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Global small data solutions for an evolution equation with structural damping and Hartree-type nonlinearity

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Abstract In this paper, we consider an evolution equation with structural damping and nonlocal nonlinearity of Hartree type, and we prove the existence of global small data solutions for supercritical powers.

1 Introduction

We consider a structurally damped evolution equation with a nonlocal nonlinearity

$$\begin{cases} u_{tt} + (-\Delta)^{\sigma} u + (-\Delta)^{\frac{\sigma}{2}} u_t = c\left(|x|^{-(n-\alpha)} * f(u)\right) g(u), \ t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

where $c \neq 0$, $\alpha \in (0,n)$ and

$$|f(u) - f(v)| \leq C\,|u-v|\left(|u|^{p-1} + |v|^{p-1}\right), \quad \text{for some } p \geq 1, \tag{2}$$

$$|g(u) - g(v)| \leq C\,|u-v|\left(|u|^{q-1} + |v|^{q-1}\right), \quad \text{for some } q \geq 1. \tag{3}$$

A nonlinearity as in (1) generalizes Hartree-type nonlinearities [13, 14, 15] which appear in several physical models. We address the interested reader to [1, 3, 16] and the references therein.

We stress that, with no loss of generality, we may fix

$$c = c_{n,\alpha} = \pi^{\frac{n}{2}}\, 2^{\alpha}\, \frac{\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)},$$

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in (1), so that

$$c_{n,\alpha}|x|^{-(n-\alpha)} * f = I_\alpha f$$

where I_α denotes the Riesz potential applied to f. The equation in (1) is then obtained by the following system:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^{\frac{\sigma}{2}} u_t = U\, g(u), \\ (-\Delta)^{\frac{\alpha}{2}} U = f(u)\,. \end{cases} \tag{4}$$

By the Hardy-Littlewood-Sobolev theorem, for any $f \in L^{r^*}$, with $r^* \in (1, n/\alpha)$, it holds

$$I_\alpha f \in L^r, \quad \|I_\alpha f\|_{L^r} \leq C\, \|f\|_{L^{r^*}}, \quad \frac{1}{r^*} = \frac{1}{r} + \frac{\alpha}{n}. \tag{5}$$

In this paper, we prove that global-in-time solutions to (1) exist, for sufficiently small data in a suitable space, if

$$p + q > 1 + \frac{2\sigma + \alpha}{n - \sigma}. \tag{6}$$

It remains open to determine if the existence exponent $1 + (2\sigma + \alpha)/(n - \sigma)$ in (6) is *critical,* that is, global-in-time solutions in general do not exist for subcritical powers. In [12], R. Filippucci and M. Gherghu studied the nonexistence exponent for quasilinear parabolic inequalities with a nonlinearity of type $(K * u^p)u^q$. In particular, for the parabolic m-Laplacian equation

$$u_t - \Delta_m u = (|x|^{-(n-\alpha)} * u^p) u^q,$$

where $\Delta_m u = \nabla \cdot (|\nabla u|^{m-2} \nabla u)$, with $0 < n - \alpha < m/2$, they proved the nonexistence of nonnegative nontrivial smooth solutions for

$$p + q < m - 1 + \frac{m + \alpha}{2n - \alpha}.$$

The critical exponent for structurally damped evolution equations with power nonlinearities

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\theta u_t = g(u), \ t > 0, \ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), \\ u_t(0, x) = u_1(x), \end{cases} \tag{7}$$

has been well-investigated recently, and it has been highlighted how it depends on the strength of damping. If the damping is effective, i.e., $0 < 2\theta < \sigma$, the critical exponent is $1+2\sigma/(n-2\theta)$, the same of the corresponding parabolic equation (see [6], see also [2] and the references therein), as it happens for the classical damped wave equation [18] (see also [10] for a deeper analysis of critical nonlinearities of type

$g(u)$), i.e., $\theta = 0$ and $\sigma = 1$. In the noneffective case $\theta \in (\sigma/2, \sigma]$, oscillations come into play [9, 17] in the asymptotic profile of the solution, and it has been recently shown [7] that the critical exponent becomes $1 + 2\sigma/(n - \sigma)$, the same of the undamped evolution equation [11], that is, the same in (6) when $\alpha = 0$.

At the threshold of effectiveness, i.e., $\theta = \sigma/2$, the equation in (7) has the simplest structure, since oscillations are very slow, and the critical exponent is $1+2\sigma/(n-\sigma)$ as in the case of noneffective damping. If the power nonlinearity is replaced by a nonlocal-in-time power nonlinearity as

$$g(t, u) = \int_0^t (t - s)^{-(1-\alpha)} |u(s, x)|^q \, ds,$$

with $\alpha \in (0, 1)$, then the critical exponent becomes (see [4], see also [8])

$$\max\{\tilde{q}, (1 - \alpha)^{-1}\}, \quad \tilde{q} = 1 + \frac{(2 + \alpha)\sigma}{n - (1 + \alpha)\sigma} .$$

In this paper, we show that the influence from a nonlocal-in-space nonlinearity is quite different.

Solutions in $C([0, \infty), L^{p_0})$

In our first result, for any given supercritical value $p + q$, we look for the less restrictive assumption on the regularity of initial data, such that we may construct a global-in-time solution to (1) in $C([0, \infty), L^{p_0})$, for some p_0.

Theorem 1 *Let $n \geq 1$ and $\sigma \in (0, n)$. Assume that p, q in* (2) *and* (3) *verify the following:*

$$\frac{\alpha}{n}\, p < q < \frac{n}{\alpha}\, p, \tag{8}$$

and that (6) *holds. Moreover, if $n > 2\sigma$ we also assume that*

$$p + q < 1 + \frac{2\sigma + \alpha}{n - 2\sigma}. \tag{9}$$

Fix

$$p_0 = \frac{n}{n + \alpha}\,(p + q), \quad \frac{1}{p_0^*} = \frac{1}{p_0} + \frac{\sigma}{n}.$$

Then there exists $\varepsilon > 0$ such that for any

$$(u_0, u_1) \in \mathcal{A} = \left(L^{m_0} \cap L^{p_0}\right) \times \left(L^{m_1} \cap L^{p_0^*}\right), \quad \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon, \tag{10}$$

where $1 \leq m_1 \leq m_0$ verify

$$p + q \geq 1 + \frac{2\sigma + \alpha}{(n/m_1) - \sigma}, \quad \frac{n}{\sigma}\left(\frac{1}{m_1} - \frac{1}{m_0}\right) \leq 1, \tag{11}$$

there is a unique global-in-time solution $u \in C([0,\infty), L^{m_0} \cap L^{p_0})$*. Moreover, the following long-time decay estimate holds*

$$\|u(t,\cdot)\|_{L^{p_0}} \leq C\,(1+t)^{1-\frac{n}{\sigma}\left(\frac{1}{m_1}-\frac{1}{p_0}\right)}\,\|(u_0,u_1)\|_{\mathcal{A}}, \tag{12}$$

and the following estimate holds

$$\|u(t,\cdot)\|_{L^{m_0}} \leq C\,(1+t)^{1-\frac{n}{\sigma}\left(\frac{1}{m_1}-\frac{1}{m_0}\right)}\,\|(u_0,u_1)\|_{\mathcal{A}}. \tag{13}$$

Remark 1 We stress that condition (11) follows as a consequence of (6) if $m_1 = 1$. In particular, in this case, (12) reads as

$$\|u(t,\cdot)\|_{L^{p_0}} \leq C\,(1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p_0}\right)}\,\|(u_0,u_1)\|_{\mathcal{A}}, \tag{14}$$

Moreover, if $m_0 = 1$, then (13) reads as:

$$\|u(t,\cdot)\|_{L^1} \leq C\,(1+t)\,\|(u_0,u_1)\|_{\mathcal{A}}. \tag{15}$$

Remark 2 We stress that $m_1 < p_0^*$ in (10), as a consequence of the fact that the exponent in the estimate in (12) is negative due to (11), that is,

$$\frac{1}{m_1} > \frac{1}{p_0} + \frac{\sigma}{n} = \frac{1}{p_0^*}.$$

An analogous reasoning shows that $m_0 < p_0$ in (10).

Solutions in $C([0,\infty), L^1 \cap L^\infty)$

We may remove assumptions (8) and (9) by strengthening the regularity assumption on the initial data. In particular, we may easily construct solutions in $L^1 \cap L^\infty$, proceeding as in [5].

Theorem 2 *Let* $n \geq 1$ *and* $\sigma \in (0,n)$*. Assume that* p, q *in* (2) *and* (3) *verifies* (6)*. Then there exists* $\varepsilon > 0$ *such that for any*

$$(u_0,u_1) \in \mathcal{A} = \left(L^1 \cap L^\infty\right) \times \left(L^1 \cap L^{\frac{n}{\sigma}}\right), \quad \|(u_0,u_1)\|_{\mathcal{A}} \leq \varepsilon, \tag{16}$$

there is a unique global-in-time solution $u \in C([0,\infty), L^1 \cap L^\infty)$*. Moreover, the following long-time decay estimate holds*

$$\|u(t,\cdot)\|_{L^\infty} \leq C\,(1+t)^{1-\frac{n}{\sigma}}\,\|(u_0,u_1)\|_{\mathcal{A}}, \tag{17}$$

and estimate (15) *holds.*

Is the exponent critical?

Following the same approach in this paper, one may clearly study a large class of evolution equations with Hartree-type nonlinearity $c\,\big(|x|^{-(n-\alpha)} * f(u)\big)g(u)$. For instance, the existence exponent for the heat equation with that nonlinearity is obtained by the solution of the following equation:

$$1 = \frac{n}{2}\left(1 - \frac{1}{p_0}\right)(p+q) = \frac{n}{2}(p+q-1) - \frac{\alpha}{2}.$$

That is, the existence exponent is a shifted Fujita exponent corresponding to the equation

$$n(p+q-1) = 2+\alpha.$$

We stress that a gap remains open, comparing this result with the nonexistence exponent obtained in [12]. The question naturally arising is then: "Is this shifted Fujita exponent critical?"

The case $q = p - 1$

A model case is when $f(u) = |u|^p$ and $g(u) = |u|^{p-2}u$, that is, $q = p-1$ for some $p \geq 2$. In this case, condition (8) holds if, and only if, $p > n/(n-\alpha)$. Condition (6) reads as

$$p > 1 + \frac{\sigma + \alpha/2}{n-\sigma},$$

and $p_0 = (2p-1)n/(n+\alpha)$. In particular, $p = 2$ in the classical Hartree type inequality, so that our result applies for $\alpha < 2(n-2\sigma)$ if $n < 4\sigma$, and for any $\alpha \in (0,n)$ if $n \geq 4\sigma$.

2 Proof of Theorem 1

In order to prove our results, we consider the linear problem

$$\begin{cases} u_{tt} + (-\Delta)^{\sigma} u + (-\Delta)^{\frac{\sigma}{2}} u_t = 0, \ t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x). \end{cases} \tag{18}$$

We address the reader to [6] and the references therein for the proof of the following estimates for the solution to (18):

$$\|u(t,\cdot)\|_{L^{q_2}} \leq C_0\, t^{-\frac{n}{\sigma}\left(\frac{1}{q_0}-\frac{1}{q_2}\right)} \|u_0\|_{L^{q_0}} + C_1\, t^{1-\frac{n}{\sigma}\left(\frac{1}{q_1}-\frac{1}{q_2}\right)} \|u_1\|_{L^{q_1}}, \tag{19}$$

where $q_2 \in [1,\infty]$ and $q_0, q_1 \in [1, q_2]$, with C_j independent of $t > 0$ and $\|u_j\|_{L^{q_j}}$, $j = 0, 1$.

Lemma 1 *Let p_0, m_0, m_1 as in the statement of Theorem 1. Then the solution to* (18) *verifies the long-time decay estimate* (12) *and the estimate* (13)*.*

Proof For $t \leq 1$, we apply (19) with $q_2 = q_0 = p_0$ and $q_1 = p_0^*$, so that we obtain

$$\|u(t,\cdot)\|_{L^{p_0}} \leq C_0 \, \|u_0\|_{L^{p_0}} + C_1 \, \|u_1\|_{L^{p_0^*}},$$

whereas, for $t \geq 1$, we apply (19) with $q_2 = p_0$, $q_0 = m_0$ and $q_1 = m_1$, so that we obtain

$$\|u(t,\cdot)\|_{L^{p_0}} \leq t^{1-\frac{n}{\sigma}\left(\frac{1}{m_1}-\frac{1}{p_0}\right)} \big(C_0 \, \|u_0\|_{L^{m_0}} + C_1 \, \|u_1\|_{L^{m_1}}\big),$$

thanks to the second half of (11). Summing the previous two estimates, we derive (12). On the other hand, if we apply (19) with $q_2 = q_0 = m_0$ and $q_1 = m_1$, we obtain

$$\|u(t,\cdot)\|_{L^{m_0}} \leq C_0 \, \|u_0\|_{L^{m_0}} + C_1 \, t^{1-\frac{n}{\sigma}\left(\frac{1}{m_1}-\frac{1}{m_0}\right)} \, \|u_1\|_{L^{m_1}},$$

from which we get (13), thanks to the second half of (11). □

The interplay between integrability and desired decay

In Lemma 1, the assumption of $u_0 \in L^{p_0}$ and $u_1 \in L^{p_0^*}$ comes into play to guarantee that we may find a solution with $u(t,\cdot) \in L^{p_0}$ for any $t \geq 0$, whereas the assumption $u_0 \in L^{m_0}$ and $u_1 \in L^{m_1}$ provides the desired decay rate as $t \to \infty$ for the solution. The assumption that $u(t,\cdot) \in L^{p_0}$ and that $\|u(t,\cdot)\|_{L^{p_0}}$ has a certain decay rate, will be crucial to treat the nonlinear problem. On the other hand, as a bonus consequence, the previous assumption that $u_0 \in L^{m_0}$ and $u_1 \in L^{m_1}$ guarantees "for free" that also $u(t,\cdot) \in L^{m_0}$ for any $t \geq 0$.

We define the solution space

$$X = u \in C([0,\infty), L^{m_0} \cap L^{p_0}),$$

equipped with the norm

$$\|u\|_X = \sup_{t\in[0,\infty)} \Big((1+t)^{-1+\frac{n}{\sigma}\left(\frac{1}{m_1}-\frac{1}{p_0}\right)} \|u(t,\cdot)\|_{L^{p_0}} + (1+t)^{-1+\frac{n}{\sigma}\left(\frac{1}{m_1}-\frac{1}{m_0}\right)} \|u(t,\cdot)\|_{L^{m_0}} \Big).$$

In particular, for any $u \in X$, the following estimate holds

$$\|u(t,\cdot)\|_{L^{p_0}} \leq (1+t)^{1-\frac{n}{\sigma}\left(\frac{1}{m_1}-\frac{1}{p_0}\right)} \|u\|_X. \tag{20}$$

As a consequence of Lemma 1, the solution u^{lin} to (18) is in X and

$$\|u^{\mathrm{lin}}\|_X \leq C_1 \, \|(u_0, u_1)\|_{\mathcal{A}}. \tag{21}$$

We now consider the operator

$$N : X \to X, \qquad Nu = \int_0^t K(t-s,\cdot)(I_\alpha f(u(s,\cdot)))g(u(s,\cdot))\, ds,$$

where $K(t,\cdot)$ is the fundamental solution to (18), that is, the solution with $u_0 = 0$ and $u_1 = \delta$.

By Duhamel's principle, a function $u \in \mathcal{C}([0,\infty), L^{m_0} \cap L^{p_0})$ is the solution to (1) if, and only if,

$$u(t,x) = u^{\mathrm{lin}}(t,x) + Nu(t,x), \quad \text{for any } t \geq 0 \text{ and for a.e. } x, \tag{22}$$

where u^{lin} is the solution to (18).

Lemma 2 *Let $u, v \in X$. Then*

$$\|Nu - Nv\|_X \leq C \, \|u - v\|_X \, (\|u\|_X^{p+q-1} + \|v\|_X^{p+q-1}). \tag{23}$$

Proof Let $r \in (n/(n-\alpha), \infty)$ and let $r' = r/(r-1)$, its Hölder conjugate. Also, $p' = p/(p-1)$ and $q' = q/(q-1)$. Then, using Hölder inequality, (2), (3) and (5), we may estimate the L^1 norm of

$$\begin{aligned} h(u,v) &= (I_\alpha f(u))\, g(u) - (I_\alpha f(v))\, g(v) \\ &= (I_\alpha (f(u) - f(v))\, g(u) + (I_\alpha f(v))\, (g(u) - g(v)) \end{aligned}$$

as follows:

$$\begin{aligned} \|h(u,v)\|_{L^1} &\leq \|I_\alpha(f(u) - f(v))\|_{L^r} \, \|g(u)\|_{L^{r'}} \\ &\quad + \|I_\alpha f(v)\|_{L^r} \, \|g(u) - g(v)\|_{L^{r'}} \\ &\leq C_1 \, \|f(u) - f(v)\|_{L^{r^*}} \, \|g(u)\|_{L^{r'}} \\ &\quad + C_1 \, \|f(v)\|_{L^{r^*}} \, \|g(u) - g(v)\|_{L^{r'}} \\ &\leq C_2 \, \|u - v\|_{L^{r^* p}} \, \||u|^{p-1} + |v|^{p-1}\|_{L^{r^* p'}} \, \||u|^q\|_{L^{r'}} \\ &\quad + C_2 \, \||v|^p\|_{L^{r^*}} \, \|u - v\|_{L^{r' q}} \, \|(|u|^{q-1} + |v|^{q-1})\|_{L^{r' q'}} \\ &\leq C_3 \, \|u - v\|_{L^{r^* p}} \, \big(\|u\|_{L^{r^* p}}^{p-1} + \|v\|_{L^{r^* p}}^{p-1}\big) \, \|u\|_{L^{r' q}}^q \\ &\quad + C_3 \, \|v\|_{L^{r^* p}}^p \, \|u - v\|_{L^{r' q}} \, \big(\|u\|_{L^{r' q}}^{q-1} + \|v\|_{L^{r' q}}^{q-1}\big). \end{aligned}$$

We fix

$$r = \frac{n}{np - \alpha q}\, (p+q),$$

so that

$$r^* p = r' q = p_0.$$

We notice that $r \in (n/(n-\alpha), \infty)$, thanks to (8).

Now, using that $u, v \in X$, by (20), replacing $p_0 = n(p+q)/(n+\alpha)$, we obtain the estimate

$$\begin{aligned}
&\|h(u,v)(t,\cdot)\|_{L^1} \\
&\qquad \leq C\,\|u-v\|_X \left(\|u\|_X^{p+q-1} + \|v\|_X^{p+q-1}\right)(1+t)^{p+q-\frac{n(p+q)}{\sigma}\left(\frac{1}{m_1}-\frac{1}{p_0}\right)} \\
&\qquad = C\,\|u-v\|_X \left(\|u\|_X^{p+q-1} + \|v\|_X^{p+q-1}\right)(1+t)^{(p+q)\left(1-\frac{n}{\sigma m_1}\right)+\frac{n+\alpha}{\sigma}},
\end{aligned}$$

for any $t \geq 0$. Using (19) with $u_0 = 0$, $q_2 = p_0$ and $q_1 = 1$, we get

$$\begin{aligned}
\|(Nu - Nv)(t,\cdot)\|_{L^{p_0}} &\leq C \int_0^t (t-s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p_0}\right)} \|h(u,v)(s,\cdot)\|_{L^1}\, ds \\
&\leq C\,\|u-v\|_X \left(\|u\|_X^{p+q-1} + \|v\|_X^{p+q-1}\right) I(t),
\end{aligned}$$

where

$$I(t) = \int_0^t (t-s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p_0}\right)} (1+s)^{(p+q)\left(1-\frac{n}{\sigma m_1}\right)+\frac{n+\alpha}{\sigma}}\, ds\,.$$

We notice that, thanks to (9),

$$1 - \frac{n}{\sigma}\left(1 - \frac{1}{p_0}\right) > -1.$$

We now distinguish two cases. If $m_1 = 1$, then, using (6) we may estimate

$$(p+q)\left(1 - \frac{n}{\sigma}\right) + \frac{n+\alpha}{\sigma} < -1.$$

Therefore (see, for instance, [7, Lemma 3.1]), we get

$$I(t) \approx (1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p_0}\right)}\,.$$

If $m_1 > 1$, thanks to the first half of (11), we may estimate

$$\begin{aligned}
(p+q)\left(1 - \frac{n}{\sigma m_1}\right) + \frac{n+\alpha}{\sigma} &\leq -\frac{n + \sigma m_1 + \alpha m_1}{\sigma m_1} + \frac{n+\alpha}{\sigma} \\
&= \frac{n}{\sigma}\left(1 - \frac{1}{m_1}\right) - 1.
\end{aligned}$$

Therefore (see, for instance, [7, Lemma 3.1]), we get

$$I(t) \leq C(1+t)^{1-\frac{n}{\sigma}\left(\frac{1}{m_1}-\frac{1}{p_0}\right)}\,.$$

We proceed similarly to estimate $\|(Nu - Nv)(t,\cdot)\|_{L^{m_0}}$, and we conclude the proof. □

We may now conclude the proof of Theorem 1.

Proof (Theorem 1) We now define

$$R = 2C_1 \, \|(u_0, u_1)\|_{\mathcal{A}},$$

where C_1 is as in (21). For sufficiently small data, $2CR^{p+q-1} \leq 1/2$, where C is as in (23). Then, by (21) it follows that the operator $u^{\rm lin}(t,x) + N$ maps the ball $B_R = \{u : \|u\|_X \leq R\}$ in itself. Due to (23), it is a contraction. Therefore, there is a unique fixed point for $u^{\rm lin}(t,x)+F$ in B_R, that is, a unique solution to (22). Moreover, $\|u\|_X \leq 2C_1 \, \|(u_0, u_1)\|_{\mathcal{A}}$, that is, we get (12) and (13). This concludes the proof. □

3 Proof of Theorem 2

To prove Theorem 2, we follow the proof of Theorem 1, with some modifications.

Lemma 3 *Let m_0, m_1 as in the statement of Theorem 1. Then the solution to* (18) *verifies the long-time decay estimate* (17) *and the estimate* (15).

Proof The proof is as the proof of Lemma 1, formally setting $p_0 = \infty$, and with $m_0 = m_1 = 1$. □

We define the solution space

$$X = u \in C([0,\infty), L^1 \cap L^\infty),$$

equipped with the norm

$$\|u\|_X = \sup_{t\in[0,\infty)} \left((1+t)^{-1+\frac{n}{\sigma}} \, \|u(t,\cdot)\|_{L^\infty} + (1+t)^{-1} \, \|u(t,\cdot)\|_{L^1} \right).$$

In particular, by interpolation, for any $u \in X$, the following estimate holds

$$\|u(t,\cdot)\|_{L^m} \leq (1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{m}\right)} \, \|u\|_X, \quad \forall m \in [1,\infty]. \tag{24}$$

As a consequence of Lemma 3, the solution $u^{\rm lin}$ to (18) is in X and (21) holds. We now look for $u \in C([0,\infty), L^1 \cap L^\infty)$ verifying (22).

The proof of Lemma 2 is now modified.

Proof (Lemma 2) Letting $r \in (n/(n-\alpha), \infty)$, $r' = r/(r-1)$, $p' = p/(p-1)$ and $q' = q/(q-1)$, and proceeding as in the proof in Section 2, we get

$$\begin{aligned} \|h(u,v)\|_{L^1} \leq C_3 \, \|u-v\|_{L^{r^*p}} \left(\|u\|_{L^{r^*p}}^{p-1} + \|v\|_{L^{r^*p}}^{p-1} \right) \|u\|_{L^{r'q}}^{q} \\ + C_3 \, \|v\|_{L^{r^*p}}^{p} \, \|u-v\|_{L^{r'q}} \left(\|u\|_{L^{r'q}}^{q-1} + \|v\|_{L^{r'q}}^{q-1} \right). \end{aligned}$$

Since we did not assume (8), we cannot fix

$$r = \frac{n}{np - \alpha q} \, (p+q),$$

in general, as we did in the proof in Section 2. However, thanks to (24) it is now not important to fix a specific value for r, and we may choose any $r \in (n/(n-\alpha), \infty)$.

Indeed, independently on the choice of r, we obtain the estimate

$$\begin{aligned}&\|h(u,v)(t,\cdot)\|_{L^1}\\&\qquad\leq C\,\|u-v\|_X\left(\|u\|_X^{p+q-1}+\|v\|_X^{p+q-1}\right)(1+t)^{(p+q)\left(1-\frac{n}{\sigma}\right)+\frac{n+\alpha}{\sigma}},\end{aligned}$$

for any $t\geq 0$. Similarly, we may derive

$$\begin{aligned}&\|h(u,v)(t,\cdot)\|_{L^{\frac{n}{\sigma}}}\\&\qquad\leq C\,\|u-v\|_X\left(\|u\|_X^{p+q-1}+\|v\|_X^{p+q-1}\right)(1+t)^{(p+q)\left(1-\frac{n}{\sigma}\right)+\frac{\alpha}{\sigma}}.\end{aligned}$$

Using (19) with $u_0=0$, $q_2=\infty$ and $q_1=1$ if $s\in[0,t/2]$ and $q_1=n/\sigma$ if $s\in[t/2,t]$, we get

$$\begin{aligned}\|(Nu-Nv)(t,\cdot)\|_{L^\infty}&\leq C\int_0^{t/2}(t-s)^{1-\frac{n}{\sigma}}\,\|h(u,v)(s,\cdot)\|_{L^1}\,ds\\&\quad+C\int_{t/2}^{t}\|h(u,v)(s,\cdot)\|_{L^{\frac{n}{\sigma}}}\,ds\\&\leq C\,\|u-v\|_X\left(\|u\|_X^{p+q-1}+\|v\|_X^{p+q-1}\right)\big(I_1(t)+I_2(t)\big),\end{aligned}$$

where

$$\begin{aligned}I_1(t)&=\int_0^{t/2}(t-s)^{1-\frac{n}{\sigma}}\,(1+s)^{(p+q)\left(1-\frac{n}{\sigma}\right)+\frac{n+\alpha}{\sigma}}\,ds,\\I_2(t)&=\int_{t/2}^{t}(1+s)^{(p+q)\left(1-\frac{n}{\sigma}\right)+\frac{\alpha}{\sigma}}\,ds\,.\end{aligned}$$

Thanks to (6), that is,

$$(p+q)\left(1-\frac{n}{\sigma}\right)+\frac{n+\alpha}{\sigma}<-1,$$

we may estimate:

$$\begin{aligned}I_1(t)&\leq C\,(1+t)^{1-\frac{n}{\sigma}}\int_0^{t/2}(1+s)^{(p+q)\left(1-\frac{n}{\sigma}\right)+\frac{n+\alpha}{\sigma}}\,ds\\&\leq C'\,(1+t)^{1-\frac{n}{\sigma}},\\I_2(t)&\leq C\,(1+t)^{1+(p+q)\left(1-\frac{n}{\sigma}\right)+\frac{\alpha}{\sigma}}\leq C\,(1+t)^{1-\frac{n}{\sigma}}\,.\end{aligned}$$

We proceed similarly to estimate $\|(Nu-Nv)(t,\cdot)\|_{L^1}$, and we conclude the proof.□

The conclusion of the proof of Theorem 2 is as the conclusion of the proof of Theorem 1.

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