ON FANO SCHEMES OF LINEAR SPACES OF GENERAL COMPLETE INTERSECTIONS

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Abstract. We consider the *Fano scheme* $F_k(X)$ of k–dimensional linear subspaces contained in a complete intersection $X \subset \mathbb{P}^n$ of multi–degree $\underline{d} = (d_1, \ldots, d_s)$. Our main result is an extension of a result of Riedl and Yang concerning Fano schemes of lines on very general hypersurfaces: we consider the case when X is a very general complete intersection and $\Pi_{i=1}^s d_i > 2$ and we find conditions on n, $\underline{d}$ and k under which $F_k(X)$ does not contain either rational or elliptic curves. At the end of the paper, we study the case $\Pi_{i=1}^s d_i = 2$.

1. Introduction

In this paper we are concerned with the *Fano scheme* $F_k(X) \subset \mathbb{G}(k, n)$, parameterizing k–dimensional linear subspaces contained in $X \subset \mathbb{P}^n$, when X is a complete intersection of multi–degree $\underline{d} = (d_1, \ldots, d_s)$, with $1 \leqslant s \leqslant n-2$. We will avoid the trivial case in which X is a linear subspace, so that $\Pi_{i=1}^s\, d_i \geqslant 2$.

Our inspiration has been the following result by Riedl and Yang concerning the case of hypersurfaces:

Theorem 1.1. *(cf.* [8, Thm. 3.3]*) If $X \subset \mathbb{P}^n$ is a very general hypersurface of degree d such that $n \leqslant \frac{(d+1)(d+2)}{6}$, then $F_1(X)$ contains no rational curves.*

This paper is devoted to generalize Riedl–Yang's result to complete intersections and to arbitrary $k \geqslant 1$:

Theorem 1.2. *Let $X \subset \mathbb{P}^n$ be a very general complete intersection of multi–degree $\underline{d} = (d_1, \ldots, d_s)$, with $1 \leqslant s \leqslant n-2$ and $\Pi_{i=1}^s d_i > 2$. Let $1 \leqslant k \leqslant n-s-1$ be an integer. Then, if*

$$n \leqslant \frac{1}{k+2}\left(k(k+1) - 2 + \sum_{i=1}^{s} \binom{d_i + k + 1}{k+1}\right), \tag{1.1}$$

$F_k(X)$ contains neither rational nor elliptic curves.

The proof is contained in Section 3. Section 4 concerns the quadric case $\Pi_{i=1}^s\, d_i = 2$.

We work over the complex field $\mathbb{C}$. As customary, the term *"general"* is used to denote a point which sits outside a union of finitely many proper closed subsets of an irreducible algebraic variety whereas the term *"very general"* is used to denote a point sitting outside a countable union of proper closed subsets of an irreducible algebraic variety.

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2. Preliminaries

Let $n \geqslant 3$, $1 \leqslant s \leqslant n-2$ and $\underline{d} = (d_1, \ldots, d_s)$ be an s–tuple of positive integers such that $\Pi_{i=1}^s\, d_i \geqslant 2$. Let $S_{\underline{d}} := \bigoplus_{i=1}^s H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i)\right)$ and consider its Zariski open subset $S_{\underline{d}}^* := \bigoplus_{i=1}^s \left(H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i)\right) \setminus \{0\}\right)$. For any $u := (g_1, \ldots, g_s) \in S_{\underline{d}}^*$, let $X_u := V(g_1, \ldots, g_s) \subset \mathbb{P}^n$ denote the closed subscheme defined by the vanishing of the polynomials $g_1, \ldots, g_s$. When $u \in S_{\underline{d}}^*$ is general, X_u is a smooth, irreducible variety of dimension $n - s \geqslant 2$, so that $S_{\underline{d}}^*$ contains an open dense subset parameterizing s–tuples u such that X_u is a smooth complete intersection.

For any integer $1 \leqslant k \leqslant n-s-1$, consider the locus

$$W_{\underline{d},k} := \left\{ u \in S_{\underline{d}}^* \,\middle|\, F_k(X_u) \neq \emptyset \right\} \subseteq S_{\underline{d}}^*$$

and set

$$t(n, k, \underline{d}) := (k+1)(n-k) - \sum_{i=1}^{s} \binom{d_i + k}{k}.$$

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If no confusion arises, we will denote by t the integer $t(n,k,\underline{d})$.

In this set–up, we remind the following results:

Result 1. *(cf., e.g.,* [1, 2, 5, 6, 7] *and* [3, Thm. 22.14, p. 294]*) Let n,k,s and $\underline{d}=(d_1,\dots,d_s)$ be integers as above.*

(a) When $\Pi_{i=1}^s d_i>2$, one has the following situation.

(i) For $t<0$, $W_{\underline{d},k}\subsetneq S^*_{\underline{d}}$ so, for $u\in S^*_{\underline{d}}$ general, $F_k(X_u)=\emptyset$.

(ii) For $t\geqslant 0$, $W_{\underline{d},k}=S^*_{\underline{d}}$ and, for $u\in S^*_{\underline{d}}$ general, $F_k(X_u)$ is smooth with $\dim(F_k(X_u))=t$ and it is irreducible when $t\geqslant 1$.

(b) When $\Pi_{i=1}^s d_i=2$, one has the following situation.

(i) For $\lfloor\frac{n-s}{2}\rfloor<k\leqslant n-s-1$, $W_{\underline{d},k}\subsetneq S^*_{\underline{d}}$; more precisely, for $u\in S^*_{\underline{d}}$ such that X_u is smooth, one has $F_k(X_u)=\emptyset$.

(ii) For $1\leqslant k\leqslant\lfloor\frac{n-s}{2}\rfloor$, $W_{\underline{d},k}=S^*_{\underline{d}}$ and, for $u\in S^*_{\underline{d}}$ such that X_u is smooth, then $F_k(X_u)$ is smooth, (equidimensional) with $\dim(F_k(X_u))=t=(k+1)(n-s-\frac{3k}{2})$. Moreover $F_k(X_u)$ is irreducible unless $\dim(X_u)=n-s$ is even and $k=\frac{n-s}{2}$, in which case $F_k(X_u)$ consists of two disjoint irreducible components. In this case, each irreducible component of $F_k(X_u)$ is isomorphic to $F_{k-1}(X'_u)$, where X'_u is a general hyperplane section of X_u.

Result 2. *(cf., e.g.,* [2, Remarques 3.2, (2)]*) Let n,k,s and $\underline{d}=(d_1,\dots,d_s)$ be integers as above with $\Pi_{i=1}^s d_i\geqslant 2$. When $u\in S^*_{\underline{d}}$ is such that $X_u\subset\mathbb{P}^n$ is a smooth, irreducible complete intersection of dimension $n-s$ and $F_k(X_u)$ is not empty, smooth and of (expected) dimension t, the canonical bundle of $F_k(X_u)$ is given by*

$$\omega_{F_k(X_u)}=\mathcal{O}_{F_k(X_u)}\left(-n-1+\sum_{i=1}^{s}\binom{d_i+k}{k+1}\right)\tag{2.1}$$

where $\mathcal{O}_{F_k(X_u)}(1)$ is the hyperplane line bundle of $F_k(X_u)$ in $\mathbb{P}^{\binom{n+1}{k+1}-1}$ via the Plücker embedding of $\mathbb{G}(k,n)$.

We will also need the following:

Result 3. *(cf.,*[8, Prop. 3.5]*) Let n,m be positive integers with $m\leqslant n$. Let $B\subset\mathbb{G}(m,n)$ be an irreducible subvariety of codimension at least $\epsilon\geqslant 1$. Let $C\subset\mathbb{G}(m-1,n)$ be a non–empty subvariety satisfying the following condition: for every $(m-1)$–plane $c\in C$, if $b\in\mathbb{G}(m,n)$ is such that $c\subset b$, then $b\in B$. Then the codimension of C in $\mathbb{G}(m-1,n)$ is at least $\epsilon+1$.*

3. THE PROOF OF THEOREM 1.2

This section is devoted to the proof of Theorem 1.2, which uses a strategy similar to the one in [8, Thm. 3.3].

Proof of Theorem 1.2. Let $\mathbb{G}:=\mathbb{G}(k,n)$ be the Grassmannian of k-linear subspaces in $\mathbb{P}^n$ and $\mathbb{H}:=\mathbb{H}_{\underline{d},n}$ be the component of the Hilbert scheme whose general point parameterizes a smooth, irreducible complete intersection $X\subset\mathbb{P}^n$ of dimension $n-s\geqslant 2$ and multi–degree $\underline{d}$ ($\mathbb{H}$ is the image of an open dense subset of $S^*_{\underline{d}}$ via the obvious map). Let us consider the incidence correspondence

$$\mathcal{U}_{k,n,\underline{d}}:=\{([\Lambda],[X])\in\mathbb{G}\times\mathbb{H}|\,\Lambda\subset X\}\subset\mathbb{G}\times\mathbb{H}$$

with the two projections

$$\mathbb{G}\xleftarrow{\pi_1}\mathcal{U}_{k,n,\underline{d}}\xrightarrow{\pi_2}\mathbb{H}.$$

Note that $\mathcal{U}_{k,n,\underline{d}}$ is smooth and irreducible, because the map π_1 is surjective and has smooth and irreducible fibres which are all isomorphic via the action of the group of projective transformations (cf. e.g. [1, § 2]).

Since $\Pi_{i=1}^s d_i>2$, from Result 1–(a), if $t\leqslant 0$ then $F_k(X)$ is either empty or a zero–dimensional scheme, so in particular the statement holds true. We can therefore assume $t\geqslant 1$. In this case, from Result 1–(a.ii), the map π_2 is dominant and the fibre over the general point $[X]\in\mathbb{H}$ is isomorphic to $F_k(X)$, hence it is smooth, irreducible of dimension t. Thus $\mathcal{U}_{k,n,\underline{d}}$ dominates $\mathbb{H}$ via π_2 and

$$\dim(\mathcal{U}_{k,n,\underline{d}})=\dim(\mathbb{H})+t.\tag{3.1}$$

Consider now

$$\mathcal{R}_{k,n,\underline{d}}:=\left\{([\Lambda],[X])\in\mathcal{U}_{k,n,\underline{d}}\ \mid\ \text{there exists a rational curve in } F_k(X) \text{ containing } [\Lambda]\right\}\subseteq\mathcal{U}_{k,n,\underline{d}}$$

and similarly

$$\mathcal{E}_{k,n,\underline{d}}:=\left\{([\Lambda],[X])\in\mathcal{U}_{k,n,\underline{d}}\ \mid\ \text{there exists an elliptic curve in } F_k(X) \text{ containing } [\Lambda]\right\}\subseteq\mathcal{U}_{k,n,\underline{d}}.$$

Notice that both $\mathcal{R}_{k,n,\underline{d}}$ and $\mathcal{E}_{k,n,\underline{d}}$ are possibly countable unions of irreducible subvarieties.

We claim that, under our assumptions, both $\mathcal{R}_{k,n,\underline{d}}$ and $\mathcal{E}_{k,n,\underline{d}}$ have codimension at least $t+1$ in $\mathcal{U}_{k,n,\underline{d}}$, which proves the statement. Indeed, consider the case of $\mathcal{R}_{k,n,\underline{d}}$ (the same reasoning applies to $\mathcal{E}_{k,n,\underline{d}}$); if $\operatorname{codim}_{\mathcal{U}_{k,n,\underline{d}}}(\mathcal{R}_{k,n,\underline{d}}) \geqslant t+1$ then, by (3.1), we have $\dim(\mathcal{R}_{k,n,\underline{d}}) \leqslant \dim(\mathbb{H}) - 1$ thus $\dim(\pi_2\left(\mathcal{R}_{k,n,\underline{d}}\right)) \leqslant \dim(\mathbb{H}) - 1$ proving the assertion.

We are therefore left to showing that both $\mathcal{R}_{k,n,\underline{d}}$ and $\mathcal{E}_{k,n,\underline{d}}$ have codimension at least $t+1$ in $\mathcal{U}_{k,n,\underline{d}}$. In what follows, we will focus on $\mathcal{R}_{k,n,\underline{d}}$ (the same arguments work for $\mathcal{E}_{k,n,\underline{d}}$).

If $\mathcal{R}_{k,n,\underline{d}} = \emptyset$, we are done. So assume $\mathcal{R}_{k,n,\underline{d}} \neq \emptyset$ and take $([\Lambda_0], [X_0]) \in \mathcal{R}_{k,n,\underline{d}}$ a very general point in a component of $\mathcal{R}_{k,n,\underline{d}}$. It suffices to find an irreducible subvariety $\mathcal{F} \subset \mathcal{U}_{k,n,\underline{d}}$ such that $([\Lambda_0], [X_0]) \in \mathcal{F}$ and $\operatorname{codim}_{\mathcal{F}}(\mathcal{R}_{k,n,\underline{d}} \cap \mathcal{F}) \geqslant t+1$.

To do so, we start with the following remark: since X is very general, from Results 1–(a.ii) and 2 and from (2.1), we have that $\omega_{F_k(X)}$ is ample if and only if

$$n \leqslant \sum_{i=1}^{s} \binom{d_i+k}{k+1} - 2.$$

We set

$$m = m(\underline{d}, k) := \sum_{i=1}^{s} \binom{d_i+k}{k+1} - 2$$

and, in view of the obvious equality $\binom{d_i+k+1}{k+1} = \binom{d_i+k}{k+1} + \binom{d_i+k}{k}$, we notice that the hypothesis (1.1) reads $m - n \geqslant t$, hence we have $m > n$.

Let $Y' \subset \mathbb{P}^m$ be a general complete intersection of multi–degree $\underline{d}$ and let $\Lambda' \subset Y'$ be a very general k-linear subspace of Y', i.e., $[\Lambda']$ corresponds to a very general point in $F_k(Y')$. By applying, if necessary, a projective transformation, we may suppose that $\Lambda' = \Lambda_0$. By Result 1 and the fact that $m > n$, we have that $F_k(Y')$ is smooth, irreducible with $\dim(F_k(Y')) = (k+1)(m-k) - \sum_{i=1}^{s} \binom{d_i+k}{k} > t \geqslant 1$. Notice that there are no rational curves in $F_k(Y')$ through $[\Lambda_0]$ since, by the choice of m, $F_k(Y')$ is smooth and of general type and $[\Lambda_0]$ is very general on $F_k(Y')$; in other words $([\Lambda'], [Y']) \in \mathcal{U}_{k,m,\underline{d}} \setminus \mathcal{R}_{k,m,\underline{d}}$, where $\mathcal{U}_{k,m,\underline{d}}$ and $\mathcal{R}_{k,m,\underline{d}}$ are defined as above for $\mathbb{P}^m$ instead of $\mathbb{P}^n$.

Take now $M \gg m > n$ an integer and let $Y'' \subset \mathbb{P}^M$ be a smooth, complete intersection of multi–degree $\underline{d}$, containing Λ_0, such that X_0 is a n-plane section of Y'' and Y' is a m-plane section of Y''.

For any integer $r \geqslant n$, let $Z_r \subset \mathbb{G}(r, M)$ be the variety of r-planes in $\mathbb{P}^M$ containing Λ_0 and let $Z'_r \subseteq Z_r$ be the subset of those r-planes $\Lambda \in Z_r$ such that the Fano scheme $F_k(Y'' \cap \Lambda)$ of the complete intersection $Y'' \cap \Lambda \subset \Lambda$ contains a rational curve through the point $[\Lambda_0] \in F_k(Y'' \cap \Lambda)$. Note that Z_r is isomorphic to $\mathbb{G}(r-k-1, M-k-1)$.

For any integer $r \geqslant n$, we have the morphism $\phi_r : Z_r \to \mathcal{U}_{k,r,\underline{d}}$ sending the r–plane Λ to $([\Lambda_0], [\Lambda \cap Y'']) \in \mathcal{U}_{k,r,\underline{d}}$. It is clear that ϕ_r maps Z_r isomorphically onto its image that we will denote by $\mathcal{F}_r$. Then the image of Z'_r in $\mathcal{U}_{k,r,\underline{d}}$ is $\mathcal{F}_r \cap \mathcal{R}_{k,r,\underline{d}}$. We will set $\mathcal{F} = \mathcal{F}_n$.

By construction of the pair $([\Lambda_0], [Y']) \in \mathcal{U}_{k,m,\underline{d}} \setminus \mathcal{R}_{k,m,\underline{d}}$, we have that $\operatorname{codim}_{\mathcal{F}_m}(\mathcal{F}_m \cap \mathcal{R}_{k,m,\underline{d}}) = \epsilon \geqslant 1$. Now we apply Result 3, to $\mathcal{F}_m \cong \mathbb{G}(r-k-1, M-k-1)$, $B = \mathcal{F}_m \cap \mathcal{R}_{k,m,\underline{d}}$ and $C = \mathcal{F}_{m-1} \cap \mathcal{R}_{k,m-1,\underline{d}}$, because clearly B and C verify the condition stated there. We deduce that $\operatorname{codim}_{\mathcal{F}_{m-1}}(\mathcal{F}_{m-1} \cap \mathcal{R}_{k,m-1,\underline{d}}) \geqslant \epsilon + 1$. Iterating this argument we see that $\operatorname{codim}_{\mathcal{F}}(\mathcal{F} \cap \mathcal{R}_{k,n,\underline{d}}) \geqslant m - n + 1$. Now, as we already noticed, $m - n + 1 \geqslant t + 1$ is equivalent to the condition (1.1), hence we have $\operatorname{codim}_{\mathcal{F}}(\mathcal{F} \cap \mathcal{R}_{k,n,\underline{d}}) \geqslant t+1$, as wanted. This completes the proof. ☐

Remark 3.1. Let $X \subset \mathbb{P}^n$ be a general complete intersection of multi–degree $\underline{d} = (d_1, \ldots, d_s)$. If

$$\sum_{i=1}^{s} \binom{d_i+k}{k+1} \leqslant n,$$

then, by (2.1), $F_k(X)$ is a smooth Fano variety and therefore by [4] it is *rationally connected*, i.e., there is a rational curve passing through two general points of it.

4. The quadric case

Here we prove the following result:

Theorem 4.1. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of multi–degree $\underline{d} = (d_1, \ldots, d_s)$, with $1 \leqslant s \leqslant n-2$ and $\Pi_{i=1}^{s} d_i = 2$. Let k be an integer such that $1 \leqslant k \leqslant n-s-1$. Then:*

(i) for $\lfloor \frac{n-s}{2} \rfloor < k \leqslant n-s-1$, $F_k(X)$ is empty, whereas

(ii) for $1 \leqslant k \leqslant \lfloor \frac{n-s}{2} \rfloor$, the single component or both components of $F_k(X)$ (see Result 1–(b.ii)) are rationally connected.

Proof. Since $\Pi_{i=1}^{s} d_i = 2$, we have that $X \subset \mathbb{P}^{n-s+1}$ is a smooth quadric hypersurface. From Result 1–(b), if $\lfloor \frac{n-s}{2} \rfloor < k \leqslant n-s-1$, $F_k(X)$ is empty.

Next we assume $1 \leqslant k \leqslant \lfloor \frac{n-s}{2} \rfloor$. From Result 2 and formula (2.1), we have

$$\omega_{F_k(X)} = \mathcal{O}_{F_k(X)}(-n+s+k).$$

Since $k \leqslant \frac{n-s}{2}$, then $-n+s+k \leqslant -1$ therefore the single component or both components of $F_k(X)$ (see Result 1–(b.ii)) are smooth Fano varieties, hence they are rationally connected by [4]. ☐

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