

Carleman estimates for singular parabolic equations with interior degeneracy and non smooth coefficients

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Abstract

We establish Carleman estimates for singular/degenerate parabolic Dirichlet problems with degeneracy and singularity occurring in the interior of the spatial domain. Our results are completely new, since this situation is not covered by previous contributions for degeneracy and singularity on the boundary. In addition, we consider non smooth coefficients, thus preventing the use of standard calculations in this framework.

Keywords: Carleman estimates, singular/degenerate equations, non smooth coefficients, Hardy–Poincaré inequality, Caccioppoli inequality, observability inequality, null controllability.

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1 Introduction

Controllability issues for parabolic problems have been a mainstream topic in recent years, and several developments have been pursued: starting from the heat equation in bounded and unbounded domain, related contributions have been found for more general situations. A common strategy in showing controllability results is to prove that certain global Carleman estimates hold true for the operator which is the adjoint of the given one.

In this paper we focus on a class of singular parabolic operators with interior degeneracy of the form

$$u_t - (a(x)u_x)_x - \frac{\lambda}{b(x)}u, \quad (1.1)$$

associated to Dirichlet boundary conditions and with $(t, x) \in Q_T := (0, T) \times (0, 1)$, $T > 0$ being a fixed number. Here a and b degenerate at the same interior point $x_0 \in (0, 1)$, and $\lambda \in \mathbb{R}$ satisfies suitable assumptions (see condition (2.11) below). The fact that both a and b degenerate at x_0 is just for the sake of simplicity and shortness: all the stated results are still valid if they degenerate at different points. The prototypes we have in mind are $a(x) = |x - x_0|^{K_1}$ and $b(x) = |x - x_0|^{K_2}$ for some $K_1, K_2 > 0$. The main goal is to establish global Carleman estimates for operators of the form given in (1.1).

Such estimates for uniformly parabolic operators without degeneracies or singularities have been largely developed (see, e.g., Fursikov–Imanuvilov [32]). Recently, these estimates have been also studied for operators which are not uniformly parabolic. Indeed, as pointed out by several authors, many problems coming from Physics (see [36]), Biology (see [21]) and Mathematical Finance (see [35]) are described by degenerate parabolic equations. In particular, new Carleman estimates (and consequently null controllability properties) were established in [1], and also in [14], [40], for the operator

$$u_t - (au_x)_x + c(t, x)u, \quad (t, x) \in Q_T,$$

where $a(0) = a(1) = 0$, $a \in C^1(0, 1)$ and $c \in L^\infty(Q_T)$ (see also [12], [13] or [26] for problems in non divergence form).

An interesting situation is the case of parabolic operators with singular inverse-square potentials. First results in this direction were obtained in [46] for the *non degenerate* singular potentials with heat-like operator

$$u_t - \Delta u - \lambda \frac{1}{|x|^2}u, \quad (t, x) \in (0, T) \times \Omega, \quad (1.2)$$

with associated Dirichlet boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N$ containing the singularity $x = 0$ in the interior (see also [45] for the wave and Schrödinger equations and [16] for boundary singularity). Similar operators of the form

$$u_t - \Delta u - \lambda \frac{1}{|x|^{K_2}}u, \quad (t, x) \in (0, T) \times \Omega,$$

arise for example in quantum mechanics (see, for example, [4], [19]), or in combustion problems (see, for example, [6], [10], [20], [33]), and is known to generate interesting phenomena. For example, in [4] and in [5] it was proved that, for all values of λ , global positive solutions exist if $K_2 < 2$, whereas instantaneous and complete blow-up occurs if $K_2 > 2$. In the critical case, i.e. $K_2 = 2$, the value of the parameter λ determines the behavior of the equation: if $\lambda \leq 1/4$ (which is the optimal constant of the Hardy inequality, see [9]) global positive solutions exist, while, if $\lambda > 1/4$, instantaneous and complete blow-up occurs (for other comments on this argument we refer to [44]). We recall that in [46], Carleman estimates were established for (1.2) under the condition $\lambda \leq \frac{1}{4}$. On the contrary, if $\lambda > \frac{1}{4}$, in [22] it was proved that null controllability fails.

We remark that the non degenerate problems studied in [4, 16, 22, 44, 45, 46] cover the multidimensional case, while here we treat the case $N = 1$, like Vancostenoble [44], who studied the operator that couples a degenerate diffusion coefficient with a singular potential. In particular, for $K_1 \in [0, 2)$ and $K_2 \leq 2 - K_1$, the author established Carleman estimates for the operator

$$u_t - (x^{K_1} u_x)_x - \lambda \frac{1}{x^{K_2}} u, \quad (t, x) \in Q_T,$$

unifying the results of [15] and [46] in the purely degenerate operator and in the purely singular one, respectively. This result was then extended in [23] and in [24] to the operators

$$u_t - (a(x) u_x)_x - \lambda \frac{1}{x^{K_2}} u, \quad (t, x) \in Q_T, \quad (1.3)$$

for $a \sim x^{K_1}$, $K_1 \in [0, 2)$ and $K_2 \leq 2 - K_1$. Here, as before, the function a degenerates at the boundary of the space domain, and Dirichlet boundary conditions are in force.

We remark the fact that all the papers cited so far, with the exception of [22], consider a singular/degenerate operator with degeneracy or singularity appearing at the boundary of the domain. For example, in (1.3) as a one can also consider the double power function

$$a(x) = x^k (1-x)^\kappa, \quad x \in [0, 1],$$

where k and κ are positive constants. To the best of our knowledge, [8], [29] and [30] are the first papers dealing with Carleman estimates (and, consequently, null controllability) for operators (in divergence and in non divergence form with Dirichlet or Neumann boundary conditions) with mere degeneracy at the interior of the space domain (for related systems of degenerate equations we refer to [7]). We also recall [28] and [27] for other type of control problems associated to parabolic operators with interior degeneracy in divergence and non divergence form, respectively.

We emphasize the fact that an interior degeneracy does not imply a simple adaptation of previous results and of the techniques used for boundary degeneracy. Indeed, imposing homogeneous Dirichlet boundary conditions, in the latter

case one knows *a priori* that any function in the reference functional space vanishes exactly at the degeneracy point. Now, since the degeneracy point is in the interior of the spatial domain, such information is not valid anymore, and we cannot take advantage of this fact.

For this reason, the present paper is devoted to study the operator defined in (1.1), that couples a *general* degenerate diffusion coefficient with a *general* singular potential with degeneracy and singularity at the *interior* of the space domain. In particular, under suitable conditions on all the parameters of the operator, we establish Carleman estimates and, as a consequence, null controllability for the associated generalized heat problem. Clearly, this result generalizes the one obtained in [29] or [30]: in fact, if $\lambda = 0$ (that is, if we consider the purely degenerate case), we recover the main contributions therein. See also [25] for the problem in non divergence form for both Dirichlet and Neumann boundary conditions.

We also remark the fact that, though we have in mind prototypes as power functions for the degeneracy and the singularity, we do not limit our investigation to these functions, which are analytic out of their zero. Indeed, in this paper, pure powers singularities and degeneracies are considered only as a by-product of our main results, which are valid for *non smooth* general coefficients. This is quite a new view-point when dealing with Carleman estimates, since in this framework it is natural to assume that all the coefficients in force are quite regular. However, though this strategy has been successful for years, it is clear that also more irregular coefficients can be considered and appear in a natural way (for instance, see [34], [37]). Nevertheless, it will be clear from the proof that Carleman estimates *do* hold without particular conditions also in the non smooth setting, while for observability (and thus controllability) another technical condition is needed; however, such a condition is trivially true for the prototypes.

For this reason, for the first time to our best knowledge, in [30] *non smooth* degenerate coefficients were treated. Continuing in this direction, here we consider operators which contain both degenerate and singular coefficients, as in [23], [24] and [44], but with low regularity.

The classical approach to study singular operators in dimension 1 relies in the validity of the Hardy–Poincaré inequality

$$\int_0^1 \frac{u^2}{x^2} dx \leq 4 \int_0^1 (u')^2 dx, \quad (1.4)$$

which is valid for every $u \in H^1(0, 1)$ with $u(0) = 0$. Similar inequalities are the starting point to prove well-posedness of the associated problems in the Sobolev spaces under consideration. In our situation, we prove an inequality related to (1.4), but with a degeneracy coefficient in the gradient term; such an estimate is valid in a suitable Hilbert space \mathcal{H} we shall introduce below, and it states the existence of $C > 0$ such that for all $u \in \mathcal{H}$ we have

$$\int_0^1 \frac{u^2}{b} dx \leq C \int_0^1 a(u')^2 dx.$$

This inequality, which is related to another weighted Hardy-Poincaré inequality (see Proposition 2.1), is the key step for the well-posedness of (1.5). Once this is done, global Carleman estimates follow, provided that an *ad hoc* choice of the weight functions is made (see Theorem 3.1).

The introduction of the space \mathcal{H} (which may coincide with the usual Sobolev space in some cases) is another feature of this paper, which is completely new with respect to all the previous approaches: including the integrability of u^2/b in the definition of \mathcal{H} has the advantage of obtaining immediately some useful functional properties, that in general could be hard to show in the usual Sobolev spaces. Indeed, solutions were already found in suitable function spaces for the “critical” and “supercritical” cases (when λ equals or exceeds the best constant in the classical Hardy-Poincaré inequality) in [45] and [47] for purely singular problems. However, as already done in the purely degenerate case ([1, 7, 8, 12, 13, 15, 23, 24, 26, 29, 30, 31]), a weighted Sobolev space must be used. For this reason, we believe that it is natural to unify these approaches in the singular/degenerate, as we do.

Now, let us consider the evolution problem

$$\begin{cases} u_t - (au_x)_x - \frac{\lambda}{b(x)}u = h(t,x)\chi_\omega(x), & (t,x) \in Q_T, \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \quad (1.5)$$

where $u_0 \in L^2(0,1)$, the control $h \in L^2(Q_T)$ acts on a non empty interval $\omega \subset (0,1)$ and χ_ω denotes the characteristic function of ω .

As usual, we say that problem (1.5) is null controllable if there exists $h \in L^2(Q_T)$ such that $u(T,x) \equiv 0$ for $x \in [0,1]$. A common strategy to show that (1.5) is null controllable is to prove Carleman estimates for any solution v of the adjoint problem of (1.5)

$$\begin{cases} v_t + (av_x)_x + \frac{\lambda}{b(x)}v = 0, & (t,x) \in Q_T, \\ v(t,0) = v(t,1) = 0, & t \in (0,T), \\ v(T,x) = v_T(x), \end{cases}$$

and then deduce an observability inequality of the form

$$\int_0^1 v^2(0,x)dx \leq C_T \int_0^T \int_\omega v^2(t,x)dxdt, \quad (1.6)$$

where $C_T > 0$ is a universal constant. In the non degenerate case this has been obtained by a well-established procedure using Carleman and Caccioppoli inequalities. In our singular/degenerate non smooth situation, we need a new suitable Caccioppoli inequality (see Proposition 4.2), as well as global Carleman estimates in the non smooth non degenerate and non singular case (see Proposition 4.3), which will be used far away from x_0 within a localization procedure

via cut-off functions. Once these tools are established, we are able to prove an observability inequality like (1.6), and then controllability results for (1.5). However, we cannot do that in all cases, since we have to exclude that both the degeneracy and the singularity are strong, see condition **(SSD)** below.

Finally, we remark that our studies with non smooth coefficients are particularly useful. In fact, though null controllability results could be obtained also in other ways, for example by a localization technique (at least when $x_0 \in \omega$), in [30] it is shown that with non smooth coefficients, even when $\lambda = 0$, this is not always the case. For this, our approach with observability inequalities is very general and permits to cover more involved situations.

The paper is organized in the following way: in Section 2 we study the well-posedness of problem (1.5), giving some general tools that we shall use several times. In Section 3 we provide one of the main results of this paper, i.e. Carleman estimates for the adjoint problem to (1.5). In Section 4 we apply the previous Carleman estimates to prove an observability inequality, which, together with a Caccioppoli type inequality, lets us derive new null controllability results for the associated singular/degenerate problem, also when the degeneracy and the singularity points are *inside* the control region.

A final comment on the notation: by c or C we shall denote *universal* positive constants, which are allowed to vary from line to line.

2 Well-posedness

The ways in which a and b degenerate at x_0 can be quite different, and for this reason we distinguish four different types of degeneracy. In particular, we consider the following cases:

Hypothesis 2.1. Doubly weakly degenerate case (WWD): there exists $x_0 \in (0, 1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0, 1] \setminus \{x_0\}$, $a, b \in W^{1,1}(0, 1)$ and there exists $K_1, K_2 \in (0, 1)$ such that $(x - x_0)a' \leq K_1 a$ and $(x - x_0)b' \leq K_2 b$ a.e. in $[0, 1]$.

Hypothesis 2.2. Weakly-strongly degenerate case (WSD): there exists $x_0 \in (0, 1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0, 1] \setminus \{x_0\}$, $a \in W^{1,1}(0, 1)$, $b \in W^{1,\infty}(0, 1)$ and there exist $K_1 \in (0, 1)$, $K_2 \geq 1$ such that $(x - x_0)a' \leq K_1 a$ and $(x - x_0)b' \leq K_2 b$ a.e. in $[0, 1]$.

Hypothesis 2.3. Strongly-weakly degenerate case (SWD): there exists $x_0 \in (0, 1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0, 1] \setminus \{x_0\}$, $a \in W^{1,\infty}(0, 1)$, $b \in W^{1,1}(0, 1)$, and there exist $K_1 \geq 1$, $K_2 \in (0, 1)$ such that $(x - x_0)a' \leq K_1 a$ and $(x - x_0)b' \leq K_2 b$ a.e. in $[0, 1]$.

Hypothesis 2.4. Doubly strongly degenerate case (SSD): there exists $x_0 \in (0, 1)$ such that $a(x_0) = b(x_0) = 0$, $a, b > 0$ on $[0, 1] \setminus \{x_0\}$, $a, b \in W^{1,\infty}(0, 1)$ and there exist $K_1, K_2 \geq 1$ such that $(x - x_0)a' \leq K_1 a$ and $(x - x_0)b' \leq K_2 b$ a.e. in $[0, 1]$.

Typical examples for the previous degeneracies and singularities are $a(x) = |x - x_0|^{K_1}$ and $b(x) = |x - x_0|^{K_2}$, with $0 < K_1, K_2 < 2$.

Remark 1. The restriction $K_i < 2$ is related to the controllability issue. Indeed, it is clear from the proof of Theorem 2.1 that such a condition is useless, for example, when $\lambda < 0$. On the other hand, concerning controllability, we will not consider the case $K_i \geq 2$, since if $a(x) = |x - x_0|^{K_1}$, $K_1 \geq 2$ and $\lambda = 0$, by a standard change of variables (see [30]), problem (1.5) may be transformed in a non degenerate heat equation on an unbounded domain, while the control remains distributed in a bounded domain. This situation is now well-understood, and the lack of null controllability was proved by Micu and Zuazua in [41].

We will use the following result several times; we state it for a , but an analogous one holds for b replacing K_1 with K_2 :

Lemma 2.1 (Lemma 2.1, [29]). *Assume that there exists $x_0 \in (0, 1)$ such that $a(x_0) = 0$, $a > 0$ on $[0, 1] \setminus \{x_0\}$, and either*

- $a \in W^{1,1}(0, 1)$ and there exist $K_1 \in (0, 1)$ such that $(x - x_0)a' \leq K_1 a$ a.e. in $[0, 1]$, or
- $a \in W^{1,\infty}(0, 1)$ and there exist $K_1 \in [1, 2)$ such that $(x - x_0)a' \leq K_1 a$ a.e. in $[0, 1]$.

1. Then for all $\gamma \geq K_1$ the map

$$x \mapsto \frac{|x - x_0|^\gamma}{a} \text{ is non increasing on the left of } x = x_0$$

and non decreasing on the right of $x = x_0$,

$$\text{so that } \lim_{x \rightarrow x_0} \frac{|x - x_0|^\gamma}{a} = 0 \text{ for all } \gamma > K_1.$$

2. If $K_1 < 1$, then $\frac{1}{a} \in L^1(0, 1)$.

3. If $K_1 \in [1, 2)$, then $\frac{1}{\sqrt{a}} \in L^1(0, 1)$ and $\frac{1}{a} \notin L^1(0, 1)$.

For the well-posedness of the problem, we start introducing the following weighted Hilbert spaces, which are suitable to study all situations, namely the (WWD), (SSD), (WSD) and (SWD) cases:

$$H_a^1(0, 1) := \left\{ u \in W_0^{1,1}(0, 1) : \sqrt{a}u' \in L^2(0, 1) \right\}$$

and

$$H_{a,b}^1(0, 1) := \left\{ u \in H_a^1(0, 1) : \frac{u}{\sqrt{b}} \in L^2(0, 1) \right\},$$

endowed with the inner products

$$\langle u, v \rangle_{H_a^1(0,1)} := \int_0^1 au'v' dx + \int_0^1 uv dx,$$

and

$$\langle u, v \rangle_{H_{a,b}^1(0,1)} = \int_0^1 au'v' dx + \int_0^1 uv dx + \int_0^1 \frac{uv}{b} dx,$$

respectively.

Note that, if $u \in H_a^1(0, 1)$, then $au' \in L^2(0, 1)$, since $|au'| \leq (\max_{[0,1]} \sqrt{a})\sqrt{a}|u'|$.

We recall the following weighted Hardy–Poincaré inequality, see [29, Proposition 2.6]:

Proposition 2.1. *Assume that $p \in C([0, 1])$, $p > 0$ on $[0, 1] \setminus \{x_0\}$, $p(x_0) = 0$ and there exists $q > 1$ such that the function*

$$x \mapsto \frac{p(x)}{|x - x_0|^q} \text{ is non increasing on the left of } x = x_0 \quad (2.7)$$

and non decreasing on the right of } x = x_0.

Then, there exists a constant $C_{HP} > 0$ such that for any function w , locally absolutely continuous on $[0, x_0) \cup (x_0, 1]$ and satisfying

$$w(0) = w(1) = 0 \text{ with } \int_0^1 p(x)|w'(x)|^2 dx < +\infty,$$

the following inequality holds:

$$\int_0^1 \frac{p(x)}{(x - x_0)^2} w^2(x) dx \leq C_{HP} \int_0^1 p(x)|w'(x)|^2 dx. \quad (2.8)$$

Remark 2. Actually, such a proposition was proved in [29] also requiring $q < 2$. However, as it is clear from the proof, the result is true without such an upper bound on q , that in [29] was used for other estimates.

Moreover, we will also need other types of Hardy’s inequalities. Let us start with the following crucial

Lemma 2.2. *If $K_1 + K_2 \leq 2$ and $K_2 < 1$, then there exists a constant $C > 0$ such that*

$$\int_0^1 \frac{u^2}{b} dx \leq C \int_0^1 a(u')^2 dx \quad (2.9)$$

for every $u \in H_a^1(0, 1)$.

Proof. We set $p(x) := \frac{(x - x_0)^2}{b}$, so that p satisfies (2.7) with $q = 2 - K_2 > 1$ by Lemma 2.1. Thus, taken $u \in H_a^1(0, 1)$, by Proposition 2.1, we get

$$\int_0^1 \frac{u^2}{b} dx = \int_0^1 \frac{p(x)}{(x - x_0)^2} u^2 dx \leq C_{HP} \int_0^1 p(x)|u'(x)|^2 dx.$$

Now, by Lemma 2.1,

$$p(x) = (x - x_0)^{2-K_1-K_2} a(x) \frac{(x - x_0)^{K_1}}{a(x)} \frac{(x - x_0)^{K_2}}{b(x)} \leq ca(x)$$

for some $c > 0$, and the claim follows. \square

Remark 3. A similar proof shows that, when $K_1 + 2K_2 \leq 2$ and $K_2 < 1/2$, then

$$\int_0^1 \frac{u^2}{b^2} dx \leq C \int_0^1 a(u')^2 dx$$

for every $u \in H_a^1(0, 1)$.

Lemma 2.2 implies that $H_a^1(0, 1) = H_{a,b}^1(0, 1)$ when $K_1 + K_2 \leq 2$ and $K_2 < 1$. However, inequality (2.9) holds in other cases, see Proposition 2.2 below. In order to prove such a proposition, we need a preliminary result:

Lemma 2.3. *If $K_2 \geq 1$, then $u(x_0) = 0$ for every $u \in H_{a,b}^1(0, 1)$.*

Proof. Since $u \in W_0^{1,1}(0, 1)$, there exists $\lim_{x \rightarrow x_0} u(x) = L \in \mathbb{R}$. If $L \neq 0$, then $|u(x)| \geq \frac{L}{2}$ in a neighborhood of x_0 , that is

$$\frac{|u(x)|^2}{b} \geq \frac{L^2}{4b} \notin L^1(0, 1)$$

by Lemma 2.1, and thus $L = 0$. \square

We also need the following result, whose proof, with the aid of Lemma 2.3, is a simple adaptation of the one given in [31, Lemma 3.2].

Lemma 2.4. *If $K_2 \geq 1$, then*

$$H_c^1(0, 1) := \left\{ u \in H_0^1(0, 1) \text{ such that } \text{supp } u \subset (0, 1) \setminus \{x_0\} \right\}$$

is dense in $H_{a,b}^1(0, 1)$.

In the spirit of [18, Lemma 5.3.1], now we are ready for the following ‘‘classical’’ Hardy inequality in the space $H_{a,b}^1(0, 1)$ for $a(x) = |x - x_0|^\alpha$ and $b(x) = |x - x_0|^{2-\alpha}$. However, note that our inequality is more interesting than the classical one, since we admit a singularity inside the interval:

Lemma 2.5. *For every $\alpha \in \mathbb{R}$ the inequality*

$$\frac{(1 - \alpha)^2}{4} \int_0^1 \frac{u^2}{|x - x_0|^{2-\alpha}} dx \leq \int_0^1 |x - x_0|^\alpha (u')^2 dx$$

holds true for every $u \in H_{|x-x_0|^\alpha, |x-x_0|^{2-\alpha}}^1(0, 1)$.

Proof. The case $\alpha = 1$ is trivial. So, take $\beta = (1 - \alpha)/2 \neq 0$ and $\varepsilon \in (0, 1 - x_0)$.
First case: $\beta < 0$ ($\alpha > 1$). In this case we have

$$\begin{aligned}
& \int_{x_0+\varepsilon}^1 (x - x_0)^\alpha (u')^2 dx \\
&= \int_{x_0+\varepsilon}^1 (x - x_0)^\alpha \left((x - x_0)^\beta ((x - x_0)^{-\beta} u)' + \beta (x - x_0)^{-1} u \right)^2 dx \\
&\geq \beta^2 \int_{x_0+\varepsilon}^1 (x - x_0)^{\alpha-2} u^2 dx + 2\beta \int_{x_0+\varepsilon}^1 (x - x_0)^{\alpha+\beta-1} u ((x - x_0)^{-\beta} u)' dx \\
&= \beta^2 \int_{x_0+\varepsilon}^1 (x - x_0)^{\alpha-2} u^2 dx + \beta \left((x - x_0)^{-\beta} u \right)^2 \Big|_{x_0+\varepsilon}^1 \quad (\text{since } \alpha + \beta - 1 = -\beta) \\
&\geq \beta^2 \int_{x_0+\varepsilon}^1 (x - x_0)^{\alpha-2} u^2 dx.
\end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$, we get that

$$\int_{x_0}^1 (x - x_0)^\alpha (u')^2 dx \geq \beta^2 \int_{x_0}^1 (x - x_0)^{\alpha-2} u^2 dx. \quad (2.10)$$

Second case: $\beta > 0$. In this situation we have $2 - \alpha > 1$. Thus, in view of Lemma 2.4 with $K_2 = 2 - \alpha$, we will prove (2.10) first if $u \in H_c^1(0, 1)$ and then, by density, if $u \in H_{|x-x_0|^\alpha, |x-x_0|^{2-\alpha}}^1(0, 1)$. Thus, take $u \in H_c^1(0, 1)$; proceeding as above, we get

$$\begin{aligned}
& \int_{x_0+\varepsilon}^1 (x - x_0)^\alpha (u')^2 dx \\
&\geq \beta^2 \int_{x_0+\varepsilon}^1 (x - x_0)^{\alpha-2} u^2 dx + \beta \left((x - x_0)^{-\alpha} u \right)^2 \Big|_{x_0+\varepsilon}^1 \\
&\geq \beta^2 \int_{x_0+\varepsilon}^1 (x - x_0)^{\alpha-2} u^2 dx,
\end{aligned}$$

since $u(x_0 + \varepsilon) = 0$ for ε small enough.

Passing to the limit as $\varepsilon \rightarrow 0^+$, and using Lemma 2.4, we get that (2.10) holds true for every $u \in H_{|x-x_0|^\alpha, |x-x_0|^{2-\alpha}}^1(0, 1)$.

Operating in a symmetric way on the left of x_0 , we get the conclusion. \square

As a corollary of the previous result, we get the following improvement of Lemma 2.2.

Proposition 2.2. *If one among Hypotheses 2.1, 2.2, 2.3 holds with $K_1 + K_2 \leq 2$, then (2.9) holds for every $u \in H_{a,b}^1(0, 1)$.*

Proof. By Lemma 2.1 and Lemma 2.5 with $\alpha = 2 - K_2$, we immediately get

that for every $u \in H_{a,b}^1(0,1)$,

$$\begin{aligned} \int_0^1 \frac{u^2}{b} dx &\leq c \int_0^1 \frac{u^2}{|x-x_0|^{K_2}} dx \leq c \int_0^1 |x-x_0|^{2-K_2} (u')^2 dx \\ &\leq c \int_0^1 |x-x_0|^{K_1} (u')^2 dx \leq c \int_0^1 a(u')^2 dx. \end{aligned}$$

□

Remark 4. It is well known that when $K_1 = K_2 = 1$, an inequality of the form (2.9) doesn't hold (see [42]). Being such an inequality fundamental for the observability inequality (see Lemma 4.2), it is no surprise if with our techniques we cannot handle this case in Section 4.

The fundamental space in which we will work is clearly the one where the Hardy–Poincaré–type inequality (2.9) holds: in view of Proposition, it is clear that such a space is

$$\mathcal{H} := H_{a,b}^1(0,1)$$

Remark 5. Under the assumptions of Proposition 2.2, the standard norm $\|\cdot\|_{\mathcal{H}}^2$ is equivalent to

$$\|u\|_{\circ}^2 := \int_0^1 a(u')^2 dx$$

for all $u \in \mathcal{H}$. Indeed, for all $u \in \mathcal{H}$, we have

$$\int_0^1 u^2 dx = \int_0^1 b \frac{u^2}{b} dx \leq c \int_0^1 a(u')^2 dx,$$

and this is enough to conclude.

Moreover, when $\lambda < 0$, an equivalent norm is given by

$$\|u\|_{\sim}^2 := \int_0^1 a(u')^2 dx - \lambda \int_0^1 \frac{u^2}{b} dx.$$

This is particularly useful if Hypothesis 2.4 holds (see the proof of Theorem 2.1).

First, let us call C^* the best constant of (2.9) in \mathcal{H} . From now on, we make the following assumptions on a , b and λ :

Hypothesis 2.5. 1. One among Hypothesis 2.1, 2.2 or 2.3 holds true with $K_1 + K_2 \leq 2$, and we assume that

$$\lambda \in \left(0, \frac{1}{C^*}\right), \quad (2.11)$$

or

2. Hypotheses 2.1, 2.2, 2.3 or 2.4 hold with $\lambda < 0$.

Observe that the assumption $\lambda \neq 0$ is not restrictive, since the case $\lambda = 0$ was already considered in [29] and in [30].

Using the previous lemmas one can prove the next inequality.

Proposition 2.3. *Assume Hypothesis 2.5. Then there exists $\Lambda \in (0, 1]$ such that for all $u \in \mathcal{H}$*

$$\int_0^1 a(u')^2 dx - \lambda \int_0^1 \frac{u^2}{b} dx \geq \Lambda \int_0^1 a(u')^2 dx.$$

Proof. If $\lambda < 0$, the result is obvious taking $\Lambda = 1$. Now, assume that $\lambda \in \left(0, \frac{1}{C^*}\right)$. Then

$$\int_0^1 a(u')^2 dx - \lambda \int_0^1 \frac{u^2}{b} dx \geq \int_0^1 a(u')^2 dx - \lambda C^* \int_0^1 a(u')^2 dx \geq \Lambda \int_0^1 a(u')^2 dx.$$

□

We recall the following definition:

Definition 2.1. Let $u_0 \in L^2(0, 1)$ and $h \in L^2(Q_T)$. A function u is said to be a (weak) solution of (1.5) if

$$u \in L^2(0, T; \mathcal{H}) \cap H^1([0, T]; \mathcal{H}^*)$$

and it satisfies (1.5) in the sense of \mathcal{H}^* -valued distributions.

Note that, by [42, Lemma 11.4], any solution belongs to $C([0, T]; L^2(0, 1))$.

Finally, we introduce the Hilbert space

$$H_{a,b}^2(0, 1) := \left\{ u \in H_a^1(0, 1) : au' \in H^1(0, 1) \text{ and } Au \in L^2(0, 1) \right\},$$

where

$$Au := (au')' + \frac{\lambda}{b}u \text{ with } D(A) = H_{a,b}^2(0, 1).$$

Remark 6. Observe that if $u \in D(A)$, then $\frac{u}{b}$ and $\frac{u}{\sqrt{b}} \in L^2(0, 1)$, so that $u \in H_{a,b}^1(0, 1)$ and inequality (2.9) holds.

We also recall the following integration by parts with functions in the reference spaces:

Lemma 2.6 (Green formula, [31], Lemma 2.3). *Assume one among the Hypotheses 2.1, 2.2, 2.3, 2.4. Then, for all $(u, v) \in H_{a,b}^2(0, 1) \times H_a^1(0, 1)$ the following identity holds:*

$$\int_0^1 (au')' v dx = - \int_0^1 au' v' dx. \quad (2.12)$$

Observe that in the non degenerate case, it is well known that the heat operator with an inverse-square singular potential

$$u_t - \Delta u - \lambda \frac{u}{|x|^2} v$$

gives rise to well-posed Cauchy-Dirichlet problems if and only if λ is not larger than the best Hardy inequality (see [5], [11], [47]). For this reason, it is not strange that we require an analogous condition for problem (1.5), by invoking Hypothesis 2.5; as a consequence, using the standard semigroup theory, we have that (1.5) is well-posed:

Theorem 2.1. *Assume Hypothesis 2.5. For every $u_0 \in L^2(0, 1)$ and $h \in L^2(Q_T)$ there exists a unique solution of problem (1.5). In particular, the operator $A : D(A) \rightarrow L^2(0, 1)$ is non positive and self-adjoint in $L^2(0, 1)$ and it generates an analytic contraction semigroup of angle $\pi/2$. Moreover, let $u_0 \in D(A)$; then*

$$\begin{aligned} h \in W^{1,1}(0, T; L^2(0, 1)) &\Rightarrow u \in C^1(0, T; L^2(0, 1)) \cap C([0, T]; D(A)), \\ h \in L^2(Q_T) &\Rightarrow u \in H^1(0, T; L^2(0, 1)). \end{aligned}$$

Proof. Observe that $D(A)$ is dense in $L^2(0, 1)$. The existence of the unique solution follows in a standard way by a Faedo-Galerkin procedure, see, e.g., [42, Theorem 11.3], or [39, Theorem 3.4.1 and Remark 3.4.3]. Let us prove the other facts.

A is non positive. By Proposition 2.3, Remark 5 and Lemma 2.6, for all $u \in D(A)$ we have

$$-\langle Au, u \rangle_{L^2(0,1)} = - \int_0^1 \left((au')' + \frac{\lambda}{b} u \right) u \, dx = \int_0^1 a(u')^2 \, dx - \lambda \int_0^1 \frac{u^2}{b} \, dx \geq C \|u\|_{\mathcal{H}}^2.$$

A is self-adjoint. Let $T : L^2(0, 1) \rightarrow L^2(0, 1)$ be the mapping defined in the following usual way: to each $h \in L^2(0, 1)$ associate the weak solution $u = T(h) \in \mathcal{H}$ of

$$\int_0^1 \left(au'v' - \lambda \frac{uv}{b} \right) \, dx = \int_0^1 hv \, dx$$

for every $v \in \mathcal{H}$. Note that T is well defined by the Lax-Milgram Lemma via Proposition 2.3, which also implies that T is continuous. Now, it is easy to see that T is injective and symmetric. Thus it is self-adjoint. As a consequence, $A = T^{-1} : D(A) \rightarrow L^2(0, 1)$ is self-adjoint (for example, see [43, Proposition A.8.2]).

A is m -dissipative. Being A non positive and self-adjoint, this is a straightforward consequence of [17, Corollary 2.4.8]. Then $(A, D(A))$ generates a cosine family and an analytic contractive semigroup of angle $\frac{\pi}{2}$ on $L^2(0, 1)$ (see, for instance, [3, Examples 3.14.16 and 3.7.5]).

The additional regularity is a consequence of [17, Lemma 4.1.5 and Proposition 4.1.6] in the first case, and of [2, 6.2.2 and 6.2.4] in the second one. \square

3 Carleman estimates for singular/degenerate problems

In this section we prove one of the main result of this paper, i.e. a new Carleman estimate with boundary terms for solutions of the singular/degenerate problem

$$\begin{cases} v_t + (av_x)_x + \frac{\lambda}{b(x)}v = h(t, x) = h, & (t, x) \in Q_T, \\ v(t, 0) = v(t, 1) = 0, & t \in (0, T), \\ v(T, x) = v_T(x), \end{cases} \quad (3.13)$$

which is the adjoint of problem (1.5).

On the degenerate function a we make the following assumption:

Hypothesis 3.1. Hypothesis 2.5 holds. Moreover, if $K_1 > \frac{4}{3}$, then there exists a constant $\theta \in (0, K_1]$ such that

$$x \mapsto \frac{a(x)}{|x - x_0|^\theta} \begin{cases} \text{is non increasing on the left of } x = x_0, \\ \text{is non decreasing on the right of } x = x_0. \end{cases} \quad (3.14)$$

In addition, when $K_1 > \frac{3}{2}$ the function in (3.14) is bounded below away from 0 and there exists a constant $\Sigma > 0$ such that

$$|a'(x)| \leq \Sigma|x - x_0|^{2\theta-3} \text{ for a.e. } x \in [0, 1]. \quad (3.15)$$

Moreover, if $\lambda < 0$ we require that

$$(x - x_0)b'(x) \geq 0 \text{ in } [0, 1]. \quad (3.16)$$

Remark 7. If $a(x) = |x - x_0|^{K_1}$, then (3.14) is clearly satisfied with $\theta = K_1$. Moreover, the additional requirements for the sub-case $K_1 > \frac{3}{2}$ are technical ones and are introduced in [30] to guarantee the convergence of some integrals (see [30, Appendix]). Of course, the prototype $a(x) = |x - x_0|^{K_1}$ satisfies again such conditions with $\theta = K_1$. Finally, (3.16) is clearly satisfied by the prototype $b(x) = |x - x_0|^{K_2}$.

To prove Carleman estimate, let us introduce the function $\varphi := \Theta\psi$, where

$$\Theta(t) := \frac{1}{[t(T-t)]^4} \quad \text{and} \quad \psi(x) := c_1 \left[\int_{x_0}^x \frac{y - x_0}{a(y)} dy - c_2 \right], \quad (3.17)$$

where $c_2 > \sup_{[0,1]} \int_{x_0}^x \frac{y - x_0}{a(y)} dy$ and $c_1 > 0$ (for the observability inequality c_1 will be taken sufficiently large, see Lemma 4.1). Observe that $\Theta(t) \rightarrow +\infty$ as $t \rightarrow 0^+, T^-$, and clearly $-c_1c_2 \leq \psi < 0$.

The main result of this section is the following

Theorem 3.1. *Assume Hypothesis 3.1. Then, there exist two positive constants C and s_0 , such that every solution v of (3.13) in*

$$\mathcal{V} := L^2(0, T; H_{a,b}^2(0, 1)) \cap H^1(0, T; \mathcal{H}) \quad (3.18)$$

satisfies, for all $s \geq s_0$,

$$\begin{aligned} & \int_{Q_T} \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dxdt \\ & \leq C \left(\int_{Q_T} h^2 e^{2s\varphi} dxdt + sc_1 \int_0^T \left[a\Theta e^{2s\varphi(t,x)} (x-x_0)(v_x)^2 dt \right]_{x=0}^{x=1} \right). \end{aligned}$$

Remark 8. In [46] the authors prove a related Carleman inequality for the *non degenerate* singular 1-D problem

$$\begin{cases} v_t + v_{xx} + \frac{\mu}{x^2} + \frac{\lambda}{x^\beta} v = h & (t, x) \in Q_T, \\ v(t, 0) = v(t, 1) = 0 & t \in (0, T), \\ v(T, x) = v_T(x) & x \in (0, 1), \end{cases} \quad (3.19)$$

where $\beta \in [0, 2)$. When $\mu = 0$ and $x_0 = 0$, such an inequality reads as follows:

$$\int_{Q_T} \left(s^3\Theta^3 x^2 v^2 + \frac{s}{2}\Theta \frac{v^2}{x^2} + \frac{s}{2}\Theta \frac{v^2}{x^{2/3}} \right) e^{2s\Psi} dxdt \leq \frac{1}{2} \int_{Q_T} h^2 e^{2s\Psi} dxdt,$$

where $\Psi(x) = \frac{x^2}{2} - 1 < 0$ in $[0, 1]$. Actually, it is proved for solutions v such that

$$v(t, x) = 0 \text{ for all } (t, x) \in (0, T) \times (1 - \eta, 1) \text{ for some } \eta \in (0, 1). \quad (3.20)$$

However, in [46, Remark 3.5] the authors say that Carleman estimates can be proved also for all solutions of (3.19) not satisfying (3.20). We think that this latter situation is much more interesting, since by the Carleman estimates, if $h = 0$, then $v \equiv 0$ even if (3.20) does not hold.

The proof of Theorem 3.1 is quite long, and several intermediate lemmas will be used. First, for $s > 0$, define the function

$$w(t, x) := e^{s\varphi(t,x)} v(t, x),$$

where v is any solution of (3.13) in \mathcal{V} ; observe that, since $v \in \mathcal{V}$ and $\varphi < 0$, then $w \in \mathcal{V}$ and satisfies

$$\begin{cases} (e^{-s\varphi} w)_t + (a(e^{-s\varphi} w)_x)_x + \lambda \frac{e^{-s\varphi} w}{b} = h, & (t, x) \in (0, T) \times (0, 1), \\ w(t, 0) = w(t, 1) = 0, & t \in (0, T), \\ w(T, x) = w(0, x) = 0, & x \in (0, 1). \end{cases} \quad (3.21)$$

As usual, we re-write the previous problem as follows: setting

$$Lv := v_t + (av_x)_x + \lambda \frac{v}{b} \quad \text{and} \quad L_s w = e^{s\varphi} L(e^{-s\varphi} w),$$

then (3.21) becomes

$$\begin{cases} L_s w = e^{s\varphi} h, \\ w(t, 0) = w(t, 1) = 0, \quad t \in (0, T), \\ w(T, x) = w(0, x) = 0, \quad x \in (0, 1). \end{cases}$$

Computing $L_s w$, one has

$$L_s w = L_s^+ w + L_s^- w,$$

where

$$L_s^+ w := (aw_x)_x + \lambda \frac{w}{b} - s\varphi_t w + s^2 a\varphi_x^2 w,$$

and

$$L_s^- w := w_t - 2sa\varphi_x w_x - s(a\varphi_x)_x w.$$

Of course,

$$\begin{aligned} 2\langle L_s^+ w, L_s^- w \rangle &\leq 2\langle L_s^+ w, L_s^- w \rangle + \|L_s^+ w\|_{L^2(Q_T)}^2 + \|L_s^- w\|_{L^2(Q_T)}^2 \\ &= \|L_s w\|_{L^2(Q_T)}^2 = \|he^{s\varphi}\|_{L^2(Q_T)}^2, \end{aligned} \quad (3.22)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(Q_T)$. As usual, we will separate the scalar product $\langle L_s^+ w, L_s^- w \rangle$ in distributed terms and boundary terms.

Lemma 3.1. *The following identity holds:*

$$\begin{aligned} &\left. \begin{aligned} &\langle L_s^+ w, L_s^- w \rangle \\ &= \frac{s}{2} \int_{Q_T} \varphi_{tt} w^2 dx dt - 2s^2 \int_{Q_T} a\varphi_x \varphi_{tx} w^2 dx dt \\ &+ s \int_{Q_T} (2a^2 \varphi_{xx} + aa' \varphi_x) (w_x)^2 dx dt \\ &+ s^3 \int_{Q_T} (2a\varphi_{xx} + a' \varphi_x) a(\varphi_x)^2 w^2 dx dt \\ &- s\lambda \int_{Q_T} \frac{a\varphi_x b'}{b^2} w^2 dx dt \end{aligned} \right\} \{D.T.\} \\ &+ \left. \begin{aligned} &\int_0^T [aw_x w_t]_{x=0}^{x=1} dt - \frac{s}{2} \int_0^1 [w^2 \varphi_t]_{t=0}^{t=T} dx + \frac{s^2}{2} \int_0^1 [a(\varphi_x)^2 w^2]_{t=0}^{t=T} dt \\ &+ \int_0^T [-s\varphi_x (aw_x)^2 + s^2 a\varphi_t \varphi_x w^2 - s^3 a^2 (\varphi_x)^3 w^2 - s\lambda \frac{a\varphi_x}{b} w^2]_{x=0}^{x=1} dt \\ &+ \int_0^T [-sa(a\varphi_x)_x w w_x]_{x=0}^{x=1} dt - \frac{1}{2} \int_0^1 [a(w_x)^2 - \lambda \frac{1}{2b} w^2]_{t=0}^{t=T} dx. \end{aligned} \right\} \{B.T.\} \end{aligned} \quad (3.23)$$

Proof. Computing $\langle L_s^+ w, L_s^- w \rangle$, one has that

$$\langle L_s^+ w, L_s^- w \rangle = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &:= \int_{Q_T} ((aw_x)_x - s\varphi_t w + s^2 a(\varphi_x)^2 w) w_t dx dt, \\ I_2 &:= \int_{Q_T} ((aw_x)_x - s\varphi_t w + s^2 a(\varphi_x)^2 w) (-2sa\varphi_x w_x) dx dt, \\ I_3 &:= \int_{Q_T} ((aw_x)_x - s\varphi_t w + s^2 a(\varphi_x)^2 w) (-s(a\varphi_x)_x w) dx dt, \end{aligned}$$

and

$$I_4 := \lambda \int_{Q_T} \frac{w}{b} (w_t - 2sa\varphi_x w_x - s(a\varphi_x)_x w) dx dt.$$

By several integrations by parts in space and in time (see [1, Lemma 3.4], [29, Lemma 3.1] or [30, Lemma 3.1]), and observing that $\int_{Q_T} a(a\varphi_x)_{xx} w w_x dx dt = 0$ (by the very definition of φ), we get

$$\begin{aligned} &I_1 + I_2 + I_3 \\ &= \frac{s}{2} \int_{Q_T} \varphi_{tt} w^2 dx dt - 2s^2 \int_{Q_T} a\varphi_x \varphi_{tx} w^2 dx dt \\ &+ s \int_{Q_T} (2a^2 \varphi_{xx} + aa' \varphi_x) (w_x)^2 dx dt \\ &+ s^3 \int_{Q_T} (2a\varphi_{xx} + a' \varphi_x) a(\varphi_x)^2 w^2 dx dt \\ &+ \int_0^T [aw_x w_t]_{x=0}^{x=1} dt - \frac{s}{2} \int_0^1 [w^2 \varphi_t]_{t=0}^{t=T} dx + \frac{s^2}{2} \int_0^1 [a(\varphi_x)^2 w^2]_{t=0}^{t=T} dt \\ &+ \int_0^T [-s\varphi_x (aw_x)^2 + s^2 a\varphi_t \varphi_x w^2 - s^3 a^2 (\varphi_x)^3 w^2]_{x=0}^{x=1} dt \\ &+ \int_0^T [-sa(a\varphi_x)_x w w_x]_{x=0}^{x=1} dt - \frac{1}{2} \int_0^1 [a(w_x)^2]_{t=0}^{t=T} dx. \end{aligned} \tag{3.24}$$

Next, we compute I_4 :

$$\begin{aligned}
I_4 &= \lambda \left(\int_{Q_T} \frac{1}{2b} (w^2)_t dx dt - 2s \int_{Q_T} \frac{a}{b} \varphi_x w_x w dx dt \right. \\
&\quad \left. - s \int_{Q_T} \frac{(a\varphi_x)_x}{b} w^2 dx dt \right) \\
&= \lambda \left(\int_0^1 \frac{1}{2b} [w^2]_{t=0}^{t=T} dx - s \int_{Q_T} \frac{a}{b} \varphi_x (w^2)_x dx dt - s \int_{Q_T} \frac{(a\varphi_x)_x}{b} w^2 dx dt \right) \\
&= \lambda \left(\int_0^1 \frac{1}{2b} [w^2]_{t=0}^{t=T} dx - s \int_0^T \left[\frac{a}{b} \varphi_x w^2 \right]_{x=0}^{x=1} dt \right. \\
&\quad \left. + s \int_{Q_T} \left(\frac{a\varphi_x}{b} \right)_x w^2 dx dt - s \int_{Q_T} \frac{(a\varphi_x)_x}{b} w^2 dx dt \right) \\
&= \lambda \left(\int_0^1 \frac{1}{2b} [w^2]_{t=0}^{t=T} dx - s \int_0^T \left[\frac{a\varphi_x}{b} w^2 \right]_{x=0}^{x=1} dt - s \int_{Q_T} \frac{a\varphi_x b'}{b^2} w^2 dx dt \right). \tag{3.25}
\end{aligned}$$

Adding (3.24)-(3.25), (3.23) follows immediately. \square

For the boundary terms in (3.23), we have:

Lemma 3.2. *The boundary terms in (3.23) reduce to*

$$-s \int_0^T [\Theta (aw_x)^2 \psi']_{x=0}^{x=1} dt.$$

Proof. As in [29] or [30], using the definition of φ and the boundary conditions on w , one has that

$$\begin{aligned}
&\int_0^T [aw_x w_t]_{x=0}^{x=1} dt - \frac{s}{2} \int_0^1 [w^2 \varphi_t]_{t=0}^{t=T} dx + \frac{s^2}{2} \int_0^1 [a(\varphi_x)^2 w^2]_{t=0}^{t=T} dt \\
&+ \int_0^T [-s\varphi_x (aw_x)^2 + s^2 a\varphi_t \varphi_x w^2 - s^3 a^2 (\varphi_x)^3 w^2]_{x=0}^{x=1} dt \\
&+ \int_0^T [-sa(a\varphi_x)_x w w_x]_{x=0}^{x=1} dt - \frac{1}{2} \int_0^1 [a(w_x)^2]_{t=0}^{t=T} dx = -s \int_0^T [\Theta (aw_x)^2 \psi']_{x=0}^{x=1} dt. \tag{3.26}
\end{aligned}$$

Moreover, since $w \in \mathcal{V}$, $w \in C([0, T]; \mathcal{H})$; thus $w(0, x)$, $w(T, x)$ are well defined, and using the boundary conditions of w , we get that

$$\int_0^1 \left[\frac{1}{2b} w^2 \right]_{t=0}^{t=T} dx = 0.$$

Now, consider the last boundary term $s\lambda \int_0^T \left[\frac{a\varphi_x}{b} w^2 \right]_{x=0}^{x=1} dt$. Using the definition of φ , this term becomes $s\lambda \int_0^T \left[\Theta \frac{a\psi'}{b} w^2 \right]_{x=0}^{x=1} dt$. By definition of ψ ,

the function $\Theta \frac{a\psi'}{b} w^2$ is bounded in $(0, T)$. Thus, by the boundary conditions on w , one has

$$s\lambda \int_0^T \left[\Theta \frac{a\psi'}{b} w^2 \right]_{x=0}^{x=1} dt = 0.$$

□

Now, the crucial step is to prove the following estimate:

Lemma 3.3. *Assume Hypothesis 3.1. Then there exist two positive constants s_0 and C such that for all $s \geq s_0$ the distributed terms of (3.23) satisfy the estimate*

$$\begin{aligned} & \frac{s}{2} \int_{Q_T} \varphi_{tt} w^2 dxdt - 2s^2 \int_{Q_T} a\varphi_x \varphi_{tx} w^2 dxdt \\ & + s \int_{Q_T} (2a^2 \varphi_{xx} + aa' \varphi_x) (w_x)^2 dxdt \\ & + s^3 \int_0^T \int_0^1 (2a\varphi_{xx} + a' \varphi_x) a (\varphi_x)^2 w^2 dxdt - s\lambda \int_{Q_T} \frac{a\varphi_x b'}{b^2} w^2 dxdt \\ & \geq \frac{C}{2} s \int_{Q_T} \Theta a (w_x)^2 dxdt + \frac{C^3}{2} s^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt. \end{aligned}$$

Proof. Proceeding as in [29, Lemma 3.2] or in [30, Lemma 4.1], one can prove that, for s large enough,

$$\begin{aligned} & \frac{s}{2} \int_{Q_T} \varphi_{tt} w^2 dxdt - 2s^2 \int_{Q_T} a\varphi_x \varphi_{tx} w^2 dxdt \\ & + s \int_{Q_T} (2a^2 \varphi_{xx} + aa' \varphi_x) (w_x)^2 dxdt \\ & + s^3 \int_{Q_T} (2a\varphi_{xx} + a' \varphi_x) a (\varphi_x)^2 w^2 dxdt \\ & \geq \frac{3C}{4} s \int_{Q_T} \Theta a (w_x)^2 dxdt + \frac{C^3}{2} s^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt, \end{aligned}$$

where C is a positive constant. Let us remark that one can assume C as large as desired, provided that s_0 increases as well. Indeed, taken $k > 0$, from

$$Cs\mathcal{A}_1 + C^3 s^3 \mathcal{A}_2 = kC \frac{s}{k} \mathcal{A}_1 + k^3 C^3 \frac{s^3}{k^3} \mathcal{A}_2,$$

we can choose $s'_0 = ks_0$ and $C' = kC$ large as needed.

Now, we estimate the term $-s\lambda \int_{Q_T} \frac{a\varphi_x b'}{b^2} w^2 dxdt$. If $\lambda < 0$, the thesis follows immediately by the previous inequality and by (3.16). Otherwise, if

$\lambda > 0$, by definition of φ and the assumption on b , one has

$$\begin{aligned} -s\lambda \int_{Q_T} \frac{a\varphi_x b'}{b^2} w^2 dxdt &= -s\lambda \int_{Q_T} \Theta \frac{a\psi' b'}{b^2} w^2 dxdt \\ &= -s\lambda c_1 \int_{Q_T} \Theta \frac{(x-x_0)b'}{b^2} w^2 dxdt \\ &\geq -s\lambda c_1 K_2 \int_{Q_T} \frac{\Theta}{b} w^2 dxdt. \end{aligned}$$

Since $w(t, \cdot) \in \mathcal{H}$ for every $t \in [0, 1]$, for $w \in \mathcal{V}$, by (2.9) we get

$$\int_{Q_T} \frac{\Theta}{b} w^2 dxdt \leq C^* \int_{Q_T} \Theta a(w_x)^2 dxdt.$$

Hence,

$$-s\lambda \int_{Q_T} \frac{a\varphi_x b'}{b^2} w^2 dxdt \geq -s\lambda c_1 K_2 C^* \int_{Q_T} \Theta a(w_x)^2 dxdt,$$

and we can assume, in view of what remarked above, that this last quantity is greater than

$$-s\frac{C}{4} \int_{Q_T} \Theta a(w_x)^2 dxdt.$$

Summing up, the distributed terms of $\int_{Q_T} L_s^+ w L_s^- w dxdt$ can be estimated as

$$\{\text{D.T.}\} \geq \frac{C}{2} s \int_{Q_T} \Theta a(w_x)^2 dxdt + \frac{C^3}{2} s^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt,$$

for s large enough and $C > 0$. \square

From Lemma 3.1, Lemma 3.2 and Lemma 3.3, we deduce immediately that there exist two positive constants C and s_0 , such that for all $s \geq s_0$,

$$\begin{aligned} \int_{Q_T} L_s^+ w L_s^- w dxdt &\geq Cs \int_{Q_T} \Theta a(w_x)^2 dxdt \\ &\quad + Cs^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt - s \int_0^T [\Theta a^2 w_x^2 \psi']_{x=0}^{x=1} dt. \end{aligned} \tag{3.27}$$

Thus, a straightforward consequence of (3.22) and of (3.27) is the next result.

Lemma 3.4. *Assume Hypothesis 3.1. Then, there exist two positive constants C and s_0 , such that for all $s \geq s_0$,*

$$\begin{aligned} &s \int_{Q_T} \Theta a(w_x)^2 dxdt + s^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt \\ &\leq C \left(\int_{Q_T} h^2 e^{2s\varphi(t,x)} dxdt + s \int_0^T [\Theta a^2 (w_x)^2 \psi']_{x=0}^{x=1} dt \right). \end{aligned} \tag{3.28}$$

Recalling the definition of w , we have $v = e^{-s\varphi} w$ and $v_x = -s\Theta\psi' e^{-s\varphi} w + e^{-s\varphi} w_x$. Thus, substituting in (3.28), Theorem 3.1 follows.

4 Observability results and application to null controllability

In this section we shall apply the just established Carleman inequalities to observability and controllability issues. For this, we assume that the control set ω satisfies the following assumption:

Hypothesis 4.1. The subset ω is such that

- (i) it is an interval which contains the degeneracy point:

$$\omega = (\alpha, \beta) \subset (0, 1) \text{ is such that } x_0 \in \omega, \quad (4.29)$$

or

- (ii) it is an interval lying on one side of the degeneracy point:

$$\omega = (\alpha, \beta) \subset (0, 1) \text{ is such that } x_0 \notin \bar{\omega}. \quad (4.30)$$

On the coefficients a and b we essentially start with the assumptions made so far, with the exception of Hypothesis 2.4, and we add another technical one. We summarize all of them in the following:

Hypothesis 4.2.

- Assume one among Hypotheses 2.1, 2.2 or 2.3 with $K_1 + K_2 \leq 2$ and $\lambda < 1/C^*$.
- If $\lambda < 0$, (3.16) holds.
- If $K_1 > 4/3$, condition (3.14) holds, and if $K_1 > 3/2$, (3.15) is satisfied.
- If Hypothesis 2.1 or 2.2 holds, there exist two functions $\mathfrak{g} \in L_{\text{loc}}^\infty([0, 1] \setminus \{x_0\})$, $\mathfrak{h} \in W_{\text{loc}}^{1, \infty}([0, 1] \setminus \{x_0\})$ and two strictly positive constants $\mathfrak{g}_0, \mathfrak{h}_0$ such that $\mathfrak{g}(x) \geq \mathfrak{g}_0$ for a.e. x in $[0, 1]$ and

$$-\frac{a'(x)}{2\sqrt{a(x)}} \left(\int_x^B \mathfrak{g}(t) dt + \mathfrak{h}_0 \right) + \sqrt{a(x)} \mathfrak{g}(x) = \mathfrak{h}(x, B) \quad (4.31)$$

for a.e. $x, B \in [0, 1]$ with $x < B < x_0$ or $x_0 < x < B$.

Remark 9. Since we require identity (4.31) far from x_0 , once a is given, it is easy to find $\mathfrak{g}, \mathfrak{h}, \mathfrak{g}_0$ and \mathfrak{h}_0 with the desired properties. For example, if $a(x) := |x - x_0|^\alpha$, $\alpha \in (0, 1)$, we can take $\mathfrak{g}(x) \equiv \mathfrak{g}_0 = \mathfrak{h}_0 = 1$ and $\mathfrak{h}(x, B) = |x - x_0|^{\frac{\alpha}{2}-1} \left[\frac{\alpha}{2} \text{sign}(x - x_0)(B + 1 - x) + |x - x_0| \right]$, for all x and $B \in [0, 1]$, with $x < B < x_0$ or $x_0 < x < B$. Clearly, $\mathfrak{g} \in L_{\text{loc}}^\infty([0, 1] \setminus \{x_0\})$ and $\mathfrak{h} \in W_{\text{loc}}^{1, \infty}([0, 1] \setminus \{x_0\}; L^\infty(0, 1))$.

Now, we associate to problem (1.5) the homogeneous adjoint problem

$$\begin{cases} v_t + (av_x)_x + \frac{\lambda}{b(x)}v = 0, & (t, x) \in Q_T, \\ v(t, 0) = v(t, 1) = 0, & t \in (0, T), \\ v(T, x) = v_T(x), \end{cases} \quad (4.32)$$

where $T > 0$ is given and $v_T(x) \in L^2(0, 1)$. By the Carleman estimate in Theorem 3.1, we will deduce the following observability inequality for all the degenerate cases:

Proposition 4.1. *Assume Hypotheses 4.1 and 4.2. Then there exists a positive constant C_T such that every solution $v \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; \mathcal{H})$ of (4.32) satisfies*

$$\int_0^1 v^2(0, x)dx \leq C_T \int_0^T \int_\omega v^2(t, x)dxdt. \quad (4.33)$$

Using the observability inequality (4.33) and a standard technique (e.g., see [38, Section 7.4]), one can prove the null controllability result for the linear degenerate problem (1.5):

Theorem 4.1. *Assume Hypotheses 4.1 and 4.2. Then, given $u_0 \in L^2(0, 1)$, there exists $h \in L^2(Q_T)$ such that the solution u of (1.5) satisfies*

$$u(T, x) = 0 \quad \text{for every } x \in [0, 1].$$

Moreover

$$\int_{Q_T} h^2 dxdt \leq C \int_0^1 u_0^2(x)dx,$$

for some positive constant C .

4.1 Proof of Proposition 4.1

In this subsection we will prove, as a consequence of the Carleman estimate proved in Section 3, the observability inequality (4.33). For this purpose, we will give some preliminary results. As a first step, consider the adjoint problem

$$\begin{cases} v_t + Av = 0, & (t, x) \in Q_T, \\ v(t, 0) = v(t, 1) = 0, & t \in (0, T), \\ v(T, x) = v_T(x) \in D(A^2), \end{cases} \quad (4.34)$$

where

$$D(A^2) = \left\{ u \in D(A) : Au \in D(A) \right\}$$

and $Au := (au_x)_x + \lambda \frac{u}{b}$. Observe that $D(A^2)$ is densely defined in $D(A)$ for the graph norm (see, for example, [9, Lemma 7.2]) and hence in $L^2(0, 1)$. As in [12], [13], [26] or [29], define the following class of functions:

$$\mathcal{W} := \left\{ v \text{ is a solution of (4.34)} \right\}.$$

Obviously (see, for example, [9, Theorem 7.5])

$$\mathcal{W} \subset C^1([0, T]; H_{a,b}^2(0, 1)) \subset \mathcal{V} \subset \mathcal{U},$$

where, \mathcal{V} is defined in (3.18) and

$$\mathcal{U} := C([0, T]; L^2(0, 1)) \cap L^2(0, T; \mathcal{H}). \quad (4.35)$$

We start with

Proposition 4.2 (Caccioppoli's inequality). *Assume Hypothesis 2.5. Let ω' and ω be two open subintervals of $(0, 1)$ such that $\omega' \subset\subset \omega \subset (0, 1)$ and $x_0 \notin \bar{\omega}'$. Let $\varphi(t, x) = \Theta(t)\Upsilon(x)$, where Θ is defined in (3.17) and*

$$\Upsilon \in C([0, 1], (-\infty, 0)) \cap C^1([0, 1] \setminus \{x_0\}, (-\infty, 0))$$

is such that

$$|\Upsilon_x| \leq \frac{c}{\sqrt{a}} \text{ in } [0, 1] \setminus \{x_0\} \quad (4.36)$$

for some $c > 0$. Then, there exist two positive constants C and s_0 such that every solution $v \in \mathcal{W}$ of the adjoint problem (4.34) satisfies

$$\int_0^T \int_{\omega'} (v_x)^2 e^{2s\varphi} dx dt \leq C \int_0^T \int_{\omega} v^2 dx dt, \quad (4.37)$$

for all $s \geq s_0$.

Of course, our prototype for Υ is the function ψ defined in (3.17), since

$$|\psi'(x)| = c_1 \sqrt{\frac{|x - x_0|^2}{a(x)}} \frac{1}{\sqrt{a(x)}} \leq c \frac{1}{\sqrt{a(x)}}$$

by Lemma 2.1.

Proof. The proof follows the one of [29, Proposition 4.2], but it is different for the presence of the singular term.

Let us consider a smooth function $\xi : [0, 1] \rightarrow \mathbb{R}$ such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & \text{for all } x \in [0, 1], \\ \xi(x) = 1, & x \in \omega', \\ \xi(x) = 0, & x \in [0, 1] \setminus \omega. \end{cases}$$

Since v solves (4.34), we have

$$\begin{aligned}
0 &= \int_0^T \frac{d}{dt} \left(\int_0^1 \xi^2 e^{2s\varphi} v^2 dx \right) dt = \int_{Q_T} 2s\xi^2 \varphi_t e^{2s\varphi} v^2 + 2\xi^2 e^{2s\varphi} v v_t dx dt \\
&= 2 \int_{Q_T} \xi^2 s \varphi_t e^{2s\varphi} v^2 dx dt + 2 \int_{Q_T} \xi^2 e^{2s\varphi} v \left(-\lambda \frac{v}{b} - (av_x)_x \right) dx dt \\
&= 2 \int_{Q_T} \xi^2 s \varphi_t e^{2s\varphi} v^2 dx dt - 2\lambda \int_{Q_T} \xi^2 e^{2s\varphi} \frac{v^2}{b} dx dt + 2 \int_{Q_T} (\xi^2 e^{2s\varphi} v)_x a v_x dx dt.
\end{aligned} \tag{4.38}$$

If $\lambda < 0$, then, differentiating the last term in (4.38), we get

$$\begin{aligned}
2 \int_{Q_T} \xi^2 e^{2s\varphi} a (v_x)^2 dx dt &= 2\lambda \int_{Q_T} \xi^2 e^{2s\varphi} \frac{v^2}{b} dx dt - 2 \int_{Q_T} \xi^2 s \varphi_t e^{2s\varphi} v^2 dx dt \\
&\quad - 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x a v v_x dx dt \\
&\leq -2 \int_{Q_T} \xi^2 s \varphi_t e^{2s\varphi} v^2 dx dt - 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x a v v_x dx dt,
\end{aligned}$$

and then one can proceed as for the proof of [29, Proposition 4.2], obtaining the claim.

Otherwise, if $\lambda > 0$, fixed $\varepsilon > 0$, by the Cauchy–Schwarz inequality, we have for $w = \xi e^{s\varphi} v$

$$\begin{aligned}
\int_0^1 \xi^2 e^{2s\varphi} \frac{v^2}{b} dx &\leq C^* \int_0^1 a (w_x)^2 dx \\
&\leq C_\varepsilon \int_0^1 a [(\xi e^{s\varphi})_x]^2 v^2 dx + \varepsilon \int_0^1 \xi^2 e^{2s\varphi} a (v_x)^2 dx
\end{aligned}$$

for some $C_\varepsilon > 0$. Moreover,

$$[(\xi e^{s\varphi})_x]^2 \leq C \chi_\omega (e^{2s\varphi} + s^2 (\varphi_x)^2 e^{s\varphi}) \leq C \chi_\omega \left(1 + \frac{1}{a} \right)$$

for some positive constant C . Indeed, $e^{2s\varphi} < 1$, while $s^2 (\varphi_x)^2 e^{2s\varphi}$ can be estimated with

$$\frac{c}{(-\max \Upsilon)^2} (\Upsilon_x)^2 \leq \frac{c}{a}$$

by (4.36), for some constants $c > 0$. Thus

$$\begin{aligned}
2\lambda \int_{Q_T} \xi^2 e^{2s\varphi} \frac{v^2}{b} dx dt &\leq 2\lambda C_\varepsilon \int_{Q_T} a [(\xi e^{s\varphi})_x]^2 v^2 dx dt \\
&\quad + 2\lambda \varepsilon \int_{Q_T} \xi^2 e^{2s\varphi} a (v_x)^2 dx dt \\
&\leq C \int_0^T \int_\omega v^2 dx dt + 2\lambda \varepsilon \int_{Q_T} \xi^2 e^{2s\varphi} a (v_x)^2 dx dt,
\end{aligned} \tag{4.39}$$

for a positive constant C depending on ε . Hence, differentiating the last term in (4.38) and using (4.39), we get

$$\begin{aligned}
2 \int_{Q_T} \xi^2 e^{2s\varphi} a(v_x)^2 dxdt &= 2\lambda \int_{Q_T} \xi^2 e^{2s\varphi} \frac{v^2}{b} dxdt - 2 \int_{Q_T} \xi^2 s\varphi_t e^{2s\varphi} v^2 dxdt \\
&\quad - 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x a v v_x dxdt \\
&\leq C \int_0^T \int_{\omega} v^2 dxdt + 2\lambda\varepsilon \int_{Q_T} \xi^2 e^{2s\varphi} a(v_x)^2 dxdt \\
&\quad - 2 \int_{Q_T} \xi^2 s\varphi_t e^{2s\varphi} v^2 dxdt - 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x a v v_x dxdt.
\end{aligned}$$

Thus, applying again the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
(2 - 2\lambda\varepsilon) \int_{Q_T} \xi^2 e^{2s\varphi} a(v_x)^2 dxdt &\leq C \int_0^T \int_{\omega} v^2 dxdt - 2 \int_{Q_T} \xi^2 s\varphi_t e^{2s\varphi} v^2 dxdt \\
&\quad - 2 \int_{Q_T} (\xi^2 e^{2s\varphi})_x a v v_x dxdt \\
&\leq C \int_0^T \int_{\omega} v^2 dxdt - 2 \int_0^T \int_{\omega} \xi^2 s\varphi_t e^{2s\varphi} v^2 dxdt + 2\varepsilon \int_0^T \int_{\omega} (\sqrt{a} \xi e^{s\varphi} v_x)^2 dxdt \\
&\quad + D_\varepsilon \int_0^T \int_{\omega} \left(\sqrt{a} \frac{(\xi^2 e^{2s\varphi})_x}{\xi e^{s\varphi}} v \right)^2 dxdt \\
&= C \int_0^T \int_{\omega} v^2 dxdt - 2 \int_0^T \int_{\omega} \xi^2 s\varphi_t e^{2s\varphi} v^2 dxdt + 2\varepsilon \int_0^T \int_{\omega} \xi^2 e^{2s\varphi} a(v_x)^2 dxdt \\
&\quad + D_\varepsilon \int_0^T \int_{\omega} \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}} a v^2 dxdt
\end{aligned}$$

for some $D_\varepsilon > 0$. Hence,

$$\begin{aligned}
2(1 - \varepsilon - \lambda\varepsilon) \int_0^T \int_{\omega} \xi^2 e^{2s\varphi} a(v_x)^2 dxdt &\leq C \int_0^T \int_{\omega} v^2 dxdt \\
&\quad - 2 \int_0^T \int_{\omega} \xi^2 s\varphi_t e^{2s\varphi} v^2 dxdt + D_\varepsilon \int_0^T \int_{\omega} \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}} a v^2 dxdt.
\end{aligned}$$

Since $x_0 \notin \bar{\omega}'$, then

$$\begin{aligned}
2(1 - \varepsilon - \lambda\varepsilon) \inf_{\bar{\omega}'} a(x) \int_0^T \int_{\omega'} e^{2s\varphi} (v_x)^2 dxdt \\
\leq 2(1 - \varepsilon - \lambda\varepsilon) \int_0^T \int_{\bar{\omega}'} \xi^2 e^{2s\varphi} a(v_x)^2 dxdt \\
\leq 2(1 - \varepsilon - \lambda\varepsilon) \int_0^T \int_{\omega} \xi^2 e^{2s\varphi} a(v_x)^2 dxdt \\
\leq C \int_0^T \int_{\omega} v^2 dxdt - 2 \int_0^T \int_{\omega} \xi^2 s\varphi_t e^{2s\varphi} v^2 dxdt + D_\varepsilon \int_0^T \int_{\omega} \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}} a v^2 dxdt.
\end{aligned}$$

Finally, we show that there exists a positive constant C (still depending on ε) such that

$$\begin{aligned} & -2 \int_0^T \int_{\omega} \xi^2 s \varphi_t e^{2s\varphi} v^2 dx dt + D_\varepsilon \int_0^T \int_{\omega} \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}} a v^2 dx dt \\ & \leq C \int_0^T \int_{\omega} v^2 dx dt, \end{aligned}$$

so that the claim will follow. Indeed,

$$|s\varphi_t e^{2s\varphi}| \leq c \frac{1}{s_0^{1/4} (-\max \Upsilon)^{1/4}},$$

$|\dot{\Theta}| \leq c\Theta^{5/4}$ and

$$|s\varphi_t e^{2s\varphi}| \leq cs(-\Upsilon)\Theta^{5/4} e^{2s\varphi} \leq \frac{c}{(s(-\Upsilon))^{5/4}}$$

for some constants $c > 0$ which may vary at every step.

On the other hand, $\frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}}$ can be estimated by

$$C(e^{2s\varphi} + s^2(\varphi_x)^2 e^{2s\varphi}) \chi_{\omega},$$

and proceeding as before, we get the claim, choosing ε small enough, namely $\varepsilon < (1 + \lambda)^{-1}$. \square

We shall also use the following

Lemma 4.1. *Assume Hypotheses 4.1 and 4.2. Then there exist two positive constants C and s_0 such that every solution $v \in \mathcal{W}$ of (4.34) satisfies, for all $s \geq s_0$,*

$$\int_{Q_T} \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dx dt \leq C \int_0^T \int_{\omega} v^2 dx dt.$$

Here Θ and φ are as in (3.17) with c_1 sufficiently large.

Using the following non degenerate classical Carleman estimate, one has that the proof of the previous lemma is a simple adaptation of the proof of [30, Lemma 5.1 and 5.2], to which we refer, also to explain why c_1 must be large.

Proposition 4.3 (Nondegenerate nonsingular Carleman estimate). *Let z be the solution of*

$$\begin{cases} z_t + (az_x)_x + \lambda \frac{z}{b} = h \in L^2((0, T) \times (A, B)), \\ z(t, A) = z(t, B) = 0, \quad t \in (0, T), \end{cases} \quad (4.40)$$

where $b \in C([A, B])$ is such that $b \geq b_0 > 0$ in $[A, B]$ and a satisfies

(a₁) $a \in W^{1,1}(A, B)$, $a \geq a_0 > 0$ in (A, B) and there exist two functions $\mathbf{g} \in L^1(A, B)$, $\mathbf{h} \in W^{1,\infty}(A, B)$ and two strictly positive constants $\mathbf{g}_0, \mathbf{h}_0$ such that $\mathbf{g}(x) \geq \mathbf{g}_0$ for a.e. x in $[A, B]$ and

$$-\frac{a'(x)}{2\sqrt{a(x)}} \left(\int_x^B \mathbf{g}(t)dt + \mathbf{h}_0 \right) + \sqrt{a(x)}\mathbf{g}(x) = \mathbf{h}(x) \quad \text{for a.e. } x \in [A, B];$$

or

(a₂) $a \in W^{1,\infty}(A, B)$ and $a \geq a_0 > 0$ in (A, B) .

Then, for all $\lambda \in \mathbb{R}$, there exist three positive constants C, r and s_0 such that for any $s > s_0$

$$\int_0^T \int_A^B (s\Theta(z_x)^2 + s^3\Theta^3 z^2) e^{2s\Phi} dxdt \leq C \left(\int_0^T \int_A^B h^2 e^{2s\Phi} dxdt - (B.T.) \right), \quad (4.41)$$

where

$$(B.T.) = \begin{cases} sr \int_0^T \left[a^{3/2} e^{2s\Phi} \Theta \left(\int_x^B \mathbf{g}(\tau)d\tau + \mathbf{h}_0 \right) (z_x)^2 \right]_{x=A}^{x=B} dt, & \text{if } (a_1) \text{ holds,} \\ sr \int_0^T [ae^{2s\Phi} \Theta e^{r\zeta} (v_x)^2]_{x=A}^{x=B} dt, & \text{if } (a_2) \text{ holds.} \end{cases}$$

Here the function Φ is defined as $\Phi(t, x) := \Theta(t)\rho(x)$, where Θ is as in (3.17),

$$\rho(x) := \begin{cases} -r \left[\int_A^x \frac{1}{\sqrt{a(t)}} \int_t^B \mathbf{g}(s)dsdt + \int_A^x \frac{\mathbf{h}_0}{\sqrt{a(t)}} dt \right] - \mathbf{c}, & \text{if } (a_1) \text{ holds,} \\ e^{r\zeta(x)} - \mathbf{c}, & \text{if } (a_2) \text{ holds,} \end{cases} \quad (4.42)$$

and

$$\zeta(x) = \mathfrak{d} \int_x^B \frac{1}{a(t)} dt.$$

Here $\mathfrak{d} = \|a'\|_{L^\infty(A, B)}$ and $\mathbf{c} > 0$ is chosen in the second case in such a way that $\max_{[A, B]} \rho < 0$.

Proof. Rewrite the equation of (4.40) as $z_t + (az_x)_x = \bar{h}$, where $\bar{h} := h - \lambda \frac{z}{b}$. Then, applying [30, Theorem 3.1], there exist two positive constants C and $s_0 > 0$, such that

$$\int_0^T \int_A^B (s\Theta(z_x)^2 + s^3\Theta^3 z^2) e^{2s\Phi} dxdt \leq C \left(\int_0^T \int_A^B \bar{h}^2 e^{2s\Phi} dxdt - (B.T.) \right), \quad (4.43)$$

for all $s \geq s_0$. Using the definition of \bar{h} , the term $\int_0^T \int_A e^{2s\Phi} \bar{h}^2 dxdt$ can be estimated in the following way:

$$\int_0^T \int_A \bar{h}^2 e^{2s\Phi} dxdt \leq 2 \int_0^T \int_A h^2 e^{2s\Phi} dxdt + 2\lambda^2 \int_0^T \int_A \frac{z^2}{b^2} e^{2s\Phi} dxdt. \quad (4.44)$$

Applying the classical Poincaré inequality to $w(t, x) := e^{s\Phi} z(t, x)$ and observing that $0 < \inf \Theta \leq \Theta \leq c\Theta^2$, one has

$$\begin{aligned} 2\lambda^2 \int_0^T \int_A \frac{z^2}{b^2} e^{2s\Phi} dxdt &= 2\lambda^2 \int_0^T \int_A \frac{w^2}{b^2} dxdt \leq 2\frac{\lambda^2}{b_0^2} C \int_0^T \int_A (w_x)^2 dxdt \\ &\leq C \int_0^T \int_A (s^2 \Theta^2 z^2 + (z_x)^2) e^{2s\Phi} dxdt \\ &\leq \int_0^T \int_A \frac{s}{2} \Theta (z_x)^2 e^{2s\Phi} dxdt + \int_0^T \int_A \frac{s^3}{2} \Theta^3 z^2 e^{2s\Phi} dxdt, \end{aligned}$$

for s large enough. Using this last inequality in (4.44), we have

$$\begin{aligned} \int_0^T \int_A \bar{h}^2 e^{2s\Phi} dxdt &\leq 2 \int_0^T \int_A e^{2s\Phi} h^2 dxdt + \int_0^T \int_A \frac{s}{2} \Theta (z_x)^2 e^{2s\Phi} dxdt \\ &\quad + \int_0^T \int_A \frac{s^3}{2} \Theta^3 z^2 e^{2s\Phi} dxdt. \end{aligned} \quad (4.45)$$

Using this inequality in (4.43), (4.41) follows immediately. \square

In order to prove Proposition 4.1, the last result that we need is the following:

Lemma 4.2. *Assume Hypotheses 4.1 and 4.2. Then there exists a positive constant C_T such that every solution $v \in \mathcal{W}$ of (4.34) satisfies*

$$\int_0^1 v^2(0, x) dx \leq C_T \int_0^T \int_\omega v^2(t, x) dxdt.$$

Proof. Multiplying the equation of (4.34) by v_t and integrating by parts over $(0, 1)$, one has

$$\begin{aligned} 0 &= \int_0^1 \left(v_t + (av_x)_x + \lambda \frac{v}{b} \right) v_t dx = \int_0^1 \left(v_t^2 + (av_x)_x v_t + \lambda \frac{v v_t}{b} \right) dx \\ &= \int_0^1 v_t^2 dx + [av_x v_t]_{x=0}^{x=1} - \int_0^1 av_x v_{tx} dx + \frac{\lambda}{2} \frac{d}{dt} \int_0^1 \frac{v^2}{b} dx \\ &= \int_0^1 v_t^2 dx - \frac{1}{2} \frac{d}{dt} \int_0^1 a(v_x)^2 dx + \frac{\lambda}{2} \frac{d}{dt} \int_0^1 \frac{v^2}{b} dx \\ &\geq -\frac{1}{2} \frac{d}{dt} \int_0^1 a(v_x)^2 dx + \frac{\lambda}{2} \frac{d}{dt} \int_0^1 \frac{v^2}{b} dx. \end{aligned}$$

Thus, the function

$$t \mapsto \int_0^1 a(v_x)^2 dx - \lambda \int_0^1 \frac{v^2}{b} dx$$

is non decreasing for all $t \in [0, T]$. In particular,

$$\begin{aligned} \int_0^1 a(v_x)^2(0, x) dx - \lambda \int_0^1 \frac{v^2(0, x)}{b(x)} dx &\leq \int_0^1 a(v_x)^2(t, x) dx - \lambda \int_0^1 \frac{v^2(t, x)}{b(x)} dx \\ &\quad (\text{by Proposition 2.2}) \\ &\leq (1 + |\lambda|C^*) \int_0^1 a(v_x)^2(t, x) dx. \end{aligned}$$

Integrating the previous inequality over $\left[\frac{T}{4}, \frac{3T}{4}\right]$, Θ being bounded therein, we find

$$\begin{aligned} &\int_0^1 a(x)(v_x)^2(0, x) dx - \lambda \int_0^1 \frac{v^2(0, x)}{b(x)} dx \\ &\leq \frac{2}{T}(1 + |\lambda|C^*) \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 a(v_x)^2 dx dt \\ &\leq C_T \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 s\Theta a(v_x)^2 e^{2s\varphi} dx dt. \end{aligned} \tag{4.46}$$

Hence, from the previous inequality and Lemma 4.1, if $\lambda < 0$

$$\int_0^1 a(v_x)^2(0, x) dx \leq \int_0^1 a(v_x)^2(0, x) dx - \lambda \int_0^1 \frac{v^2(0, x)}{b(x)} dx \leq C \int_0^T \int_{\omega} v^2 dx dt$$

for some positive constant $C > 0$.

If $\lambda > 0$, using again Lemma 4.1 and (4.46), one has

$$\int_0^1 a(v_x)^2(0, x) dx - \lambda \int_0^1 \frac{v^2(0, x)}{b(x)} dx \leq C \int_0^T \int_{\omega} v^2 dx dt. \tag{4.47}$$

Hence, by (2.9) and (4.47), we have

$$\begin{aligned} \int_0^1 a(v_x)^2(0, x) dx &\leq \lambda \int_0^1 \frac{v^2(0, x)}{b(x)} dx + C \int_0^T \int_{\omega} v^2 dx dt \\ &\leq \lambda C^* \int_0^1 a(v_x)^2(0, x) dx + C \int_0^T \int_{\omega} v^2 dx dt. \end{aligned}$$

Thus

$$(1 - \lambda C^*) \int_0^1 a(v_x)^2(0, x) dx \leq C \int_0^T \int_{\omega} v^2 dx dt,$$

for a positive constant C . In every case, there exists $C > 0$ such that

$$\int_0^1 a(v_x)^2(0, x) dx \leq C \int_0^T \int_{\omega} v^2 dx dt. \tag{4.48}$$

The Hardy- Poincaré inequality (see Proposition 2.1) and (4.48) imply that

$$\begin{aligned}
\int_0^1 \left(\frac{a}{(x-x_0)^2} \right)^{1/3} v^2(0,x) dx &\leq \int_0^1 \frac{p}{(x-x_0)^2} v^2(0,x) dx \\
&\leq C_{HP} \int_0^1 p(v_x)^2(0,x) dx \\
&\leq cC_{HP} \int_0^1 a(v_x)^2(0,x) dx \\
&\leq C \int_0^T \int_{\omega} v^2 dx dt,
\end{aligned}$$

for a positive constant C . Here $p(x) = (a(x)|x-x_0|^4)^{1/3}$ if $K_1 > \frac{4}{3}$, while $p(x) = |x-x_0|^{4/3} \max_{[0,1]} a^{1/3}$ otherwise, and c, C are obtained by Lemma 2.1.

Again by Lemma 2.1, we have

$$\left(\frac{a(x)}{(x-x_0)^2} \right)^{1/3} \geq C_3 := \min \left\{ \left(\frac{a(1)}{(1-x_0)^2} \right)^{1/3}, \left(\frac{a(0)}{x_0^2} \right)^{1/3} \right\} > 0.$$

Hence

$$C_3 \int_0^1 v(0,x)^2 dx \leq C \int_0^T \int_{\omega} v^2 dx dt$$

and the claim follows. \square

Proof of Proposition 4.1. It follows by a density argument as for the proof of [29, Proposition 4.1]. \square

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