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Radial symmetry for a quasilinear elliptic equation with a critical Sobolev growth and Hardy potential

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A B S T R A C T

We consider weak positive solutions to the critical p-Laplace equation with Hardy potential in $\mathbb{R}^N$

$$-\Delta_p u - \frac{\gamma}{|x|^p} u^{p-1} = u^{p^*-1}$$

where $1 < p < N$, $0 \leqslant \gamma < \left(\frac{N-p}{p}\right)^p$ and $p^* = \frac{Np}{N-p}$.
The main result is to show that all the solutions in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ are radial and radially decreasing about the origin.

R É S U M É

Nous considérons des solutions positives faibles á l'équation critique p -Laplace avec un potentiel Hardy dans $\mathbb{R}^N$

$$-\Delta_p u - \frac{\gamma}{|x|^p} u^{p-1} = u^{p^*-1}$$

oú $1 < p < N$, $0 \leqslant \gamma < \left(\frac{N-p}{p}\right)^p$ et $p^* = \frac{Np}{N-p}$.
Le principal résultat est de montrer que toutes les solutions $\mathcal{D}^{1,p}(\mathbb{R}^N)$ sont radiaux et radialement décroissants autour de l'origine.

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1. Introduction and statement of the main result

We study the doubly critical problem

$$\begin{cases} -\Delta_p u - \dfrac{\gamma}{|x|^p} u^{p-1} = u^{p^*-1} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \end{cases} \tag{1.1}$$

where $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p < N$, $0 \leqslant \gamma < \gamma_p := \left(\frac{N-p}{p}\right)^p$ and $p^* := \frac{Np}{N-p}$ is the critical exponent for the Sobolev embedding. Here $\mathcal{D}^{1,p}(\mathbb{R}^N)$ denotes the completition of $C_0^\infty(\mathbb{R}^N)$, the space of smooth functions with compact support, with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{\frac{1}{p}}.$$

By standard regularity theory, see [12,23], it follows that solutions to (1.1) are of class $C^{1,\alpha}$ far from the origin.

We address the study of the classification of positive solutions to (1.1). As we shall discuss later on, this is a crucial issue since problem (1.1) naturally appears in the study of p-Hardy-Sobolev inequalities as well as it appears as a limiting problem in many applications. Our main effort is to show that all the positive solutions to (1.1) are radial (and radially decreasing) about the origin. Once the radial symmetry of the solution is proved it is easy to derive the associated ordinary differential equation fulfilled by the solution $u = u(r)$. The classification result reduces therefore to an ODE analysis that has been already carried out in [1] where the radial symmetry of the solutions was an assumption.

Let us start discussing the simpler case $\gamma = 0$. In this case the problem reduces to the following critical one

$$\begin{cases} -\Delta_p u = u^{p^*-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N). \end{cases} \tag{1.2}$$

For such a problem a huge literature is available and the classification of positive weak solutions of (1.2) is well understood. Indeed, for $\delta > 0$ and $x_0 \in \mathbb{R}^N$, an explicit family of solutions to (1.2) is given by

$$V_{\delta,x_0}(x) := \left(\frac{\delta^{\frac{1}{p-1}}\alpha_{N,p}}{\delta^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}, \tag{1.3}$$

where $\alpha_{N,p} := N^{\frac{1}{p}}\left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}$. The family of functions given by (1.3) are the minimizers to

$$\mathcal{S}_p := \inf_{\substack{\varphi\in\mathcal{D}^{1,p}(\mathbb{R}^N) \\ \varphi\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla\varphi|^p}{\left(\displaystyle\int_{\mathbb{R}^N} \varphi^{p*}\right)^{\frac{p}{p*}}} \tag{1.4}$$

and the classification of the minimizers (see [21]) follows via symmetrization arguments. Note that such a technique can be applied in the same way both in the semilinear case $p = 2$ and in the quasilinear case $1 < p < \infty$.

Furthermore, if we restrict the attention to the class of *radial* solutions, then the analysis carried out in [14] shows that all the *regular radial* solutions to (1.2) are given by (1.3).

For $p = 2$ all the solutions to the equation are classified by (1.3) as a consequence of the results in [2] where the Kelvin transform is strongly exploited. A Kelvin type transformation is not applicable for the quasilinear case and this fact causes that a different proof is needed. When no a priori assumption are imposed, the classification of all the positive solutions to (1.2) (showing that all the solutions to (1.2) are given by (1.3)) has been in fact an open and challenging problem recently solved in [7,19,24] (see also [8,9]). The techniques used are mainly based on a fine asymptotic analysis at infinity and refined versions of the *moving plane procedure*, see [13,20].

Let us now turn to the case $0 < \gamma < \gamma_p$ but in the case $p = 2$ so that γ_2 is the best constant in the Hardy-Sobolev inequality for $p = 2$. For

$$\mathcal{S}_{2,\gamma} = \inf_{\substack{\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ \varphi \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} \left(|\nabla \varphi|^2 - \frac{\gamma}{|x|^2}\varphi^2 \right)}{\left(\displaystyle\int_{\mathbb{R}^N} |\varphi|^{2^*} \right)^{\frac{2}{2^*}}},$$

it is known that $\mathcal{S}_{2,\gamma}$ is attained and extremals for $S_{2,\gamma}$ have the form (up to a multiplicative constant)

$$U_\delta(x) = \delta^{-\frac{N-2}{2}} U\left(\frac{x}{\delta}\right) = \frac{\alpha_N \delta^{\Gamma}}{|x|^{\beta_-}(\delta^{\frac{4\Gamma}{N-2}} + |x|^{\frac{4\Gamma}{N-2}})^{\frac{N-2}{2}}}, \quad \delta > 0, \tag{1.5}$$

where

$$U(x) = \frac{\alpha_N}{|x|^{\beta^-}(1 + |x|^{\frac{4\Gamma}{N-2}})^{\frac{N-2}{2}}} = \frac{\alpha_N}{\left(|x|^{\frac{2}{N-2}\beta^-} + |x|^{\frac{2}{N-2}\beta^+}\right)^{\frac{N-2}{2}}}$$

with

$$\Gamma = \sqrt{\frac{(N-2)^2}{4} - \gamma}, \quad \beta_\pm = \frac{N-2}{2} \pm \Gamma, \quad \alpha_N = \left[\frac{4\Gamma^2 N}{N-2}\right]^{\frac{N-2}{4}},$$

see [3,4,22]. Moreover (1.5) gives all the solutions of the problem (1.1) for $p = 2$ and $\gamma \in (0, \gamma_2)$ and this has been proved in the celebrated paper [22]. In the case $p = 2$ it is also known that when $\gamma < 0$ then $\mathcal{S}_{2,\gamma}$ is not attained even if (1.5) are still solutions of the problem.

Here we are concerned with the quasilinear doubly critical case $1 < p < N$ and $\gamma \in (0, \gamma_p)$. It is worth recalling that in [1] the authors considered minimization problem:

$$\mathcal{S}_{p,\gamma} = \inf_{\substack{\varphi \in \mathcal{D}^{1,p}(\mathbb{R}^N) \\ \varphi \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} \left(|\nabla \varphi|^p - \frac{\gamma}{|x|^p}\varphi^p \right)}{\left(\displaystyle\int_{\mathbb{R}^N} |\varphi|^{p^*} \right)^{\frac{p}{p^*}}}. \tag{1.6}$$

It follows that $0 < \mathcal{S}_{p,\gamma} < \mathcal{S}_p$ where $\mathcal{S}_p$ is defined in (1.4) and $\mathcal{S}_{p,\gamma}$ is attained by a function $u_0(x)$ which is not explicit. It has been proved in [1] that all minimizers of (1.6) are radial. Also uniqueness up to scaling of the *radial* solutions as well as the asymptotic behavior are proved showing in particular that, given a *radial* solution $u = u(r)$ to (1.1), then

$$\lim_{r\to 0} r^{\gamma_1} u(r) = C_1, \qquad \lim_{r\to +\infty} r^{\gamma_2} u(r) = C_2$$

and

$$\lim_{r\to 0} r^{\gamma_1+1} |u'(r)| = C_1\gamma_1, \qquad \lim_{r\to +\infty} r^{\gamma_2+1} |u'(r)| = C_2\gamma_2,$$

for some positive constants C_1, C_2. Here and hereafter $\gamma_1, \gamma_2 \in [0, +\infty)$, $\gamma_1 < \gamma_2$ are defined as the two roots of the equation

$$\mu^{p-2}\left[(p-1)\mu^2 - (N-p)\mu\right] + \gamma = 0. \tag{1.7}$$

We remark (for later use) that

$$0 \leqslant \gamma_1 < \frac{N-p}{p} < \gamma_2 \leqslant \frac{N-p}{p-1}.$$

Note that when $p = 2$ then $\gamma_1 = \beta_-$ and $\gamma_2 = \beta_+$. Instead, when $p \neq 2$ but $\gamma = 0$ then $\gamma_1 = 0$ and $\gamma_2 = \frac{N-p}{p-1}$. Moreover in [25,26] the author extends the results on the asymptotic behavior proved for radial solutions in [1] to all weak positive solutions of (1.1).

We shall prove here that actually all positive solutions to (1.1) are *radially symmetric* thus allowing to deduce that the characterization of the solutions described here above do apply to all positive solutions. In particular, as a consequence of our result, we deduce uniqueness up to scaling of the positive solutions as well as their asymptotic behavior at the origin and at infinity.

Our main result is the following:

Theorem 1.1. *Assume $\gamma \in (0, \gamma_p)$ and let u be an energy solution to (1.1). Then u is radial and radially decreasing with respect to the origin.*

All the proofs of the classification results described above are based on the use of the *moving plane method*. When $p \neq 2$ this is completely not trivial because of the nonlinear degenerate nature of the operator. In our case, when trying to adapt the techniques developed in [9,10,19], an obstruction occurs due to the homogeneity of the Hardy potential. In particular this fact is related to the nonlinear nature of the operator that also obstructs the application of the techniques introduced in [7,22]. In fact, to face this fact, we exploit a different test function technique that, on the other hand, introduces several difficulties as the reader shall see. Let us also stress that, for the absence of the Kelvin transformation, an analysis on the behavior at infinity is needed. We will in fact exploit the results in [25,26] and in particular our Theorem 3.3.

1.1. Notations

Throughout the paper, we denote by Ω^c the complement of a domain $\Omega \subset \mathbb{R}^N$ in $\mathbb{R}^N$, by

$$C_0^k(\mathbb{R}^N) = \left\{u \in C^k(\mathbb{R}^N) \,:\, u(x) \to 0 \text{ as } |x| \to +\infty\right\},$$

and by $B_R(x_0)$ the ball of radius R centered at $x_0 \in \mathbb{R}^N$.
Moreover χ_Ω is the characteristic function of the set Ω, $(v-w)^+ := \max\{v-w, 0\}$ and $(v-w)^- :=$

$\min\{v-w,0\}$.
Finally we underline that we will denote by C, C_i, c_i several constants whose value may change from line to line and, sometimes, on the same line. However these values will be not relevant in the proofs.

We remark that the potential $|x|^{-p}$ is related to the Hardy-Sobolev inequality. More precisely, for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, one has

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \leqslant \frac{1}{\gamma_p} \int_{\mathbb{R}^N} |\nabla u|^p, \tag{1.8}$$

where γ_p^{-1} is optimal and never achieved.
As a consequence of a Pohozaev type identity, one can see that problem (1.1) does not have non-trivial solutions in any bounded starshaped domain with respect to the origin (Lemma 3.7 in [15]).

2. Preliminaries and known technical results

In this section we first recall useful results such as the strong comparison principle, a weighted Hardy-Sobolev inequality and decay estimates.

Let us start the discussion on the *strong comparison principles* recalling the following

Theorem 2.1 *(Theorem 1.4 of [11]). Let $u, v \in C^1(\bar{\Omega})$ where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $\frac{2N+2}{N+2} < p < 2$ or $p > 2$. Suppose that either u or v is a weak solution of*

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

with $f : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is a continuous function which is positive and of class C^1 in $\Omega \times (0,\infty)$. Assume that

$$-\Delta_p u + \Lambda u \leqslant -\Delta_p v + \Lambda v \quad \text{and} \quad u \leqslant v \quad \text{in } \Omega,$$

where $\Lambda \in \mathbb{R}$. Then $u \equiv v$ in Ω unless $u < v$ in Ω.

Actually the assumption that u or v fulfill the zero Dirichlet boundary datum can be removed and local versions of Theorem 2.1 are available, see [17,18]. On the contrary there are no results removing the assumption $p > \frac{2N+2}{N+2}$. Therefore in some cases we could prefer to exploit also the following result:

Theorem 2.2 *(Theorem 1.4 of [5]). Suppose Ω is a domain in $\mathbb{R}^N$ and let $u, v \in C^1(\Omega)$ weakly satisfy*

$$-\Delta_p u + \Lambda u \leqslant -\Delta_p v + \Lambda v \quad \text{and} \quad u \leqslant v \quad \text{in } \Omega,$$

$1 < p < \infty$ and denote by $Z_v^u := \{x \in \Omega : \nabla u(x) = \nabla v(x) = 0\}$. Then if there exists $x_0 \in \Omega \setminus Z_v^u$ with $u(x_0) = v(x_0)$, then $u \equiv v$ in the connected component of $\Omega \setminus Z_v^u$ containing x_0. The same result holds if more generally

$$-\Delta_p u - f(u) \leqslant -\Delta_p v - f(v) \quad \text{and} \quad u \leqslant v \quad \text{in } \Omega,$$

with $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous.

In the spirit of the moving plane procedure we shall exploit the *strong comparison principle* together with the *weak comparison principle* (that actually will be included in the proofs and we refer the readers to [10]) and improved Hardy inequalities proved in [16]. For convenience we summarize the following

Theorem 2.3 *(Proposition 1.1 of [16]). Let $r \geqslant 1$, $\tau > 0$, $\alpha, \gamma \in \mathbb{R}$ such that*

$$\frac{1}{\tau} + \frac{\gamma}{N} = \frac{1}{r} + \frac{\alpha - 1}{N},$$

and with

$$0 \leqslant \alpha - \gamma \leqslant 1.$$

Let $u \in C_0^1(\mathbb{R}^N \setminus \{0\})$ and let $\frac{1}{\tau} + \frac{\gamma}{N} < 0$ then it holds

$$\left(\int_{\mathbb{R}^N} |x|^{\gamma\tau} |u|^\tau \right)^{\frac{1}{\tau}} \leqslant C \left(\int_{\mathbb{R}^N} |x|^{r\alpha} |\nabla u|^r \right)^{\frac{1}{r}}$$

where C is a positive constant independent of u.

Remark 2.4. In Theorem 2.3 it is assumed that $u \in C_0^1(\mathbb{R}^N \setminus \{0\})$. Actually it is clear from the proof, and via density arguments, that the same result applies if u is defined in exterior domains and has the right decay properties at infinity.

To exploit Theorem 2.3 for weak positive solutions to problem (1.1) we need to know the asymptotic behavior of the solution at infinity. Let us start recalling some results from [25,26].

Theorem 2.5. *Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak positive solution to equation (1.1). Then there exist positive constants C, c depending on N, p, γ and the solution u such that*

$$c|x|^{-\gamma_1} \leqslant u(x) \leqslant C|x|^{-\gamma_1} \qquad \text{for } |x| < R_0, \tag{2.2}$$

and

$$c|x|^{-\gamma_2} \leqslant u(x) \leqslant C|x|^{-\gamma_2} \qquad \text{for } |x| > R_1. \tag{2.3}$$

Moreover

$$|\nabla u(x)| \leqslant c|x|^{-(\gamma_1+1)} \qquad \text{for } |x| < R_0, \tag{2.4}$$

and

$$|\nabla u(x)| \leqslant c|x|^{-(\gamma_2+1)} \qquad \text{for } |x| > R_1. \tag{2.5}$$

Here γ_1, γ_2 are roots of (1.7) and such that

$$0 \leqslant \gamma_1 < \frac{N-p}{p} < \gamma_2 \leqslant \frac{N-p}{p-1},$$

while $0 < R_0 < 1 < R_1$ are constants depending on N, p, γ and the solution u.

Finally, we recall the following regularity result for solutions of (1.1).

Theorem 2.6 *([1,12,23]). Let u be any solution of* (1.1)*, then $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{0\})$ with $0 < \alpha < 1$.*

3. Asymptotic estimates

Here we shall prove some new gradient estimates that we will use in the next section in order to apply the moving plane method. The moving plane procedure is strongly related to the use of suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to the use of Hardy type inequalities that involves the classical radial weights. Since our problem has a natural associated weight $|\nabla u|^{p-2}$, we will need to relate the weight $|\nabla u|^{p-2}$ with the weights appearing in Theorem 2.3. To do this, especially for the hardest case $p > 2$, a further information is required, namely estimates from below on the modulus of the gradient of the solution. This is what we prove in this section starting from the following:

Lemma 3.1. *Let u, v be positive and C^1-functions in a neighborhood of some point $x_0 \in \mathbb{R}^N$. Then it holds*

$$\begin{aligned} &|\nabla u|^{p-2}\nabla u \cdot \nabla \left(u - \frac{v^p}{u^p} u \right) + |\nabla v|^{p-2}\nabla v \cdot \nabla \left(v - \frac{u^p}{v^p} v \right) \\ &\geqslant C_p \min\{v^p, u^p\} \left(|\nabla \log u| + |\nabla \log v| \right)^{p-2} |\nabla \log u - \nabla \log v|^2, \end{aligned} \tag{3.1}$$

near x_0 for some constant C_p depending only on p.

Proof. The estimate (3.1) for $1 < p < 2$ can be found in Lemma 3.1 of [25]. Then we just need to prove (3.1) for $p \geqslant 2$.
By making some simple computations we find that

$$\begin{aligned} T :&= |\nabla u|^{p-2}\nabla u \cdot \nabla \left(u - \frac{v^p}{u^p} u \right) + |\nabla v|^{p-2}\nabla v \cdot \nabla \left(v - \frac{u^p}{v^p} v \right) \\ &= |\nabla u|^p + |\nabla v|^p - v^p \underbrace{\left(|\nabla \log u|^p + p|\nabla \log u|^{p-2}\nabla \log u \cdot (\nabla \log v - \nabla \log u) \right)}_{(I)} \\ &\quad - u^p \underbrace{\left(|\nabla \log v|^p + p|\nabla \log v|^{p-2}\nabla \log v \cdot (\nabla \log u - \nabla \log v) \right)}_{(II)}. \end{aligned} \tag{3.2}$$

Now let $f(t) = |a + t(b - a)|^p$ for $a, b \in \mathbb{R}^N$ then one has

$$f(1) = f(0) + f'(0) + \int_0^1 (1 - t) f''(t),$$

which gives (recall that $p \geqslant 2$)

$$\begin{aligned} |b|^p &= |a|^p + p|a|^{p-2} a \cdot (b - a) \\ &\quad + p(p-2) \int_0^1 (1 - t)|a + t(b - a)|^{p-4} \left((a + t(b - a)) \cdot (b - a) \right)^2 dt \\ &\quad + p \int_0^1 (1 - t)|a + t(b - a)|^{p-2} |b - a|^2 \, dt \end{aligned} \tag{3.3}$$

$$\geqslant |a|^p + p|a|^{p-2}a\cdot(b-a) + \int_0^1 (1-t)p|a+t(b-a)|^{p-2}|b-a|^2\,dt.$$

We apply (3.3) to (I) with $a = \nabla\log u$ and $b = \nabla\log v$ and to (II) with $a = \nabla\log v$ and $b = \nabla\log u$. Hence we get

$$\begin{aligned}
T &\geqslant v^p\left[\int_0^1 (1-t)p\,|\nabla\log u + t(\nabla\log v - \nabla\log u)|^{p-2}\,|\nabla\log u - \nabla\log v|^2\,dt\right]\\
&+ u^p\left[\int_0^1 (1-t)p\,|\nabla\log v + t(\nabla\log u - \nabla\log v)|^{p-2}\,|\nabla\log u - \nabla\log v|^2\,dt\right]\\
&\geqslant \frac{3}{4}pv^p|\nabla\log u - \nabla\log v|^2\left[\int_0^{\frac14} |\nabla\log u + t(\nabla\log v - \nabla\log u)|^{p-2}\,dt\right]\\
&+ \frac{3}{4}pu^p|\nabla\log u - \nabla\log v|^2\left[\int_0^{\frac14} |\nabla\log v + t(\nabla\log u - \nabla\log v)|^{p-2}\,dt\right].
\end{aligned} \tag{3.4}$$

Now suppose that $|\nabla\log u| \geqslant |\nabla\log v|$. In order to estimate the first term on the right hand side of (3.4) we distinguish two cases.
First of all let $|\nabla\log v - \nabla\log u| \leqslant \frac12|\nabla\log u|$ then (recall $0 < t < 1$)

$$\begin{aligned}
|\nabla\log u + t(\nabla\log v - \nabla\log u)| &\geqslant |\nabla\log u| - |\nabla\log v - \nabla\log u|\\
&\geqslant \frac12|\nabla\log u| \geqslant \frac14\left(|\nabla\log u| + |\nabla\log v|\right),
\end{aligned}$$

namely

$$|\nabla\log u + t(\nabla\log v - \nabla\log u)|^{p-2} \geqslant \left(\frac14\right)^{p-2}\left(|\nabla\log u| + |\nabla\log v|\right)^{p-2}.$$

Otherwise if $|\nabla\log v - \nabla\log u| > \frac12|\nabla\log u|$ then we let

$$t_0 := \frac{|\nabla\log u|}{|\nabla\log v - \nabla\log u|} \in (0,2).$$

Hence

$$\begin{aligned}
|\nabla\log u + t(\nabla\log v - \nabla\log u)| &\geqslant \big||\nabla\log u| - t|\nabla\log u - \nabla\log v|\big|\\
&= \big|t_0|\nabla\log u - \nabla\log v| - t|\nabla\log u - \nabla\log v|\big|\\
&= |t_0 - t||\nabla\log u - \nabla\log v| \geqslant \frac12|t_0 - t||\nabla\log u|\\
&\geqslant \frac14|t_0 - t|\left(|\nabla\log u| + |\nabla\log v|\right),
\end{aligned}$$

since we are assuming that $|\nabla\log u| \geqslant |\nabla\log v|$. Therefore

$$|\nabla \log u + t(\nabla \log v - \nabla \log u)|^{p-2} \geqslant \left(\frac{1}{4}\right)^{p-2} |t_0 - t|^{p-2} \left(|\nabla \log u| + |\nabla \log v|\right)^{p-2}.$$

Then, observing that $\int_0^{\frac{1}{4}} |t_0 - t|^{p-2} \geqslant C_p$, one has

$$\begin{aligned} &\frac{3}{4} p v^p |\nabla \log u - \nabla \log v|^2 \left[\int_0^{\frac{1}{4}} |\nabla \log u + t(\nabla \log v - \nabla \log u)|^{p-2} \, dt \right] \\ &\geqslant C_p v^p \left(|\nabla \log u| + |\nabla \log v|\right)^{p-2} |\nabla \log u - \nabla \log v|^2. \end{aligned}$$

In the case $|\nabla \log u| \leqslant |\nabla \log v|$, arguing in the same way, we deduce that

$$\begin{aligned} &\frac{3}{4} p u^p |\nabla \log u - \nabla \log v|^2 \left[\int_0^{\frac{1}{4}} |\nabla \log v + t(\nabla \log u - \nabla \log v)|^{p-2} \, dt \right] \\ &\geqslant C_p u^p \left(|\nabla \log u| + |\nabla \log v|\right)^{p-2} |\nabla \log u - \nabla \log v|^2, \end{aligned}$$

which concludes the proof. □

As we have already observed, a key tool in our proofs is the moving plane technique. To exploit it we need the following notations. We will study the symmetry of the solutions in the $\nu-$ direction for any $\nu \in S^{N-1}$ (i.e. $|\nu| = 1$). Since the problem is invariant up to rotations we fix $\nu = e_1$ and we let

$$\begin{aligned} T_\lambda &= \left\{x \in \mathbb{R}^N \, : \, x_1 = \lambda\right\}, \\ \Sigma_\lambda &= \left\{x \in \mathbb{R}^N \, : \, x_1 < \lambda\right\}, \\ x_\lambda = R_\lambda(x) &= (2\lambda - x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}, \\ u_\lambda(x) &= u(x_\lambda). \end{aligned}$$

Now we state a result that will be used afterwards.

Theorem 3.2. *Let $1 < p < N$ and let $v \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{0\})$ with $0 < \alpha < 1$ be a positive solution to*

$$-\Delta_p v - \frac{\gamma}{|x|^p} v^{p-1} = 0 \quad in \quad \mathbb{R}^N \setminus \{0\}, \tag{3.5}$$

such that

$$\lim_{|x| \to 0} v(x) = \infty \qquad \lim_{|x| \to +\infty} v(x) = 0. \tag{3.6}$$

Then v is a radial (strict) decreasing function.

Proof. To prove that v is a radial non-increasing function we apply the moving plane technique. We fix a direction $\nu = e_1$ and, for $\lambda < 0$ and $\varepsilon > 0$ (small), we take as test function $\varphi_{1,\lambda} = v^{1-p}(v^p - (v_\lambda + \varepsilon)^p)^+ \chi_{\Sigma_\lambda}$ and $\varphi_{2,\lambda} = (v_\lambda + \varepsilon)^{1-p}(v^p - (v_\lambda + \varepsilon)^p)^+ \chi_{\Sigma_\lambda}$ in the weak formulation solved, respectively, by v and v_λ. We note that v_λ solves

$$-\Delta_p v_\lambda - \frac{\gamma}{|x_\lambda|^p} v_\lambda^{p-1} = 0. \tag{3.7}$$

We also remark that, by using (3.6),

$$\operatorname{supp}(\varphi_{j,\lambda}) \subset\subset \Sigma_\lambda \setminus \{0_\lambda\} \qquad j = 1, 2.$$

Furthermore, since $\varphi_{1,\lambda}$, $\varphi_{2,\lambda}$ have compact support far from the singularities, we can use the weak formulations of (3.5), (3.7) and, taking the difference, we deduce that

$$\begin{aligned} &\underbrace{\int_{\Sigma_\lambda} |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi_{1,\lambda} - |\nabla v_\lambda|^{p-2}\nabla v_\lambda \cdot \nabla\varphi_{2,\lambda}}_{(I)} \\ &+ \underbrace{\gamma \int_{\Sigma_\lambda} -\frac{1}{|x|^p}(v^p - (v_\lambda+\varepsilon)^p)^+}_{(II)} + \gamma \int_{\Sigma_\lambda} \frac{1}{|x_\lambda|^p}\left(\frac{v_\lambda}{v_\lambda+\varepsilon}\right)^{p-1}(v^p - (v_\lambda+\varepsilon)^p)^+ = 0. \end{aligned} \tag{3.8}$$

Now by exploiting (3.1)

$$\begin{aligned} (I) &= \int_{\Sigma_\lambda} |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi_{1,\lambda} - |\nabla(v_\lambda+\varepsilon)|^{p-2}\nabla(v_\lambda+\varepsilon) \cdot \nabla\varphi_{2,\lambda} \\ &\geqslant C_p \int_{\Sigma_\lambda\cap\{v\geqslant v_\lambda+\varepsilon\}} (v_\lambda+\varepsilon)^p \left(|\nabla \log v| + |\nabla \log(v_\lambda+\varepsilon)|\right)^{p-2} |\nabla \log v - \nabla \log(v_\lambda+\varepsilon)|^2 \end{aligned} \tag{3.9}$$

and

$$(II) \geqslant \int_{\Sigma_\lambda} -\frac{1}{|x|^p}(v^p - v_\lambda^p)^+ \tag{3.10}$$

Then

$$\begin{aligned} &C_p \int_{\Sigma_\lambda\cap\{v\geqslant v_\lambda+\varepsilon\}} (v_\lambda+\varepsilon)^p \left(|\nabla \log v| + |\nabla \log(v_\lambda+\varepsilon)|\right)^{p-2} |\nabla \log v - \nabla \log(v_\lambda+\varepsilon)|^2 \\ &+ \gamma \int_{\Sigma_\lambda} -\frac{1}{|x|^p}(v^p - v_\lambda^p)^+ + \gamma \int_{\Sigma_\lambda} \frac{1}{|x_\lambda|^p}\left(\frac{v_\lambda}{v_\lambda+\varepsilon}\right)^{p-1}(v^p - (v_\lambda+\varepsilon)^p)^+ \leqslant 0. \end{aligned} \tag{3.11}$$

We are in position to apply Fatou's lemma as $\varepsilon \to 0$ getting

$$\begin{aligned} &C_p \int_{\Sigma_\lambda\cap\{v\geqslant v_\lambda\}} v_\lambda^p \left(|\nabla \log v| + |\nabla \log v_\lambda|\right)^{p-2} |\nabla \log v - \nabla \log v_\lambda|^2 \\ &\gamma \int_{\Sigma_\lambda} \left(-\frac{1}{|x|^p} + \frac{1}{|x_\lambda|^p}\right)(v^p - v_\lambda^p)^+ \leqslant 0, \end{aligned} \tag{3.12}$$

and, since $|x| > |x_\lambda|$ in Σ_λ, the second term on the left hand side of (3.12) is nonnegative. Then, it follows that

$$C_p \int_{\Sigma_\lambda\cap\{v\geqslant v_\lambda\}} v_\lambda^p \left(|\nabla \log v| + |\nabla \log v_\lambda|\right)^{p-2} |\nabla(\log v - \log v_\lambda)|^2 = 0$$

which implies that $\log v - \log v_\lambda$ is constant $\Sigma_\lambda \cap \{v \geqslant v_\lambda\}$ and since $\log v - \log v_\lambda = 0$ on T_λ we have $v = v_\lambda$ on $\Sigma_\lambda \cap \{v \geqslant v_\lambda\}$ for any $\lambda < 0$ Hence $v \leqslant v_\lambda$ on Σ_λ for any $\lambda < 0$. We repeat the same argument in the $-e_1$ direction deducing that v is symmetric with respect to the e_1-direction. This procedure can be clearly performed in any direction $\nu \in \mathcal{S}^{N-1}$ whence one gets the radial monotone nonincreasing behavior of v.

A simple application of the Hopf Lemma (that can be applied since the level sets are spheres) shows now that v has no critical points and in particular the radial derivative is strictly negative. □

Next we provide the corresponding lower bound for the decay rate of $|\nabla u|$ of Theorem 2.5.

Theorem 3.3. *Let $1 < p < N$ and let u be a solution of (1.1). Then there exists $R_2 > 0$ and a constant $\bar{C} > 0$ such that*

$$|\nabla u(x)| \geqslant \frac{\bar{C}}{|x|^{\gamma_2+1}} \quad \textit{for } |x| > R_2. \tag{3.13}$$

Proof. Once that Theorem 3.2 is in force we can carry out the proof borrowing some ideas from Theorem 2.2 of [19]. We sketch it for the sake of completeness.
By contradiction let us assume that there exist sequences of radii R_n and points x_n with $R_n \to +\infty$ as $n \to +\infty$ and $|x_n| = R_n$, such that

$$|\nabla u(x_n)| \leqslant \frac{\theta_n}{|R_n|^{\gamma_2+1}}, \tag{3.14}$$

with $\theta_n \to 0$ as $n \to +\infty$. Without loss of generality we suppose $R_n > 1$ for any n and we set $w_{R_n}(x) := R_n^{\gamma_2} u(R_n x)$. One can observe that for fixed $0 < a < A$ then $||w_{R_n}||_{L^\infty(B_A \setminus B_a)}$ is bounded with respect to n. Otherwise if $|x| > \frac{R_1}{R_n}$ one deduces by Theorem 2.5 that

$$\frac{\bar{c}}{A^{\gamma_2}} \leqslant w_{R_n}(x) \leqslant \frac{\bar{C}}{a^{\gamma_2}},$$

and that

$$\begin{cases} w_{R_n}(x) \leqslant \dfrac{\bar{C}}{A^{\gamma_2}} & x \in \partial B_A, \\[2mm] w_{R_n}(x) \geqslant \dfrac{\bar{c}}{a^{\gamma_2}} & x \in \partial B_a. \end{cases} \tag{3.15}$$

Therefore, the above bound in $L^\infty(B_A \setminus B_a)$ implies that w_{R_n} is also uniformly bounded in $C^{1,\alpha}(K)$ with $0 < \alpha < 1$ for any compact set $K \subset B_A \setminus B_a$. Finally, since $a > 0$, without loss of generality we suppose that the $C^{1,\alpha}$ estimates hold in the closure of $B_A \setminus B_a$. Hence, for $x \in B_A \setminus B_a$ and up to subsequences, one gets that $w_{R_n}(x) \longrightarrow w_{a,A}(x)$ in $C^{1,\alpha'}$ for $0 < \alpha' < \alpha$. We also underline that $w_{a,A}(x)$ satisfies (3.15). Furthermore, since

$$-\Delta_p w_{R_n} - \frac{\gamma}{|x|^p} w_{R_n}^{p-1} = \frac{w_{R_n}^{p^*-1}}{R_n^{(p^*-p)\gamma_2 - p}} \quad \text{in} \quad \mathbb{R}^N,$$

then

$$-\Delta_p w_{a,A} - \frac{\gamma}{|x|^p} w_{a,A}^{p-1} = 0 \qquad \text{in } B_A \setminus \overline{B_a}. \tag{3.16}$$

Now, for $j \in \mathbb{N}$, one can take $a_j = \frac{1}{j}$ and $A_j = j$ and reasoning as above one constructs w_{a_j,A_j}. Then, for $j \to \infty$, a diagonal argument implies the existence of a limiting profile w_∞ such that $w_\infty \equiv w_{a_j,A_j}$ in $B_{A_j} \setminus B_{a_j}$. In particular from (3.16) read for w_{a_j,A_j} one has

$$-\Delta_p w_\infty - \frac{\gamma}{|x|^p} w_\infty^{p-1} = 0 \qquad \text{in } \mathbb{R}^N \setminus \{0\}.$$

From (3.15) with $a = a_j$ and $A = A_j$, one gets that the limiting profile w_∞ is such that

$$\lim_{|x|\to+\infty} w_\infty(x) = 0 \qquad \text{and} \qquad \lim_{|x|\to 0} w_\infty(x) = +\infty.$$

Therefore Theorem 3.2 can be applied providing that w_∞ is radial with negative radial derivative.

To conclude let now x_n be as in (3.14) and set $y_n = \frac{x_n}{R_n}$. Then, by (3.14), it follows that $|\nabla w_{R_n}(y_n)|$ tends to zero as $n \to +\infty$. Up to subsequences, since $|y_n| = 1$, we have that $y_n \to \bar{y} \in \partial B_1$. Consequently, by the uniform convergence of the gradients one has that $\nabla w_\infty(\bar{y}) = 0$, which is in contradiction with the definition of w_∞, since, by Theorem 3.2, this cannot happen. □

4. Proof of the symmetry result

We are now able to prove Theorem 1.1. First of all we underline that it is easy to see that u_λ solves

$$-\Delta_p u_\lambda - \frac{\gamma}{|x_\lambda|^p} u_\lambda^{p-1} = u_\lambda^{p^*-1} \quad \text{in } \mathbb{R}^N. \tag{4.1}$$

In what follows we set

$$\Lambda^- = \{\lambda < 0 \,:\, u \leqslant u_\mu \text{ in } \Sigma_\mu,\ \forall\, \mu \leqslant \lambda\}, \quad \Lambda^+ = \{\lambda > 0 \,:\, u \geqslant u_\mu \text{ in } \Sigma_\mu,\ \forall\, \mu \leqslant \lambda\}.$$

If $\Lambda^- \neq \emptyset$ and $\Lambda^+ \neq \emptyset$ we denote by $\lambda_0^- := \sup \Lambda^-$ and by $\lambda_0^+ := \inf \Lambda^+$.

Roughly speaking, the moving plane method consists of two main steps: first in reflecting the domain about a fixed hyperplane and proving that the value the solution at each reflected point is larger than the value at the point itself and secondly in moving the hyperplane to a critical position; finally the solution results to be symmetric with respect to this limit hyperplane.

Proof of Theorem 1.1. We prove the result by analyzing, sometimes in different ways, the case $1 < p < 2$ and the case $p > 2$. For $p = 2$ we refer to [22]. We divide the proof in two steps.

Step 1: $\Lambda^- \neq \emptyset$ and $\Lambda^+ \neq \emptyset$.

We only prove $\Lambda^- \neq \emptyset$, which is the existence of $\lambda < 0$ with $|\lambda|$ sufficiently large such that $u \leqslant u_\mu$ in Σ_μ for every $\mu \leqslant \lambda$. The proof of the fact that $\Lambda^+ \neq \emptyset$ is analogous and, at the end of the step, we outline the main changes in the proof in order to conclude it.

For the entire proof we denote by R_0, R_1 and R_2 the radii given by (2.2), (2.3) and (3.13) and we firstly observe that for $|\overline{\lambda}| > \max(R_1, R_2)$ one has, by (2.2) and (2.3), that there exists $\tilde{R}_0 := \tilde{R}_0(\overline{\lambda})$ such that $\tilde{R}_0 < R_0$, $B_{\tilde{R}_0}(0_{\overline{\lambda}}) \subset \Sigma_{\overline{\lambda}}$ and

$$\sup_{x \in B_{\tilde{R}_0}(0_{\overline{\lambda}})} u(x) < \inf_{x \in B_{\tilde{R}_0}(0_{\overline{\lambda}})} u_{\overline{\lambda}}(x). \tag{4.2}$$

Therefore, exploiting also (2.3), we deduce that

$$\sup_{x \in B_{\tilde{R}_0}(0_\lambda)} u(x) \leq \inf_{x \in B_{\tilde{R}_0}(0_\lambda)} u_\lambda(x),$$

which gives that $u < u_\lambda$ in $B_{\tilde{R}_0}(0_\lambda) \subset \Sigma_\lambda$ for every $\lambda \leqslant \overline{\lambda}$ and with $\tilde{R}_0$ independent of λ. Moreover we also denote by $\eta \in C_0^\infty(B_{2R}(0))$ a cut-off function such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B_R(0)$ and $|\nabla \eta| \leqslant \frac{2}{R}$.
In what follows we employ the following notation: $\Sigma'_\lambda = \Sigma_\lambda \setminus B_{\tilde{R}_0}(0_\lambda)$ and $\hat{B}_\rho := B_\rho(0) \cap \Sigma'_\lambda$ for $\rho > 0$.
If $\alpha > \max\{2, p\}$ and $\lambda \leqslant \overline{\lambda}$, we consider

$$\varphi_{1,\lambda} = \eta^\alpha u^{1-p}(u^p - u_\lambda^p)^+ \chi_{\Sigma_\lambda}, \qquad \varphi_{2,\lambda} = \eta^\alpha u_\lambda^{1-p}(u^p - u_\lambda^p)^+ \chi_{\Sigma_\lambda}. \tag{4.3}$$

We remark that $\mathrm{supp}(\varphi_{j,\lambda}) \subset \hat{B}_{2R}$ for $j = 1, 2$. Then we take $\varphi_{1,\lambda}$ as a test function in (1.1), $\varphi_{2,\lambda}$ in (4.1) and we subtract. Hence, denoting by $\psi_\lambda := (u^p - u_\lambda^p)^+$ and by $\varphi_\lambda := (u - u_\lambda)^+$ one gets

$$\begin{aligned} &\int_{\hat{B}_{2R}} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi_{1,\lambda} - |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla \varphi_{2,\lambda}\right) + \gamma \int_{\hat{B}_{2R}} \left(-\frac{1}{|x|^p} + \frac{1}{|x_\lambda|^p}\right)\eta^\alpha \psi_\lambda \\ &= \int_{\hat{B}_{2R}} (u^{p^*-p} - u_\lambda^{p^*-p})\eta^\alpha \psi_\lambda, \end{aligned} \tag{4.4}$$

and, since $|x| \geqslant |x_\lambda|$ in Σ_λ, one has that the second term on the left hand side of (4.4) is nonnegative. Hence

$$\begin{aligned} &\underbrace{\int_{\hat{B}_{2R}} \eta^\alpha \left(|\nabla u|^{p-2}\nabla u \cdot \nabla(u^{1-p}\psi_\lambda) - |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla(u_\lambda^{1-p}\psi_\lambda)\right)}_{I_1} \\ &\leqslant \underbrace{-\alpha \int_{\hat{B}_{2R}} \eta^{\alpha-1}u^{1-p}\psi_\lambda |\nabla u|^{p-2}\nabla u \cdot \nabla \eta}_{I_2} + \underbrace{\alpha \int_{\hat{B}_{2R}} \eta^{\alpha-1}u_\lambda^{1-p}\psi_\lambda |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla \eta}_{I_3} \\ &+ \underbrace{\int_{\hat{B}_{2R}} (u^{p^*-p} - u_\lambda^{p^*-p})\eta^\alpha \psi_\lambda}_{I_4}. \end{aligned} \tag{4.5}$$

We start by estimating I_1. By using (3.1) it yields that for $p > 2$ one has

$$\begin{aligned} I_1 &\geqslant C_p \int_{\hat{B}_{2R}\cap\{u \geqslant u_\lambda\}} \eta^\alpha u_\lambda^p \left(|\nabla \log u| + |\nabla \log u_\lambda|\right)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 \\ &\geqslant C_p \int_{\hat{B}_{2R}\cap\{u \geqslant u_\lambda\}} \eta^\alpha \left(\frac{u_\lambda}{u}\right)^p u^2 \left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 \\ &\geqslant c_1 \int_{\hat{B}_{2R}\cap\{u \geqslant u_\lambda\}} \eta^\alpha u^2 \left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2, \end{aligned} \tag{4.6}$$

while for $1 < p < 2$ we obtain

$$\begin{aligned} I_1 &\geqslant C_p \int_{\hat{B}_{2R}\cap\{u \geqslant u_\lambda\}} \eta^\alpha u_\lambda^p \frac{|\nabla \log u - \nabla \log u_\lambda|^2}{(|\nabla \log u| + |\nabla \log u_\lambda|)^{2-p}} \\ &\geqslant C_p \int_{\hat{B}_{2R}\cap\{u \geqslant u_\lambda\}} \eta^\alpha u_\lambda^2 \frac{|\nabla \log u - \nabla \log u_\lambda|^2}{(|\nabla u| + |\nabla u_\lambda|)^{2-p}}. \end{aligned} \tag{4.7}$$

We remark that in (4.6) we used that

$$\frac{u_\lambda}{u} \geqslant \tilde{c} \quad \text{in } \Sigma_\lambda, \tag{4.8}$$

and $c_1 := C_p \tilde{c}^p$. Indeed if $x \in \Sigma_\lambda \setminus B_{R_1}(0_\lambda)$ then from (2.2) and (2.3) one has (recall that $|x| \geqslant |x_\lambda|$)

$$\frac{u_\lambda}{u} \geqslant \tilde{c}_1 \frac{|x|^{\gamma_2}}{|x_\lambda|^{\gamma_2}} \geqslant \tilde{c}_1.$$

Otherwise if $x \in \Sigma_\lambda \cap B_{R_1}(0_\lambda)$ then

$$\frac{u_\lambda}{u} \geqslant \tilde{c}_1 |\overline{\lambda}|^{\gamma_2} \inf_{x \in B_{R_1}(0)} u(x) \geqslant \tilde{c}_2,$$

and we set $\tilde{c} = \min(\tilde{c}_1, \tilde{c}_2)$. Now it follows from (2.3) and (2.5) that

$$\begin{aligned} I_2 &\leqslant \alpha \int\limits_{\hat{B}_{2R} \cap \{u \geqslant u_\lambda\}} \eta^{\alpha-1} u \left(1 - \left(\frac{u_\lambda}{u}\right)^p\right) |\nabla u|^{p-1} |\nabla \eta| \\ &\leqslant \frac{2\alpha}{R} \int\limits_{\hat{B}_{2R} \setminus \hat{B}_R} u |\nabla u|^{p-1} \leqslant \frac{C}{R} \int\limits_{\hat{B}_{2R} \setminus \hat{B}_R} \frac{1}{|x|^{(\gamma_2+1)(p-1)+\gamma_2}} \leqslant \frac{C}{R^\beta} \end{aligned} \tag{4.9}$$

where, from here on, $\beta := p\gamma_2 + p - N$ which is strictly positive since $\gamma_2 > \frac{N-p}{p}$. For I_3, using (2.3) and (4.8), we deduce that

$$\begin{aligned} I_3 &\leqslant C \int\limits_{\hat{B}_{2R} \setminus \hat{B}_R} \alpha \eta^{\alpha-1} u_\lambda^{1-p} \left(u^p - u_\lambda^p\right)^+ |\nabla u_\lambda|^{p-1} |\nabla \eta| \\ &\leqslant \frac{2}{R} \int\limits_{\hat{B}_{2R} \setminus \hat{B}_R \cap \{u \geqslant u_\lambda\}} u_\lambda \left(\left(\frac{u}{u_\lambda}\right)^p - 1\right) |\nabla u_\lambda|^{p-1} \\ &\leqslant \frac{2}{R} \int\limits_{\hat{B}_{2R} \setminus \hat{B}_R \cap \{u \geqslant u_\lambda\}} u_\lambda \left(\frac{u}{u_\lambda}\right)^p |\nabla u_\lambda|^{p-1} \leqslant \frac{C}{R} \int\limits_{\hat{B}_{2R} \setminus \hat{B}_R \cap \{u \geqslant u_\lambda\}} u |\nabla u_\lambda|^{p-1} \\ &\leqslant \frac{C}{R} \left(\int\limits_{\mathbb{R}^N} |\nabla u_\lambda|^p\right)^{\frac{p-1}{p}} \left(\int\limits_{\hat{B}_{2R} \setminus \hat{B}_R} u^p\right)^{\frac{1}{p}} \leqslant \frac{C}{R} \left(\int\limits_{\hat{B}_{2R} \setminus \hat{B}_R} \frac{1}{|x|^{\gamma_2 p}}\right)^{\frac{1}{p}} \leqslant \frac{C}{R^{\frac{\beta}{p}}}. \end{aligned} \tag{4.10}$$

For the term I_4 we first note that (since $u \geqslant u_\lambda$)

$$I_4 = \int\limits_{\hat{B}_{2R}} \left(\frac{u^{p^*-1}}{u^{p-1}} - \frac{u_\lambda^{p^*-1}}{u_\lambda^{p-1}}\right) \eta^\alpha \psi_\lambda \leqslant \int\limits_{\hat{B}_{2R}} \frac{1}{u^{p-1}} \left(u^{p^*-1} - u_\lambda^{p^*-1}\right) \eta^\alpha \psi_\lambda,$$

then applying twice the Lagrange Theorem and using (2.3) one has that in case $p^* \geqslant 2$

$$I_4 \leqslant c_p \int\limits_{\hat{B}_{2R}} u^{p^*-2} \eta^\alpha \varphi_\lambda^2 \leqslant c_p \int\limits_{\hat{B}_{2R}} \frac{1}{|x|^{\gamma_2(p^*-2)}} \eta^\alpha \varphi_\lambda^2,$$

while for $1 < p^* < 2$ (recall (4.8))

$$I_4 \leqslant c_p \int\limits_{\hat{B}_{2R}} \frac{\eta^\alpha \varphi_\lambda^2}{u_\lambda^{2-p^*}} = \int\limits_{\hat{B}_{2R}} \left(\frac{u}{u_\lambda}\right)^{2-p^*} \frac{\eta^\alpha \varphi_\lambda^2}{u^{2-p^*}} \leqslant c_p \int\limits_{\hat{B}_{2R}} \frac{\eta^\alpha \varphi_\lambda^2}{u^{2-p^*}} \leqslant c_p \int\limits_{\hat{B}_{2R}} \frac{1}{|x|^{\gamma_2(p^*-2)}} \eta^\alpha \varphi_\lambda^2,$$

which gives for any $p > 1$

$$I_4 \leqslant c_p \int\limits_{\hat{B}_{2R}} \frac{1}{|x|^{\gamma_2(p^*-2)}} \eta^\alpha \varphi_\lambda^2. \tag{4.11}$$

Let us now consider $f(t) = \log(a + t(b-a))$ where $a, b > 0$ $(b \geqslant a)$ then

$$\log b = \log a + (b-a) \int\limits_0^1 \frac{1}{a + t(b-a)},$$

and since $t \in [0, 1]$ we get

$$b - a = \frac{\log b - \log a}{\int_0^1 \frac{1}{a+t(b-a)}} \leqslant b(\log b - \log a). \tag{4.12}$$

We use (4.12) with $b = u$ and $a = u_\lambda$ and estimate the right hand side of (4.11) (by using also (2.3)) as

$$\begin{aligned} I_4 &\leqslant C \int\limits_{\hat{B}_{2R} \cap \{u \geqslant u_\lambda\}} \frac{1}{|x|^{\gamma_2(p^*-2)}} \eta^\alpha u^2 \left(\log u - \log u_\lambda\right)^2 \\ &\leqslant C \int\limits_{\hat{B}_{2R}} \frac{1}{|x|^{\gamma_2 p^*}} \eta^\alpha \left((\log u - \log u_\lambda)^+\right)^2. \end{aligned}$$

Moreover

$$\begin{aligned} I_4 &\leqslant C \int\limits_{\hat{B}_{2R}} \frac{1}{|x|^{\beta^* - 2\alpha + 2}} \left(\eta^{\frac{\alpha}{2}} (\log u - \log u_\lambda)^+\right)^2 \\ &\leqslant \frac{C}{|\lambda|^{\beta^*}} \int\limits_{\hat{B}_{2R}} |x|^{2\alpha - 2} \left(\eta^{\frac{\alpha}{2}} (\log u - \log u_\lambda)^+\right)^2, \end{aligned} \tag{4.13}$$

where

$$\beta^* := \gamma_2(p^* - p) - p; \qquad 2\alpha := -[(\gamma_2 + 1)(p - 2) + 2\gamma_2].$$

We underline that $\beta^* - 2\alpha + 2 = \gamma_2 p^*$ and that $\beta^* > 0$ since $\gamma_2 > \frac{N-p}{p}$. For the right hand side of (4.13) we can apply Theorem 2.3 where $r = 2, \tau = 2$ which implies that

$$\gamma := \alpha - 1 = -\frac{(\gamma_2 + 1)p}{2}$$

and that

$$\frac{1}{2} + \frac{\gamma}{N} = \frac{N - \gamma_2 p - p}{2N} < 0$$

since $\gamma_2 > \frac{N-p}{p}$. Hence we obtain

$$I_4 \leqslant \frac{C}{|\lambda|^{\beta^*}} \int\limits_{\hat{B}_{2R}} |x|^{2\alpha} |\nabla(\eta^{\frac{\alpha}{2}} (\log u - \log u_\lambda)^+)|^2, \tag{4.14}$$

and now, in order to estimate the right hand side of (4.14), we distinguish between the case $p > 2$ and the case $1 < p < 2$. From (4.14) and for $p > 2$ we get

$$\begin{aligned} I_4 &\leqslant \frac{C}{|\lambda|^{\beta^*}} \int\limits_{\hat{B}_{2R} \cap \{u \geqslant u_\lambda\}} \frac{1}{|x|^{(\gamma_2+1)(p-2)}} \eta^\alpha u^2 |\nabla \log u - \nabla \log u_\lambda|^2 \\ &+ \frac{C}{|\lambda|^{\beta^*}} \int\limits_{\hat{B}_{2R} \cap \{u \geqslant u_\lambda\}} |x|^{2\alpha} \left(\log u - \log u_\lambda\right)^2 |\nabla \eta|^2 \\ &\leqslant \frac{C}{|\lambda|^{\beta^*}} \int\limits_{\hat{B}_{2R} \cap \{u \geqslant u_\lambda\}} \eta^\alpha u^2 |\nabla u|^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 + \frac{C}{|\lambda|^{\beta^*} R^2} \int\limits_{\hat{B}_{2R} \setminus \hat{B}_R} |x|^{2\alpha} \\ &\leqslant \frac{C}{|\lambda|^{\beta^*}} \int\limits_{\hat{B}_{2R} \cap \{u \geqslant u_\lambda\}} \eta^\alpha u^2 \left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 + \frac{C}{|\lambda|^{\beta^*} R^\beta}. \end{aligned} \tag{4.15}$$

Then, by using the estimates (4.6), (4.9), (4.10) and (4.15) in (4.5), we

$$\left(c_1 - \frac{C}{|\lambda|^{\beta^*}}\right) \int\limits_{\hat{B}_{2R} \cap \{u \geqslant u_\lambda\}} \eta^\alpha u^2 \left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 \leqslant \frac{C}{R^{\frac{\beta}{p}}} + \frac{C}{|\lambda|^{\beta^*} R^\beta} + \frac{C}{R^\beta}.$$

For $|\lambda|$ sufficiently large, as R goes to $+\infty$, we deduce that

$$\begin{aligned} &\int\limits_{\Sigma'_\lambda \cap \{u \geqslant u_\lambda\}} u^2 \left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 \\ &= \lim_{R \to +\infty} \int\limits_{\hat{B}_R \cap \{u \geqslant u_\lambda\}} u^2 \left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 \leqslant 0. \end{aligned}$$

Now we have to estimate the right hand side of (4.14) in the case $1 < p < 2$.
We first remark that $2\alpha < 0$ (for $N > 2$) and, since $|x| \geqslant |x_\lambda|$, one has that $|x|^{2\alpha} \leqslant |x_\lambda|^{2\alpha}$. Then

$$\begin{aligned} I_4 &\leqslant \frac{C}{|\lambda|^{\beta^*}} \int\limits_{\hat{B}_{2R} \cap \{u \geqslant u_\lambda\}} |x_\lambda|^{2\alpha} \eta^\alpha |\nabla \log u - \nabla \log u_\lambda|^2 \\ &+ \frac{C}{|\lambda|^{\beta^*}} \int\limits_{\hat{B}_{2R} \cap \{u \geqslant u_\lambda\}} |x|^{2\alpha} \left(\log u - \log u_\lambda\right)^2 |\nabla \eta|^2. \end{aligned} \tag{4.16}$$

Let $\bar{R} = \max\{R_1, R_2\}$ and let $A_{\bar{R},\tilde{R}_0} = \overline{B_{\bar{R}}(0_\lambda) \setminus B_{\tilde{R}_0}(0_\lambda)}$. Then we get

$$\hat{B}_{2R} = \hat{A}_{\bar{R},\tilde{R}_0} \cup \left(\hat{B}_{2R} \setminus \hat{A}_{\bar{R},\tilde{R}_0}\right).$$

Exploiting (2.3) we deduce that

$$
\begin{aligned}
&\int\limits_{\hat{B}_{2R}\setminus\hat{A}_{\bar{R},\tilde{R}_0}} |x_\lambda|^{2\alpha}\eta^\alpha|\nabla\log u-\nabla\log u_\lambda|^2 \\
&\leqslant C\int\limits_{\hat{B}_{2R}\setminus\hat{A}_{\bar{R},\tilde{R}_0}} |x_\lambda|^{(\gamma_2+1)(2-p)}|x_\lambda|^{-2\gamma_2}\eta^\alpha|\nabla\log u-\nabla\log u_\lambda|^2 \\
&\leqslant C\int\limits_{\hat{B}_{2R}\setminus\hat{A}_{\bar{R},\tilde{R}_0}} u_\lambda^2\eta^\alpha\frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}.
\end{aligned}
\tag{4.17}
$$

In $A_{\bar{R},\tilde{R}_0}$ it holds that $|x_\lambda|\geqslant\tilde{R}_0$ and, since we are far from 0_λ, we also get that $|\nabla u_\lambda|$ is bounded. Let $L:=\inf\limits_{B_{\bar{R}}(0)\setminus B_{\tilde{R}_0}(0)} u$. Hence we get (by using (4.8) and the fact that $(|\nabla u|+|\nabla u_\lambda|)^{2-p}\leqslant C$ away from $0,0_\lambda$)

$$
\begin{aligned}
\int\limits_{A_{\bar{R},\tilde{R}_0}} |x_\lambda|^{2\alpha}\eta^\alpha|\nabla\log u-\nabla\log u_\lambda|^2 &\leqslant C\tilde{R}_0^{2\alpha}\int\limits_{A_{\bar{R},\tilde{R}_0}}\eta^\alpha|\nabla\log u-\nabla\log u_\lambda|^2 \\
&\leqslant \frac{C\tilde{R}_0^{2\alpha}}{L^2}\int\limits_{A_{\bar{R},\tilde{R}_0}} u_\lambda^2\eta^\alpha\frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}.
\end{aligned}
\tag{4.18}
$$

Gathering (4.17) and (4.18) in the first term of (4.16) and reasoning as in (4.15) for the second term of (4.16) one yields to

$$
I_4\leqslant\frac{C}{|\lambda|^{\beta^*}}\int\limits_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} u_\lambda^2\eta^\alpha\frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}+\frac{C}{|\lambda|^{\beta^*}R^\beta}. \tag{4.19}
$$

Hence, by collecting (4.7), (4.9), (4.10) and (4.19) in (4.5), we get

$$
\left(c_1-\frac{C}{|\lambda|^{\beta^*}}\right)\int\limits_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}}\eta^\alpha u_\lambda^2\frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}\leqslant\frac{C}{R^{\frac{\beta}{p}}}+\frac{C}{R^\beta}+\frac{C}{|\lambda|^{\beta^*}R^\beta}.
$$

Once again we can choose $|\lambda|$ large enough so that, as R goes to $+\infty$, it yields

$$
\int\limits_{\Sigma'_\lambda\cap\{u\geqslant u_\lambda\}} u_\lambda^2\frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}=\limsup_{R\to+\infty}\int\limits_{\hat{B}_R} u_\lambda^2\frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}\leqslant 0.
$$

Hence, in both cases, $\log u-\log u_\lambda$ is constant and since $\log u-\log u_\lambda=0$ on T_λ then $\log u-\log u_\lambda=0$ on the set $\Sigma'_\lambda\cap\{u\geqslant u_\lambda\}$. Therefore we get $u\leqslant u_\lambda$ on Σ_λ. Hence $\Lambda^-\neq\emptyset$ and λ_0^- exists and it is also finite. In order to show that $\Lambda^+\neq\emptyset$ then we take as test functions

$$
\phi_{1,\lambda}=u^{1-p}(u^p-u_\lambda^p)^-\chi_{\Sigma_\lambda},\ \ \phi_{2,\lambda}=u_\lambda^{1-p}(u^p-u_\lambda^p)^-\chi_{\Sigma_\lambda}
$$

and, analogously to what already done, we are able to prove the claim so that there exists λ_0^+ which is also finite.

Step 2: $\lambda_0^-=\lambda_0^+=0$.

We argue by contradiction assuming that $\lambda_0^-\neq 0$. Arguing as in the proof of Step 1 we will get the contradiction proving that $u\leqslant u_{\lambda_0^-+\varepsilon}$ in $\Sigma_{\lambda_0^-+\varepsilon}$ for all $0\leq\varepsilon\leqslant\bar{\varepsilon}$ for some $\bar{\varepsilon}>0$.

In what follows we shall exploit the *strong comparison principle*. To do this we start noticing that from Step 1 and by continuity it holds that

$$u \leqslant u_{\lambda_0^-} \quad \text{in} \quad \Sigma_{\lambda_0^-}\,.$$

By Theorem 2.2 we deduce that $u \equiv u_{\lambda_0^-}$ or $u < u_{\lambda_0^-}$ in any connected component $\mathcal{C}$ of $\Sigma_{\lambda_0^-} \setminus Z_u$ ($Z_u = \{\nabla u = 0\}$). We will frequently use the fact that Z_u has zero Lebesgue measure [10].

Assume first that $\Sigma_{\lambda_0^-} \setminus Z_u$ has only one connected component. We observe that $u \equiv u_{\lambda_0^-}$ is not possible in this case since, by (2.2), there exists $B_{\tilde{R}_0}(0_{\lambda_0^-})$ where $u < u_{\lambda_0^-}$; this means that $u < u_{\lambda_0^-}$ in $\Sigma_{\lambda_0^-} \setminus Z_u$.

Assume now that there are at least two connected components of $\Sigma_{\lambda_0^-} \setminus Z_u$. Our Theorem 3.3 implies that Z_u is bounded so that only one component can be unbounded. We refer to such a unbounded connected component as $\mathcal{C}_1$ and set

$$\mathcal{C}_\lambda \;:=\; (\mathcal{C}_1^c \cap \Sigma_{\lambda_0^-}) \cup R_\lambda(\mathcal{C}_1^c \cap \Sigma_{\lambda_0^-})$$

If $u \equiv u_{\lambda_0^-}$ in $\mathcal{C}_1$ it is easy to see that, by symmetry, $\mathcal{C}_\lambda$ contains at least one connected component of $\mathbb{R}^N \setminus Z_u$. But this is not possible as it has been shown in [10, Theorem 1.4] and [6, Lemma 5]. If else $u \equiv u_{\lambda_0^-}$ in $\mathcal{C}_2$ for some bounded component $\mathcal{C}_2$, then in this case we set

$$\mathcal{C}_\lambda \;:=\; \mathcal{C}_2 \cup R_\lambda(\mathcal{C}_2)\,,$$

and also in this case, by symmetry, $\mathcal{C}_\lambda$ would contain at least one connected component of $\mathbb{R}^N \setminus Z_u$ thus providing a contradiction. Resuming we just proved that

$$u < u_{\lambda_0^-} \quad \text{in} \quad \Sigma_{\lambda_0^-} \setminus Z_u\,.$$

Now, recalling that Z_u is bounded by Theorem 3.3, we fix $\overline{R} > 0$ in such a way that

$$Z_u \subset B_{\overline{R}}(0)\,,$$

and, for $\tau > 0$, we let Z_u^τ be an open set containing Z_u such that $\mathcal{L}(Z_u^\tau) < \tau$ (that exists since $\mathcal{L}(Z_u) = 0$). Then, for $\delta, \varepsilon, \overline{R}, \tau > 0$, we denote by

$$B_{\overline{R},\varepsilon} := B^c_{\overline{R}}(0) \cap \Sigma_{\lambda_0^- + \varepsilon}, \quad S_\delta^\varepsilon := \Big((\Sigma_{\lambda_0^- + \varepsilon} \setminus \Sigma_{\lambda_0^- - \delta}) \cap B_{\overline{R}}(0) \Big) \cup (Z_u^\tau \cap \Sigma_{\lambda_0^- - \delta}),$$
$$K_\delta := \overline{B_{\overline{R}}(0) \cap \Sigma_{\lambda_0^- - \delta}} \cap (Z_u^\tau)^c,$$

where $\delta \leqslant \overline{\delta}$ so that K_δ is nonempty. We underline that this construction gives

$$\Sigma_{\lambda_0^- + \varepsilon} = B_{\overline{R},\varepsilon} \cup S_\delta^\varepsilon \cup K_\delta.$$

We also remark that, since K_δ is compact, then by the uniform continuity of u and u_λ, for $\overline{\varepsilon} > 0$ small enough one has that $u < u_{\lambda_0^- + \varepsilon}$ in K_δ for every $\varepsilon \leqslant \overline{\varepsilon}$. Moreover we underline the existence of $\tilde{R}_0$ such that $u < u_{\lambda_0^- + \varepsilon}$ in $B_{\tilde{R}_0}(0_{\lambda_0^- + \varepsilon}) \subset \Sigma_{\lambda_0^- + \varepsilon}$ for every $\varepsilon \leqslant \overline{\varepsilon}$ and with $\tilde{R}_0$ independent of ε as done in Step 1.

From now on, for $R > \overline{R}$, we consider $\eta \in C_0^\infty(B_{2R}(0))$ a cut-off function with $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B_R(0)$ and $|\nabla \eta| \leqslant \frac{2}{R}$. Then, letting $\alpha > \max\{2, p\}$, we consider the following test functions

$$\varphi_{1,\lambda_0^- + \varepsilon} = \eta^\alpha u^{1-p} \left(u^p - u^p_{\lambda_0^- + \varepsilon} \right)^+ \chi_{\Sigma_{\lambda_0^- + \varepsilon}}, \qquad \varphi_{2,\lambda_0^- + \varepsilon} = \eta^\alpha u^{1-p}_{\lambda_0^- + \varepsilon} \left(u^p - u^p_{\lambda_0^- + \varepsilon} \right)^+ \chi_{\Sigma_{\lambda_0^- + \varepsilon}},$$

and, analogously to Step 1, $\psi_{\lambda_0^-+\varepsilon} := (u^p - u^p_{\lambda_0^-+\varepsilon})^+$ and by $\varphi_{\lambda_0^-+\varepsilon} := (u - u_{\lambda_0^-+\varepsilon})^+$.
Let us take $\varphi_{1,\lambda_0^-+\varepsilon}$ as a test function in (1.1), $\varphi_{2,\lambda_0^-+\varepsilon}$ in (4.1) and, reasoning as in Step 1, one yields to

$$\begin{aligned} c_1 \int\limits_{\hat{B}_{2R}\cap\{u\geqslant u_{\lambda_0^-+\varepsilon}\}} & \eta^\alpha u^2 \left(|\nabla u| + |\nabla u_{\lambda_0^-+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2 \\ \leqslant \int\limits_{\hat{B}_{2R}\cap B_{\overline{R},\varepsilon}} & (u^{p^*-p} - u^{p^*-p}_{\lambda_0^-+\varepsilon})\eta^\alpha \psi_\lambda + \int\limits_{\hat{B}_{2R}\cap S_\delta^\varepsilon} (u^{p^*-p} - u^{p^*-p}_{\lambda_0^-+\varepsilon})\eta^\alpha \psi_{\lambda_0^-+\varepsilon} + \frac{C}{R^{\frac{\beta}{p}}} + \frac{C}{R^\beta}. \end{aligned} \tag{4.20}$$

Here we have used once again the fact that $\frac{u_{\lambda_0^-+\varepsilon}}{u} \geqslant \tilde{c}$ for every $0 \leqslant \varepsilon \leqslant \bar{\varepsilon}$ as to deduce (4.8).
In order to estimate the first term on the right hand side of (4.20) we argue exactly as to estimate I_4 in (3.2) (taking into account Remark 2.4) where here $\overline{R}$ plays the role of λ in Step 1. Hence we get

$$\begin{aligned} & \int\limits_{\hat{B}_{2R}\cap B_{\overline{R},\varepsilon}} (u^{p^*-p} - u^{p^*-p}_{\lambda_0^-+\varepsilon})\eta^\alpha \psi_\lambda \leqslant \frac{C}{R^\beta} \\ & + \frac{C}{\overline{R}^{\beta^*}} \int\limits_{\hat{B}_{2R}\cap B_{\overline{R},\varepsilon}\cap\{u\geqslant u_{\lambda_0^-+\varepsilon}\}} \eta^\alpha u^2 \left(|\nabla u| + |\nabla u_{\lambda_0^-+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2. \end{aligned}$$

For the second term on the right hand side of (4.20) we reason as in Step 1, getting

$$\int\limits_{\hat{B}_{2R}\cap S_\delta^\varepsilon} (u^{p^*-p} - u^{p^*-p}_{\lambda_0^-+\varepsilon})\eta^\alpha \psi_\lambda \leqslant C_u \int\limits_{\hat{B}_{2R}\cap S_\delta^\varepsilon\cap\{u\geqslant u_{\lambda_0^-+\varepsilon}\}} (\log u - \log u_{\lambda_0^-+\varepsilon})^2, \tag{4.21}$$

where

$$C_u := \begin{cases} \sup\limits_{S_{\overline{\delta}}^{\overline{\varepsilon}}} u^{p^*-2} & \text{if } p^* \geqslant 2, \\ \inf\limits_{S_{\overline{\delta}}^{\overline{\varepsilon}}} u^{p^*-2} & \text{if } p^* < 2. \end{cases}$$

Now we need to divide the estimate by the value of p; indeed if $p > 2$ we apply a suitable weighted Poincaré inequality to the right hand side of (4.21) which can be found in Theorem 3.2 of [10]. Hence in this case one has

$$\begin{aligned} & \int\limits_{\hat{B}_{2R}\cap S_\delta^\varepsilon} (u^{p^*-p} - u^{p^*-p}_{\lambda_0^-+\varepsilon})\eta^\alpha \psi_\lambda \\ & \leqslant C_p^2(S_\delta^\varepsilon) C_u \int\limits_{\hat{B}_{2R}\cap S_\delta^\varepsilon\cap\{u\geqslant u_{\lambda_0^-+\varepsilon}\}} |\nabla u|^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2 \\ & \leqslant \frac{C_p^2(S_\delta^\varepsilon) C_u}{\inf\limits_{S_{\overline{\delta}}^{\overline{\varepsilon}}} u^2} \int\limits_{\hat{B}_{2R}\cap S_\delta^\varepsilon\cap\{u\geqslant u_{\lambda_0^-+\varepsilon}\}} u^2 \left(|\nabla u| + |\nabla u_{\lambda_0^-+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2, \end{aligned}$$

where $C_p(E)$ is the Poincaré constant which goes to zero as $|E| \to 0$. Otherwise if $1 < p < 2$ one can apply the classical Poincaré inequality in order to deduce

$$\begin{aligned}
&\int\limits_{\hat{B}_{2R}\cap S_\delta^\varepsilon} (u^{p^*-p} - u_{\lambda_0^-+\varepsilon}^{p^*-p})\eta^\alpha \psi_\lambda \\
&\leqslant C_p^2(S_\delta^\varepsilon) C_u \int\limits_{\hat{B}_{2R}\cap S_\delta^\varepsilon \cap \{u \geqslant u_{\lambda_0^-+\varepsilon}\}} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2 \\
&\leqslant \frac{C C_p^2(S_\delta^\varepsilon) C_u}{\inf\limits_{S_\delta^{\bar{\varepsilon}}} u^2} \int\limits_{\hat{B}_{2R}\cap S_\delta^\varepsilon \cap \{u \geqslant u_{\lambda_0^-+\varepsilon}\}} u^2 \left(|\nabla u| + |\nabla u_{\lambda_0^-+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2,
\end{aligned}$$

which can be deduced since in $\Sigma_{\lambda_0^-+\varepsilon} \setminus B_{\tilde{R}_0}(0_{\lambda_0^-+\varepsilon})$ one has that

$$(|\nabla u| + |\nabla u_\lambda|)^{2-p} \leqslant C,$$

for some constant C which does not depend on $\varepsilon \leqslant \bar{\varepsilon}$. Hence in both cases one has that

$$\begin{aligned}
&c_1 \int\limits_{\hat{B}_{2R}\cap \{u \geqslant u_{\lambda_0^-+\varepsilon}\}} \eta^\alpha u^2 \left(|\nabla u| + |\nabla u_{\lambda_0^-+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2 \\
&\leqslant \frac{C}{R^{\frac{\beta}{p}}} + \frac{C}{R^\beta} + \frac{C}{\bar{R}^{\beta^*}} \int\limits_{(\hat{B}_{2R}\cap B_{\bar{R},\varepsilon})\cap \{u \geqslant u_{\lambda_0^-+\varepsilon}\}} \eta^\alpha u^2 \left(|\nabla u| + |\nabla u_{\lambda_0^-+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2 \\
&+ \frac{C C_p^2(S_\delta^\varepsilon) C_u}{\inf\limits_{S_\delta^{\bar{\varepsilon}}} u^2} \int\limits_{S_\delta^\varepsilon \cap \{u \geqslant u_{\lambda_0^-+\varepsilon}\}} \left(|\nabla u| + |\nabla u_{\lambda_0^-+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2.
\end{aligned}$$

Now we take care of the variable parameters $\bar{R}, \delta, \bar{\varepsilon}$. First we fix $\bar{R}$ large such that

$$\frac{C}{c_1 \bar{R}^{\beta^*}} < 1.$$

Then, since $C_p^2(\Omega)$ goes to zero if the Lebesgue measure of Ω goes to zero, we choose $\delta, \bar{\varepsilon}, \tau$ small so that

$$\frac{C C_p^2(S_\delta^\varepsilon) C_u}{c_1 \inf\limits_{S_\delta^{\bar{\varepsilon}}} u^2} < 1$$

for every $0 \leqslant \varepsilon \leqslant \bar{\varepsilon}$. Hence it follows that

$$\int\limits_{\hat{B}_{2R}\cap \{u \geqslant u_{\lambda_0^-+\varepsilon}\}} u^2 \left(|\nabla u| + |\nabla u_{\lambda_0^-+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2 \leqslant \frac{C}{R^{\frac{\beta}{p}}} + \frac{C}{R^\beta}$$

getting again (as $R \to +\infty$)

$$\int\limits_{\Sigma_{\lambda_0^-+\varepsilon}\cap \{u \geqslant u_{\lambda_0^-+\varepsilon}\}} u^2 \left(|\nabla u| + |\nabla u_{\lambda_0^-+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0^-+\varepsilon}|^2 = 0,$$

which gives that $u \leqslant u_{\lambda_0^-+\varepsilon}$ in $\Sigma_{\lambda_0^-+\varepsilon}$ which contradicts the definition of λ_0^-. This proves that $\lambda_0^- = 0$. In an analogous way we deduce that $\lambda_0^+ = 0$, which gives the symmetry of u along the e_1-direction. Repeating the same arguments in the remaining $N-1$ linearly independent directions of $\mathbb{R}^N$ then one deduces that u is symmetric about the origin and that is a radially decreasing function. $\square$

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