# CONES OF LINES HAVING HIGH CONTACT WITH GENERAL HYPERSURFACES AND APPLICATIONS

FRANCESCO BASTIANELLI, CIRO CILIBERTO, FLAMINIO FLAMINI, AND PAOLA SUPINO

Abstract. Given a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2$, we study the cones $V_p^h \subset \mathbb{P}^{n+1}$ swept out by lines having contact order $h \geqslant 2$ at a point $p \in X$. In particular, we prove that if $X$ is general, then for any $p \in X$ and $2 \leqslant h \leqslant \min\{n+1, d\}$, the cone $V_p^h$ has dimension exactly $n+2-h$. Moreover, when $X$ is a very general hypersurface of degree $d \geqslant 2n+2$, we describe the relation between the cones $V_p^h$ and the degree of irrationality of $k$–dimensional subvarieties of $X$ passing through a general point of $X$. As an application, we give some bounds on the least degree of irrationality of $k$–dimensional subvarieties of $X$ passing through a general point of $X$, and we prove that the connecting gonality of $X$ satisfies $d - \left\lfloor \frac{\sqrt{16n+25}-3}{2} \right\rfloor \leqslant \operatorname{conn.gon}(X) \leqslant d - \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor$.

## 1. Introduction

Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface of degree $d \geqslant 2$. Given a point $p \in X$ and an integer $h \geqslant 2$, we consider the cone $V_p^h \subset \mathbb{P}^{n+1}$ swept out by lines having intersection multiplicity at least $h$ with $X$ at $p$. These cones reflect the geometry of hypersurfaces and occur both in the local geometry of hypersurfaces (see e.g. [11, 13, 14]) and in the study of their global properties, such as unirationality ([6]) and their covering gonality, i.e. the least gonality of curves passing through a general point of $X$ ([4]).

In this paper, we study the cones $V_p^h$ of general hypersurfaces $X \subset \mathbb{P}^{n+1}$, and we apply our results to achieve some bounds concerning the degree of irrationality of $k$–dimensional subvarieties of $X$ passing through general points of $X$, where we recall that the degree of irrationality of an irreducible variety $Y$ of dimension $k$ is the least degree of a dominant rational map $Y \dashrightarrow \mathbb{P}^k$.

Given a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2$, a point $p \in X$, and an integer $2 \leqslant h \leqslant d$, the cone $V_p^h \subset \mathbb{P}^{n+1}$ is defined by the vanishing of $h-1$ polynomials of degree $1, 2, \ldots, h-1$, respectively, where the linear polynomial defines the tangent hyperplane of $X$ at $p$ (cf. Section 2). When $p \in X$ is a general point, then $V_p^h$ is a complete intersection defined by those polynomials, i.e. $\dim V_p^h = n+2-h$ (cf. [13]). However, it may happen that for some special point of $X$, the cone $V_p^h$ fails to be a complete intersection of multi–degree $(1, 2, \ldots, h-1)$ and its dimension is larger than expected. We prove that when $X \subset \mathbb{P}^{n+1}$ is a general hypersurface of degree $d \geqslant 2$, this is not the case.

**Theorem 1.1.** *Let $n \geqslant 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geqslant 2$. Then, for any point $p \in X$ and for any integer $2 \leqslant h \leqslant \min\{n+1, d\}$, the cone $V_p^h$ has dimension*

$$\dim(V_p^h) = n+2-h.$$

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The authors are members of GNSAGA of INdAM.

In recent years there has been a great deal of interest concerning measures of irrationality of projective varieties, that is birational invariants which somehow measure the failure of a given variety to be rational (see e.g. [3, 5, 15, 8, 10, 16, 17]), and several interesting results have been obtained in this direction for very general hypersurfaces of large degree (cf. [2, 3, 4, 18]).

Given an irreducible complex projective variety $X$ of dimension $n$ and an integer $k$ such that $1 \leqslant k \leqslant n$, we are interested in the following birational invariants. According to [4, Section 5.3], we define the *k–irrationality degree* of $X$ as the integer

$$\operatorname{irr}_k(X) := \min \left\{ c \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{Given a general point } p \in X,\ \exists \text{ an irreducible subvariety } Z \subseteq X \\ \text{of dimension } k \text{ such that } p \in Z \text{ and there is a dominant rational} \\ \text{map } Z \dashrightarrow \mathbb{P}^k \text{ of degree } c \end{array} \right\}$$

and, in line with [3], we define the *connecting gonality* of $X$ as the integer

$$\operatorname{conn.gon}(X) := \min \left\{ c \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{Given two general points } q, q' \in X,\ \exists \text{ an irreducible} \\ \text{curve } C \subset X \text{ such that } q, q' \in C \text{ and } \operatorname{gon}(C) = c \end{array} \right\}.$$

Therefore, $\operatorname{conn.gon}(X)$ can be thought as a measure of the failure of $X$ to be rationally connected, whereas $\operatorname{irr}_k(X)$ measures how $X$ is far from being covered by $k$–dimensional rational varieties. We note further that $\operatorname{irr}(X) := \operatorname{irr}_n(X)$ is the *degree of irrationality* of $X$ and $\operatorname{cov.gon}(X) := \operatorname{irr}_1(X)$ is the *covering gonality* of $X$. Moreover, these invariants satisfy the obvious inequalities

$$\operatorname{cov.gon}(X) \leqslant \operatorname{conn.gon}(X) \leqslant \operatorname{irr}(X) \quad \text{and} \quad \operatorname{irr}_1(X) \leqslant \operatorname{irr}_2(X) \leqslant \cdots \leqslant \operatorname{irr}_n(X). \tag{1.1}$$

In [8], it has been proved that $\operatorname{irr}_k(A) \geqslant k + \frac{1}{2}(\dim A + 1)$, provided that $A$ is a very general abelian variety of dimension at least 3 and $1 \leqslant k \leqslant \dim A$. Apart from this result and the cases $k \in \{1, n\}$, very little is known about the $k$–irrationality degrees and the connecting gonality of projective varieties. When $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d \geqslant 2n+2$, it follows from [3, Theorem C] and [4, Theorem 1.1] that

$$\operatorname{irr}_n(X) = d-1 \quad \text{and} \quad d - \left\lfloor \frac{\sqrt{16n+9}-1}{2} \right\rfloor \leqslant \operatorname{irr}_1(X) \leqslant d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor, \tag{1.2}$$

where the latter relation is often a chain of equalities. Under the same assumption on $X \subset \mathbb{P}^{n+1}$, we are concerned with its connecting gonality and the $k$–irrationality degree, with $2 \leqslant k \leqslant n-1$.

In this direction, we extend [4, Proposition 2.12], which relates curves of low gonality covering $X$ to the cones $V_p^h$ of lines having high contact with $X$. In particular, we prove that if $Z \subset X$ is a $k$–dimensional subvariety passing through a general point, endowed with a dominant rational map $\varphi\colon Z \dashrightarrow \mathbb{P}^k$ of degree $c \leqslant d-3$, then $Z \subset V_p^{d-c}$ for some $p \in X$, and the map $\varphi$ is the projection from $p$ (cf. Proposition 4.1). Combining Theorem 1.1 and the latter result, we achieve bounds on the $k$–irrationality degrees of $X$.

**Theorem 1.2.** *Let $n \geqslant 3$ and let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geqslant 2n+2$. Then*

$$\operatorname{irr}_k(X) \geqslant d - 1 - n + k \quad \text{for } 1 \leqslant k \leqslant n. \tag{1.3}$$

*Moreover, equality holds for $n-2 \leqslant k \leqslant n$, that is*

$$\operatorname{irr}_{n-2}(X) = d-3, \quad \operatorname{irr}_{n-1}(X) = d-2 \quad \text{and} \quad \operatorname{irr}_n(X) = d-1.$$

We point out that the assertion for $k = n$ is given by [3, Theorem C]. Furthermore, the larger the value of $k$ is, the more significant Theorem 1.2 becomes, since for small values of $k$ the bound (1.3) is superseded by (1.2) and (1.1).

In order to discuss the connecting gonality of $X$, we prove further that for any pair of general points $q, q' \in X$ and for any $2 \leqslant h \leqslant \frac{n}{2}+1$, there exists a general point $p \in X$ such that $q, q' \in V_p^h$ (cf. Lemma 3.2). Then, using the fact that for $h = \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor$ the locus $V_p^h$ is a cone over a rationally connected variety, we bound from above the connecting gonality of a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of large degree.

**Theorem 1.3.** *Let $n \geqslant 4$ and let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geqslant 2n+2$. Then*

$$\operatorname{conn.gon}(X) \leqslant d - \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor. \tag{1.4}$$

Finally, using our results and the Grassmannian techniques introduced in [20], we also obtain a lower bound on the connecting gonality of very general hypersurfaces, which slightly improves the bound descending from (1.2) and (1.1).

**Theorem 1.4.** *Let $n \geqslant 4$ and let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geqslant 2n+2$. Then*

$$\operatorname{conn.gon}(X) \geqslant d - \left\lfloor \frac{\sqrt{16n+25}-3}{2} \right\rfloor. \tag{1.5}$$

*In particular,*

$$\operatorname{conn.gon}(X) > \operatorname{cov.gon}(X)$$

$$\forall n \in \mathbb{Z}_{\geqslant 4} \smallsetminus \left\{ 4a^2+3a, 4a^2+5a, 4a^2+5a+1, 4a^2+7a+2, 4a^2+9a+4, 4a^2+11a+6 \,\middle|\, a \in \mathbb{N} \right\}.$$

In Example 4.5 we also discuss the cases $1 \leqslant n \leqslant 3$, which turn out to satisfy equality in (1.5). We note that the second part of the statement of Theorem 1.4 is obtained from (1.2) by determining the values of $n$ such that $\left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor \neq \left\lfloor \frac{\sqrt{16n+25}-3}{2} \right\rfloor$. We believe that the bound (1.5) is far from being sharp. However, for any $4 \leqslant n \leqslant 16$ with $n \neq 9, 13, 14$, the right–hand sides of (1.4) and (1.5) do coincide, hence Theorems 1.3 and 1.4 compute the connecting gonality of $X$ in these cases (cf. Example 4.5).

The paper is organized as follows. In Section 2, we recall some basic facts on the cones of lines of high contact with a general hypersurface $X \subset \mathbb{P}^{n+1}$ and we prove Theorem 1.1.

In Section 3, we consider polar hypersurfaces of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, in order to discuss when any pair $q, q' \in X$ lies on a cone $V_p^h$ for some $p \in X$.

Finally, Section 4 is concerned with the applications to measures of irrationality. In particular, after describing the relation between the cones $V_p^h$ and the degree of irrationality of $k$–dimensional subvarieties of $X$ passing through a general point, we prove Theorems 1.2, 1.3 and 1.4, and we discuss the behavior of the connecting gonality for small values of $n = \dim X$.

**Notation.** We work throughout over the field $\mathbb{C}$ of complex numbers. By *variety* we mean a complete reduced algebraic variety $X$, and by *curve* we mean a variety of dimension 1. We say that a property holds for a *general* (resp. *very general*) point $x \in X$ if it holds on a Zariski open nonempty subset of $X$ (resp. on the complement of the countable union of proper subvarieties of $X$).

## 2. Dimension of cones of lines having high contact

Let $n \geqslant 2$ be an integer, and let $X = V(F) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface defined by the vanishing of a homogeneous polynomial $F \in \mathbb{C}[x_0, \dots, x_{n+1}]$ of degree $d \geqslant 2$. Given a point $p \in X$ and an integer $2 \leqslant h \leqslant d$, we define the *cone* $V_p^h = V_{p,X}^h \subset \mathbb{P}^{n+1}$ *of tangent lines of order* $h$ *at* $p$ as the Zariski closure of the locus swept out by lines $\ell \subset \mathbb{P}^{n+1}$ such that either $\ell \subset X$ or $\ell \cdot X \geqslant hp$. Therefore $V_p^h$ is a cone with vertex at $p$, defined by the vanishing of the following $h-1$ polynomials occurring in the Taylor expansion of $F$ at $p$

$$\begin{aligned} G_k(x_0, \dots, x_{n+1}) &:= \left( x_0 \frac{\partial}{\partial x_0} + \cdots + x_{n+1} \frac{\partial}{\partial x_{n+1}} \right)^{(k)} F(p) \\ &= \sum_{i_0 + \cdots + i_{n+1} = k} \frac{k!}{i_0! \cdots i_{n+1}!} x_0^{i_0} \cdots x_{n+1}^{i_{n+1}} \frac{\partial^k F}{\partial x_0^{i_0} \cdots \partial x_{n+1}^{i_{n+1}}}(p) \end{aligned} \tag{2.1}$$

where $(-)^{(k)}$ denotes the usual symbolic power, $1 \leqslant k \leqslant h-1$, and $\deg G_k = k$ (cf. [6, p. 186]). In particular, the cone $V_p^2$ coincides with the (projective) tangent hyperplane $T_pX \subset \mathbb{P}^{n+1}$ to $X$ at $p$. When instead $h \geqslant 3$, we denote by $\Lambda_p^h = \Lambda_{p,X}^h$ the intersection of $V_p^h$ with a general hyperplane $H \subset \mathbb{P}^{n+1}$ not containing $p$, so that $\Lambda_p^h$ is defined in $H \cap T_pX \cong \mathbb{P}^{n-1}$ by $h-2$ polynomial equations of degree $2, 3, \dots, h-1$, respectively, and $V_p^h$ is the cone over $\Lambda_p^h$ with vertex at $p$.

For any $2 \leqslant h \leqslant \min\{n+1, d\}$, it follows from this description of $V_p^h$ that

$$\dim V_p^h \geqslant n + 2 - h. \tag{2.2}$$

When $X$ is a general hypersurface of degree $d \geqslant 2$ and $p \in X$ is a general point, [4, Lemma 2.2] guarantees that $V_p^h \subset \mathbb{P}^{n+1}$ is a general complete intersection of multi–degree $(1, 2, \dots, h-1)$ and, in particular, (2.2) is an equality.

According to the assertion of Theorem 1.1, we want to prove that equality in (2.2) holds for *any* point $p \in X$. The case $h = 2$ is trivial since $X$ is smooth and hence $V_p^2 = T_pX \cong \mathbb{P}^n$. The assertion for $h = 3$ is implied by the following result.

**Lemma 2.1.** *Let $n \geqslant 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geqslant 2$. Then, for any point $p \in X$, the intersection of $X$ with the tangent hyperplane $T_pX$ has multiplicity 2 at $p$.*

*Proof.* Let $\mathcal{L}$ be the linear system of all hypersurfaces of degree $d$ in $\mathbb{P}^{n+1}$, which has dimension

$$\dim \mathcal{L} = \binom{d+n+1}{n+1} - 1.$$

Consider the variety $\mathcal{V}$ consisting of all triples $(p, \Pi, Y)$ where $\Pi \subset \mathbb{P}^{n+1}$ is a hyperplane, $p \in \Pi$ and $Y \subset \Pi$ is a hypersurface of degree $d$ with a point of multiplicity at least 3 at $p$. Then we define the variety

$$\mathcal{Z} := \{ (p, \Pi, Y, X) \in \mathcal{V} \times \mathcal{L} | \, Y \subset X \}.$$

endowed with the projection $\pi_1 \colon \mathcal{Z} \longrightarrow \mathcal{V}$, whose fibers are all isomorphic to linear systems of hypersurfaces of degree $d$ of the same dimension $\binom{d+n}{n+1}$. Looking at the map $\mathcal{V} \longrightarrow \mathbb{P}^{n+1} \times \left(\mathbb{P}^{n+1}\right)^*$ given by $(p, \Pi, Y) \longmapsto (p, \Pi)$, it is easy to see that $\mathcal{V}$ is irreducible of dimension

$$\dim \mathcal{V} = 2n + \binom{d+n}{n} - \binom{n+2}{2}.$$

Hence $\mathcal{Z}$ is also irreducible, having dimension

$$\dim \mathcal{Z} = 2n + \binom{d+n}{n} - \binom{n+2}{2} + \binom{d+n}{n+1} = 2n + \binom{d+n+1}{n+1} - \binom{n+2}{2}.$$

Consider now the projection $\pi_2 \colon \mathcal{Z} \longrightarrow \mathcal{L}$, whose image is the locus $\mathcal{T}$ of all hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $d$ having a point $p \in X$ and a hyperplane $\Pi \subset \mathbb{P}^{n+1}$ such that the intersection scheme $X \cap \Pi$ has a point of multiplicity at least 3 at $p$. Hence

$$\dim \mathcal{T} \leqslant \dim \mathcal{Z} = \dim \mathcal{L} + 2n + 1 - \binom{n+2}{2}$$

and, since $\binom{n+2}{2} > 2n+1$ as soon as $n \geqslant 2$, we conclude that $\mathcal{T}$ is a proper closed subset of $\mathcal{L}$, as wanted. □

We notice that the double point at $p \in X$ of the intersection of $X$ with $T_pX$ as in Lemma 2.1 does not need to be an ordinary double point, and the locus where the singularity is worse than an ordinary double point is the intersection of $X$ with its Hessian hypersurface.

Now we prove Theorem 1.1.

*Proof of Theorem 1.1.* We already discussed the trivial case $h = 2$. Moreover, Lemma 2.1 ensures that for any $p \in X$, a general line $\ell \subset T_pX$ tangent to $X$ at $p$ intersects $X$ at $p$ with multiplicity exactly 2. Hence $V_p^3$ is a proper subvariety of $T_pX$, and (2.2) is an equality. Thus we assume hereafter $h \geqslant 4$.

Let $[x_0 : \ldots : x_{n+1}]$ be homogeneous coordinates in $\mathbb{P}^{n+1}$ and, for any positive integer $k$, we set

$$S_k := \mathbb{C}[x_0, \ldots, x_{n+1}]_k \quad \text{and} \quad S_k^* := \mathbb{C}[x_0, \ldots, x_{n+1}]_k \setminus \{0\}. \tag{2.3}$$

For any $F \in S_d^*$, we denote by $V(F) \subset \mathbb{P}^{n+1}$ the hypersurface defined by the vanishing of $F$, and for any $\mathbf{G} := (G_1, \ldots, G_{h-1}) \in \prod_{k=1}^{h-1} S_k$, we denote by $V(\mathbf{G})$ the intersection scheme of the hypersurfaces $V(G_k)$ for $1 \leqslant k \leqslant h-1$.

For $F \in S_d^*$ and $p \in V(F) \subset \mathbb{P}^{n+1}$, let $G_k = G_{p,k}(F)(x_0, \ldots, x_{n+1})$ be the homogeneous polynomial of degree $k$ defined in (2.1), where $1 \leqslant k \leqslant h-1$, and let

$$\mathbf{G} = \mathbf{G}_p(F) := (G_{p,1}(F), \ldots, G_{p,h-1}(F)) \in \prod_{k=1}^{h-1} S_k.$$

Therefore $V_p^h$ is the cone $V(\mathbf{G}_p(F))$ with vertex at $p$, and (2.2) fails to be an equality if and only if $\mathbf{G}_p(F)$ is not a *regular sequence*. In order to prove the assertion, we show that if $F \in S_d^*$ is general, then the sequence $\mathbf{G}_p(F)$ is regular for all points $p \in V(F)$.

To this aim, let $U_d \subset S_d^*$ be the open dense subset parametrizing those $F \in S_d^*$ such that $V(F)$ is smooth and let

$$\mathcal{J} := \left\{ (p, F, \mathbf{G}) \in \mathbb{P}^{n+1} \times U_d \times \prod_{k=1}^{h-1} S_k \,\middle|\, p \in V(F) \text{ and } \mathbf{G} = \mathbf{G}_p(F) \right\},$$

which is endowed with the two projections $\pi_1 \colon \mathcal{J} \longrightarrow U_d$ and $\pi_2 \colon \mathcal{J} \longrightarrow \mathbb{P}^{n+1} \times \prod_{k=1}^{h-1} S_k$. The map $\pi_1$ is surjective, and for any $F \in U_d$, the fiber $\pi_1^{-1}(F)$ is isomorphic to $V(F)$, which is irreducible of dimension $n$. Thus $\mathcal{J}$ is irreducible of dimension

$$\dim \mathcal{J} = \dim(U_d) + n = \binom{n+1+d}{d} + n. \tag{2.4}$$

Let us define $\mathcal{W} := \pi_2(\mathcal{J}) \subset \mathbb{P}^{n+1} \times \prod_{k=1}^{h-1} S_k$, which is irreducible too.

**Claim 2.2.** *All fibers of $\pi_2 \colon \mathcal{J} \longrightarrow \mathcal{W}$ have dimension*

$$f = \binom{n+h}{h} + \binom{n+h+1}{h+1} + \cdots + \binom{n+d}{d} = \binom{n+d+1}{d} - \binom{n+h}{h-1}.$$

*Proof of Claim 2.2.* Let $(p, \mathbf{G}) = \pi_2(p, F, \mathbf{G}) \in \mathcal{W}$, with $(p, F, \mathbf{G}) \in \mathcal{J}$, $\mathbf{G} = (G_1, \ldots, G_{h-1})$ and $G_k = G_{p,k}(F)$ for $1 \leqslant k \leqslant h-1$. Up to projective transformations, we may assume $p = [1 : 0 : \ldots : 0]$ and $T_pV(F) = V(x_{n+1})$. Then $F$ is of the form

$$F(x_0, \ldots, x_{n+1}) = cx_{n+1}x_0^{d-1} + F_2(x_1, \ldots, x_{n+1})x_0^{d-2} + \cdots + F_d(x_1, \ldots, x_{n+1}), \tag{2.5}$$

where $c$ is a non–zero constant and each $F_i \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ is homogeneous of degree $i$. Easy computations show that

$$\begin{aligned} G_1 =& cx_{n+1} \\ G_2 =& 2(d-1)cx_{n+1}x_0 + 2F_2 \\ G_3 =& 3(d-1)(d-2)cx_{n+1}x_0^2 + 6(d-2)x_0F_2 + 6F_3 \\ &\cdots \\ G_{h-1} =& (h-1)\frac{(d-1)!}{(d-h+1)!}cx_{n+1}x_0^{h-2} + \sum_{i=2}^{h-1} \frac{(h-1)!}{(h-i-1)!}\frac{(d-i)!}{(d-h+1)!}x_0^{h-i-1}F_i. \end{aligned} \tag{2.6}$$

To determine the fiber of $\pi_2$ over $(p, \mathbf{G})$, we have to find all forms $F' \in U_d$ such that $(p, F', \mathbf{G}) \in \mathcal{J}$. As in (2.5), we have $F' = c'x_{n+1}x_0^{d-1} + F_2'x_0^{d-2} + \cdots + F_d'$, which satisfies the corresponding equations in (2.6). Therefore, Equations (2.6) imply $c = c'$ and $F_k' = F_k$ for any $2 \leqslant k \leqslant h-1$. Thus $F'$ may differ from $F$ by the terms $F_i'$ for $h \leqslant i \leqslant d$, which can be chosen arbitrarily in $\mathbb{C}[x_1, \ldots, x_{n+1}]_i$. □

We deduce from the claim and (2.4) that

$$\dim \mathcal{W} = \dim \mathcal{J} - f = \binom{n+h}{h-1} + n.$$

Moreover, according to the description of (2.6), this equality implies that $\mathcal{W}$ coincides with the set of all $h$–tuples $(p, G_1, ..., G_{h-1})$ where $p \in \mathbb{P}^{n+1}$ is arbitrary, $G_1$ is an arbitrary homogeneous polynomial of degree 1 vanishing at $p$, and for any $k = 2, \ldots, h-1$, $G_k$ is a homogeneous polynomial of degree $k$ such that modulo $G_1, \ldots, G_{k-1}$ is arbitrary.

Then we set

$$\mathcal{W}_0 := \{ (p, \mathbf{G}) \in \mathcal{W} | \, \mathbf{G} \text{ is not a regular sequence}\} \subset \mathcal{W},$$

so that proving the assertion is equivalent to showing that $\pi_1\left(\pi_2^{-1}(\mathcal{W}_0)\right)$ is a proper closed subset of $U_d$. In particular, it suffices to prove that

$$\operatorname{codim}_{\mathcal{W}} \mathcal{W}_0 > n, \tag{2.7}$$

because in this case Claim 2.2 and (2.4) yield

$$\begin{aligned} \dim \pi_1\left(\pi_2^{-1}(\mathcal{W}_0)\right) \leqslant \dim \pi_2^{-1}(\mathcal{W}_0) = \dim \mathcal{W}_0 + f = \dim \mathcal{W}_0 + \dim \mathcal{J} - \dim \mathcal{W} = \\ = \dim \mathcal{J} - \operatorname{codim}_{\mathcal{W}} \mathcal{W}_0 < \dim \mathcal{J} - n = \dim U_d, \end{aligned}$$

so that the assertion holds. Hence we focus on proving (2.7).

To this aim, let $\mathcal{Z}$ be an irreducible component of $\mathcal{W}_0$ and let $(p, \mathbf{G}) \in \mathcal{Z}$ be a general point, with $\mathbf{G} = (G_1, \ldots, G_{h-1})$. Then $\mathbf{G}$ is not a regular sequence, and we may define $\alpha = \alpha_{\mathcal{Z}}$ to be the greatest integer such that $(G_1, \dots, G_\alpha)$ is a regular sequence. Since we already showed that $V_p^3 = V(G_1, G_2)$ has dimension $n-1$, we have $2 \leqslant \alpha < h-1$. By maximality of $\alpha$, there exists an irreducible component $Y$ of $V(G_1, G_2, \dots, G_\alpha)$ of dimension $m := \dim Y = n - \alpha + 1$ and contained in $V(G_{\alpha+1})$. Therefore, in view of the above interpretation of $\mathcal{W}$ in terms of $h$–tuples $(p, G_1, ..., G_{h-1})$, in order to prove (2.7), it is enough to show that the *Hilbert function* $h_Y \colon \mathbb{N} \longrightarrow \mathbb{N}$ of $Y$ satisfies $h_Y(\alpha+1) > n$.

For any $\ell = 0, \dots, m$, let $Y_\ell$ denote a general linear section of $Y$ of dimension $\ell$. In particular, $Y_m = Y$ and $Y_{m-1}$ is a general hyperplane section of $Y$. Then, for any positive integer $t$, we have

$$h_Y(t) - h_Y(t-1) \geqslant h_{Y_{m-1}}(t)$$

(see [12, Lemma 3.1]). Analogously, for any $1 \leqslant \ell \leqslant m$ and for any integer $t > 0$, we obtain

$$h_{Y_\ell}(t) \geqslant h_{Y_\ell}(t-1) + h_{Y_{\ell-1}}(t), \tag{2.8}$$

and we deduce by iteration that for any integer $t > 0$,

$$h_Y(t) \geqslant \sum_{\ell=1}^{m} h_{Y_\ell}(t-1) + h_{Y_0}(t) \geqslant \sum_{\ell=1}^{m} h_{Y_\ell}(t-1) + 1. \tag{2.9}$$

**Claim 2.3.** *For any $1 \leqslant \ell \leqslant m$ and for any integer $t > 0$,*

$$h_{Y_\ell}(t) \geqslant \binom{\ell+t}{t}.$$

*Proof of Claim 2.3.* We recall that $Y_\ell$ is a general linear section of $Y \subset V(G_1) \cong \mathbb{P}^n$ of dimension $\ell$, so that $Y_\ell$ is irreducible and it sits in a projective space $\Lambda_\ell \cong \mathbb{P}^{n+\ell-m}$, with $1 \leqslant \ell \leqslant m$. If $t = 1$, the claim is true as the linear span of $Y_\ell \subset \Lambda_\ell$ has dimension at least $\ell = \dim Y_\ell$, i.e. $Y_\ell$ contains at least $\ell + 1$ independent points of $\Lambda_\ell$. Then we argue by induction on $t$, and using inequality (2.9) applied to $Y_\ell$, we obtain

$$h_{Y_\ell}(t) \geqslant \sum_{j=1}^{\ell} h_{Y_j}(t-1) + 1 \geqslant \sum_{j=1}^{\ell} \binom{j+t-1}{t-1} + 1 = \sum_{j=0}^{\ell} \binom{j+t-1}{t-1} = \binom{\ell+t}{t}$$

as desired. □

Finally, setting $t = \alpha + 1$ and $\ell = m = n - \alpha + 1$, Claim 2.3 ensures that

$$h_Y(\alpha+1) \geqslant \binom{m+\alpha+1}{\alpha+1} = \binom{n+2}{\alpha+1}.$$

By assumption, we have $\alpha < h - 1 \leqslant n$, and hence $\binom{n+2}{\alpha+1} \geqslant n+1$. Thus $h_Y(\alpha+1) > n$, which concludes the proof of Theorem 1.1. □

## 3. Polar hypersurfaces and cones of lines having high contact

Let $n \geqslant 2$ be an integer and let $X := V(F) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geqslant 2$. Given a point $q = [q_0 : \dots : q_{n+1}] \in \mathbb{P}^{n+1}$ and an integer $0 \leqslant s \leqslant d$, we introduce the *$s$–th polar hypersurface of $X$ with respect to $q$* as the hypersurface $\Delta_q^s = \Delta_q^s(X) \subset \mathbb{P}^{n+1}$ defined by the vanishing of the polynomial of degree $d - s$

$$\mathrm{Pol}_q^s(F)(x_0, \dots, x_{n+1}) := \left( q_0 \frac{\partial}{\partial x_0} + \dots + q_{n+1} \frac{\partial}{\partial x_{n+1}} \right)^{(s)} F(x_0, \dots, x_{n+1}), \tag{3.1}$$

where $(-)^{(s)}$ denotes the usual symbolic power and $\mathrm{Pol}_q^0(F) = F$, that is $\Delta_q^0 = X$ for any $q \in \mathbb{P}^{n+1}$. Furthermore, we define the intersection scheme

$$\Delta_{q,h}(X) := \bigcap_{s=0}^{h-1} \Delta_q^s. \tag{3.2}$$

In this section, we use polar hypersurfaces of $X$ and Theorem 1.1 to show that for general $q, q' \in X$ and for any $2 \leqslant h \leqslant \frac{n}{2} + 1$, there exists a general point $p \in X$ such that $q, q' \in V_p^h$. In particular, this fact is crucial in order to prove Theorem 1.3. To start, we prove the following.

**Lemma 3.1.** *Let $n \geqslant 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geqslant 2$. For any integer $2 \leqslant h \leqslant \frac{n}{2}+1$ and for any $q, q' \in X$, there exists a point $p \in X$ such that $q, q' \in V_p^h$.*

*Proof.* We point out that for $q \in \mathbb{P}^{n+1}$, the intersection $X \cap \Delta_q^1$ consists of the points $p \in X$ such that $q \in T_pX$, i.e. the line $\langle q, p \rangle$ intersects $X$ with multiplicity at least 2 at $p$, provided that $p \neq q$. Similarly, given two points $q \in \mathbb{P}^{n+1}$ and $p \in X$ with $p \neq q$, the line $\langle q, p \rangle$ intersects $X$ with multiplicity at least $h$ at $p$—that is $q \in V_p^h$—if and only if $p$ belongs to $\Delta_{q,h}(X)$ defined in (3.2).

Therefore, proving the statement is equivalent to showing that for any $q, q' \in X$, the intersection of $\Delta_{q,h}(X)$ and $\Delta_{q',h}(X)$ is not empty. Since both $\Delta_{q,h}(X)$ and $\Delta_{q',h}(X)$ lie on $X = \Delta_q^0 = \Delta_{q'}^0$, then $\Delta_{q,h}(X) \cap \Delta_{q',h}(X)$ is the intersection of $2h-1$ hypersurfaces of $\mathbb{P}^{n+1}$, which is non–empty because of the assumption $h \leqslant \frac{n}{2} + 1$. □

In addition, when both $X \subset \mathbb{P}^{n+1}$ and the points $q, q' \in X$ are assumed to be general, the following holds.

**Lemma 3.2.** *Let $n \geqslant 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geqslant 2$. Then, for any integer $2 \leqslant h \leqslant \frac{n}{2} + 1$ and for general $q, q' \in X$, there exists a general point $p \in X$ such that $q, q' \in V_p^h$.*

*Proof.* Set $2 \leqslant h \leqslant \frac{n}{2} + 1$ and consider the variety

$$\mathcal{P}_h := \overline{\left\{(q, q', p) \in X \times X \times X \,|\, q \neq q' \text{ and } p \in \Delta_{q,h} \cap \Delta_{q',h}\right\}}$$

endowed with the projections

$$X \times X \xleftarrow{\pi_{12}} \mathcal{P}_h \xrightarrow{\pi_3} X.$$

Thanks to Lemma 3.1, the map $\pi_{12}$ is surjective. Let $\mathcal{Z} \subset \mathcal{P}_h$ be an irreducible component dominating $X \times X$. Therefore, the proof of Lemma 3.1 gives that for any $(q, q') \in X \times X$,

$$\dim\left(\Delta_{q,h}(X) \cap \Delta_{q',h}(X)\right) \geqslant n + 2 - 2h \quad \text{and hence} \quad \dim \mathcal{Z} \geqslant 3n + 2 - 2h. \tag{3.3}$$

Moreover, given any point $p \in \pi_3(\mathcal{Z})$ and setting $Z_p := V_p^h \cap X$, we have

$$\left(\pi_{3|\mathcal{Z}}\right)^{-1}(p) \subseteq \pi_3^{-1}(p) \cong Z_p \times Z_p. \tag{3.4}$$

In order to prove that for general $q, q' \in X$, there exists a general point $p \in X$ such that $q, q' \in V_p^h$, it is enough to prove that $\mathcal{Z}$ dominates $X$ via $\pi_3$. We assume by contradiction that $\pi_{3|\mathcal{Z}}$ is not dominant, i.e. $\dim \pi_3(\mathcal{Z}) < n = \dim X$. If $p \in \pi_3(\mathcal{Z})$ is a general point, then (3.3) and (3.4) give

$$\dim(Z_p \times Z_p) \geqslant \dim\left(\left(\pi_{3|\mathcal{Z}}\right)^{-1}(p)\right) = \dim \mathcal{Z} - \dim \pi_3(\mathcal{Z}) > 2n + 2 - 2h. \tag{3.5}$$

It follows that $\dim Z_p > n + 1 - h$ and hence $\dim V_p^h > n + 2 - h$. Since the latter inequality contradicts Theorem 1.1, we conclude that $\pi_{3|\mathcal{Z}}$ is dominant. □

## 4. $k$–IRRATIONALITY DEGREE AND CONNECTING GONALITY OF GENERAL HYPERSURFACES

In this section we apply the previous results in order to bound the $k$–irrationality degree and the covering gonality of a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2n+2$.

According to Section 2, we recall that if $V_p^h \subset \mathbb{P}^{n+1}$ is the cone of tangent lines of order $h$ at $p \in X$, we denote by $\Lambda_p^h$ a general hyperplane section of $V_p^h$. The link between the cones $V_p^h \subset \mathbb{P}^{n+1}$ and the invariants $\mathrm{irr}_k(X)$ and $\mathrm{conn.\,gon}(X)$ is expressed by the following result, which extends [4, Proposition 2.12] to higher dimensional subvarieties of $X$.

**Proposition 4.1.** *Let $n \geqslant 3$ and $1 \leqslant k \leqslant n-1$ be integers. Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geqslant 2n+2$. Suppose that for a general point $q \in X$, there exist a $k$–dimensional irreducible subvariety $Z \subset X$ containing $q$ and a dominant rational map $\varphi \colon Z \dashrightarrow \mathbb{P}^k$ of degree $c \leqslant d-3$. Then:*

(i) *there exists a point $p \in Z$ such that $Z \subset V_p^{d-c} \cap X$;*
(ii) *the map $\varphi \colon Z \dashrightarrow \mathbb{P}^k$ of degree $c$ is the projection from $p$.*

*In particular, the image of $Z$ under $\varphi$ is a $k$–dimensional rational variety $R \subset \Lambda_p^{d-c}$.*

*Proof.* The case $k=1$ is covered by [4, Proposition 2.12], so we assume $2 \leqslant k \leqslant n-1$. Since $d \geqslant 2n+2$, we have that $c \leqslant d-3 < 2d-2n-1$. Let $z \in Z$ be a general point and let $\ell \subset \mathbb{P}^k$ be a general line passing through $\varphi(z)$. Consider the curve $C_\ell \subset \varphi^{-1}(\ell)$ which is the union of all irreducible components of curves in $Z$ which dominate $\ell$ via $\varphi$. We claim that $C_\ell$ is irreducible. Indeed, if $C'$ and $C$ were two irreducible components, then $\mathrm{gon}(C_{\mathrm{red}}) + \mathrm{gon}(C'_{\mathrm{red}}) \leqslant \deg(\varphi_{|C_\ell})$. Being $\varphi_{|C_\ell} \colon C_\ell \dashrightarrow \ell \cong \mathbb{P}^1$ a map of degree $c$, either $C_{\mathrm{red}}$ or $C'_{\mathrm{red}}$ would have gonality at most $\frac{c}{2} < d-n-\frac{1}{2}$. By varying $\ell \subset \mathbb{P}^k$, $z \in Z$ and $q \in X$, we conclude that $X$ is covered by curves of gonality smaller than $d-n$, which is impossible (cf. [3, Theorem A]). The same argument shows that $C_\ell$ is reduced.

Therefore, we may define a family $\mathcal{C} \xrightarrow{\pi} U \subset \mathbb{G}(1,k)$ of curves with a map of degree $c$ to $\mathbb{P}^1$, where $U \cong \mathbb{P}^{k-1}$ parametrizes lines through $\varphi(z)$, and for any $[\ell] \in U$, the corresponding curve is $C_\ell$. As we vary $q \in X$ (and hence $Z$), we may define a family of curves covering $X$, each endowed with a $c$–gonal map. Thus [4, Proposition 2.12] ensures that for general $[\ell] \in U$, there exists a point $x_\ell \in C_\ell$ such that $C_\ell \subset X \cap V_{x_\ell}^{d-c}$ and the degree $c$ map $\varphi_{|C_\ell} \colon C_\ell \dashrightarrow \ell \cong \mathbb{P}^1$ is the projection from $x_\ell$.

Next we need to show that all the points $x_\ell$ coincide with some fixed point $p \in Z$. For this we consider the map $\psi \colon U \dashrightarrow Z \subset X$ sending $[\ell] \in U$ to the corresponding point $x_\ell$. Since $U \cong \mathbb{P}^{k-1}$, the image of $\psi$ is unirational. As $X$ does not contain rational curves (see e.g. [9]), we conclude that $\psi(U)$ is a point $p \in Z$. Thus $Z$ is covered by the curves $C_\ell \subset V_p^{d-c}$ for general $[\ell] \in U$, and the degree $c$ map $\varphi_{|C_\ell} \colon C_\ell \dashrightarrow \ell \cong \mathbb{P}^1$ is the projection from $p$. The assertion follows. □

**Remark 4.2.** Let $X$ be an irreducible projective variety of dimension $n$ and let $\mathcal{Z} \xrightarrow{\pi} T$ be a family of $k$–dimensional subvarieties of $X$. If $\mathcal{Z} \xrightarrow{\pi} T$ is a *covering family* (i.e. for general $q \in X$, there exists $t \in T$ such that $q \in Z_t = \pi^{-1}(t)$), then $\dim(T) \geqslant n-k$. Indeed, the map $f \colon \mathcal{Z} \longrightarrow X$ must be dominant and hence $\dim(\mathcal{Z}) = \dim(T) + k \geqslant n$.

If in addition $\mathcal{Z} \xrightarrow{\pi} T$ is a *connecting family* (i.e. for general $q, q' \in X$, there exists $t \in T$ such that $q, q' \in Z_t$), then $\dim(T) \geqslant 2n-2k$. Indeed the map $\mathcal{Z} \times_T \mathcal{Z} \longrightarrow X \times X$ induced by $f \colon \mathcal{Z} \longrightarrow X$ must be dominant, hence $\dim(\mathcal{Z} \times_T \mathcal{Z}) = 2k + \dim(T) \geqslant 2n$.

**Remark 4.3.** If $X \subset \mathbb{P}^{n+1}$ and $Y \subset \mathbb{P}^{m+1}$ are very general hypersurfaces of degree $d$, with $n \leqslant m$, then

$$\operatorname{irr}_k(Y) \leqslant \operatorname{irr}_k(X) \text{ for any } 1 \leqslant k \leqslant n \quad \text{and} \quad \operatorname{conn.gon}(Y) \leqslant \operatorname{conn.gon}(X).$$

Indeed, the section of $Y$ by a general $(n+1)$–plane of $\mathbb{P}^{m+1}$ is a very general hypersurface of $\mathbb{P}^{n+1}$.

We can now prove Theorem 1.2.

*Proof of Theorem 1.2.* When $k = n$, the assertion is covered by [3, Theorem C] and $\operatorname{irr}_n(X) = d-1$.

If $k = n-1$, we claim that $\operatorname{irr}_{n-1}(X) \leqslant d-2$. Indeed, tangent hyperplane sections $Z = X \cap T_pX$ of $X$ are $(n-1)$–dimensional varieties of degree $d$ having a double point at $p$ (see Lemma 2.1), so that the projection from $p$ is a dominant rational map $Z \dashrightarrow \mathbb{P}^{n-1}$ of degree $d-2$. On the other hand, suppose by contradiction that $\operatorname{irr}_{n-1}(X) = c \leqslant d-3$. Proposition 4.1 ensures that any $(n-1)$–dimensional subvariety $Z \subset X$ computing $\operatorname{irr}_{n-1}(X)$ is contained in $X \cap V_p^{d-c}$ for some $p \in X$. Thanks to (1.1) and [3, Theorem A], we have $c \geqslant \operatorname{irr}_1(X) \geqslant d-n$, so that $3 \leqslant d-c \leqslant n$. Then Theorem 1.1 gives that $\dim V_p^{d-c} = n+2-(d-c)$. In order to cover $X$ by $(n-1)$–dimensional varieties cut out by the cones $V_p^{d-c}$, we must have that $\dim(X \cap V_p^{d-c}) = n+1-(d-c) \geqslant n-1$ and hence $c \geqslant d-2$, a contradiction. Thus $\operatorname{irr}_{n-1}(X) = d-2$.

If $1 \leqslant k \leqslant n-2$, we claim that $\operatorname{irr}_k(X) \leqslant d-3$. To see this, we note that for any $p \in X$, Theorem 1.1 ensures that $V_p^3$ is a cone in $T_pX \cong \mathbb{P}^n$ over a quadric $\Lambda_p^3 \subset \mathbb{P}^{n-1}$ (cf. Section 2). Then the variety $Z = X \cap V_p^3$ has dimension $n-2$ and the projection from $p$ is a dominant map $Z \dashrightarrow \Lambda_p^3$ of degree $d-3$ to a rational variety. Thus $\operatorname{irr}_1(X) \leqslant \cdots \leqslant \operatorname{irr}_{n-2}(X) \leqslant d-3$.

Finally, any $k$–dimensional subvariety $Z \subset X$ computing $c := \operatorname{irr}_k(X)$ is contained in some $X \cap V_p^{d-c}$ by Proposition 4.1. As above, we deduce $d-c \leqslant n$ and for any $p \in X$, Theorem 1.1 gives $\dim(X \cap V_p^{d-c}) = n+1-(d-c)$. Thus, in order to cover $X$ by $k$–dimensional varieties in $X \cap V_p^{d-c}$, we must have that $n+1-(d-c) \geqslant k$, that is $c \geqslant d-1-n+k$.

For $k = n-2$, the latter inequality gives $\operatorname{irr}_{n-2}(X) \geqslant d-3$, so the assertion follows. □

Now, we prove Theorem 1.3.

*Proof of Theorem 1.3.* By [4, Lemma 2.2], if $p \in X$ is a general point and $3 \leqslant h \leqslant \min\{n+1, d\}$, then $\Lambda_p^h$ is a general complete intersection of type $(2, 3, \ldots, h-1)$ in $\mathbb{P}^{n-1}$. If $\Lambda_p^h$ is a Fano variety, then it is rationally connected (see [19]). The canonical bundle of $\Lambda_p^h \subset \mathbb{P}^{n-1}$ is $\mathcal{O}_{\Lambda_p^h}(\sum_{i=2}^{h-1} i - n)$. Therefore, $\Lambda_p^h$ is a Fano variety if and only if

$$\sum_{i=2}^{h-1} i \leqslant n-1 \iff \frac{h(h-1)}{2} - 1 \leqslant n-1 \iff h \leqslant \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor. \tag{4.1}$$

We note that $d > n+1 \geqslant \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \geqslant 3$ and we assume hereafter $h := \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor$, so that the general $\Lambda_p^h$ is a smooth, rationally connected, complete intersection, whose dimension is $n+1-h$.

Setting $Z_p := V_p^h \cap X$, the projection $\varphi\colon Z_p \dashrightarrow \Lambda_p^h$ from $p$ has degree $d-h \leqslant d-3$. Given two general points $q, q' \in Z_p$, let $D \subset \Lambda_p^h$ be a rational curve connecting $\varphi(q)$ and $\varphi(q')$, and let $C := \varphi^{-1}(D)$. By arguing as for the curves $C_\ell$ in the proof of Proposition 4.1, we deduce that $C$ is integral. Then $C$ is an irreducible curve passing through two general points $q, q' \in Z_p$ endowed with a map $\varphi_{|C}\colon C \dashrightarrow D$ of degree $d-h$. Thus $\operatorname{conn.gon}(Z_p) \leqslant d-h = d - \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor$.

Since $n \geqslant 4$, we have $h = \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \leqslant \frac{n}{2} + 1$. So Lemma 3.2 ensures that for general $q, q' \in X$, there exists a general point $p \in X$ such that $q, q' \in Z_p$, i.e. the varieties $Z_p$ produce a connecting family. Thus $\operatorname{conn.gon}(X) \leqslant \operatorname{conn.gon}(Z_p) \leqslant d - \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor$. □

Let us consider integers $n, d \geqslant 2$ and $2 \leqslant h \leqslant \min\{n+1, d\}$. Before proving Theorem 1.4, we aim at introducing a suitable parameter space $\Theta_h^{n+1}$ for 4–tuples $(p, \ell_1, \ell_2, X)$, where $X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree $d$, $p \in X$ and $\ell_1, \ell_2 \subset V_{p,X}^h$ are lines having intersection multiplicity at least $h$ with $X$ at $p$.

To this aim, we define $S_d := \mathbb{C}[x_0, \dots, x_{n+1}]_d$ and $S_d^* := S_d \setminus \{0\}$ as in (2.3), and we set

$$\mathbb{P} := \mathbb{P}^{n+1}, \quad \mathbb{G} := \mathbb{G}(1, n+1) \quad \text{and} \quad N+1 := \dim_{\mathbb{C}}(S_d) = \binom{d+n+1}{d}.$$

Let $\mathcal{P} \subset \mathbb{P} \times \mathbb{G}$ be the universal family of lines over $\mathbb{G}$, endowed with the projections $\mathbb{P} \xleftarrow{\pi_1} \mathcal{P} \xrightarrow{\pi_2} \mathbb{G}$. The morphism $\pi_1$ makes $\mathcal{P}$ a $\mathbb{P}^n$–bundle over $\mathbb{P}$, whereas $\pi_2$ makes $\mathcal{P}$ a $\mathbb{P}^1$-bundle over $\mathbb{G}$, so that $\dim(\mathcal{P}) = 2n+1$. Consider the fibred product

$$\mathcal{P} \times_{\mathbb{P}} \mathcal{P} := \{(p, [\ell_1], [\ell_2]) \mid p \in \ell_1 \cap \ell_2\} \subset \mathbb{P} \times \mathbb{G} \times \mathbb{G}$$

and its *diagonal locus*

$$\Delta := \{(p, [\ell], [\ell]) \in \mathcal{P} \times_{\mathbb{P}} \mathcal{P} \mid p \in \ell\} \cong \mathcal{P}.$$

Let $\widetilde{\mathcal{P}}$ denote the blow-up of $\mathcal{P} \times_{\mathbb{P}} \mathcal{P}$ along $\Delta$ and let $\widetilde{\Delta}$ be the exceptional divisor. Thus,

$$\dim(\widetilde{\mathcal{P}}) = 3n + 1 = \dim(\widetilde{\Delta}) + 1.$$

Moreover, as a set, we have

$$\widetilde{\Delta} = \{(p, [\ell], [\Pi]) \mid p \in \ell \subset \Pi\} \subset \mathbb{P} \times \mathbb{G} \times \mathbb{G}(2, n+1).^1$$

Given a polynomial $F \in S_d^*$, let $X_F := V(F) \subset \mathbb{P}$ denote its vanishing locus. Then we define the variety $\Theta_h^{n+1} \subset \widetilde{\mathcal{P}} \times \mathbb{P}(S_d)$ as

$$\Theta_h^{n+1} := \overline{\left\{ (p, [\ell_1], [\ell_2], [F]) \in \widetilde{\mathcal{P}} \times \mathbb{P}(S_d) \;\middle|\; \begin{array}{l} \ell_1 \neq \ell_2 \text{ and for } 1 \leqslant i \leqslant 2, \\ \text{either } \ell_i \subset X_F \text{ or } X_F \cdot \ell_i \geqslant hp \end{array} \right\}}. \tag{4.2}$$

**Lemma 4.4.** *For any $2 \leqslant h \leqslant \min\{n+1, d\}$, $\Theta_h^{n+1}$ is smooth, irreducible, of dimension $3n + 2 + N - 2h$, dominating both $\widetilde{\mathcal{P}}$ and $\mathbb{P}(S_d)$ via the projection maps*

$$\widetilde{\mathcal{P}} \xleftarrow{\Psi} \Theta_h^{n+1} \xrightarrow{\Phi} \mathbb{P}(S_d).$$

*Proof.* Let $(p, [\ell_1], [\ell_2]) \in \widetilde{\mathcal{P}} \setminus \widetilde{\Delta}$ and $F \in S_d^*$. Requiring that $(p, [\ell_1], [\ell_2], [F]) \in \Theta_h^{n+1}$ amounts to impose $2h - 1$ independent linear conditions to $F$, corresponding to the conditions $\ell_1 \cdot X_F \geqslant hp$ and $\ell_2 \cdot X_F \geqslant hp$.

[1]This fact follows from a standard argument, but we sketch it for the sake of completeness. The exceptional divisor $\widetilde{\Delta}$ is the projectivization of the normal bundle of $\Delta$ in $\mathcal{P} \times_{\mathbb{P}} \mathcal{P}$. Let $\Gamma$ denote the algebraic set $\{(p, [\ell], [\Pi]) | p \in \ell \subset \Pi\}$ on the right–hand side. There is an obvious morphism $\xi \colon \Gamma \longrightarrow \widetilde{\Delta}$, that maps a triple $(p, [\ell], [\Pi]) \in \Gamma$ to the point of $\widetilde{\Delta}$ corresponding to the deformation of $(p, [\ell], [\ell])$ to a pair $(p, [\ell], [\ell'])$, where $\ell'$ moves in the pencil of lines passing through $p$ inside the plane $\Pi$. The map $\xi$ is clearly injective. It is also surjective, because it has an inverse $\eta \colon \widetilde{\Delta} \longrightarrow \Gamma$ defined as follows. A point $x \in \widetilde{\Delta}$ that lies over $(p, [\ell], [\ell])$ corresponds to a deformation $\{(p_t, [\ell_{1,t}], [\ell_{2,t}])\}$ of $(p, [\ell], [\ell])$, with $t$ moving in a disc with centre 0. Then $\eta$ associates to $x$ the point $(p, [\ell], [\Pi]) \in \Gamma$, where $\Pi$ is the flat limit of the plane spanned by $\ell_{1,t}$ and $\ell_{2,t}$, when $t \to 0$. This shows that $\Gamma$ and $\widetilde{\Delta}$ are set–theoretically the same.

We claim that the same happens at $(p,[\ell],[\Pi]) \in \widetilde{\Delta}$, when we require $(p,[\ell],[\Pi],[F]) \in \Theta_h^{n+1}$. In fact, choose affine coordinates $(\eta,\zeta)$ on $\Pi$ such that $p=(0,0)$ and $\ell = V(\eta)$. Write $F|_{\Pi} = F_0+F_1+\cdots+F_d$, where $F_i = \sum_{0\leqslant j\leqslant i} a_{i,j}\eta^{i-j}\zeta^j$ is a homogeneous polynomial of degree $i$. Imposing the condition $X_F \cdot \ell \geqslant hp$ gives $a_{i,i}=0$, for any $0 \leqslant i \leqslant h-1$. Now, consider a general line $\ell' := V(\eta - t\zeta)$ in $\pi$ through $p=(0,0)$. Imposing the condition $X\cdot \ell' \geqslant hp$ and letting $\ell'$ approach $\ell$, i.e. letting $t$ approach zero, gives $a_{i,i-1}=0$ for any $1 \leqslant i \leqslant h-1$. Therefore, there are again $2h-1$ independent conditions for the coefficients of $F$ in order to have $(p,[\ell],[\Pi],[F]) \in \Theta_h^{n+1}$.

Hence the projection $\Psi\colon \Theta_h^{n+1} \longrightarrow \widetilde{\mathcal{P}}$ is onto, and its fibers are parameterized by $(2h-1)$–codimensional linear subspaces of $\mathbb{P}(S_d)$. Therefore, $\Theta_h^{n+1}$ is smooth, irreducible, of dimension

$$\dim(\Theta_h^{n+1}) = \dim(\widetilde{\mathcal{P}}) + \dim(\mathbb{P}(S_d)) - (2h-1) = 3n+1+N-2h+1.$$

The surjectivity of $\Phi\colon \Theta_h^{n+1} \longrightarrow \mathbb{P}(S_d)$ is clear; indeed, for any $F \in S_d^*$ and any $p \in X_F$, we have that $\dim(V^h_{p,X_F}) \geqslant n+2-h > 0$, as $h \leqslant n+1$ by assumption. □

For a general polynomial $F \in S_d^*$, we set

$$\Theta^{n+1}_{h,F} := \Phi^{-1}([F]),$$

which is smooth, equidimensional, of dimension $3n+2-2h$.

Now, we argue as in [20, Proof of Theorem 2.3] and we prove Theorem 1.4.

*Proof of Theorem 1.4.* Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geqslant 2n+2$ and let $h \in \mathbb{N}$ such that $\operatorname{conn.gon}(X) = d-h$. It follows from Theorem 1.3 that $h \geqslant \left\lfloor \frac{1+\sqrt{8n+1}}{2} \right\rfloor$. We note that if $h = \left\lfloor \frac{1+\sqrt{8n+1}}{2} \right\rfloor$, we are done because $\left\lfloor \frac{1+\sqrt{8n+1}}{2} \right\rfloor \leqslant \left\lfloor \frac{\sqrt{16n+25}-3}{2} \right\rfloor$ for any $n \geqslant 4$. Hence we assume hereafter

$$h > \left\lfloor \frac{1+\sqrt{8n+1}}{2} \right\rfloor. \tag{4.3}$$

Given two general points $x_1, x_2 \in X$, there exists an irreducible curve $C \subset X$ containing $x_1$ and $x_2$ such that $\operatorname{gon}(C) = \operatorname{conn.gon}(X) = d-h$. Since $h \geqslant 3$, by Proposition 4.1 there exists a point $p \in X$ such that $C \subset V_p^h$, and the projection $\pi_p\colon V_p^h \dashrightarrow \Lambda_p^h$ from $p$ maps $C$ to a rational curve $D \subset \Lambda_p^h$. In particular, if $\ell_1, \ell_2 \subset V_p^h$ denote the lines connecting the vertex $p$ to the points $x_1, x_2 \in X$, respectively, then $D$ passes through the corresponding points $\ell_1 \cap \Lambda_p^h$ and $\ell_2 \cap \Lambda_p^h$. So, for any integer $t \geqslant n+1$, we consider the variety $\Theta_h^t$ defined as in (4.2) (using the obvious modifications $\mathbb{P} = \mathbb{P}^t$, $\mathbb{G} = \mathbb{G}(1,t)$, etc.), and we introduce the locus $\mathcal{R}_t \subset \Theta_h^t$ given by

$$\mathcal{R}_t := \overline{\left\{ (p,[\ell_1],[\ell_2],[F]) \in \Theta_h^t \,\middle|\, \begin{array}{l} \ell_1 \neq \ell_2 \text{ and } \exists \text{ a rational curve } D \subset \Lambda^h_{p,X_F} \text{ passing} \\ \text{through the points } \ell_1 \cap \Lambda^h_{p,X_F} \text{ and } \ell_2 \cap \Lambda^h_{p,X_F} \end{array} \right\}}. \tag{4.4}$$

In particular, we are interested in the case $t = n+1$.

It follows from [4, Lemma 2.2] that for $F \in S_d^*$ general and $p \in X_F$ general, the variety $\Lambda_p^h = \Lambda^h_{p,X_F}$ is a general complete intersection of type $(1,1,2,\dots,h-1)$ in $\mathbb{P}^{n+1}$. Thus its canonical bundle is isomorphic to $\mathcal{O}_{\Lambda_p^h}(\sum_{i=2}^{h-1} i - n)$, which is effective by (4.1) and (4.3). In particular, $\Lambda_p^h$ is not covered by rational curves, so that $\mathcal{R}_{n+1}$ consists of (at most) countably many proper closed subsets of $\Theta_h^{n+1}$.

Let $F \in S_d^*$ be the polynomial defining the very general hypersurface $X \subset \mathbb{P}^{n+1}$. According to the discussion above, for general $x_1, x_2 \in X$, we may find (at least) one 4–tuple $(p,[\ell_1],[\ell_2],[F]) \in \mathcal{R}_{n+1}$, where $\ell_i = \langle p, x_i\rangle$ for $i=1,2$. Since $\ell_i \cap X$ consists of finitely many points, as we vary the pair

$(x_1, x_2) \in X \times X$, the corresponding 4–tuples $(p, [\ell_1], [\ell_2], [F])$ describe a subset of $\mathcal{R}_{n+1} \cap \Theta^{n+1}_{h,F}$ having dimension at least $2n$. In particular, $\dim\left(\mathcal{R}_{n+1} \cap \Theta^{n+1}_{h,F}\right) \geqslant 2n$ and as $F \in S^*_d$ is very general, we deduce that $\dim \mathcal{R}_{n+1} \geqslant 2n + N$. Therefore,

$$\operatorname{codim}_{\Theta^{n+1}_h} \mathcal{R}_{n+1} \leqslant n + 2 - 2h. \tag{4.5}$$

We point out that for any subfamily $\mathcal{F} \subset \Theta^{n+1}_h$ such that $(p, [\ell_1], [\ell_2], [F]) \in \mathcal{F}$, we have $\operatorname{codim}_{\mathcal{F}}(\mathcal{R}_{n+1} \cap \mathcal{F}) \leqslant \operatorname{codim}_{\Theta^{n+1}_h} \mathcal{R}_{n+1}$. Hence (4.5) gives

$$\operatorname{codim}_{\mathcal{F}}(\mathcal{R}_{n+1} \cap \mathcal{F}) \leqslant n + 2 - 2h. \tag{4.6}$$

We construct a subfamily $\mathcal{F} \subset \Theta^{n+1}_h$ with $(p, [\ell_1], [\ell_2], [F]) \in \mathcal{F}$, as follows. Let

$$m := \frac{h(h-1)}{2} - 2$$

and let $(p', [\ell'_1], [\ell'_2], [F']) \in \Theta^{m+2}_h$ be a 4–tuple such that $Y' := V(F') \subset \mathbb{P}^{m+2}$ is a very general hypersurface of degree $d$, $p' \in Y'$ is a very general point, $\ell'_1$ is very general among lines in $V^h_{p',Y'}$ passing through $p'$, and $\ell'_2 \neq \ell'_1$. Moreover, we deduce from (4.3) that $m + 2 \geqslant n + 1$.

Let $M \geqslant m + 2 \geqslant n + 1$ and let $(p'', [\ell''_1], [\ell''_2], [F'']) \in \Theta^M_h$, where $Y'' := V(F'') \subset \mathbb{P}^M$ is a hypersurface of degree $d$ such that $X$ is a $(n+1)$–plane section and $Y'$ is a $(m+2)$–plane section, with $p = p' = p''$, $\ell_1 = \ell'_1 = \ell''_1$, and $\ell_2 = \ell'_2 = \ell''_2$.

Now, for any $r \geqslant n + 1$, let $Z_r \subset \operatorname{Hom}\left(\mathbb{P}^r, \mathbb{P}^M\right)$ be the set of parameterized $r$–planes in $\mathbb{P}^M$ containing the plane $\langle \ell_1, \ell_2 \rangle$, and let $Z'_r \subset Z_r$ be the subset of parameterized $r$–planes $\Lambda \subset \mathbb{P}^M$ such that $(p, [\ell_1], [\ell_2], [F_\Lambda]) \in \mathcal{R}_r$, where $F_\Lambda$ is a polynomial defining the section of $Y''$ by $\Lambda$ as a hypersurface in $\Lambda$.[2]

We point out that for any $\Lambda \in Z_r$, we have $(p, [\ell_1], [\ell_2], [F_\Lambda]) \in \Theta^r_h$. To see this fact, consider the hypersurface $Y := V(F_\Lambda) = \Lambda \cap Y''$ of degree $d$ in $\Lambda \cong \mathbb{P}^r$. For $i = 1, 2$, we have $\ell_i \subset \Lambda$, so the intersection schemes $\ell_i \cdot Y$ and $\ell_i \cdot Y''$ are supported on the same 0–cycle of degree $d$, i.e. $\operatorname{mult}_q(\ell_i \cdot Y) = \operatorname{mult}_q(\ell_i \cdot Y'')$ for any $q \in Y \cap \ell_i$. In particular, $Y \cdot \ell_i \geqslant hp$ for $i = 1, 2$, so that $(p, [\ell_1], [\ell_2], [F_\Lambda]) \in \Theta^r_h$.

As in [20], let $\mathcal{F}$ be the image of $Z_{n+1}$ in $\Theta^{n+1}_h$ under the map sending a $(n+1)$–plane $\Lambda \in Z_{n+1}$ to the point $(p, [\ell_1], [\ell_2], [F_\Lambda]) \in \Theta^{n+1}_h$. Thus $\mathcal{R}_{n+1} \cap \mathcal{F}$ is the image of $Z'_{n+1}$. According to (4.6), we have

$$\operatorname{codim}_{Z_{n+1}} Z'_{n+1} \leqslant n + 2 - 2h. \tag{4.7}$$

Let $\varepsilon_r := \operatorname{codim}_{Z_r} Z'_r$. Since $Y' \subset \mathbb{P}^{m+2}$ and $p \in Y'$ are very general, then $\Lambda^h_{p,Y'}$ is a smooth complete intersection of type $(1, 1, 2, \ldots, h-1)$ in $\mathbb{P}^{m+2}$ by [4, Lemma 2.2]. Hence its canonical bundle is $\mathcal{O}_{\Lambda^h_{p,Y'}}(\sum_{i=2}^{h-1} i - m - 1)$, which is trivial by the choice of $m$. In particular, $\Lambda^h_{p,Y'}$ is not covered by rational curves and as $\ell_1 \subset V^h_{p,Y'}$ is a very general line through $p$, there are no rational curves of $\Lambda^h_{p,Y'}$ passing through the point $\ell_1 \cap \Lambda^h_{p,Y'}$. Thus $(p, [\ell_1], [\ell_2], [F']) \notin \mathcal{R}_{m+2}$ and $\varepsilon_{m+2} \geqslant 1$.

Applying [20, Proposition 2.5], we obtain

$$\varepsilon_{m+1} = \operatorname{codim}_{Z_{m+1}} Z'_{m+1} \geqslant \varepsilon_{m+2} + 1 \geqslant 2,$$

[2] As in [20], we consider the locus $Z_r \subset \operatorname{Hom}\left(\mathbb{P}^r, \mathbb{P}^M\right)$—rather than its counterpart in $\mathbb{G}(r, M)$—because we need to fix homogeneous coordinates $[y_0 : \ldots : y_r]$ on each $r$–plane $\Lambda \subset \mathbb{P}^M$, in order to define properly the polynomial $F_\Lambda \in \mathbb{C}[y_0, \ldots, y_r]_d \smallsetminus \{0\}$ (up to scalar multiplication).

and by recursion

$$\varepsilon_{n+1} = \operatorname{codim}_{Z_{n+1}} Z'_{n+1} \geqslant m - n + 2.$$

By (4.7), we must have $m - n + 2 \leqslant n + 2 - 2h$, and as $m := \frac{h(h-1)}{2} - 2$, we deduce

$$\frac{h(h-1)}{2} - n \leqslant n + 2 - 2h, \quad \text{so that} \quad h \leqslant \frac{-3+\sqrt{16n+25}}{2}.$$

Thus the connecting gonality of $X$ satisfies $\operatorname{conn.gon}(X) \geqslant d - \left\lfloor \frac{-3+\sqrt{16n+25}}{2} \right\rfloor$.

The final part of the statement is achieved by using (1.2) and noting that $\left\lfloor \frac{-1+\sqrt{16n+1}}{2} \right\rfloor = \left\lfloor \frac{-3+\sqrt{16n+25}}{2} \right\rfloor$ if and only if $n$ belongs to the set

$$\left\{ 4a^2+3a, 4a^2+5a, 4a^2+5a+1, 4a^2+7a+2, 4a^2+9a+4, 4a^2+11a+6 \,\middle|\, a \in \mathbb{N} \right\}.$$

□

Finally, we discuss the values of $\operatorname{conn.gon}(X)$, when the hypersurface $X$ has small dimension.

**Example 4.5.** Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geqslant 2n+2$, with $1 \leqslant n \leqslant 16$ and $n \neq 9, 13, 14$.

**Case *n*=1.** When $X$ is a plane curve, $\operatorname{conn.gon}(X)$ equals the gonality of $X$, which is $\operatorname{gon}(X) = d-1$ (cf. [7, Teorema 3.14]).

**Case *n*=2.** The connecting gonality of very general surfaces $X \subset \mathbb{P}^3$ of degree $d \geq 5$ is computed by tangent hyperplane sections $X \cap T_pX$, so that $\operatorname{conn.gon}(X) = d-2$ (see e.g. [1]).

**Case *n*=3.** When $n=3$, we have $\operatorname{conn.gon}(X) = d-2$. To see this fact, notice that $\operatorname{conn.gon}(X) \leqslant d-2$ by Remark 4.3 and case $n=2$ above. On the other hand, $\operatorname{conn.gon}(X) \geqslant \operatorname{cov.gon}(X) = d-3$ by (1.1) and (1.2). Suppose by contradiction that there exists a connecting family $\mathcal{C} \xrightarrow{\pi} T$ of $(d-3)$–gonal curves. Then Proposition 4.1 ensures that the general curve $C_t := \pi^{-1}(t)$ lies on $X \cap V_p^3$ for some $p \in X$. By Theorem 1.1, the varieties $Z_p := X \cap V_p^3$ are curves and, as we vary $p \in X$, we obtain a 3–dimensional family. However, according to Remark 4.2, the family $\mathcal{C} \xrightarrow{\pi} T$ should have dimension at least 4, a contradiction.

**Cases 4⩽*n*⩽16 with *n*≠9,13,14.** For all these values of $n$, we may apply Theorems 1.3 and 1.4, and the bounds included therein coincide. Thus

$$\operatorname{conn.gon}(X) = \begin{cases} d-3 & \text{if } n = 4,5 \\ d-4 & \text{if } n = 6,7,8 \\ d-5 & \text{if } n = 10,11,12 \\ d-6 & \text{if } n = 15,16. \end{cases}$$


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Francesco Bastianelli, Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via Edoardo Orabona 4, 70125 Bari – Italy
*Email address*: francesco.bastianelli@uniba.it

Ciro Ciliberto, Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Viale della Ricerca Scientifica 1, 00133 Roma – Italy
*Email address*: cilibert@mat.uniroma2.it

Flaminio Flamini, Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Viale della Ricerca Scientifica 1, 00133 Roma – Italy
*Email address*: flamini@mat.uniroma2.it

Paola Supino, Dipartimento di Matematica e Fisica, Università degli Studi "Roma Tre", Largo S. L. Murialdo 1, 00146 Roma – Italy
*Email address*: supino@mat.uniroma3.it