t}w_t + \frac{\tilde{\nu}}{(1+t)^2}w = (1+t)^{\tilde{s}}|w|^p, \quad t \geq 0, x \in \mathbb{R}^n$$

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if there exists a continuous function  $f(\mu, \nu) \in \mathbb{R} \times \mathbb{R} \rightarrow f(\mu, \nu) \in \mathbb{R}$  such that

$$\begin{aligned}\tilde{\mu} &= \mu - 2f(\mu, \nu), \\ \tilde{\nu} &= \nu - f(\mu, \nu)(\mu - f(\mu, \nu) - 1), \\ \tilde{s} &= s - f(\mu, \nu)(p - 1).\end{aligned}$$

We resume the first two equations as

$$\bar{\nu} = \nu - \frac{\mu - \bar{\mu}}{2} \left( \frac{\mu + \bar{\mu}}{2} - 1 \right).$$

In particular, the equation 4 is equivalent to another equation without mass if and only if there is a  $\bar{\mu}$  solution of the equation  $\nu - \frac{\mu - \bar{\mu}}{2} \left( \frac{\mu + \bar{\mu}}{2} - 1 \right) = 0$ . Such  $\bar{\mu}$  exists if and only if  $(\mu - 1)^2 - 4\nu \geq 0$ ; for this reason the quantity

$$\delta := (\mu - 1)^2 - 4\nu$$

plays a special role in the classification of the existence results for 1.

The idea of treating equivalent equations works well. Indeed, we can discuss global existence for 1 by using the most simpler equivalent equations.

**Lemma 1.2.** *Suppose that the equations in the Cauchy problems 1 and 3 with  $r = f(\mu, \nu)$  are equivalent. If 3 admits a global solution  $v : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$  for  $p > C(\tilde{\mu}, \tilde{\nu}, \tilde{s})$ , then the equation 1 admits a global solution  $u : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$  for all  $p > 1$  satisfying*

$$p > C(\mu - 2f(\mu, \nu), \nu - f(\mu, \nu)(\mu - f(\mu, \nu) - 1), s - f(\mu, \nu)(p - 1)).$$

**Definition 1.3.** We say that 4 is a *special scale invariant damped wave equation* if

$$\nu = \frac{\mu}{2} \left( \frac{\mu}{2} - 1 \right).$$

This condition corresponds to the case  $\delta = 1$ . We underline that the mass is positive,  $\nu \geq 0$ , if and only if  $\mu \geq 2$  or  $\mu \leq 0$ .

The most important example of special scale invariant wave equation has parameters  $\mu = 2$  and  $\nu = 0$ . Fixed the transformation 2, if 4 is special scale invariant, any equivalent equation is special scale invariant. In particular, via  $f(\mu, \nu) = \mu/2$ , the special scale invariant damped wave equations are equivalent to

$$u_{tt} - \Delta u = (1 + t)^{s - \frac{\mu}{2}(p-1)} |u|^p.$$

Clearly for  $\mu = 0$  and  $\nu = 0$  we have that the wave operator is special scale invariant. Many papers concern these equations with  $s = 0$ . In the next table we only recall the global existence results for 1, with the usual notation

$$\begin{aligned}p_F(d) &= 1 + \frac{2}{d}, \quad \text{with } d > 0, \\ p_S^2(d) &\text{ nonnegative solution to } (d-1)p^2 - (d+1)p - 2 = 0, \text{ if } d > 1 \\ p_{So}(d) &= 1 + \frac{2}{d-2} \text{ if } d > 3, \quad p_{So}(d) = +\infty \text{ if } 1 \leq d \leq 2.\end{aligned}$$

When  $d = n$  is the space dimension, these exponents are, respectively, the Fujita, Strauss and Sobolev critical exponent for the semilinear wave equation. The aim of this paper is to add other cases to the following analysis. In particular for  $n = 2, 3$  we consider the case  $s \neq 0$  and  $\mu \neq 2$  without radial assumptions.

Our more complete result is the following:

**Theorem 1.4.** *Let  $n = 2$ ,  $s \in \mathbb{R}$  and  $p \geq 2$  such that  $2s + \mu < p(\mu - 1)$ . Let  $u_0, u_1 \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$  with compact support. Then there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0)$  the Cauchy problem 1 admits a unique global small data solution  $u \in C^1([0, \infty), H^2)$  with  $u_t \in C([0, \infty), H^1)$ .*

Dimension	Main conditions	Paper
$n = 1$	$\mu > 5/3, \quad \nu = 0, \quad p > p_F(n)$	[1]
$n = 2$	$\mu \geq 3, \quad \nu = 0, \quad p > p_F(n)$ $\mu = 2 \quad \nu = 0, \quad p > p_S(2 + \mu)$	[1] [3]
$n \geq 1$	$\mu \geq n + 2, \quad \nu = 0, \quad p_F(n) < p < p_{S_o}(n)$	[1]
$n = 3$	$\mu = 2, \quad \nu = 0, \quad p > p_S(3 + \mu)$ radial solution smooth solution	[3] [7]
$n \geq 4$	$\mu = 2, \quad \nu = 0, \quad p_S(n + \mu) < p < p_{S_o}(n)$ radial solution, $n = 5, 7$	[2]
$n \geq 1$	$\mu \geq 2, \delta = 1 \quad p > p_S(n + \mu)$ radial solution, even $n$ radial solution, odd $n$	[11] [12]
$n \leq 4$	$\mu \neq 2, \delta \leq 0$ or $\delta \geq (n + 1)^2$ , suitable range of $p$ energy solution	[4]
$n \leq 1$	$\mu \neq 2, \delta \leq 0$ or $\delta \geq (n + 1)^2$ , suitable range of $p$ exponentially weighted decay data energy solution less regular solution	[10] [4] [13]

Our result coincides with the one given in [3] for  $\mu = 2$  and  $s = 0$ , that is global existence for  $p > 2$ , but we can also multiply by the potential  $(1 + t)^s$  that interacts with the critical exponent. Let  $\mu \neq 2$ ; for some  $\mu$  only decreasing potentials are admissible, for others  $\mu$ , increasing potentials can be considered and the range of  $p$  depends on  $s$ . The sharp discussion is given in Section 4.1. Here we do not give a blow up results that can assert when the condition  $2s + \mu < p(\mu - 1)$  is critical, this will be object of next studies.

For  $n = 3$  our result is weaker than [7], since we do not reach the critical exponent even for  $\mu = 2$ . More precisely we need

$$s + 1 < \frac{\mu}{2}(p - 1) \quad \text{and} \quad 5/3 \leq p \leq 3,$$

see the discussion in Section 4.2. Comparing our result with [7], we see that we get well posedness in energy space while in that paper the existence is given in a more regular setting. Moreover we believe it is interesting to give a hint for  $n = 4, 5$  where the technique used in [7] is not available since it is based on [6], while we prefer Klainerman's approach [8] that avoids the fundamental solutions of the wave equations. The simplest 4D case, that is  $p = 2$  is treated in Section 4.3.

Comparing with Palmieri's results [11] and [12], we avoid the radial assumption but also give a shorter proof. Also the results in [4], [13], [10] avoid radial condition but in those papers  $\delta \neq 1$ , so we try to cover the special scale invariant case.

**Scheme of the paper.** In Section 2 we give the main tools of the proof of a global existence result of 1 with  $(\mu, \nu, s) = (0, 0, \gamma)$ . These will be obtained in Section 3 for  $n = 2, 3$  and the remarkable case  $n = 4, p = 2$ . Finally, in Section 4 we apply these results for special scale invariant wave equations.

**2. Main tools.** First, we recall some useful results on the non-homogeneous wave equations:

$$\begin{cases} u_{tt}(t, x) - \Delta u(t, x) = F(t, x), & t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (5)$$

with  $\varepsilon > 0$ . We assume

$$\text{supp}(u_0), \text{supp}(u_1) \subset B_R(0)$$

for suitable  $R > 0$ .

**2.1. Klainerman fields.** Let  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ , by Klainerman fields we mean the vector fields

$$\begin{aligned} \Gamma &= (D, L_0, L_j, \Omega_{ij}), \\ D &= (\partial_t, \partial_j), \\ L_0 &= (1 + t)\partial_t + x \cdot \nabla, \\ L_j &= (1 + t)\partial_j + x_j \partial_t, \\ \Omega_{ij} &= x_i \partial_j - x_j \partial_i. \end{aligned}$$

Given a multi-index  $\gamma$ , the following relations hold with suitable constants  $a_\beta, b_\beta$ :

$$[\square, \Gamma^\gamma] = \sum_{|\beta| \leq |\gamma| - 1} a_\beta \Gamma^\beta \square, \quad (6)$$

$$[D, \Gamma^\gamma] = \sum_{|\beta| \leq |\gamma| - 1} b_\beta \Gamma^\beta D. \quad (7)$$

We consider  $u(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ . Fixed  $t \in \mathbb{R}_+$ , these fields define the space  $H_\Gamma^N(\mathbb{R}^n)$  by means of the the norm

$$\|u(t, \cdot)\|_{\Gamma, N, 2} = \sum_{|k| \leq N} \|\Gamma^k u(t, \cdot)\|_2, \quad N \in \mathbb{N}.$$

When  $u(t, x) = f(x)$  is independent of  $t$ , we have

$$\|f\|_{\Gamma, N, 2} \simeq \|f\|_{H^1(\mathbb{R}^N)} + \|x \cdot \nabla f\|_{L^2(\mathbb{R}^N)}.$$

Due to 7, any permutation of  $\Gamma$  fields gives equivalent norms, uniform with respect to  $t \geq 0$ .

From [8] and [14], one has the following Sobolev-type inequalities in these generalized Sobolev spaces:

$$\|w(t, \cdot)\|_{\infty} \lesssim (1+t)^{-\frac{n-1}{2}} \|w\|_{\Gamma, s, 2} \quad \text{if } s > n/2, \quad (8)$$

$$\|w(t, \cdot)\|_{\infty} \lesssim (1+t)^{1-\frac{n-1}{2}} (\|w\|_{\Gamma, s, 2} + \|Dw\|_{\Gamma, s, 2}) \quad \text{if } s+1 > n/2, \quad (9)$$

$$\|w(t, \cdot)\|_q \lesssim (1+t)^{-(n-1)(\frac{1}{2}-\frac{1}{q})} \|w\|_{\Gamma, s, 2} \quad \text{if } 2 \leq q < \infty; \frac{1}{q} \geq \frac{1}{2} - \frac{s}{n} \geq 0, \quad (10)$$

for any  $t > 0$  and any  $w(t, \cdot)$  such that the right-hand sides are well-defined.

**2.2. Energy estimate.** In this section we consider the functional

$$E_L[u](t) = \frac{1}{2} \int_{\mathbb{R}^n} |u_t(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(t, x)|^2 dx.$$

Deriving this quantity and having in mind the finite propagation speed, for the solution of 5, we have

$$\|Du(t)\|_2 \lesssim \varepsilon \|\nabla u_0\|_2 + \varepsilon \|u_1\|_2 + \int_0^t \|F(\tau, \cdot)\|_2 d\tau.$$

Since 6 holds, we get

$$\|Du\|_{\Gamma, k, 2} \lesssim \varepsilon \|\nabla u_0\|_{\Gamma, k, 2} + \varepsilon \|u_1\|_{\Gamma, k, 2} + \int_0^t \|F(\tau, x)\|_{\Gamma, k, 2} d\tau,$$

for any  $k \in \mathbb{N}$ .

**2.3. Decomposition in polar coordinates.** Let  $m, \ell \in [1, +\infty]$ . For any  $x \neq 0$ , we put  $x = r\omega$  with  $r = |x|$  and  $\omega \in S^{n-1}$ , we define

$$\|g\|_{m, \ell} = \|r^{\frac{n-1}{m}} \|g(r \cdot)\|_{L_{\omega}^{\ell}(S^{n-1})}\|_{L_r^m([0, +\infty))}$$

for any function  $g : \mathbb{R} \rightarrow \mathbb{R}$  with finite right-hand side. For  $m = \infty$ , in the previous expression one reads  $r^{\frac{n-1}{m}} = 1$ . Let  $g = g(t, \cdot)$ , we put

$$\|g\|_{\Gamma, N, m, \ell} = \sum_{|k| \leq N} \|\Gamma^k g\|_{m, \ell}, \quad N \in \mathbb{N}.$$

Any permutation of  $\Gamma$  fields gives equivalent norms, uniform with respect to  $t \geq 0$ .

We put  $\|f\|_{\Gamma, N, 2, 2} = \|f\|_{\Gamma, N, 2}$

Following [9] we can prove the next estimate for the fundamental solution of the wave equation.

**Lemma 2.1.** *Let  $n = 2$  and  $1 < q \leq 2$  or  $n \geq 3$  and  $\frac{2n}{n+2} \leq q \leq 2$ . Then*

$$\left\| \frac{\sin(t|\xi|)}{|\xi|} \hat{g}(\xi) \right\|_{L^2} \lesssim t^{n(\frac{1}{2}-\frac{1}{q})+1} \|g\|_{q, 2},$$

for any  $g \in L^2$  and  $t \geq 0$ .

**2.4. Estimate for the zero order term.** We can apply the previous lemma to

$$\hat{u}(t, \xi) = \varepsilon \cos(t|\xi|)\hat{u}_0 + \varepsilon \frac{\sin(t|\xi|)}{|\xi|}\hat{u}_1 + \int_0^t \frac{\sin(t|\xi|)}{|\xi|}\hat{F}(s, \xi) ds.$$

Having in mind [6](#), we have the estimate for the zero order term.

**Proposition 1.** *Let*

$$n = 2, \quad q = 1 + \varepsilon, \quad \delta(q) = \frac{2\varepsilon}{1 + \varepsilon},$$

or

$$n \geq 3, \quad q = \frac{2n}{n + 2}, \quad \delta(q) = 0.$$

Then

$$\|u(t, \cdot)\|_{\Gamma, k, 2} \lesssim \varepsilon \|u_0\|_{\Gamma, k, 2} + \varepsilon t^{\delta(q)} \|u_1\|_{\Gamma, k, q, 2} + \int_0^t (t - \tau)^{\delta(q)} \|F(\tau, \cdot)\|_{\Gamma, k, q, 2} d\tau,$$

for any  $k \in \mathbb{N}$ .

**3. Global existence theorem for the not-damped case.** Now we come back to the semilinear Cauchy problem

$$\begin{cases} u_{tt}(t, x) - \Delta u(t, x) = (1 + t)^\gamma |u|^p, & t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (11)$$

For any  $T > 0$ , we introduce the space  $X_{\delta, k}(T)$  with the norm

$$\|u\|_{X_{\delta, k}(T)} = \sup_{t \in [0, T]} ((1 + t)^{-\delta} \|u\|_{\Gamma, k, 2} + \|Du\|_{\Gamma, k, 2}),$$

where  $\delta = \delta(q)$  and  $q \geq 1$  are given by [Proposition 1](#).

For any  $\omega \in X_{\delta, k}(T)$  let  $u = S[\omega]$  be the solution to

$$\begin{cases} u_{tt} - \Delta u = (1 + t)^\gamma |\omega|^p, \\ u(0, x) = \varepsilon u_0(x), \\ u_t(0, x) = \varepsilon u_1(x) \end{cases}$$

with compactly supported data. Thanks to [Proposition 1](#), we may estimate

$$\|u(t, \cdot)\|_{\Gamma, k, 2} \lesssim \varepsilon \|u_0\|_{\Gamma, k, 2} + \varepsilon t^\delta \|u_1\|_{\Gamma, k, q, 2} + \int_0^t (t - \tau)^\delta \|(1 + \tau)^\gamma |\omega(\tau, \cdot)|^p\|_{\Gamma, k, q, 2} d\tau.$$

Since

$$\begin{aligned} [\partial_\tau, (1 + \tau)^\gamma] &= \gamma \frac{1}{(1 + \tau)} (1 + \tau)^\gamma, \\ [L_0, (1 + \tau)^\gamma] &= \gamma (1 + \tau)^\gamma, \\ [L_j, (1 + \tau)^\gamma] &= \gamma \frac{x_j}{(1 + \tau)} (1 + \tau)^\gamma \end{aligned}$$

and thanks to the finite speed of propagation, we have  $|x| \lesssim (1 + t)$  on  $\text{supp} u$ . We arrive at

$$\|u(t, \cdot)\|_{\Gamma, k, 2} \lesssim \varepsilon \|u_0\|_{\Gamma, k, 2} + t^\delta \varepsilon \|u_1\|_{\Gamma, k, q, 2} + \int_0^t (t - \tau)^\delta (1 + \tau)^\gamma \|\omega(\tau, \cdot)\|_{\Gamma, k, q, 2}^p d\tau.$$

Due to the compact support, being  $q \leq 2$ , we can control  $\|u_1\|_{\Gamma,k,q,2} \lesssim \|u_1\|_{\Gamma,k,2}$ . We can conclude

$$\|Du\|_{\Gamma,k,2} \lesssim \varepsilon \|\nabla u_0\|_{\Gamma,k,2} + \varepsilon \|u_1\|_{\Gamma,k,2} + \int_0^t (1+\tau)^\gamma \|\omega\|^p \| \omega\|_{\Gamma,k,2} d\tau.$$

We look for some condition on  $p > 1$  such that

$$\|\omega\|^p(s) \| \omega\|_{\Gamma,k,q,2} \lesssim (1+s)^\alpha \|\omega\|_{X_{\delta,k}(T)}^p \quad (12)$$

and

$$\|\omega\|^p(s) \| \omega\|_{\Gamma,k,2} \lesssim (1+s)^\beta \|\omega\|_{X_{\delta,k}(T)}^p \quad (13)$$

with

$$\delta + \gamma + \alpha < -1 \text{ and } \gamma + \beta < -1. \quad (14)$$

so that  $S[\omega]$  has a unique fixed point in  $X_{\delta,k}(T)$ , that is  $u = S[u]$ , such that

$$\|Du(t)\|_{\Gamma,k,2} \lesssim 1 \text{ and } \|u(t)\|_{\Gamma,k,2} \lesssim (1+t)^\delta$$

for any  $t \leq T$ . This gives local existence and small data global existence as well.

It remains to obtain 12, 13 provided 14. First by Hölder inequality we get

$$\|\omega(\tau, \cdot)\|^p \| \omega\|_{\Gamma,1,q,2} \lesssim \|\omega\|^{p-1} \| \omega\|_{\bar{q},\infty} \|\omega(\tau, \cdot)\|_{\Gamma,1,2}, \quad \frac{1}{q} = \frac{1}{2} + \frac{1}{\bar{q}}.$$

Moreover, assuming  $p \geq 2/q$  that is  $p - 1 - 2/\bar{q} \geq 0$ , we can estimate

$$\|\omega(\tau, \cdot)\|^{p-1} \| \omega\|_{\bar{q},\infty} \lesssim \|\omega(\tau, \cdot)\|_{\infty}^{p-1-2/\bar{q}} \|\omega(\tau, \cdot)\|_{2,\infty}^{2/\bar{q}}.$$

Applying Sobolev embeddings on the unit sphere  $S^{n-1}$  leads to

$$\|\omega(\tau, \cdot)\|_{2,\infty} \lesssim \|\omega(\tau, \cdot)\|_{H^{k+1}(\mathbb{R}^n)} \leq \|\omega(\tau, \cdot)\|_{H^k(\mathbb{R}^n)} + \|D\omega(\tau, \cdot)\|_{\Gamma,k,2},$$

provided  $k+1 > \frac{n-1}{2}$  that is  $n = 2, 3, 4$  if  $k = 1$  and  $n = 5, 6$  if  $k = 2$ . In such cases we have

$$\|\omega(\tau, \cdot)\|_{2,\infty} \lesssim (1+t)^\delta X_{\delta,k}(T).$$

Thanks to 9, we find

$$\|\omega(\tau, \cdot)\|_{\infty} \lesssim (1+\tau)^{1-\frac{n-1}{2}} (\|\omega(\tau, \cdot)\|_{\Gamma,k,2} + \|D\omega(\tau, \cdot)\|_{\Gamma,k,2}).$$

assuming  $k+1 > n/2$ , that is  $k = 1$  if  $n = 2, 3$  and  $k = 2$  if  $n = 4, 5$ . Summarizing,

$$\|\omega\|^p(s) \| \omega\|_{\Gamma,1,q,2} \lesssim (1+s)^\alpha \|\omega\|_{X_{\delta,k}(T)}^p$$

for  $k = 1$  if  $n = 2, 3$  and  $k = 2$  if  $n = 4, 5$ , provided

$$\begin{aligned} \alpha &= (p-1) \left(1 - \frac{n-1}{2}\right) - \left(1 - \frac{n-1}{2}\right) \left(\frac{2}{q} - 1\right) + \delta p \\ &= \left(1 - \frac{n-1}{2}\right) \left(p - \frac{2}{q}\right) + \delta p. \end{aligned}$$

In particular, for  $k = 1$  we have 12 provided  $n = 2, 3$ .

Now we turn to 13. Proceeding as before we take  $r, q_1 \geq 2$  such that

$$\|\omega(\tau, \cdot)\|^p \| \omega\|_{\Gamma,1,2} \lesssim \|\omega(\tau, \cdot)\|^{p-1} \| \omega\|_r \|\omega(\tau, \cdot)\|_{\Gamma,1,q_1}$$

provided

$$\frac{1}{2} = \frac{1}{r} + \frac{1}{q_1}.$$

By Sobolev embedding, we get

$$\|w(\tau, \cdot)\|_{\Gamma, 1, q_1} \lesssim \|w(\tau, \cdot)\|_{\Gamma, 1, 2} + \|Dw(\tau, \cdot)\|_{\Gamma, 1, 2}$$

assuming that

$$1 \geq n \left( \frac{1}{2} - \frac{1}{q_1} \right),$$

that is trivially satisfied for  $n = 2$ , otherwise

$$2 \leq q_1 \leq \frac{2n}{n-2}.$$

On the other hand, by Sobolev embedding 10, we get

$$\begin{aligned} \| |w(\tau, \cdot)|^{p-1} \|_r &\leq \|w(\tau, \cdot)\|_{r(p-1)}^{p-1} \\ &\lesssim (1+\tau)^{-(n-1)\left(\frac{1}{2} - \frac{1}{r(p-1)}\right)(p-1)} \|w(\tau, \cdot)\|_{\Gamma, 1, 2}^{p-1}, \end{aligned}$$

once we check

$$\frac{1}{r(p-1)} \geq \frac{1}{2} - \frac{1}{n} \geq 0 \quad \text{and} \quad 2 \leq r(p-1) < +\infty.$$

As a conclusion 13 holds with

$$\beta = -(n-1) \left( \frac{1}{2} - \frac{1}{r(p-1)} \right) (p-1) + \delta p$$

for any  $r \in [2/(p-1), \infty)$  if  $n = 2$ , otherwise  $r \geq n$  and  $2 \leq r(p-1) \leq \frac{2n}{n-2}$ .

We conclude rewriting the conditions 14:

$$\begin{cases} \gamma + \left(1 - \frac{n-1}{2}\right) \left(p - \frac{2}{q}\right) + \delta(p+1) < -1, \\ \gamma - (n-1) \left(\frac{p-1}{2} - \frac{1}{r}\right) + \delta p < -1. \end{cases} \quad (15)$$

**3.1. 2D Wave equation with potential.** Let  $n = 2$ . Due to Proposition 1, let us choose  $k = 1$  and  $\delta = \delta(q)$ ,  $q = 1 + \epsilon$  with  $\epsilon > 0$  so that  $\delta(q) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . We know that in  $X_{\delta, k}(T)$  we have a fixed point for 1 with parameter  $(\mu, \nu, s) = (0, 0, \gamma)$  provided  $p \geq 2/q$  and

$$\begin{cases} \gamma + \frac{p}{2} < -1 + \frac{1}{1+\epsilon} - \delta(p+1) \\ \gamma - \frac{p}{2} < -\frac{3}{2} - \frac{1}{r} - \delta p \end{cases} \quad (16)$$

with  $r \in [2/(p-1), \infty)$ . In particular if

$$2\gamma < \min\{-p, p-3\},$$

then we can find  $r \rightarrow \infty$  and  $\delta \rightarrow 0$  such that 16 is satisfied. Since  $p \geq 2$  then it suffices to take

$$2\gamma < -p.$$

We can conclude with the following result

**Theorem 3.1.** *Let  $n = 2$ ,  $p \geq 2$  and  $\gamma < -p/2$ . Let  $u_0, u_1 \in H_{\Gamma}^2(\mathbb{R}^n) \times H_{\Gamma}^1(\mathbb{R}^n)$  with compact support. Then there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0)$  the Cauchy problem 11 admits a unique global small data solution*

$$u \in \mathcal{C}([0, \infty), H_{\Gamma}^2) \cap \mathcal{C}^1([0, \infty), H_{\Gamma}^1).$$



**3.2. 3D Wave equation with potential.** Let  $n = 3$ . Let us choose  $k = 1$  and  $\delta = 0$  from Proposition 1 so that in  $X_{0,1}(T)$  we have a fixed point once 15 holds. This leads to

$$p > 2 + \gamma + \frac{2}{r} \quad \text{and} \quad \gamma < -1, \quad (17)$$

but we also need

$$1 + \frac{2}{r} \leq p \leq 1 + \frac{6}{r}, \quad r \geq 3, \quad p \geq \frac{2}{q} \quad \text{with} \quad q = \frac{6}{5}.$$

The best choice is given by  $r = 3$  that leads to the following result:

**Theorem 3.2.** *Let  $n = 3$ ,  $\gamma < -1$  and  $5/3 \leq p \leq 3$ . Let  $u_0, u_1 \in H_{\Gamma}^2(\mathbb{R}^n) \times H_{\Gamma}^1(\mathbb{R}^n)$  with compact support. Then there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0)$  the Cauchy problem 11 admits a unique global small data solution*

$$u \in \mathcal{C}([0, \infty), H_{\Gamma}^2) \cap \mathcal{C}^1([0, \infty), H_{\Gamma}^1).$$

**3.3. 4D Wave equation with potential.** We shall prove 12, 13 provided 14 with  $p = 2$ ,  $k = 2$ . From Proposition 1, we have  $q = 4/3$  and  $\delta = 0$ . Due to the simple relation  $\Gamma^2 w^2 = w(w + 2\Gamma w + \Gamma^2 w) + (\Gamma w)^2$ , we have to estimate

$$\|\omega\|^2_{\Gamma, 2, 4, 2} \lesssim \sum_{\Gamma, |\alpha| \leq 2} \|w\Gamma^{\alpha} w\|_{4/3, 2} + \sum_{\Gamma} \|\Gamma w\|_{2/3, 2}^2 := I + II,$$

and

$$\|\omega\|^2_{\Gamma, 2, 2} \lesssim \sum_{\Gamma, |\alpha| \leq 2} \|w\Gamma^{\alpha} w\|_2 + \sum_{\Gamma} \|\Gamma w\|_4^2 := III + IV.$$

By using Hölder only in radial coordinates, with  $r^{n-1}$  as a weight, we find

$$I \lesssim \|w\|_{4, 2} \|w\|_{\Gamma, 2, 2} \lesssim \|w\|_4 \|w\|_{\Gamma, 2, 2}.$$

By using standard Sobolev embedding we can conclude  $I \lesssim \|w\|_{\Gamma, 2, 2}^2$  that is  $\alpha = 0$  is a good candidate for 12. Similarly by using 9, we arrive at

$$III \lesssim \|w\|_{\infty} \|w\|_{\Gamma, 2, 2} \lesssim (1+t)^{-1/2} X_2^2(T).$$

Also for  $IV$  we take advantage of Klainerman's inequality since

$$IV \lesssim (1+t)^{-3/2} \|\Gamma w\|_{\Gamma, 1, 2}^2 \lesssim (1+t)^{-3/2} X_2^2(T)$$

being 9 valid for  $n = 4, q = 4, s = 1$ . In any case we have  $\beta = -1/2$ . It remains to find  $II \lesssim X_2^2(T)$ . This holds since

$$II \lesssim \sum_{\Gamma} \|\Gamma w\|_{4, 2} \|w\|_{\Gamma, 1, 2} \lesssim \|w\|_{\Gamma, 1, 2} \sum_{\Gamma} \|\Gamma w\|_4 \lesssim X_2^2(T)$$

again by standard Sobolev embedding. We can conclude the following.

**Theorem 3.3.** *Let  $n = 4$ ,  $\gamma < -1$  and  $p = 2$ . Let  $u_0, u_1 \in H_{\Gamma}^3(\mathbb{R}^n) \times H_{\Gamma}^2(\mathbb{R}^n)$  with compact support. Then there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0)$  the Cauchy problem 11 admits a unique global small data solution*

$$u \in \mathcal{C}([0, \infty), H_{\Gamma}^3) \cap \mathcal{C}^1([0, \infty), H_{\Gamma}^2).$$

#### 4. Equivalent nonlinear special scale invariant damped wave equations.

Here we come back to 1 with  $\nu = \frac{\mu}{2} (\frac{\mu}{2} - 1)$  and we discuss the conditions on  $(\mu, \nu, s, p)$  such data global existence holds. We simply look for the equivalent equation with parameters  $(0, 0, s - \frac{\mu}{2}(p-1))$  and find condition such that Theorem 3.1 and Theorem 3.2 holds for 11.

4.1. **2D case.** Here we deduce Theorem 1.4 from Theorem 3.1 that requires

$$s - \frac{\mu}{2} < -\frac{p}{2} \quad \text{and} \quad p \geq 2. \quad (18)$$

This means

$$2s + \mu < (\mu - 1)p \quad \text{if} \quad p \geq 2. \quad (19)$$

Let us compare this condition with the known results.

In [3] we take  $(\mu, \nu, s) = (2, 0, 0)$ , condition 18 is empty, while 19 gives the critical case  $p > 2$ . Indeed in [3] for  $p \leq 2$  we gained blow-up.

In the present paper we extend the result in [3] only for a potential  $(1+t)^s$  behind the forcing term  $|u|^p$ . However we can understand the important role of this potential for special scale invariant wave equations by writing more explicitly 18. A first result is given by global existence for 3 when

$$\mu = 1, \quad \nu = -1/4, \quad p \geq 2, \quad \text{for any } s < -1/2.$$

We see that  $s$  interacts with the  $\mu$  that is with the damping term and increasing potentials ( $s \geq 0$ ) are non admissible.

Now we turn to the main studied case, of large damping coefficients, the global existence result holds for

$$\begin{aligned} \mu > 1, \quad \nu &= \frac{\mu}{2} \left( \frac{\mu}{2} - 1 \right), \quad p > \max \left\{ 2, \frac{2s + \mu}{\mu - 1} \right\} \quad \forall s \in \mathbb{R}, \\ \mu > 1, \quad \nu &= \frac{\mu}{2} \left( \frac{\mu}{2} - 1 \right), \quad p = 2, \quad \text{for any } s < 2 - \frac{\mu}{2}. \end{aligned}$$

In these relations we see that the potential interacts with the source term, that is the range of  $p$ . It remains to discuss the case of small  $\mu$ :

$$\mu < 1, \quad \nu = \frac{\mu}{2} \left( \frac{\mu}{2} - 1 \right), \quad \forall p \geq 2, \quad \text{for any } s < -\frac{\mu}{2}.$$

In particular for  $\mu = \nu = 0$  we see that our result does not cover the 2D wave equation without potential since it needs  $s < 0$ . In such case the critical exponent is  $p_{Str}(2) = \frac{3+\sqrt{17}}{2}$  and it has to be reached by others technique, see [5].

4.2. **3D case.** The discussion for 3D is simpler than the 2D case, but the result is weaker, indeed  $p \geq 5/3$  is far from the prevent critical value that is of Strauss type and it can be seen as a Fujita exponent. The classical wave equation cannot be obtained since  $\gamma = 0$  is not admissible in Theorem 3.2. Taking parameters  $(2, 0, 0)$  in 1, like in [3], the equation is equivalent to  $(0, 0, -(p-1))$  so that we need  $p > 2$ ; in addition we are requiring  $p \leq 3$ . This result is not sharp since in [3] the critical exponent is given by  $p_S(5) < 2$ , but here the radial assumption is avoided by means of Klainerman fields. In [7] the radial assumption is avoided by using 3D fundamental solution of wave equations, here we try a simpler approach that works also in higher dimension. In any case for variable potentials and special scale invariant case we get some new results. More precisely, considering 1 with parameters  $(\mu, \frac{\mu}{2} (\frac{\mu}{2} - 1), s)$  in the equivalent version with parameters  $(0, 0, s - \frac{\mu}{2}(p-1))$ , rewriting 17 with  $r = 3$ , we get global existence of this Cauchy problem c

$$\begin{aligned} \mu \leq 0, \quad 5/3 < p \leq 3, \quad \text{for all } s < -1, \\ \max\{0, s + 1\} < \mu \leq 3(s + 1), \quad \max \left\{ 1, \frac{2(s + 1)}{\mu} + 1 \right\} < p \leq 3, \\ \mu > \max\{0, 3(s + 1)\}, \quad 2 < p \leq 3. \end{aligned}$$

We underline that increasing potentials can be considered for  $\mu > 1$ .

**4.3. 4D case.** From Theorem 3.3, we know that if  $n = 4$  the case  $(0, 0, \gamma)$  has global solution for  $p = 2$  without radial assumption provided  $\gamma < -1$ . After equivalence between equations, we can say that the special scale invariant wave equation in 4D with  $p = 2$  and parameters  $(\mu, \frac{\mu}{2} (\frac{\mu}{2} - 1), s)$  has a global solution provided  $s < \frac{\mu}{2} - 1$ . Clearly something more general can be obtained for  $n = 4$ , we prefer to give this case to show the possibility of avoid the radial assumption also for high dimension by means of Klainerman fields.

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#### REFERENCES

- [1] M. D'Abbico, The threshold of effective damping for semilinear wave equations. *Math. Methods Appl. Sci.*, **38** (2015), 1032–1045.
- [2] M. D'Abbico, S. Lucente, NLWE with a special scale invariant damping in odd space dimension. *Discrete Contin. Dyn. Syst. Dynamical systems, differential equations and applications*. 10th AIMS Conference. Suppl. (2015), 312–319.
- [3] M. D'Abbico, S. Lucente, M. Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping. *J. Differential Equations*, **259** (2015), 5040–5073.
- [4] W. Nunes do Nascimento, A. Palmieri, M. Reissig. Semi-linear wave models with power non-linearity and scale-invariant time-dependent mass and dissipation. *Math. Nachr.*, **290** (2017), 1779–1805.
- [5] R. Glassey, Existence in the large for  $(\partial_t^2 - \Delta)u = F(u)$  in two space dimensions. *Math. Z.*, **178** (1981), 233–261.
- [6] F. John, Blow-up of Solutions of Nonlinear Wave Equations in Three Space Dimensions. *Manuscripta Math.*, **28** (1979), 235–268.
- [7] M. Kato, M. Sakuraba, Global existence and blow-up for semilinear damped wave equations in three space dimensions. *Nonlinear Anal.*, **182** (2019), 209–225.
- [8] S. Klainerman, Uniform decay estimates and Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, **38** (1985), 321–332.
- [9] Li T., Xin Y., Lifespan of classical solutions to fully nonlinear wave equations. *Comm. Partial Differential Equations*, **16** (1991), 909–940.
- [10] A. Palmieri, Global existence of solutions for semi-linear wave equation with scale-invariant damping and mass in exponentially weighted spaces. *Journal of Mathematical Analysis and Applications*, **461** (2018), 1215–1240.
- [11] A. Palmieri, A global existence result for a semilinear scale-invariant wave equation in even dimension. *Mathematical Methods in the Applied Sciences*, **42** (2019), 2680–2706.
- [12] A. Palmieri, Global Existence Results for a Semilinear Wave Equation with Scale-Invariant Damping and Mass in Odd Space Dimension. *New Tools for Nonlinear PDEs and Application*, 2019, 305–369.
- [13] A. Palmieri, M. Reissig, Semi-linear wave models with power non-linearity and scale-invariant time-dependent mass and dissipation II. *Mathematische Nachrichten*, **291** (2018), 1859–1892.
- [14] Y. Zhou, Cauchy problem for semilinear wave equations in four space dimensions with small initial data. *J. Differential Equations*, **8** (1995), 135–144.

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