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Positive solutions for some generalized p–Laplacian type problems*

Anna Maria Candela[1] and Addolorata Salvatore[2]
Dipartimento di Matematica
Università degli Studi di Bari Aldo Moro
Via E. Orabona 4, 70125 Bari, Italy
[1] *annamaria.candela@uniba.it,* [2] *addolorata.salvatore@uniba.it*

Dedicated to Patrizia Pucci on the occasion of her 65th birthday

Abstract

In this paper, we prove the existence of nontrivial weak bounded solutions of the nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = f(x,u) & \text{in } \Omega,\\ u \geq 0 & \text{in } \Omega,\\ u \ = \ 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $N \geq 3$, and $A(x,t,\xi)$, $f(x,t)$ are given functions, with $A_t = \frac{\partial A}{\partial t}$, $a = \nabla_\xi A$.

To this aim, we use variational arguments which are adapted to our setting and exploit a weak version of the Cerami–Palais–Smale condition.

Furthermore, if $A(x,t,\xi)$ grows fast enough with respect to t, then the nonlinear term related to $f(x,t)$ may have also a supercritical growth.

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1 Introduction

During the past years there has been a considerable amount of research in investigating the existence of solutions of the quasilinear elliptic problem

$$(GP) \qquad \begin{cases} -\,\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) \ = \ f(x,u) & \text{in } \Omega,\\ u \ = \ 0 & \text{on } \partial\Omega, \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $N \geq 3$, $f(x,t)$ is a given Carathéodory function on $\Omega \times \mathbb{R}$ and $A(x,t,\xi)$ is a C^1 Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$, with

$$A_t(x,t,\xi) = \tfrac{\partial A}{\partial t}(x,t,\xi), \quad a(x,t,\xi) = (\tfrac{\partial A}{\partial \xi_1}(x,t,\xi), \dots, \tfrac{\partial A}{\partial \xi_N}(x,t,\xi)). \tag{1.1}$$

Setting $F(x,t) = \int_0^t f(x,s)ds$, under suitable growth assumptions on $A(x,t,\xi)$ and $F(x,t)$ problem (GP) can be associated to the functional

$$\mathcal{J}(u) \ = \ \int_\Omega A(x,u,\nabla u)dx \ - \ \int_\Omega F(x,u)\ dx \tag{1.2}$$

whose natural domain $\mathcal{D}$ is a subset of a suitable Sobolev space W and contains $W \cap L^\infty(\Omega)$.

Taking $p > 1$, a family of model problems is given by

$$A_p(x,t,\xi) \ = \ \left(\sum_{i,j=1}^N a_{i,j}(x,t)\xi_i\xi_j\right)^{\frac{p}{2}},$$

where $(a_{i,j}(x,t))_{1\leq i,j\leq N}$ is an elliptic matrix. In particular, if $\mathcal{A} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function such that

$$a_{i,j}(x,t) \ = \ \left(\frac{1}{p}\mathcal{A}(x,t)\right)^{\frac{2}{p}} \delta_i^j,$$

then $A_p(x,t,\xi) = \frac{1}{p}\mathcal{A}(x,t)|\xi|^p$ and the equation in (GP) reduces to the quasi–p-linear equation

$$(P) \qquad - \operatorname{div}(\mathcal{A}(x,u)\,|\nabla u|^{p-2}\,\nabla u) + \frac{1}{p}\,\mathcal{A}_t(x,u)\,|\nabla u|^p \ = \ f(x,u) \qquad \text{in } \Omega,$$

with related functional

$$J_{\mathcal{A}}(u) \ = \ \frac{1}{p}\int_\Omega \mathcal{A}(x,u)\ |\nabla u|^p\,dx \ - \ \int_\Omega F(x,u)\ dx$$

defined in a natural domain $\mathcal{D} \subset W_0^{1,p}(\Omega)$.

We note that, if $\mathcal{A}(x,t)$ is constant, then (P) becomes the classical p–Laplacian equation

$$-\Delta_p u \ = \ f(x,u) \qquad \text{in } \Omega$$

and the related functional is defined in the whole Sobolev space $W_0^{1,p}(\Omega)$.

On the other hand, if $\mathcal{A}(x,t)$ depends on t, even in the simplest case $F(x,t) \equiv 0$, with $\mathcal{A}(x,t)$ smooth, bounded and far away from zero, functional $J_{\mathcal{A}}$ is defined in $W_0^{1,p}(\Omega)$ but is Gâteaux differentiable only along directions of the Banach space $X = W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Anyway, we are able to prove that $J_{\mathcal{A}}$, or more in general $\mathcal{J}$, is C^1 in X equipped with the "intersection norm" $\|\cdot\|_X$ equal to the sum of the classical

$W_0^{1,p}$–norm, namely $\|\cdot\|_W$, and the standard L^∞ one, namely $|\cdot|_\infty$ (see Proposition 3.1), so problem (GP) has a variational structure and its weak solutions are critical points of $\mathcal{J}$ in X.

Usually, in order to apply variational methods, a compactness assumption on the critical points set such as the classical Palais–Smale condition is considered. Unluckily, in our setting Palais–Smale sequences of $\mathcal{J}$ in X may not have converging subsequences, as they may be unbounded in the L^∞–norm and so in $\|\cdot\|_X$ (see [11, Example 4.3]). However, since X is continuously embedded in $W_0^{1,p}(\Omega)$, making use of the weaker norm $\|\cdot\|_W$ we can introduce weaker versions of the Palais–Smale condition and of its Cerami's variant (see Definitions 2.1 and 2.3) and, consequently, we can give some modified versions of classical variational theorems (see Section 2).

Hence, existence and multiplicity results of weak bounded solutions of (GP), i.e. of critical points for $\mathcal{J}$ in X, have been stated under different growth assumptions in previous papers (see, e.g., [6]–[15]). We recall that other existence results have been obtained also in [2, 3, 16, 18], but by using a different definition of critical point for $\mathcal{J}$ and/or nonsmooth variational techniques, while other problems involving Banach spaces equipped with an "intersection norm" have been studied with a different approach in [4, 5, 19].

The aim of this paper is to give a summary of the abstract variational arguments useful for this kind of problems, as developed in [8, 10, 11], and, in particular, a weaker version of the Ambrosetti–Rabinowitz Mountain Pass Theorem (see Theorem 2.5). Furthermore, in Section 4 we will apply our abstract setting to search for positive solutions of problem (GP), thus improving the previous result in [2, Theorem 3.3] .

2 Abstract setting

We denote $\mathbb{N} = \{1, 2, \dots\}$ and, throughout this section, we assume that:

- $(X, \|\cdot\|_X)$ is a Banach space with dual $(X', \|\cdot\|_{X'})$,
- $(W, \|\cdot\|_W)$ is a Banach space such that $X \hookrightarrow W$ continuously, i.e. $X \subset W$ and a constant $\sigma_0 > 0$ exists such that

$$\|u\|_W \ \le\ \sigma_0\ \|u\|_X \qquad \text{for all } u \in X, \tag{2.1}$$

- $J : \mathcal{D} \subset W \to \mathbb{R}$ and $J \in C^1(X, \mathbb{R})$ with $X \subset \mathcal{D}$,
- $K^J = \{u \in X :\ dJ(u) = 0\}$ is the set of the critical points of J in X.

Furthermore, fixing β, β_1, $\beta_2 \in \mathbb{R}$, we denote

- $K^J_\beta = \{u \in X :\ J(u) = \beta,\ dJ(u) = 0\}$ the set of the critical points of J in X at the critical level β,
- $J^\beta = \{u \in X :\ J(u) \le \beta\}$ the sublevel of J with respect to the level β,

- $J_{\beta_1}^{\beta_2} = \{u \in X : \ \beta_1 \leq J(u) \leq \beta_2\}$ the closed "strip" between β_1 and β_2.

Anyway, in order to avoid any ambiguity and simplify, when possible, the notations, from now on by X we denote the space equipped with its given norm $\|\cdot\|_X$ while, if a different norm is involved, we write it down explicitely.

For simplicity, taking $\beta \in \mathbb{R}$, we say that a sequence $(u_n)_n \subset X$ is

- a *Palais–Smale sequence at level* β, briefly $(PS)_\beta$*–sequence*, if
$$\lim_{n\to+\infty} J(u_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'} = 0;$$
- a *Cerami–Palais–Smale sequence at level* β, briefly $(CPS)_\beta$*–sequence*, if
$$\lim_{n\to+\infty} J(u_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) = 0.$$

We say that the functional J satisfies the *Palais–Smale condition at level* β in X, briefly $(PS)_\beta$ *condition*, if every $(PS)_\beta$–sequence converges in $(X, \|\cdot\|_X)$, up to subsequences, while J satisfies the *Cerami's variant of Palais–Smale condition at level* β in X, briefly $(CPS)_\beta$ *condition*, if every $(CPS)_\beta$–sequence has a converging subsequence in $(X, \|\cdot\|_X)$.

By the way, $(CPS)_\beta$–sequences are $(PS)_\beta$–sequences which are bounded or cannot be "too fast" if diverging; hence, $(CPS)_\beta$ condition is weaker than $(PS)_\beta$ condition.

As already pointed out in Section 1, in our setting $(PS)_\beta$ condition may not be satisfied, so the following weaker version of the $(PS)_\beta$ condition can be introduced.

Definition 2.1. The functional J satisfies a *weak version of the Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(wPS)_\beta$ *condition*, if, for every $(PS)_\beta$–sequence $(u_n)_n$, a point $u \in X$ exists, such that

(i) $\lim_{n\to+\infty} \|u_n - u\|_W = 0$ (up to subsequences),

(ii) $J(u) = \beta$, $dJ(u) = 0$.

If J satisfies $(wPS)_\beta$ at each level $\beta \in I$, I real interval, we say that J satisfies (wPS) condition in I.

We note that if J satisfies $(wPS)_\beta$ condition at a level $\beta \in \mathbb{R}$, then K_β^J is compact with respect to $\|\cdot\|_W$. Moreover, a quite general Deformation Lemma can be proved (see [11, Proposition 2.4]).

Proposition 2.2. *Let* $J : X \to \mathbb{R}$ *be a* C^1 *functional which satisfies* (wPS) *in* $\mathbb{R}$. *Taking* $\beta \in \mathbb{R}$, *for any fixed* $\varrho > 0$ *and* $\varepsilon_0 > 0$ *a constant* $\varepsilon^* > 0$, $2\varepsilon^* < \varepsilon_0$, *exists, such that for each* $\varepsilon \in]0, \varepsilon^*]$ *a homeomorphism* $\Psi : X \to X$ *exists which satisfies the following conditions:*

(i) $\Psi(u) = u$ *for all* $u \notin J_{\beta-\varepsilon_0}^{\beta+\varepsilon_0}$;

(ii) $\Psi(J^{\beta+\varepsilon} \setminus N_\varrho^W(K_\beta^J)) \subset J^{\beta-\varepsilon}$ *and* $\Psi(J^{\beta+\varepsilon}) \subset J^{\beta-\varepsilon} \cup N_\varrho^W(K_\beta^J)$,

with $N_\varrho^W(K_\beta^J) = \{u \in X : \inf_{v \in K_\beta^J} \|u - v\|_W < \varrho\}$.
Furthermore, Ψ *can be chosen odd if* J *is even.*

By following standard ideas, Proposition 2.2 is enough for proving not only existence results but also multiplicity ones even if more than one critical point at the same critical level occurs (see [11, Theorems 2.7 and 2.10]).

As in the classical setting, we can introduce the following variant of (wPS)–condition.

Definition 2.3. The functional J satisfies the *weak Cerami–Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ *condition*, if for every $(CPS)_\beta$–sequence $(u_n)_n$, a point $u \in X$ exists, such that

(i) $\lim_{n \to +\infty} \|u_n - u\|_W = 0$ (up to subsequences),

(ii) $J(u) = \beta$, $dJ(u) = 0$.

If J satisfies the $(wCPS)_\beta$ condition at each level $\beta \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I.

Also condition $(wCPS)_\beta$ implies that K_β^J is compact with respect to $\|\cdot\|_W$ and this weaker "compactness" assumption allows one to prove the following weaker Deformation Lemma (see [8, Lemma 2.3]).

Lemma 2.4. *Let* $J \in C^1(X, \mathbb{R})$ *and consider* $\beta \in \mathbb{R}$ *such that*

- J *satisfies the* $(wCPS)_\beta$ *condition,*
- $K_\beta^J = \emptyset$.

Then, fixing any $\varepsilon_0 > 0$*, there exist a constant* $\varepsilon > 0$, $2\varepsilon < \varepsilon_0$*, and a homeomorphism* $\psi : X \to X$ *such that*

(i) $\psi(u) = u$ *for all* $u \notin J_{\beta-\varepsilon_0}^{\beta+\varepsilon_0}$;

(ii) $\psi(J^{\beta+\varepsilon}) \subset J^{\beta-\varepsilon}$.

In particular, if J *is even then* ψ *can be chosen odd.*

From Lemma 2.4 it follows that one can state only existence theorems for critical points at level β, in particular it implies a suitable generalization of the Mountain Pass Theorem (see [13, Theorem 2.3] and compare it with [8, Theorem 1.7] and the classical statement in [1, Theorem 2.1]).

Theorem 2.5. *Let* $J \in C^1(X, \mathbb{R})$ *be such that* $J(0) = 0$ *and* $(wCPS)$ *condition holds in* $\mathbb{R}_+$.
Moreover, assume that a continuous map $\ell : X \to \mathbb{R}$*, some constants* r_0, $\varrho_0 > 0$*, and a point* $e \in X$ *exist, such that*

(i) $\ell(0) = 0$ *and* $\ell(u) \geq \|u\|_W$ *for all* $u \in X$*;*

(ii) $u \in X, \quad \ell(u) = r_0 \quad \Longrightarrow \quad J(u) \geq \varrho_0$*;*

(iii) $\|e\|_W > r_0$ *and* $J(e) < \varrho_0$.

Then, J has a Mountain Pass critical point $u_0 \in X$ such that $J(u_0) \geq \varrho_0$.

3 Variational setting and first properties

We denote by:

- $L^q(\Omega)$ the Lebesgue space with norm $|u|_q = \left(\int_\Omega |u|^q dx\right)^{1/q}$ if $1 \leq q < +\infty$;
- $L^\infty(\Omega)$ the space of Lebesgue–measurable and essentially bounded functions $u : \Omega \to \mathbb{R}$ with norm $|u|_\infty = \operatorname*{ess\,sup}_\Omega |u|$;
- $W_0^{1,p}(\Omega)$ the classical Sobolev space with norm $\|u\|_W = |\nabla u|_p$ if $p \geq 1$.

From now on, let $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that, using the notations in (1.1), the following conditions hold:

(H_0) $A(x,t,\xi)$ is a C^1 Carathéodory function, i.e.,
$A(\cdot,t,\xi) : x \in \Omega \mapsto A(x,t,\xi) \in \mathbb{R}$ is measurable for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$,
$A(x,\cdot,\cdot) : (t,\xi) \in \mathbb{R} \times \mathbb{R}^N \mapsto A(x,t,\xi) \in \mathbb{R}$ is C^1 for a.e. $x \in \Omega$;

(H_1) a real number $p > 1$ and some positive continuous functions Φ_1, $\phi_1 : \mathbb{R} \to \mathbb{R}$ exist such that

$$|a(x,t,\xi)| \;\leq\; \Phi_1(t) + \phi_1(t)\ |\xi|^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(H_2) some positive continuous functions Φ_2, $\phi_2 : \mathbb{R} \to \mathbb{R}$ exist such that

$$|A_t(x,t,\xi)| \;\leq\; \Phi_2(t) + \phi_2(t)\ |\xi|^{p} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(h_0) $f(x,t)$ is a Carathéodory function, i.e.,
$f(\cdot,t) : x \in \Omega \mapsto f(x,t) \in \mathbb{R}$ is measurable for all $t \in \mathbb{R}$,
$f(x,\cdot) : t \in \mathbb{R} \mapsto f(x,t) \in \mathbb{R}$ is continuous for a.e. $x \in \Omega$;

(h_1) a_1, $a_2 > 0$ and $q \geq 1$ exist such that

$$|f(x,t)| \leq a_1 + a_2|t|^{q-1} \qquad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}.$$

In order to investigate the existence of weak solutions of the nonlinear problem (GP), the notations introduced for the abstract setting at the beginning of Section 2 are referred to our problem with $W = W_0^{1,p}(\Omega)$ and the Banach space $(X, \|\cdot\|_X)$ defined as

$$X := W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \qquad \|u\|_X = \|u\|_W + |u|_\infty \tag{3.1}$$

(here and in the following, $|\cdot|$ will denote the standard norm on any Euclidean space as the dimension of the considered vector is clear and no ambiguity arises). Moreover, from the Sobolev Imbedding Theorem, for any $r \in [1, p^*[$ ($p^* = \frac{pN}{N-p}$ if $N > p$ otherwise $p^* = +\infty$) a constant $\sigma_r > 0$ exists, such that

$$|u|_r \ \leq \ \sigma_r \|u\|_W \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

and the imbedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact.

From definition, $X \hookrightarrow W_0^{1,p}(\Omega)$ and $X \hookrightarrow L^\infty(\Omega)$ with continuous imbeddings and (2.1) holds with $\sigma_0 = 1$. If $p > N$, then $W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, so $X = W_0^{1,p}(\Omega)$.

Now, we consider the functional $\mathcal{J} : X \to \mathbb{R}$ defined as in (1.2).

We note that, since in assumption (h_1) no upper bound on q is actually required, the following regularity proposition extends [7, Proposition 3.1] in which $F(x,t)$ has a subcritical growth (for the proof, see [13, Proposition 3.2]).

Proposition 3.1. *Let us assume that conditions (H_0)–(H_2), (h_0)–(h_1) hold and two positive continuous functions Φ_0, $\phi_0 : \mathbb{R} \to \mathbb{R}$ exist such that*

$$|A(x,t,\xi)| \leq \Phi_0(t) + \phi_0(t)\ |\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{3.2}$$

If $(u_n)_n \subset X$, $\bar{u} \in X$ are such that

$$\|u_n - \bar{u}\|_W \to 0, \quad u_n \to \bar{u} \quad \text{a.e. in } \Omega \quad \text{if } n \to +\infty,$$

$$M > 0 \text{ exists so that } |u_n|_\infty \leq M \text{ for all } n \in \mathbb{N},$$

then

$$\mathcal{J}(u_n) \to \mathcal{J}(\bar{u}) \quad \text{and} \quad \|d\mathcal{J}(u_n) - d\mathcal{J}(\bar{u})\|_{X'} \to 0 \quad \text{if} \ \ n \to +\infty,$$

where for any u, $v \in X$ it is

$$\langle d\mathcal{J}(u), v\rangle \ = \ \int_\Omega (a(x,u,\nabla u)\cdot \nabla v + A_t(x,u,\nabla u)v)dx \ - \ \int_\Omega f(x,u)v dx.$$

Hence, $\mathcal{J}$ is a C^1 functional on $(X, \|\cdot\|_X)$.

In order to prove more properties of functional $\mathcal{J}$ in (1.2), we require that some constants $\alpha_i > 0$, $i \in \{1,2,3\}$, $\eta_j > 0$, $j \in \{1,2\}$, and $s \geq 0$, $\mu > p$, $R_0 \geq 1$, exist such that the following hypotheses are satisfied:

(H_3) $A(x,t,\xi) \leq \eta_1 a(x,t,\xi)\cdot\xi$ a.e. in Ω if $|(t,\xi)| \geq R_0$;

(H_4) $|A(x,t,\xi)| \leq \eta_2$ a.e. in Ω if $|(t,\xi)| \leq R_0$;

(H_5) $a(x,t,\xi)\cdot\xi \geq \alpha_1(1 + |t|^{ps})|\xi|^p$ a.e. in Ω, for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$;

(H_6) $a(x,t,\xi)\cdot\xi + A_t(x,t,\xi)t \geq \alpha_2 a(x,t,\xi)\cdot\xi$ a.e. in Ω if $|(t,\xi)| \geq R_0$;

(H_7) $\mu A(x,t,\xi) - a(x,t,\xi)\cdot\xi - A_t(x,t,\xi)t \geq \alpha_3 a(x,t,\xi)\cdot\xi$ a.e. in Ω if $|(t,\xi)| \geq R_0$;

(H_8) for all ξ, $\xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$, it is

$$[a(x,t,\xi) - a(x,t,\xi^*)] \cdot [\xi - \xi^*] > 0 \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

(h_2) $f(x,t)$ satisfies the Ambrosetti–Rabinowitz condition, i.e.

$$0 < \mu F(x,t) \le f(x,t)t \quad \text{for a.e. } x \in \Omega \text{ if } |t| \ge R_0.$$

Remark 3.2. From (H_1)–(H_7), for a.e. $x \in \Omega$ and all $|(t,\xi)| \in \mathbb{R} \times \mathbb{R}^N$ we have

$$\begin{aligned} |A(x,t,\xi)| &\le \eta_1\ (\Phi_1(t) + \phi_1(t))\ |\xi|^p \ + \ \eta_1 \Phi_1(t) + \eta_2, \\ A(x,t,\xi) &\ge \alpha_1\ \frac{\alpha_2 + \alpha_3}{\mu}\ (1 + |t|^{ps})\ |\xi|^p - \eta_3 \end{aligned}$$

for a suitable $\eta_3 > 0$; moreover,

$$A(x,t,\xi) \ \le \ \eta_4\ |t|^{\mu - \frac{1+\alpha_3}{\eta_1}}\ |\xi|^p \quad \text{a.e. in } \Omega, \text{ if } |t| \ge 1 \text{ and } |\xi| \ge R_0$$

(for more details, see Remarks 3.3, 3.4 and 3.5 in [13]). Hence, growth condition (3.2) holds and Proposition 3.1 applies. Furthermore, if

$$0 \le ps < \mu,$$

choosing η_1 large enough, we have

$$0 \ \le \ ps \ \le \ \mu - \frac{1 + \alpha_3}{\eta_1}.$$

Remark 3.3. In the model case $A(x,t,\xi) = \frac{1}{p}\mathcal{A}(x,t)|\xi|^p$ conditions (H_3) and (H_8) are trivially verified.

Remark 3.4. Conditions (h_0)–(h_2) imply that a function $\gamma \in L^\infty(\Omega)$, $\gamma(x) > 0$ a.e. in Ω, and some constants a_3, a_4, $a_5 \ge 0$ exist such that

$$\gamma(x)\ |t|^\mu - a_3 \le F(x,t) \le a_4 + a_5|t|^q \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}.$$

Hence, it follows that

$$ps < \mu \le q,$$

where from the hypotheses it results $1 < p < \mu$.

In this set of hypotheses, the following statement can be proved (for the proof, see [7, Proposition 4.6] and [13, Proposition 3.10]).

Proposition 3.5. *Assume that hypotheses (H_0)–(H_8), (h_0)–(h_2) hold. If*

$$q < p^*(s+1) \quad \text{when } 1 < p < N, \tag{3.3}$$

then the functional $\mathcal{J}$ satisfies the $(wCPS)$ condition in $\mathbb{R}$.

Remark 3.6. The statement in Proposition 3.5 holds even if we replace assumption (h_2) with the weaker condition

(h_2') $0 \le \mu F(x,t) \le f(x,t)t$ for a.e. $x \in \Omega$ if $|t| \ge R_0$.

Remark 3.7. Without loss of generality, if (3.3) holds we can always assume q large enough such that

$$p(s+1) \ < \ q \ < \ p^*(s+1). \tag{3.4}$$

4 Existence of positive solutions

In this section we want to apply Theorem 2.5 for proving the existence of a nontrivial weak positive solution of (GP), i.e. a nontrivial weak solution of problem

$$(GP^+) \qquad \begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) \; = \; f(x,u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u \; = \; 0 & \text{on } \partial\Omega. \end{cases}$$

Now, we are ready to state our main results.

Theorem 4.1. *Assume that $A(x,t,\xi)$ satisfies hypotheses (H_0)–(H_1), (H_3)–(H_8) and is such that*

(H_2') *a positive continuous function $\phi_2 : \mathbb{R} \to \mathbb{R}$ exists such that*

$$|A_t(x,t,\xi)| \; \leq \; \phi_2(t) \; |\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(H_9) *$\alpha_4 > 0$ exists such that*

$$A(x,t,\xi) \geq \alpha_4(1 + |t|^{ps})|\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N,$$

with $s \geq 0$ as in (H_5).

Moreover, suppose that $f(x,t)$ verifies conditions

(h_0^+) *$f(x,t)$ is a Carathéodory function such that $f(x,0) = 0$ for a.e. $x \in \Omega$;*

(h_1^+) *a_1, $a_2 > 0$ and $q \geq 1$ exist such that*

$$|f(x,t)| \leq a_1 + a_2 t^{q-1} \qquad \text{a.e. in } \Omega, \text{ for all } t \geq 0;$$

(h_2^+) *taking $\mu > p$ and $R_0 > 0$ as in (H_7), it is*

$$0 < \mu F(x,t) \leq f(x,t)t \quad \text{for a.e. } x \in \Omega \text{ if } t \geq R_0;$$

(h_3^+) $\displaystyle\limsup_{t \to 0^+} \frac{f(x,t)}{t^{p-1}} \; < \; p \; \alpha_4 \; \lambda_1$ *uniformly with respect to a.e. $x \in \Omega$, where λ_1 is the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$.*

Then, if (3.3) *holds, problem (GP^+) has at least one nontrivial weak bounded solution.*

Remark 4.2. It is known that the first eigenvalue λ_1 of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ admits a unique eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_1 > 0 \text{ in } \Omega, \quad \int_\Omega |\varphi_1|^p dx = 1 \quad \text{and} \quad \int_\Omega |\nabla\varphi_1|^p dx = \lambda_1$$

(see, e.g., [17]). Furthermore, it is also $\varphi_1 \in L^\infty(\Omega)$, hence $\varphi_1 \in X$, and

$$\int_\Omega |u|^p dx \leq \frac{1}{\lambda_1} \int_\Omega |\nabla u|^p dx \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{4.1}$$

We note that the solution u^*, found in Theorem 4.1, is such that $u^* \geq 0$ in Ω but $u^* \not\equiv 0$. Anyway, the following statement proves that if stronger assumptions hold, then it has to be $u^* > 0$ in Ω.

Corollary 4.3. *In the hypotheses of Theorem 4.1 but replacing (H_1) with the stronger condition*

(H_1') *some positive continuous functions Φ_1, $\phi_1 : \mathbb{R} \to \mathbb{R}$ exist such that*

$$|a(x,t,\xi)| \;\leq\; \Phi_1(t)|t|^p + \phi_1(t)\ |\xi|^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N,$$

and (h_3^+) with the stronger limit

$$\lim_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} = \lambda < \ p\ \alpha_4\ \lambda_1 \qquad \text{uniformly with respect to a.e. } x \in \Omega, \tag{4.2}$$

then there exists at least a nontrivial weak bounded solution u^* of problem (GP^+) such that $u^* > 0$ in Ω.

From now on, assume that hypotheses (H_0)–(H_1), (H_2'), (H_3)–(H_9) and (h_0^+)–(h_3^+) hold.

We introduce the new function $f_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined as

$$f_+(x,t) \;=\; \begin{cases} f(x,t) & \text{for a.e. } x \in \Omega \text{ and all } t \geq 0, \\ 0 & \text{for a.e. } x \in \Omega \text{ and all } t < 0, \end{cases}$$

and the related primitive

$$F_+(x,t) \;=\; \int_0^t f_+(x,s)ds \;=\; \begin{cases} F(x,t) & \text{for a.e. } x \in \Omega \text{ and all } t \geq 0, \\ 0 & \text{for a.e. } x \in \Omega \text{ and all } t < 0. \end{cases}$$

Remark 4.4. By assumptions (h_0^+)–(h_2^+) it follows that $f_+(x,t)$ satisfies conditions (h_0), (h_1) and (h_2'). Moreover, as in Remark 3.4 a function $\gamma \in L^\infty(\Omega)$, $\gamma(x) > 0$ a.e. in Ω, and some constants a_3, a_4, $a_5 \geq 0$ exist such that

$$\gamma(x)\ t^\mu - a_3 \leq F_+(x,t) \leq a_4 + a_5 t^q \quad \text{a.e. in } \Omega, \text{ for all } t \geq 0. \tag{4.3}$$

From Remark 4.4 and Proposition 3.1 the corresponding functional

$$\mathcal{J}_+(u) \;=\; \int_\Omega A(x,u,\nabla u)dx \;-\; \int_\Omega F_+(x,u)\ dx$$

is of class C^1 on the Banach space X in (3.1), where for any u, $v \in X$ it is

$$\langle d\mathcal{J}_+(u), v\rangle = \int_\Omega (a(x,u,\nabla u)\cdot \nabla v + A_t(x,u,\nabla u)v)dx - \int_\Omega f_+(x,u)vdx. \tag{4.4}$$

Firstly, as in [2, Lemma 1.3], we prove that each critical point of $\mathcal{J}_+$ in X is positive.

Proposition 4.5. *If $u \in X$ is a critical point of $\mathcal{J}_+$, then $u \geq 0$ a.e. in Ω. Hence, $\mathcal{J}(u) = \mathcal{J}_+(u)$ and $d\mathcal{J}(u) = 0$.*

Proof. Let $u \in X$ be a critical point of $\mathcal{J}_+$. Then, taking

$$k_u \;=\; \max_{|t|\leq |u|_\infty} \phi_2(t) \quad \text{with } \phi_2(t) \text{ as in } (H_2'),$$

define the real map

$$\psi(t) \;=\; te^{\eta t^2} \qquad \text{with } \eta > \left(\frac{k_u}{2\alpha_1}\right)^2,$$

where α_1 is as in (H_5). By definition, ψ is odd in $\mathbb{R}$ and

$$\alpha_1 \psi'(t) - k_u|\psi(t)| > \frac{\alpha_1}{2} \qquad \text{for all } t \in \mathbb{R}. \tag{4.5}$$

Since $d\mathcal{J}_+(u) = 0$, choosing $v = \psi(-u_-)$ with $u_- = \max\{0, -u\}$, from (4.4) and $f_+(x,u)\psi(-u_-) = 0$ for a.e. $x \in \Omega$ it follows that

$$\int_\Omega (\psi'(-u_-)a(x,u,\nabla u)\cdot \nabla(-u_-) + A_t(x,u,\nabla u)\psi(-u_-))dx \;=\; 0.$$

Hence, as $u = -u_-$ in $\Omega_- = \{x \in \Omega : u(x) \leq 0\}$ and $u_- = 0$ a.e. in $\Omega \setminus \Omega_-$, from (H_2'), (H_5) and (4.5) we have

$$\begin{aligned} 0 &= \int_{\Omega_-} (\psi'(-u_-)a(x,-u_-,\nabla(-u_-))\cdot\nabla(-u_-) + A_t(x,-u_-,\nabla(-u_-))\psi(-u_-))dx \\ &\geq \int_{\Omega_-} (\alpha_1\psi'(u_-) - k_u|\psi(u_-)|)|\nabla u_-|^p dx \geq \frac{\alpha_1}{2}\int_{\Omega_-} |\nabla u_-|^p dx \;=\; \frac{\alpha_1}{2}\|u_-\|_W^p \end{aligned}$$

which implies $u_- = 0$ a.e. in Ω. □

Then, let us recall the following Harnack type inequality for weak solutions of p–Laplacian type equations (see [20, Theorem 1.1]).

Lemma 4.6. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the equation*

$$-\mathrm{div}(a(x,u,\nabla u)) \;=\; h(x,u,\nabla u) \qquad \text{in a cube } K(3r) \subset \Omega. \tag{4.6}$$

Assume that $M > 0$ exists such that $0 \leq u(x) < M$ for all $x \in K(3r)$. If (H_1'), (H_5) hold and some positive constants d_i exist such that

$$|h(x,t,\xi)| \;\leq\; d_1|\xi|^p + d_2|\xi|^{p-1} + d_3|t|^{p-1} \tag{4.7}$$

for a.e. $x \in \Omega$ and all $(t,\xi) \in]-M, M[\times\mathbb{R}^N$, then

$$\max_{x\in K(r)} u(x) \;\leq\; C \min_{x\in K(r)} u(x),$$

where C depends only on p, N, M and the constants which appear in the hypotheses.

Remark 4.7. A statement similar to Lemma 4.6 holds for any weak bounded solution $u \in W_0^{1,p}(\Omega)$ of (4.6) which is $u \leq 0$ a.e. in Ω.

Now, we are ready to prove our main results.

Proof of Theorem 4.1. From Remark 4.4, Proposition 3.5 and Remark 3.6 the C^1 functional $\mathcal{J}_+$ satisfies the $(wCPS)$ condition in $\mathbb{R}$. Moreover, without loss of generality we can assume $\int_\Omega A(x,0,0)dx = 0$, so it is $\mathcal{J}_+(0) = 0$.
Now, in order to prove the geometric assumptions in Theorem 2.5, we define

$$\ell_{W,s}(u) \; = \; \max\{\|u\|_W, \||u|^s u\|_W\} \quad \text{for all } u \in X.$$

From (3.1) it follows that the map $u \mapsto \||u|^s u\|_W$ is well–defined and continuous in $(X, \|\cdot\|_X)$; thus, also $\ell_{W,s} : X \to \mathbb{R}$ is continuous with respect to $\|\cdot\|_X$ and verifies assumption (i) in Theorem 2.5.
On the other hand, from (h_3^+), we can take $\bar{\lambda} \in \mathbb{R}$ so that

$$\limsup_{t \to 0^+} \frac{f(x,t)}{t^{p-1}} < \bar{\lambda} < p\alpha_4\lambda_1. \tag{4.8}$$

Hence, from (h_1^+) and direct computations it follows that

$$F_+(x,t) \; \leq \; \frac{\bar{\lambda}}{p} t^p + b_1 t^q \quad \text{for a.e. } x \in \Omega, \text{ all } t \geq 0,$$

for a suitable $b_1 > 0$. Then, from (H_9), (3.3), (4.1), (4.8) and reasoning as in the proof of [13, Theorem 4.1], we obtain

$$\mathcal{J}_+(u) \; \geq \; b_2 \left[\ell_{W,s}(u)\right]^p - b_3 \left[\ell_{W,s}(u)\right]^{\frac{q}{s+1}} \quad \text{for all } u \in X,$$

for suitable constants b_2, $b_3 > 0$. Whence, from (3.4) condition (ii) in Theorem 2.5 holds.
At last, in order to prove that (iii) in Theorem 2.5 is satisfied, consider the positive eigenfunction $\varphi_1 \in X$ as in Remark 4.2 and fix any $\tau > 0$.
Then, from [7, Lemma 6.5] and direct computations we have

$$\int_\Omega A(x, \tau\varphi_1, \tau\nabla\varphi_1)dx \; \leq \; b_4 \, \tau^{\mu - \frac{\alpha_3}{\eta_1}} \; + \; b_5 \, \tau^p \; + \; b_6 \tag{4.9}$$

for suitable b_4, b_5, $b_6 > 0$.
On the other hand, as $\tau\varphi_1 > 0$ in Ω, from (4.3) it follows

$$\int_\Omega F_+(x, \tau\varphi_1)dx \geq \tau^\mu \int_\Omega \gamma(x)|\varphi_1|^\mu dx - b_7 = b_8\tau^\mu - b_7, \tag{4.10}$$

for suitable b_7, $b_8 > 0$. Hence, (4.9) and (4.10) imply that

$$\mathcal{J}_+(\tau\varphi_1) \to -\infty \quad \text{as } \tau \to +\infty.$$

Thus, condition (iii) in Theorem 2.5 holds with $e = \tau\varphi_1$ for τ large enough.
Finally, Theorem 2.5 applies and $\mathcal{J}_+$ has at least a nontrivial critical point; hence, from Proposition 4.5 problem (GP^+) has at least one nontrivial weak bounded solution. $\square$

Proof of Corollary 4.3. Assume that Ω is a connected bounded domain in $\mathbb{R}^N$. Then, from Theorem 4.1 a nontrivial weak solution $u^* \in X$ of (GP^+) exists, i.e., $u^* \in X$ is a weak solution of

$$-\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) \;=\; f_+(x,u) \quad \text{in } \Omega, \qquad u^* \geq 0 \text{ in } \Omega.$$

Now, we can apply Lemma 4.6 with $h(x,t,\xi) = -A_t(x,t,\xi) + f_+(x,t)$.
Indeed, taking $M > |u^*|_\infty$ and fixing $d_1 = \max\{\phi_2(t) : |t| \leq M\}$, $d_2 = 0$, from (4.2), (h_1^+) and direct computations a constant $d_3 > 0$ exists such that

$$|f_+(x,t)| \leq d_3 |t|^{p-1} \qquad \text{for a.e. } x \in \Omega, \text{ all } t \in]-M, M[,$$

which, together with (H_2'), implies (4.7).
Hence, Lemma 4.6 and standard arguments imply $u^* > 0$ in Ω. $\square$

Remark 4.8. By replacing assumptions (h_1^+)–(h_3^+) with corresponding conditions given for $t \leq 0$, and by using arguments similar to those ones in the proof of Theorem 4.1, respectively of Corollary 4.3, we are able to prove the existence of at least one nontrivial weak bounded solution of problem (GP) which is negative, respectively strictly negative, in Ω.

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