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ON THE DIAGONALIZABILITY AND FACTORIZABILITY OF QUADRATIC BOSON FIELDS

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Abstract. We provide a necessary and sufficient condition on the coefficient matrices A, C for the diagonalizability of quadratic fields of the form,

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) \ ,$$

where the a's and $a^\dagger$'s are the generators of the multi-dimensional Schrödinger Lie algebra. We also consider the Fock vacuum characteristic function

$$\left\langle \Phi, e^{i\,s\,X}\,\Phi \right\rangle \ ,$$

of X and study its factorizability/decomposability and how it relates to the commutativity of the simple quadratic components of X.

1. Introduction

We consider quadratic homogeneous polynomials of degree 2 in the boson Fock creation and annihilation operators $a_i, a_j^\dagger$ of a system with $n \in \mathbb{N}$ degrees of freedom. In the physics literature these polynomials are called *quadratic Hamiltonians* even in the absence of the requirement of boundedness from below of their spectra. In the following we call them *boson quadratic fields* for the reasons explained in [5].

There is a wide literature on the problem of finding conditions for the diagonalizability of boson quadratic fields, or more generally of finding their canonical forms under various groups of transformations, in particular the group of Bogolyubov transformations, [6]-[19].

The connections between quantization and the theory of orthogonal polynomials, in particular the problem of quadratic quantization (see [5] for a short review), naturally leads to the study of the related, but non–equivalent problem of finding conditions for the vacuum factorizability of boson quadratic fields, i.e. the possibility to express the vacuum characteristic function of such a field as the product of vacuum characteristic functions of other fields (see Definition 4.2) below. We show that diagonalizability implies vacuum factorizability and in the second part

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of this paper we produce examples showing that the converse implication does not hold.

In the first part of the paper we study the problem of diagonalizability under the action of the sub–group of Bogolyubov transformations implemented by second quantizations of unitary transformations of the 1–particle space of the given Fock space. This sub–group is relatively small, but it is the one that naturally appears in the vacuum factorization problem.

Our main result is a necessary and sufficient criterium, to our knowledge new, for diagonalizability of boson Fock quadratic fields under the action of this group. A consequence of this result is that, in the investigation of the factorizability problem, one can restrict one's attention to those quadratic field which do not fulfill the above mentioned criterium. We follow precisely this strategy in the construction of the examples discussed in the second part of the paper.

2. A Necessary and Sufficient Condition for Diagonalizability

For $i, j = 1, 2, ...$, let $a_i, a_j^\dagger$ be the generators of the multi-dimensional Schrödinger Lie algebra $\mathcal{S}$ with commutation relations

$$\left[a_i, a_j^\dagger\right] = \delta_{i,j} \quad , \quad \left[a_i^\dagger, a_j^\dagger\right] = [a_i, a_j] = 0 \ , \tag{2.1}$$

where $\delta_{i,j}$ is Kronecker's delta, and for $n \in \mathbb{N}$ let $M_n(\mathbb{C})$ denote the set of $n \times n$ $\mathbb{C}$–matrices. A **homogeneous quadratic boson field in standard form** [3], is a self–adjoint element X of $\mathcal{S}$, i.e. a quantum random variable, of the form

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) \ , \tag{2.2}$$

where the matrix $A = (A_{ij}) \in M_n(\mathbb{C})$ is symmetric and the matrix $C = (C_{ij}) \in M_n(\mathbb{C})$ is Hermitian, i.e.,

$$A_{ij} = A_{ji} \quad , \quad \overline{C_{ij}} = C_{ji} \quad , \quad \forall i, j \in \{1, 2, ..., n\}. \tag{2.3}$$

We use equivalently the notation:

$$X = (A, C) \ . \tag{2.4}$$

The following is a more explicit formulation of Definition 2 of [3].

Definition 2.1. A pair $X = (A, C) \in M_n(\mathbb{C}) \times M_n(\mathbb{C})$ with A symmetric and C Hermitian is called **diagonalizable** if there exists a unitary matrix U such that $M := UAU^T \in M_n(\mathbb{C})$ and $L := UCU^* \in M_n(\mathbb{R})$ are diagonal matrices.

Remark 2.2. $M = UAU^T$ is equivalent to $A = U^* M \bar{U}$ and $L = UCU^*$ is equivalent to $C = U^* L U$.

Remark 2.3. Sylvester's theorem (see Theorem 2.4.4.1 of [11]) states that if $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ are given, then the equation $AX - XB = C$ has a unique $n \times m$ matrix solution X, for each $n \times m$ matrix C, if and only if A and B have disjoint spectrums. A corollary of Sylvester's theorem (see Corollary 2.4.4.2 of [11]) is that if $B, C \in M_n(\mathbb{C})$ are block diagonal matrices of the form

$$B = B_1 \oplus \cdots \oplus B_k \quad , \quad C = C_1 \oplus \cdots \oplus C_k \ , \tag{2.5}$$

for some $k \le n$, where the blocks B_i, C_i have disjoint spectrums for each $i = 1, 2, ..., k$, and if $A \in M_n(\mathbb{C})$ satisfies with B, C the intertwining relation $AB = CA$, then A is also of the block diagonal form

$$A = A_1 \oplus \cdots \oplus A_k \ , \tag{2.6}$$

with $A_i B_i = C_i A_i$ for each $i = 1, 2, ..., k$.

The proof of the following theorem is adapted from [11] (see Theorem 4.5.15 of [11]).

Theorem 2.4. *A pair $X = (A, C) \in M_n(\mathbb{C}) \times M_n(\mathbb{C})$ with A symmetric and C Hermitian is diagonalizable if and only if CA is symmetric, i.e. if and only if $CA = A\overline{C}$.*

Proof. Suppose that the pair (A, C) is diagonalizable. Then $UAU^T = M$ and $UCU^* = L$ where L, M are diagonal and U is unitary. Using $A = U^* M \bar{U}$ and $C = U^* L U$ we have

$$CA = U^* L U U^* M \bar{U} = U^* L M \bar{U} \ , \tag{2.7}$$

and, since $(LM)^T = M^T L^T = ML = LM$, we have

$$(CA)^T = (U^* L M \bar{U})^T = U^* L M \bar{U} = CA \ , \tag{2.8}$$

so CA is symmetric. Conversely, suppose that CA is symmetric and let $\lambda_1, ..., \lambda_d \in \mathbb{R}$, where $d \le n$, be the distinct eigenvalues of C with multiplicities $n_i, ..., n_d$ respectively, with $n_1 + ... + n_d = n$. Then C can be spectrally decomposed as $C = W^* L W$, where W is a unitary matrix and the diagonal matrix L has the block diagonal form

$$L = \begin{pmatrix} \lambda_1 I_{n_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_d I_{n_d} \end{pmatrix} = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_d I_{n_d} \ , \tag{2.9}$$

where, for $i = 1, ..., d$, I_{n_i} denotes the $n_i \times n_i$ identity matrix. Then

$$\begin{aligned} CA = (CA)^T \iff CA = A\bar{C} \iff W^* L W A = A W^T L \bar{W} \\ \iff LWA = WAW^T L \bar{W} \iff L(WAW^T) = (WAW^T)L \end{aligned} \tag{2.10}$$

which by Remark 2.3 implies that WAW^T is of block diagonal form

$$WAW^T = A_1 \oplus \cdots \oplus A_d \ , \tag{2.11}$$

with symmetric blocks $A_j \in M_{n_j}(\mathbb{C})$, $j = 1, 2, ..., d$, since A is symmetric. Therefore,

$$A = W^* (A_1 \oplus \cdots \oplus A_d) \bar{W} \ . \tag{2.12}$$

By the Autonne-Takagi factorization theorem (see Corollary 4.4.4 (c) of [11]), for each $j = 1, 2, ..., d$, there exists a unitary matrix $V_j \in M_{n_j}(\mathbb{C})$ and a nonnegative diagonal matrix $M_j \in M_{n_j}(\mathbb{C})$ such that

$$A_j = V_j^* M_j \bar{V}_j \ . \tag{2.13}$$

Letting

$$V = V_1 \oplus \cdots \oplus V_d \quad , \quad M = M_1 \oplus \cdots \oplus M_d \quad , \quad U = VW \ , \tag{2.14}$$

we see that

$$A_1 \oplus \cdots \oplus A_d = V^* M \bar{V} \ . \tag{2.15}$$

Moreover, since for each $j = 1, 2, ..., d$,

$$\left[V_j, \lambda_j I_{n_j}\right] = 0 \ , \tag{2.16}$$

we obtain

$$[V, L] = [V_1 \oplus \cdots \oplus V_d, \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_d I_{n_d}] = 0 \ , \tag{2.17}$$

which, since L is a real diagonal matrix, implies that

$$[L, V^*] = [V, L]^* = 0 \ . \tag{2.18}$$

We have

$$UU^* = VW(VW)^* = VWW^*V^* = VV^* = I \ , \tag{2.19}$$

so U is unitary, and

$$U^*LU = W^*V^*LVW = W^*LV^*VW = W^*LW = C \ , \tag{2.20}$$

$$U^*M\bar{U} = (VW)^*M(\bar{VW}) = W^*(V^*M\bar{V})\bar{W} = W^*(A_1 \oplus \cdots \oplus A_d)\bar{W} = A \ . \tag{2.21}$$

Therefore, the pair (A, C) is diagonalizable. □

3. Truncated Virasoro Fields

Important examples of non diagonalizable quadratic fields are provided by the homogeneous quadratic parts of the **truncated Virasoro fields**, introduced in [4], defined by:

$$Y_n(w) = \alpha_1 \mathbf{I} + \sum_{j=1}^n \alpha_2^j a_j^\dagger + \sum_{j=1}^n \alpha_3^j a_j + \frac{1}{2} \sum_{j,k=1}^n \alpha_4^{j,k} a_j^\dagger a_k^\dagger \tag{3.1}$$

$$+ \frac{1}{2} \sum_{j,k=1}^n \alpha_5^{j,k} \left(a_j^\dagger a_k + a_k a_j^\dagger \right) + \frac{1}{2} \sum_{j,k=1}^n \alpha_6^{j,k} a_j a_k \ ,$$

where $n \geq 1$,

$$\alpha_1 = \frac{w_0}{2} \left(\mu^2 - \frac{n(n+1)}{2} \right) \quad , \quad \alpha_2^j = \alpha_3^j = \mu \, w_j \sqrt{j} \ , \tag{3.2}$$

$$\alpha_4^{j,k} = a_6^{j,k} = \chi_{n]}(j+k) w_{j+k} \sqrt{jk} \quad , \quad \alpha_5^{j,k} = \chi_{n]}(j+k) w_{|k-j|} \sqrt{jk} \ , \tag{3.3}$$

where

$$\chi_{n]}(a) = \begin{cases} 1 & \text{if } a \leq n \\ \\ 0 & \text{otherwise} \end{cases} \ , \tag{3.4}$$

and $w_m \in \mathbb{R}$ so that the coefficient $\alpha_5^{j,k}$ is common for $a_j^\dagger a_k$ and $a_k a_j^\dagger$. The quadratic part of $Y_n(w)$ is

$$Q_n(w) = \frac{1}{2} \sum_{j,k=1}^n \alpha_4^{j,k} a_j^\dagger a_k^\dagger + \frac{1}{2} \sum_{j,k=1}^n \alpha_5^{j,k} \left(a_j^\dagger a_k + a_k a_j^\dagger \right) + \frac{1}{2} \sum_{j,k=1}^n \alpha_6^{j,k} a_j a_k \ . \tag{3.5}$$

Using

$$\left[a_k, a_j^\dagger\right] = \delta_{k,j} \ , \tag{3.6}$$

we see that the homogeneous quadratic part of $Y_n(w)$ is

$$X_n(w) = \frac{1}{2}\sum_{j,k=1}^{n} \alpha_4^{j,k} a_j^\dagger a_k^\dagger + \sum_{j,k=1}^{n} \alpha_5^{j,k}\, a_j^\dagger a_k + \frac{1}{2}\sum_{j,k=1}^{n} \alpha_6^{j,k} a_j a_k \ , \tag{3.7}$$

so

$$A = (A_{j,k}) = \left(\frac{1}{2}\alpha_4^{j,k}\right) \quad , \quad C = (C_{j,k}) = \left(\alpha_5^{j,k}\right) \ . \tag{3.8}$$

Since both A, C are real, the condition on the diagonalizability of the pair (A, C) reduces to the commutativity of A and C, i.e., to $[A, C] = 0$ which is equivalent to $[\alpha_4, \alpha_5] = 0$. In general, the matrices α_4, α_5 do not commute. But they do commute, in fact they are equal, when w_m is constant for all m. For example, for $n = 4$,

$$\alpha_4 = \begin{pmatrix} w_2 & \sqrt{2}w_3 & \sqrt{3}w_4 & 0 \\ \sqrt{2}w_3 & 2w_4 & 0 & 0 \\ \sqrt{3}w_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \alpha_5 = \begin{pmatrix} w_0 & \sqrt{2}w_1 & \sqrt{3}w_2 & 0 \\ \sqrt{2}w_1 & 2w_0 & 0 & 0 \\ \sqrt{3}w_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ ,$$

and $[\alpha_4, \alpha_5]$ is

$$\begin{pmatrix} 0 & -\sqrt{2}(2w_1w_4 - w_1w_2 - w_0w_3) & -\sqrt{3}(w_0w_4 - w_2^2) & 0 \\ \sqrt{2}(2w_1w_4 - w_1w_2 - w_0w_3) & 0 & -\sqrt{6}(w_1w_4 - w_2w_3) & 0 \\ \sqrt{3}(w_0w_4 - w_2^2) & \sqrt{6}(w_1w_4 - w_2w_3) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

4. Connection with Factorizable Measures

We will show that the probability measure associated with a diagonalizable multi-dimensional quadratic field is factorizable into a product of probability measures of one-dimensional quadratic fields .

Theorem 4.1. *If the pair (A, C) is diagonalizable then the quadratic field*

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) \ , \tag{4.1}$$

takes the form

$$X = \sum_{i=1}^{n} \left(m_{ii} \left(b_i^\dagger\right)^2 + \bar{m}_{ii} b_i^2 + l_{ii} b_i^\dagger b_i \right) \ , \tag{4.2}$$

where

$$b_i^\dagger = \sum_{k=1}^{n} \overline{u_{ik}} a_k^\dagger \quad , \quad b_i = \sum_{k=1}^{n} u_{ik} a_k \ , \tag{4.3}$$

and the matrices

$$U = (u_{ij}) \quad , \quad L = (l_{ij}) \quad , \quad M = (m_{ij}) \ , \tag{4.4}$$

are as in Definition 2.1. Moreover, for $i, j \in \{1, 2, ..., n\}$,

$$\left[b_i, b_j^\dagger\right] = \delta_{i,j} \quad , \quad \left[b_i^\dagger, b_j^\dagger\right] = [b_i, b_j] = 0 \ . \tag{4.5}$$

Proof. We notice that

$$X = a^\dagger A \left(a^\dagger\right)^T + a\bar{A}a^T + a^\dagger C a^T \ , \tag{4.6}$$

where

$$a^\dagger = \begin{pmatrix} a_1^\dagger & \cdots & a_n^\dagger \end{pmatrix} \quad , \quad a = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \ . \tag{4.7}$$

In the notation of Theorem 2.4,

$$A = U^* M \bar{U} \quad , \quad C = U^* L U \ , \tag{4.8}$$

so

$$\begin{aligned} X =& a^\dagger U^* M \bar{U} \left(a^\dagger\right)^T + a\overline{U^* M \bar{U}} a^T + a^\dagger U^* L U a^T \\ =& a^\dagger U^* M \bar{U} \left(a^\dagger\right)^T + a U^T \bar{M} U a^T + a^\dagger U^* L U a^T \\ =& b^\dagger M (b^\dagger)^T + b\bar{M} b^T + b^\dagger L b^T \ , \end{aligned} \tag{4.9}$$

where

$$b^\dagger = a^\dagger U^* \quad , \quad b = a U^T \ , \tag{4.10}$$

i.e.,

$$b^\dagger = \begin{pmatrix} b_1^\dagger & \cdots & b_n^\dagger \end{pmatrix} \quad , \quad b = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} \ , \tag{4.11}$$

where, for $i \in \{1, 2, ..., n\}$,

$$b_i^\dagger = \sum_{k=1}^n \overline{u_{ik}} a_k^\dagger \quad , \quad b_i = \sum_{k=1}^n u_{ik} a_k \ . \tag{4.12}$$

Since L, M are diagonal, we obtain

$$X = \sum_{i=1}^n \left(m_{ii} \left(b_i^\dagger\right)^2 + \bar{m}_{ii} b_i^2 + l_{ii} b_i^\dagger b_i \right) \ . \tag{4.13}$$

Moreover,

$$\left[b_i, b_j^\dagger\right] = \sum_{p,q=1}^n u_{ip} \overline{u_{jq}} \left[a_p, a_q^\dagger\right] = \sum_{p,q=1}^n u_{ip} \overline{u_{jq}} \delta_{p,q} = \sum_{p=1}^n u_{ip} \overline{u_{jp}} = (UU^*)_{ij} = \delta_{i,j} \ , \tag{4.14}$$

and

$$\left[a_p^\dagger, a_q^\dagger\right] = [a_p, a_q] = 0 \implies \left[b_i^\dagger, b_j^\dagger\right] = [b_i, b_j] = 0 \ . \tag{4.15}$$

□

In what follows, where a vacuum characteristic function appears, we consider $a_i^\dagger$ and a_i, $i \in \{1, 2, ..., n\}$, in their reperesentation as unbounded operators on the Heisenberg Fock space.

Definition 4.2. The quadratic field X is **quadratic vacuum factorizable** if there exist quadratic fields X_j, $j = 1, 2, ..., k$, $k \geq 2$, such that for all s sufficiently close to zero,

$$\left\langle \Phi, e^{i\,s\,X}\, \Phi \right\rangle = \prod_{j=1}^k \left\langle \Phi, e^{i\,s\,X_j}\, \Phi \right\rangle \ , \tag{4.16}$$

and at least two of the X_j's have a non-degenerate (i.e. non-delta) vacuum distribution.

The above definition is a special case of the following:

Definition 4.3. The quadratic field X is **vacuum decomposable** if its vacuum characteristic function can be written as the product of at least two non-trivial (i.e. not corresponding to degenerate distributions) characteristic functions, i.e., for all s sufficiently close to zero,

$$\left\langle \Phi, e^{i\,s\,X}\,\Phi \right\rangle = \prod_{j=1}^{k} f_j(s) \ . \tag{4.17}$$

where, for each $j = 1, 2, ..., k$, f_j is a characteristic function.

Theorem 4.4. *Let Φ be a Fock vacuum unit vector, i.e., $\|\Phi\| = 1$ and $a_j\Phi = 0$ for all $j \in \{1, 2, ..., n\}$. In the notation of Theorem 2.4, let*

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) \ , \tag{4.18}$$

be diagonalizable and for each $j = 1, 2, ..., n$, let

$$X_j = m_{jj} \left(b_j^\dagger \right)^2 + \bar{m}_{jj} b_j^2 + l_{jj} b_j^\dagger b_j \ . \tag{4.19}$$

Then, for suitably small $s \in \mathbb{R}$ (so that a local homeomorphism exists between a neighborhood of the identity element of the Lie group and a neighborhood of the zero vector of its Lie algebra) and $i^2 = -1$, assuming that $m_{jj} \neq 0$ for all j, the vacuum characteristic function of X is

$$\left\langle \Phi, e^{i\,s\,X}\,\Phi \right\rangle = \prod_{j=1}^{n} e^{-\frac{i l_{jj} s}{2}} \sqrt{\frac{\cos L_j}{\cos(iK_j s + L_j)}} \ , \tag{4.20}$$

where, for each $j = 1, 2, ..., n$,

$$K_j = \sqrt{4\,|m_{jj}|^2 - l_{jj}^2} \quad , \quad L_j = \arctan\left(\frac{l_{jj}}{K_j}\right) \ . \tag{4.21}$$

If for some j, $m_{jj} = 0$, then the corresponding factor in the vacuum characteristic function is equal to 1.

Proof. By Lemma 3.3, Proposition 3.4 and Remark 3.6 of [1] (for, in the notation of [1], $a = 0, b = m_{jj}, \lambda = -\frac{l_{jj}}{2}, \nu = l_{jj}$), for each $j = 1, 2, ..., n$, since $a_j\Phi = 0$ implies $b_j\Phi = 0$, we have

$$e^{i\,s\,X_j}\,\Phi = p_j(s) e^{q_j(s)\left(b_j^\dagger\right)^2}\,\Phi \ , \tag{4.22}$$

and

$$\left\langle \Phi, e^{i\,s\,X_j}\,\Phi \right\rangle = p_j(s) \ , \tag{4.23}$$

where,

$$p_j(s) = e^{-\frac{i l_{jj} s}{2}} \sqrt{\frac{\cos L_j}{\cos(iK_j s + L_j)}} \quad , \quad q_j(s) = \frac{K_j \tan(iK_j s + L_j) - l_{jj}}{4\bar{m}_{jj}} \ , \tag{4.24}$$

$$K_j = \sqrt{4\,|m_{jj}|^2 - l_{jj}^2} \quad , \quad L_j = \arctan\left(\frac{l_{jj}}{K_j}\right) \ . \tag{4.25}$$

By Theorem 2.4, $X = \sum_{j=1}^{n} X_j$ where $[X_j, X_k] = 0$ for $j \neq k$. Thus

$$\left\langle \Phi, e^{i\,s\,X}\,\Phi \right\rangle = \left\langle \Phi, e^{i\,s\,\sum_{j=1}^{n} X_j}\,\Phi \right\rangle = \left\langle \Phi, \prod_{j=1}^{n} e^{i\,s\,X_j}\,\Phi \right\rangle \tag{4.26}$$

$$= \left\langle \Phi, \left(\prod_{j=1}^{n-1} e^{i\,s\,X_j}\right) e^{i\,s\,X_n}\,\Phi \right\rangle = p_n(s) \left\langle \Phi, \left(\prod_{j=1}^{n-1} e^{i\,s\,X_j}\right) e^{q_n(s)\left(b_n^\dagger\right)^2}\,\Phi \right\rangle$$

$$= p_n(s) \left\langle \Phi, \prod_{j=1}^{n-1} e^{i\,s\,X_j}\,\Phi \right\rangle = p_n(s) \left\langle \Phi, \left(\prod_{j=1}^{n-2} e^{i\,s\,X_j}\right) e^{i\,s\,X_{n-1}}\,\Phi \right\rangle$$

$$= p_n(s) p_{n-1}(s) \left\langle \Phi, \prod_{j=1}^{n-2} e^{i\,s\,X_j}\,\Phi \right\rangle = ... = \prod_{j=1}^{n} p_j(s) \langle \Phi, \Phi \rangle = \prod_{j=1}^{n} p_j(s)$$

$$= \prod_{j=1}^{n} \langle \Phi, e^{i\,s\,X_j}\,\Phi \rangle = \prod_{j=1}^{n} e^{-\frac{i l_{jj} s}{2}} \sqrt{\frac{\cos L_j}{\cos(iK_j s + L_j)}} \ .$$

If for some j, $m_{jj} = 0$, then $e^{i\,s\,X_j}\,\Phi = \Phi$. □

A converse to Theorem 4.4 is provided in the following.

Theorem 4.5. *For $j \in \{1, 2, ..., n\}$ let l_{jj}, m_{jj} be complex numbers and let*

$$K_j = \sqrt{4\,|m_{jj}|^2 - l_{jj}^2} \quad , \quad L_j = \arctan\left(\frac{l_{jj}}{K_j}\right) \ . \tag{4.27}$$

There exist quadratic fields

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) \ , \tag{4.28}$$

whose vacuum characteristic function, for suitably small $s \in \mathbb{R}$, is

$$\left\langle \Phi, e^{i\,s\,X}\,\Phi \right\rangle = \prod_{j=1}^{n} f_j(s) \ , \tag{4.29}$$

where,

$$f_j(s) = \begin{cases} e^{-\frac{i l_{jj} s}{2}} \sqrt{\frac{\cos L_j}{\cos(iK_j s + L_j)}} & \text{if } m_{jj} \neq 0 \\ 1 & \text{if } m_{jj} \neq 0 \end{cases} \ . \tag{4.30}$$

The quadratic field X is diagonalizable if and only if l_{jj} is real for all $j \in \{1, 2, ..., n\}$.

Proof. Let M and L be diagonal matrices with diagonal entries m_{jj}, l_{jj}, respectively, where $j \in \{1, 2, ..., n\}$. Define $n \times n$ matrices A, C by

$$A = U^* M \overline{U} \quad , \quad C = U^* L U \ , \tag{4.31}$$

where U is an arbitrary $n \times n$ unitary matrix. As in the proof of Theorems 4.1 and 4.4, the quadratic field

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) \ , \tag{4.32}$$

has vacuum characteristic function, for suitably small $s \in \mathbb{R}$,

$$\left\langle \Phi, e^{i\,s\,X}\, \Phi \right\rangle = \prod_{j=1}^{n} f_j(s) \ . \tag{4.33}$$

The above field X is diagonalizable if and only if A is symmetric, C is Hermitian and $CA = A\overline{C}$. In view of (4.31), these conditions are satisfied if and only if L is real. □

Remark 4.6. Theorem 4.4 implies that the probability measure μ_X associated via Bochner's theorem with the quadratic field X is quadratic vacuum factorizable, i.e., $\mu_X = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$, where μ_{X_j} is the probability measure associated with the quadratic field X_j, $j \in \{1, 2, ..., n\}$. Clearly, quadratic vacuum factorizability implies decomposability in the classical sense (see Section 5.1 of [14]) where a characteristic function is termed **decomposable** if, sans a trivial exponential factor, it can be written as a product of the characteristic functions of two non-degenerate distributions.

Remark 4.7. We define the Fourier transform of a function f by

$$\hat{f}(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i\lambda t} f(\lambda) d\lambda, \tag{4.34}$$

and the inverse Fourier transform of $\hat{f}$ by

$$f(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\lambda t} \hat{f}(t) dt, \tag{4.35}$$

so the inverse Fourier transform of

$$(2\pi)^{-1/2} \left\langle \Phi, e^{isX} \Phi \right\rangle \ , \tag{4.36}$$

gives the probability density function $p(\lambda)$ of X.

Remark 4.8. In the notation of Theorem 4.4, for $m_{jj} = 1$ and $l_{jj} = 0$, $j \in \{1, 2, ..., n\}$,

$$\left\langle \Phi, e^{i\,s\,X_j}\, \Phi \right\rangle = (\operatorname{sech} 2s)^{1/2} \ , \tag{4.37}$$

with corresponding, via Bochner's theorem, probability density function, obtained as an inverse Fourier transform with the use of Mathematica v.12,

$$p(\lambda) = \frac{1-i}{8\pi} \left(e^{-\frac{\pi\lambda}{4}}\, B\left(-1; \frac{1-i\lambda}{4}, \frac{1}{2}\right) + e^{\frac{\pi\lambda}{4}}\, B\left(-1; \frac{1+i\lambda}{4}, \frac{1}{2}\right)\right) \ , \tag{4.38}$$

where

$$B(z; a, b) = \int_0^z t^{a-1}(1-t)^{b-1}\, dt = z^a \sum_{n=0}^{\infty} \frac{(1-b)_n}{n!\,(a+n)} z^n \ , \tag{4.39}$$

is the *incomplete Beta function* and, for $x \in \mathbb{R}$, $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$. We also remark that

$$\left\langle \Phi, e^{i\,s\,\frac{1}{2}X_j}\,\Phi \right\rangle = (\operatorname{sech} s)^{1/2} \ , \tag{4.40}$$

with corresponding probability density function (inverse Fourier transform)

$$p(\lambda) = \frac{1-i}{4\pi}\left(e^{-\frac{\pi\lambda}{2}}\,B\left(-1;\frac{1-2i\lambda}{4},\frac{1}{2}\right) + e^{\frac{\pi\lambda}{2}}\,B\left(-1;\frac{1+2i\lambda}{4},\frac{1}{2}\right)\right) \ . \tag{4.41}$$

Notice that, replacing s by $as/2$, where a in any real number different from zero, we still obtain a characteristic function of a probability measure whose density is given by a re-scaled incomplete Beta function. In fact, in that case, the corresponding probability density function (inverse Fourier transform) is

$$p(\lambda) = \frac{1-i}{a\sqrt{2\pi}}\left(e^{-\frac{\pi\lambda}{a}}\,B\left(-1;\frac{a-4i\lambda}{4a},\frac{1}{2}\right) + e^{\frac{\pi\lambda}{a}}\,B\left(-1;\frac{a+4i\lambda}{4a},\frac{1}{2}\right)\right) \ . \tag{4.42}$$

5. n-Dimensional Angular Momentum Fields

Important examples of diagonalizable quadratic fields are provided by the **n-dimensional angular momentum fields**,

$$X = \sum_{k,m=1}^{n} w_{km} L_{k,m} \ , \tag{5.1}$$

where, $W = (w_{km}) \in M_n(\mathbb{R})$, and the $L_{k,m}$'s are the Hermitian (non linearly independent) generators of the **n-dimensional angular momentum** Lie algebra, introduced in [13], with commutation relations

$$[L_{k,m}, L_{k',m'}] = i\left(\delta_{k,k'}L_{m,m'} + \delta_{m,m'}L_{k,k'} - \delta_{k,m'}L_{m,k'} - \delta_{m,k'}L_{k,m'}\right) \ , \tag{5.2}$$

and skew-symmetry, duality properties,

$$L_{k,k} = 0 \quad , \quad L_{k,m} = -L_{m,k} \quad , \quad L^*_{k,m} = L_{k,m} \ . \tag{5.3}$$

Using multi-index notation

$$x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n} \ , \tag{5.4}$$

a representation on polynomials

$$p(x) = \sum_\alpha a_\alpha x^\alpha \ , \tag{5.5}$$

is provided by the operators

$$L_{k,m} = X_k P_m - X_m P_k \ , \tag{5.6}$$

where, for $j \in \{1, 2, ..., n\}$,

$$X_j p(x) = x_j p(x) \quad , \quad P_j p(x) = -i\partial_{x_j} p(x) \ , \tag{5.7}$$

are the classical position and momentum operators, with commutation relations, for $k, m \in \{1, 2, ..., n\}$,

$$[X_k, P_m] = i\delta_{m,k} \quad , \quad [X_k, X_m] = [P_k, P_m] = 0 \ , \tag{5.8}$$

and duality relations

$$X_k^* = X_k \quad , \quad P_m^* = P_m \ . \tag{5.9}$$

Theorem 5.1. *There exists a Hermitian matrix $C = (c_{km})$, with main diagonal entries equal to zero, for which the field X in (5.1) can be put in the form*

$$X = \sum_{\substack{k,m=1 \\ k \neq m}}^{n} c_{km} a_m^\dagger a_k \ , \tag{5.10}$$

where, for $k, m \in \{1, 2, ..., n\}$,

$$\left[a_k, a_m^\dagger\right] = \delta_{m,k}, [a_k, a_m] = \left[a_k^\dagger, a_m^\dagger\right] = 0, a_k^* = a_k^\dagger, \left(a_m^\dagger\right)^* = a_m \ , \tag{5.11}$$

i.e, X is a diagonalizable homogeneous quadratic boson field in standard form $X = (\mathbf{0}, C)$.

Proof. Letting, for each $j = 1, 2, ..., n$,

$$X_j = \frac{a_j + a_j^\dagger}{\sqrt{2}} \quad , \quad P_j = \frac{a_j - a_j^\dagger}{\sqrt{2}i}, \tag{5.12}$$

i.e.,

$$a_j = \frac{X_j + iP_j}{\sqrt{2}} \quad , \quad a_j^\dagger = \frac{X_j - iP_j}{\sqrt{2}}, \tag{5.13}$$

it is well-known that, for $k, m \in \{1, 2, ...\}$,

$$\left[a_k, a_m^\dagger\right] = \delta_{m,k}, [a_k, a_m] = \left[a_k^\dagger, a_m^\dagger\right] = 0, a_k^* = a_k^\dagger, \left(a_m^\dagger\right)^* = a_m \ , \tag{5.14}$$

Moreover,

$$\begin{aligned} L_{k,m} &= X_k P_m - X_m P_k \qquad (5.15) \\ &= \frac{1}{2i}\left((a_k + a_k^\dagger)(a_m - a_m^\dagger) - (a_m + a_m^\dagger)(a_k - a_k^\dagger)\right) \\ &= -\frac{i}{2}\left(a_k a_m - a_k a_m^\dagger + a_k^\dagger a_m - a_k^\dagger a_m^\dagger - a_m a_k + a_m a_k^\dagger - a_m^\dagger a_k + a_m^\dagger a_k^\dagger\right) \ , \end{aligned}$$

from which, using

$$a_k a_m^\dagger = \delta_{k,m} + a_m^\dagger a_k \quad , \quad a_m a_k^\dagger = \delta_{k,m} + a_k^\dagger a_m \ , \tag{5.16}$$

we obtain, after cancelations,

$$L_{k,m} = i\left(a_m^\dagger a_k - a_k^\dagger a_m\right) \ . \tag{5.17}$$

Thus, the field X takes the form

$$X = \sum_{k,m=1}^{n} w_{km} L_{k,m} = \sum_{\substack{k,m=1 \\ k \neq m}}^{n} w_{km} L_{k,m} \tag{5.18}$$

$$= \sum_{\substack{k,m=1 \\ k > m}}^{n} w_{km} L_{k,m} + \sum_{\substack{k,m=1 \\ k < m}}^{n} w_{km} \left(-L_{m,k}\right)$$

$$= \sum_{\substack{k,m=1 \\ k > m}}^{n} w_{km} L_{k,m} - \sum_{\substack{k,m=1 \\ k > m}}^{n} w_{mk} L_{k,m} \ .$$

Letting

$$r_{km} = w_{km} - w_{mk} \quad , \quad c_{km} = i r_{km} \quad , \quad c_{km} = -c_{mk} \ , \tag{5.19}$$

we have

$$X = \sum_{\substack{k,m=1 \\ k > m}}^{n} r_{km} L_{k,m} = \sum_{\substack{k,m=1 \\ k > m}}^{n} c_{km} \left(a_m^\dagger a_k - a_k^\dagger a_m\right) \tag{5.20}$$

$$= \sum_{\substack{k,m=1 \\ k > m}}^{n} c_{km} a_m^\dagger a_k - \sum_{\substack{k,m=1 \\ k > m}}^{n} c_{km} a_k^\dagger a_m$$

$$= \sum_{\substack{k,m=1 \\ k > m}}^{n} c_{km} a_m^\dagger a_k + \sum_{\substack{k,m=1 \\ k > m}}^{n} c_{mk} a_k^\dagger a_m$$

$$= \sum_{\substack{k,m=1 \\ k > m}}^{n} c_{km} a_m^\dagger a_k + \sum_{\substack{k,m=1 \\ k < m}}^{n} c_{km} a_m^\dagger a_k$$

$$= \sum_{\substack{k,m=1 \\ k \neq m}}^{n} c_{km} a_m^\dagger a_k \ .$$

from which we conclude that X is a homogeneous quadratic boson field in standard form $X = (\mathbf{0}, C)$, where the main diagonal entries of the Hermitian matrix C are equal to zero. By Theorem 2.4, X is diagonalizable. □

Corollary 5.2. *The vacuum characteristic function of X is*

$$\left\langle \Phi, e^{i\,s\,X}\, \Phi \right\rangle = 1 \ , \tag{5.21}$$

i.e., X has a degenerate (i.e. delta function) probability distribution and is therefore neither quadratic vacuum factorizable nor vacuum decomposable).

Proof. By Theorem 4.1, with $M = \mathbf{0}$ and U any unitary matrix that diagonalizes C, in the notation of Theorem 4.1,

$$X = \sum_{i=1}^{n} l_{ii} b_i^\dagger b_i \ . \tag{5.22}$$

Therefore, the vacuum characteristic function of X is

$$\left\langle \Phi, e^{i\,s\,X}\,\Phi \right\rangle = \left\langle \Phi, e^{i\,s\,\sum_{j=1}^{n} l_{ii} b_i^\dagger b_i}\,\Phi \right\rangle = \left\langle \Phi, \prod_{j=1}^{n} e^{i\,s\,l_{ii} b_i^\dagger b_i}\,\Phi \right\rangle = \langle \Phi, \Phi \rangle = 1 \ , \tag{5.23}$$

since $e^{i\,s\,l_{ii} b_i^\dagger b_i}\,\Phi = \Phi$, for each j. Therefore, X has a degenerate (delta function) probability distribution. Since,

- the product of two characteristic functions is always a characteristic function,
- the only characteristic functions whose reciprocals are also characteristic functions belong to degenerate distributions,

we see that the n-dimensional angular momentum field X is not quadratic (or any other non-degenerate way) vacuum factorizable and it is not vacuum decomposable. □

6. Non-Diagonalizable does not Imply non-Quadratically Vacuum Factorizable

We consider the quadratic field (4.1), i.e.

$$X = \sum_{i,j=1}^{2} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) \ , \tag{6.1}$$

where, for $a, b, c, l \in \mathbb{C}$ and $k, m \in \mathbb{R}$,

$$A = (A_{ij}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad , \quad C = (C_{ij}) = \begin{pmatrix} k & l \\ \bar{l} & m \end{pmatrix} \ . \tag{6.2}$$

The diagonalizability condition $CA = A\overline{C}$ is equivalent to the equation

$$kb + lc = a\bar{l} + bm \ . \tag{6.3}$$

Choosing $a = c = 1$, $b = 0$, $k = m = 0$, and $l = i$ where $i^2 = -1$, we have the matrices

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad C = iS = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{6.4}$$

corresponding to the non-diagonalizable quadratic field

$$Z = \left(a_1^\dagger \right)^2 + \left(a_2^\dagger \right)^2 + a_1^2 + a_2^2 + i \left(a_1^\dagger a_2 - a_2^\dagger a_1 \right) \ . \tag{6.5}$$

We will show that Z is quadratic vacuum factorizable. We prove the result in several steps. First we compute the vacuum characteristic function of Z.

Theorem 6.1. *For real t sufficiently close to zero, the vacuum characteristic function of Z is*

$$\left\langle \Phi, e^{i\,t\,Z}\,\Phi \right\rangle = \operatorname{sech} 2t \ . \tag{6.6}$$

Proof. In the notation of Theorem 2 of [2], with the identification $a_i = D_i, a_i^\dagger = x_i$, we have:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = 2A = 2I, \alpha_6 = 2\bar{A} = 2I, \alpha_5 = C = iS \ . \tag{6.7}$$

Therefore

$$v = \begin{pmatrix} \alpha_5 & \alpha_4 \\ -\alpha_6 & -\alpha_5^T \end{pmatrix} = \begin{pmatrix} C & 2A \\ -2\bar{A} & -C^T \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & i & 2 & 0 \\ -i & 0 & 0 & 2 \\ \hline -2 & 0 & 0 & i \\ 0 & -2 & -i & 0 \end{array}\right) , \tag{6.8}$$

and

$$\begin{aligned} e^{tv} &= \begin{pmatrix} P(t) & Q(t) \\ -R(t) & S(t) \end{pmatrix} \\ &= \left(\begin{array}{cc|cc} \cosh t\,\cos 2t & i\,\cos 2t\,\sinh t & \cosh t\,\sin 2t & i\,\sin 2t\,\sinh t \\ -i\,\cos 2t\,\sinh t & \cos 2t\,\cosh t & -i\,\sin 2t\,\sinh t & \cosh t\,\sin 2t \\ \hline -\cosh t\,\sin 2t & -i\,\sin 2t\,\sinh t & \cos 2t\,\cosh t & i\,\cos 2t\,\sinh t \\ i\,\sin 2t\,\sinh t & -\cosh t\,\sin 2t & -i\,\cos 2t\,\sinh t & \cos 2t\,\cosh t \end{array}\right) . \end{aligned} \tag{6.9}$$

Thus,

$$A_1(t) = A_2(t) = A_3(t) = 0 \ , \tag{6.10}$$

and

$$A_4(t) = Q(t)S(t)^{-1}, A_5(t) = -\log\left(S(t)^T\right), A_6(t) = S(t)^{-1}R(t) \ . \tag{6.11}$$

Thus, by Theorem 2 of [2] and the matrix identity

$$e^{\mathrm{Tr}(\log A)} = \det A \ , \tag{6.12}$$

for non-singular matrices A, we have

$$\begin{aligned} \left\langle \Phi, e^{i\,t\,Z}\,\Phi \right\rangle &= e^{\frac{1}{2}\sum_{j=1}^{2} A_5^{j,j}(it)} = e^{\frac{1}{2}\mathrm{Tr}A_5(it)} \\ &= \left(e^{\mathrm{Tr}\log\left(S(it)^T\right)}\right)^{-1/2} = (\det S(it))^{-1/2} \\ &= \left(\cosh^2 2t\right)^{-1/2} = \operatorname{sech} 2t \ . \end{aligned} \tag{6.13}$$

□

Corollary 6.2. *The vacuum characteristic function of Z is quadratic vacuum factorizable.*

Proof. Let

$$Z_j = \left(a_j^\dagger\right)^2 + a_j^2 \quad , \quad j = 1, 2 \ . \tag{6.14}$$

Then Z_j is diagonalizable for $j = 1, 2$ and, as in the proof of Theorem 6.1,

$$v_1 = \left(\begin{array}{cc|cc} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad , \quad v_2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right) \quad , \tag{6.15}$$

$$e^{tv_1} = \begin{pmatrix} P_1(t) & Q_1(t) \\ -R_1(t) & S_1(t) \end{pmatrix} = \left(\begin{array}{cc|cc} \cos 2t & 0 & \sin 2t & 0 \\ 0 & 1 & 0 & 0 \\ \hline -\sin 2t & 0 & \cos 2t & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad , \tag{6.16}$$

$$e^{tv_2} = \begin{pmatrix} P_2(t) & Q_2(t) \\ -R_2(t) & S_2(t) \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \cos 2t & 0 & \sin 2t \\ \hline 0 & 0 & 1 & 0 \\ 0 & -\sin 2t & 0 & \cos 2t \end{array}\right) \quad . \tag{6.17}$$

Thus, for $j = 1, 2$,

$$\begin{aligned} \left\langle \Phi, e^{i\,t\,Z_j}\,\Phi \right\rangle &= (\det S_j(it))^{-1/2} \\ &= (\cosh\,2t)^{-1/2} = (\operatorname{sech} 2t)^{1/2} \quad . \end{aligned} \tag{6.18}$$

Therefore, using the result of Theorem 6.1,

$$\left\langle \Phi, e^{i\,t\,Z}\,\Phi \right\rangle = \left\langle \Phi, e^{i\,t\,Z_1}\,\Phi \right\rangle \left\langle \Phi, e^{i\,t\,Z_2}\,\Phi \right\rangle \;, \tag{6.19}$$

so Z is quadratic vacuum factorizable. □

Recall that

$$Z = \left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2 + a_1^2 + a_2^2 + i\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) \quad . \tag{6.20}$$

Lemma 6.3.

$$\left[\left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2 + a_1^2 + a_2^2, i\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)\right] = 0 \;. \tag{6.21}$$

Proof. We have

$$\left[a_1, a_1^\dagger a_2 - a_2^\dagger a_1\right] = a_2 \;, \left[a_2, a_1^\dagger a_2 - a_2^\dagger a_1\right] = -a_2 \;, \tag{6.22}$$

$$\begin{aligned} \left[a_1^2, a_1^\dagger a_2 - a_2^\dagger a_1\right] &= a_1 \left[a_1, a_1^\dagger a_2 - a_2^\dagger a_1\right] + \left[a_1, a_1^\dagger a_2 - a_2^\dagger a_1\right] a_1 \\ &= a_1 a_2 + a_2 a_1 = 2 a_1 a_2 \;, \end{aligned} \tag{6.23}$$

$$\begin{aligned} \left[a_2^2, a_1^\dagger a_2 - a_2^\dagger a_1\right] &= a_2 \left[a_2, a_1^\dagger a_2 - a_2^\dagger a_1\right] + \left[a_2, a_1^\dagger a_2 - a_2^\dagger a_1\right] a_2 \\ &= -a_2 a_1 - a_1 a_2 = -2 a_1 a_2 \;. \end{aligned} \tag{6.24}$$

Summing the last two identities, one has

$$\left[a_1^2 + a_2^2, a_1^\dagger a_2 - a_2^\dagger a_1\right] = 2 a_1 a_2 - 2 a_1 a_2 = 0 \;. \tag{6.25}$$

Consequently,

$$0 = \left[a_1^2 + a_2^2, a_1^\dagger a_2 - a_2^\dagger a_1\right]^* = \left[a_1 a_2^\dagger - a_2 a_1^\dagger, \left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2\right] \tag{6.26}$$
$$= -\left[a_1^\dagger a_2 - a_1 a_2^\dagger, \left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2\right] = \left[\left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2, a_1^\dagger a_2 - a_1 a_2^\dagger\right] .$$

Therefore,

$$\left[a_1^2 + a_2^2 + \left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2, a_1^\dagger a_2 - a_2^\dagger a_1\right] = 0 , \tag{6.27}$$

which is equivalent to (6.21). □

Corollary 6.4. *For t sufficiently small, one has:*

$$e^{itZ} = e^{it\left(\left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2 + a_1^2 + a_2^2\right)} e^{-t\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)} . \tag{6.28}$$

Proof. (6.28) follows from (6.21). □

Remark 6.5. Corollary 6.4 means that the decomposition

$$Z = \left(\left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2 + a_1^2 + a_2^2\right) + i\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) , \tag{6.29}$$

expresses Z as a sum of two vacuum independent classical random variables. This does not contradict Corollary 6.2 because, by (6.28),

$$e^{itZ} = e^{it\left(\left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2 + a_1^2 + a_2^2\right)} e^{-t\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)} \tag{6.30}$$
$$= e^{it\left(\left(a_1^\dagger\right)^2 + a_1^2\right)} e^{it\left(\left(a_2^\dagger\right)^2 + a_2^2\right)} e^{-t\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)} ,$$

and

$$e^{-t\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)} \Phi = \Phi , \tag{6.31}$$

implies that

$$\left\langle \Phi, e^{-t\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)} \Phi \right\rangle = 1 . \tag{6.32}$$

This means that the vacuum distribution of the quadratic field $i\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)$ is the δ–measure at 0.

7. Connection with Commutativity

The non-diagonalizable, quadratic vacuum factorizable, quadratic field Z of (6.5) can be written as

$$Z = X_1 + X_2 + X_3 , \tag{7.1}$$

where,

$$X_1 = \left(a_1^\dagger\right)^2 + \left(a_2^\dagger\right)^2 \quad , \quad X_2 = a_1^2 + a_2^2 \quad , \quad X_3 = i\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) , \tag{7.2}$$

with, by Lemma 6.3,

$$[X_1 + X_2, X_3] = 0 . \tag{7.3}$$

In this Section, we describe, for $n = 2$, the most general quadratic fields for which a commutativity condition of this type is compatible with non-diagonalizability.

Theorem 7.1. *Let*

$$X = X_1 + X_2 + X_3 \ , \tag{7.4}$$

where

$$X_1 = a\left(a_1^\dagger\right)^2 + \bar{a}\, a_1^2 + k\, a_1^\dagger a_1 \ , \tag{7.5}$$

$$X_2 = c\left(a_2^\dagger\right)^2 + \bar{c}\, a_2^2 + m\, a_2^\dagger a_2 \ , \tag{7.6}$$

$$X_3 = 2\, b\, a_1^\dagger\, a_2^\dagger + 2\bar{b}\, a_1 a_2 + l\, a_1^\dagger\, a_2 + \bar{l}\, a_2^\dagger a_1 \ , \tag{7.7}$$

where $a, b, c, l \in \mathbb{C} \quad , \quad k, m \in \mathbb{R}$, *i.e.,* X *is of the form*

$$X = \sum_{i,j=1}^{2} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) \ , \tag{7.8}$$

where

$$A = (A_{ij}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad , \quad C = (C_{ij}) = \begin{pmatrix} k & l \\ \bar{l} & m \end{pmatrix} \ . \tag{7.9}$$

Then

$$[X_1, X_2] = 0 \ , \tag{7.10}$$

$$\begin{aligned}[X_1, X_3] =& 2(kb - a\bar{l})a_1^\dagger a_2^\dagger + 2(\bar{a}l - k\bar{b})a_1 a_2 \\ &+ (kl - 4a\bar{b})a_1^\dagger a_2 + (4\bar{a}b - k\bar{l})a_2^\dagger a_1 \ ,\end{aligned} \tag{7.11}$$

$$\begin{aligned}[X_2, X_3] =& 2(mb - cl)a_1^\dagger a_2^\dagger + 2(\bar{c}\bar{l} - m\bar{b})a_1 a_2 \\ &+ (4b\bar{c} - ml)a_1^\dagger a_2 + (m\bar{l} - 4\bar{b}c)a_2^\dagger a_1 \ .\end{aligned} \tag{7.12}$$

Proof. Equation (7.10) follows from the commutativity of everything that has index 1 with everything that has index 2. Using commutation relations (2.1) to put terms in **normal order**, i.e., $a^\dagger$'s on the left and a's on the right, we obtain the commutation relations:

$$\left[\left(a_1^\dagger\right)^2, a_1^\dagger\, a_2^\dagger\right] = \left[\left(a_1^\dagger\right)^2, a_1^\dagger\, a_2\right] = \left[a_1^2, a_1 a_2\right] = \left[a_1^2, a_2^\dagger a_1\right] = 0 \ , \tag{7.13}$$

$$\left[\left(a_1^\dagger\right)^2, a_1 a_2\right] = -2a_1^\dagger a_2 \ , \left[\left(a_1^\dagger\right)^2, a_2^\dagger a_1\right] = -2a_1^\dagger a_2^\dagger \ , \tag{7.14}$$

$$\left[a_1^2, a_1^\dagger a_2^\dagger\right] = 2a_2^\dagger a_1 \ , \left[a_1^2, a_1^\dagger a_2\right] = 2a_1 a_2 \ , \tag{7.15}$$

$$\left[a_1^\dagger a_1, a_1^\dagger a_2^\dagger\right] = a_1^\dagger a_2^\dagger \ , \left[a_1^\dagger a_1, a_1 a_2\right] = -a_1 a_2 \ , \tag{7.16}$$

$$\left[a_1^\dagger a_1, a_1^\dagger a_2\right] = a_1^\dagger a_2 \ , \left[a_1^\dagger a_1, a_2^\dagger a_1\right] = -a_2^\dagger a_1 \ , \tag{7.17}$$

and

$$\left[\left(a_2^\dagger\right)^2, a_1^\dagger a_2^\dagger\right] = \left[\left(a_2^\dagger\right)^2, a_2^\dagger a_1\right] = \left[a_2^2, a_1 a_2\right] = \left[a_2^2, a_1^\dagger a_2\right] = 0 \ , \tag{7.18}$$

$$\left[\left(a_2^\dagger\right)^2, a_1 a_2\right] = -2a_2^\dagger a_1 \ , \left[\left(a_2^\dagger\right)^2, a_1^\dagger a_2\right] = -2a_1^\dagger a_2^\dagger \ , \tag{7.19}$$

$$\left[a_2^2, a_1^\dagger\, a_2^\dagger\right] = 2a_1^\dagger a_2 \ , \left[a_2^2, a_2^\dagger\, a_1\right] = 2a_1 a_2 \ , \tag{7.20}$$

$$\left[a_2^\dagger a_2, a_1^\dagger a_2^\dagger\right] = a_1^\dagger a_2^\dagger \ , \left[a_2^\dagger a_2, a_1 a_2\right] = -a_1 a_2 \ , \tag{7.21}$$

$$\left[a_2^\dagger a_2, a_1^\dagger a_2\right] = -a_1^\dagger a_2 \ , \left[a_2^\dagger a_2, a_2^\dagger a_1\right] = a_2^\dagger a_1 \ . \tag{7.22}$$

Using the above commutation relations and the bilinearity of the Lie bracket, we obtain (7.11) and (7.12). □

Corollary 7.2. *In the notation of Theorem 7.1,* $[X_1 + X_2, X_3] = 0$ *if and only if*

$$(kb - a\bar{l}) + (mb - cl) = 0 \ , \tag{7.23}$$

$$(kl - 4a\bar{b}) + (4b\bar{c} - ml) = 0 \ . \tag{7.24}$$

In particular, if $[X_1 + X_2, X_3] = 0$ *then* X *is quadratic vacuum factorizable.*

Proof. Using (7.11), (7.12), and the linear independence of

$$a_1^\dagger a_2^\dagger, a_1 a_2, a_1^\dagger a_2, a_1 a_2^\dagger \ , \tag{7.25}$$

we find that $[X_1 + X_2, X_3] = 0$ if and only if

$$2(kb - a\bar{l}) + 2(mb - cl) = 0 \ , \tag{7.26}$$

$$2(\bar{a}l - k\bar{b}) + 2(\bar{c}\bar{l} - m\bar{b}) = 0 \ , \tag{7.27}$$

$$(kl - 4a\bar{b}) + (4b\bar{c} - ml) = 0 \ , \tag{7.28}$$

$$(4\bar{a}b - k\bar{l}) + (m\bar{l} - 4\bar{b}c) = 0 \ , \tag{7.29}$$

from which, since the second and fourth equations are the conjugates of the first and third equation respectively, we obtain (7.23) and (7.24). Since $[X_1 + X_2, X_3] = 0$ and $[X_1, X_2] = 0$,

$$\begin{aligned}\left\langle \Phi, e^{i\,s\,X}\,\Phi\right\rangle &= \left\langle \Phi, e^{i\,s\,(X_1+X_2)} e^{i\,s\,X_3}\,\Phi\right\rangle = \left\langle \Phi, e^{i\,s\,X_1} e^{i\,s\,X_2} e^{i\,s\,X_3}\,\Phi\right\rangle \\ &= \left\langle \Phi, e^{i\,s\,X_1} e^{i\,s\,X_2}\,\Phi\right\rangle = \left\langle \Phi, e^{i\,s\,X_1}\,\Phi\right\rangle \left\langle \Phi, e^{i\,s\,X_2}\,\Phi\right\rangle \ ,\end{aligned} \tag{7.30}$$

as in the proof of Theorem 4.4, so X is quadratic vacuum factorizable. □

Theorem 7.3. *Let*

$$X = Y_1 + Y_2 + Y_3 \ , \tag{7.31}$$

where

$$Y_1 = a\,\left(a_1^\dagger\right)^2 + \bar{a}\,a_1^2 \ , \tag{7.32}$$

$$Y_2 = c\,\left(a_2^\dagger\right)^2 + \bar{c}\,a_2^2 \ , \tag{7.33}$$

$$Y_3 = k\,a_1^\dagger a_1 + m\,a_2^\dagger a_2 + 2\,b\,a_1^\dagger\,a_2^\dagger + 2\bar{b}\,a_1 a_2 + l\,a_1^\dagger\,a_2 + \bar{l}\,a_2^\dagger a_1 \ , \tag{7.34}$$

where $a, b, c, l \in \mathbb{C}$, $k, m \in \mathbb{R}$, *i.e., X is of the form*

$$X = \sum_{i,j=1}^{2} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) , \tag{7.35}$$

where

$$A = (A_{ij}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad , \quad C = (C_{ij}) = \begin{pmatrix} k & l \\ \bar{l} & m \end{pmatrix} . \tag{7.36}$$

Then

$$[Y_1, Y_2] = 0 , \tag{7.37}$$

$$[Y_1, Y_3] = -2ak \left(a_1^\dagger\right)^2 - 4a\bar{b}\, a_1^\dagger\, a_2 - 2a\bar{l}\, a_1^\dagger\, a_2^\dagger \tag{7.38}$$
$$+ 2\bar{a}k\, a_1^2 + 4\bar{a}b\, a_2^\dagger\, a_1 + 2\bar{a}l\, a_1 a_2 ,$$

$$[Y_2, Y_3] = -2cm \left(a_2^\dagger\right)^2 - 4c\bar{b}\, a_2^\dagger\, a_1 - 2c\bar{l}\, a_1^\dagger\, a_2^\dagger \tag{7.39}$$
$$+ 2\bar{c}m\, a_2^2 + 4\bar{c}b\, a_1^\dagger\, a_2 + 2\bar{c}\bar{l}\, a_1 a_2 .$$

Proof. The proof is similar to that of theorem 7.1, using the additional commutation relations:

$$\left[a_2^2, a_1^\dagger\, a_1\right] = \left[\left(a_2^\dagger\right)^2, a_1^\dagger\, a_1\right] = 0 , \tag{7.40}$$

$$\left[\left(a_1^\dagger\right)^2, a_1^\dagger a_1\right] = -2\left(a_1^\dagger\right)^2 , \left[\left(a_2^\dagger\right)^2, a_2^\dagger a_2\right] = -2\left(a_2^\dagger\right)^2 , \tag{7.41}$$

$$\left[a_1^2, a_1^\dagger\, a_1\right] = 2a_1^2 , \left[a_2^2, a_2^\dagger\, a_2\right] = 2a_2^2 . \tag{7.42}$$

□

Corollary 7.4. *In the notation of Theorem 7.3,* $[Y_1 + Y_2, Y_3] = 0$ *if and only if*

$$ak = cm = (a + c)l = \bar{c}b - a\bar{b} = 0 . \tag{7.43}$$

In particular, if $[X_1 + X_2, X_3] = 0$ *then X is quadratic vacuum factorizable.*

Proof. Using (7.38), (7.39), and the linear independence of

$$\left(a_1^\dagger\right)^2, \left(a_2^\dagger\right)^2, a_1^2, a_2^2, a_1^\dagger a_2^\dagger, a_1 a_2, a_1^\dagger a_2, a_2^\dagger a_1 , \tag{7.44}$$

we find that $[Y_1 + Y_2, Y_3] = 0$ if and only if (7.43) is true. The factorizability of X follows as in the Proof of Corollary 7.2. □

8. An Example of a Decomposable Quadratic Field That Admits a non-Quadratic Factorization

Since quadratic vacuum factorization follows naturally from diagonalizability and/or commutativity (see Theorem 4.4 and Corollaries 7.2, 7.4), it is reasonable to investigate , starting with the simple 2×2 case, quadratic fields X of the form

$$X = \sum_{i,j=1}^{2} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) , \tag{8.1}$$

where

$$A = (A_{ij}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad , \quad C = (C_{ij}) = \begin{pmatrix} k & l \\ \bar{l} & m \end{pmatrix} \ , \tag{8.2}$$

with a, b, c, k, l, m **not satisfying**, at least one of, the diagonalizabilty and commutativity conditions

$$(kb - a\bar{l}) + (lc - bm) = 0 \ , \tag{8.3}$$

$$(kb - a\bar{l}) + (mb - cl) = (kl - 4a\bar{b}) + (4b\bar{c} - ml) = 0 \ , \tag{8.4}$$

$$ak = cm = (a + c)l = \bar{c}b - a\bar{b} = 0 \ . \tag{8.5}$$

In what follows we present an example of a decomposable quadratic field X which admits a non-quadratic factorization.

Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad , \quad C = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{8.6}$$

corresponding to the quadratic field

$$X = \left(a_2^\dagger\right)^2 + a_2^2 + 2\left(a_1^\dagger a_2^\dagger + a_1 a_2\right) + i\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) \ . \tag{8.7}$$

Then, as in the proof of Theorem 6.1,

$$v = \left(\begin{array}{cc|cc} 0 & i & 0 & 2 \\ -i & 0 & 2 & 2 \\ \hline 0 & -2 & 0 & i \\ -2 & -2 & -i & 0 \end{array}\right) \ , \tag{8.8}$$

$$\begin{aligned} P(t) &= S(t) \\ &= \begin{pmatrix} \cos^3 t & \frac{1}{2}(\cos 3t + (-1 + i\sin 2t)\cos t) \\ \frac{1}{2}(\cos 3t + (-1 - i\sin 2t)\cos t) & \frac{1}{4}(\cos t + 3\cos 3t) \end{pmatrix} \ , \end{aligned} \tag{8.9}$$

$$Q(t) = \begin{pmatrix} -\sin^3 t & \sin t\, \cos t\,(2\cos t + i\sin t) \\ \sin t\, \cos t\,(2\cos t - i\sin t) & \frac{1}{4}(-\sin t + 3\sin 3t) \end{pmatrix} \ , \tag{8.10}$$

$$R(t) = \begin{pmatrix} -\sin^3 t & -\frac{i}{4}(\cos 3t + (-1 + 4i\sin 2t)\cos t) \\ \sin t\, \cos t\,(2\cos t - i\sin t) & \frac{1}{4}(-\sin t + 3\sin 3t) \end{pmatrix} \ , \tag{8.11}$$

and

$$\left\langle \Phi, e^{i\,t\,X}\,\Phi \right\rangle = (\det S(it))^{-1/2} = \operatorname{sech} t \frac{\sqrt{2}}{\sqrt{-3 + 5\cosh 2t}}. \tag{8.12}$$

We know from Theorem 6.1 that $\operatorname{sech} t$ is the characteristic function of a probability measure. Therefore, to show that X is decomposable, it suffices to show that

$$f(t) = \frac{\sqrt{2}}{\sqrt{-3 + 5\cosh 2t}} \ , \tag{8.13}$$

satisfies $f(0) = 1$ (which is obvious) and is positive definite. Equivalently, by Bochner's theorem, that it is the Fourier transform of a nonnegative function $g(x)$ defined on the real line. Using Mathematica v.12, we find that

$$g(x) = \frac{\sqrt{2}}{\sqrt{5\pi}(1+x^2)} \left((1+ix)F_1\left(\frac{1-ix}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3-ix}{2}; \frac{3+4i}{5}, \frac{3-4i}{5}\right) \right. \tag{8.14}$$
$$\left. +(1-ix)F_1\left(\frac{1+ix}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3+ix}{2}; \frac{3+4i}{5}, \frac{3-4i}{5}\right)\right) \,,$$

where the *Appell hypergeometric function* F_1 is defined, for $|x| < 1, |y| < 1$, by the infinite series

$$F(a; b_1, b_2; c; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!} \,, \tag{8.15}$$

where, for $z \in \mathbb{R}$, $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$. For other values of x, y it is defined by analytic continuation. The bell-shaped function $g(x)$ is non-negative. Thus $f(t)$ is positive definite, so X is decomposable. We will know examine if X is quadratic vacuum factorizable, i.e., if there exist quadratic fields Y and Z with

$$\left\langle \Phi, e^{i\,t\,X}\,\Phi\right\rangle = \left\langle \Phi, e^{i\,t\,Y}\,\Phi\right\rangle \left\langle \Phi, e^{i\,t\,Z}\,\Phi\right\rangle. \tag{8.16}$$

By Remark 4.6,

$$Y = \frac{1}{2}\left(\left(a_1^\dagger\right)^2 + a_1^2 + \left(a_2^\dagger\right)^2 + a_2^2\right) \,, \tag{8.17}$$

since, as in the proof of Theorem 4.4,

$$\left\langle \Phi, e^{i\,t\,Y}\,\Phi\right\rangle = \left\langle \Phi, e^{i\,t\,\frac{1}{2}\left(\left(a_1^\dagger\right)^2 + a_1^2\right)}\,\Phi\right\rangle \left\langle \Phi, e^{i\,t\,\left(\left(a_2^\dagger\right)^2 + a_2^2\right)}\,\Phi\right\rangle = \operatorname{sech} t \,. \tag{8.18}$$

If there existed a quadratic field Z with

$$\left\langle \Phi, e^{i\,t\,Z}\,\Phi\right\rangle = \frac{\sqrt{2}}{\sqrt{-3+5\cosh 2t}} \,, \tag{8.19}$$

then, as stated in the proof of Theorem 4.4, there would exist constants l, L, K such that

$$e^{-\frac{ilt}{2}}\sqrt{\frac{\cos L}{\cos(iKt+L)}} = \frac{\sqrt{2}}{\sqrt{-3+5\cosh 2t}} \,, \tag{8.20}$$

which is equivalent to

$$g(t) := e^{-ilt}\frac{\cos L}{\cos(iKt+L)} - \frac{2}{-3+5\cosh 2t} = 0 \,, \tag{8.21}$$

for all t. Since $g'(0)$ must be equal to zero, we find that $l = K\tan L$. Since $g''(0)$ must be equal to zero, we find that $K = \pm\sqrt{10}\cos L$. In both cases, since $g'''(0)$ must be equal to zero, we find that $\sin L = 0$, so $L = k\pi$, $k \in \mathbb{Z}$. Therefore

$$g(t) = \operatorname{sech}\left(\sqrt{10}t\right) - \frac{2}{-3+5\cosh 2t} \,, \tag{8.22}$$

which is not identically equal to zero, in fact, it has two negative humps to the left and right of zero and dies out as $x \to \pm\infty$. Thus X admits a non-quadratic factorization.

Remark 8.1. It is still possible to find homogeneous quadratic fields X and Y such that

$$\operatorname{sech} t \frac{\sqrt{2}}{\sqrt{-3+5\cosh 2t}} = \left\langle \Phi, e^{i\,t\,Y}\,\Phi \right\rangle \left\langle \Phi, e^{i\,t\,Z}\,\Phi \right\rangle, \tag{8.23}$$

where both $\left\langle \Phi, e^{i\,t\,Y}\,\Phi \right\rangle$ and $\left\langle \Phi, e^{i\,t\,Z}\,\Phi \right\rangle$ are different from $\operatorname{sech} t$. But it can be shown that there are no complex numbers $l_1, l_2, L_1, L_2, K_1, K_2$ such that the equality

$$\operatorname{sech} t \frac{\sqrt{2}}{\sqrt{-3+5\cosh 2t}} = e^{-\frac{il_1 t}{2}} \sqrt{\frac{\cos L_1}{\cos(iK_1 t + L_1)}}\, e^{-\frac{il_2 t}{2}} \sqrt{\frac{\cos L_2}{\cos(iK_2 t + L_2)}}. \tag{8.24}$$

holds for arbitrary t. This leads to the fact that there is no vacuum factorization

$$\left\langle \Phi, e^{i\,t\,X}\,\Phi \right\rangle = \left\langle \Phi, e^{i\,t\,Y}\,\Phi \right\rangle \left\langle \Phi, e^{i\,t\,Z}\,\Phi \right\rangle. \tag{8.25}$$

for arbitrary one-dimensional factors (by one-dimensional factors we mean that $n = 1$ for homogeneous quadratic boson fields Y and Z).

Let us also remark that in this example, in the definition of X, n is fixed to $n = 2$. Because if one interprets

$$X = \sum_{i,j=1}^{2} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right), \tag{8.26}$$

as

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right), \tag{8.27}$$

with $A_{ij} = C_{ij} = 0$ for $i \neq 1, 2$ or $j \neq 1, 2$, then the vacuum factorizabilty condition (into one-dimensional factors (otherwise, the situation gets even more complicated) takes the form

$$\left\langle \Phi, e^{i\,t\,X}\,\Phi \right\rangle = \prod_{j=1}^{n} e^{-\frac{il_j t}{2}} \sqrt{\frac{\cos L_j}{\cos(iK_j t + L_j)}}, \tag{8.28}$$

for some l_j, L_j, K_j, for arbitrary t and for some (sufficiently large) n. As we do not consider such an interpretation in this work, we do not prove the violation of this condition for arbitrary n.

References

1. Accardi, L., Boukas, A.: On the characteristic function of random variables associated with Boson Lie algebras, *Communications on Stochastic Analysis*, **4** (4), (2010) 493–504.
2. Accardi, L., Boukas, A. : Fourier transform of random variables associated with the multi-dimensional Heisenberg Lie algebra, *Proc. Amer. Math. Soc.*, **143** (2015) 4095–4101.
3. Accardi, L., Boukas, A. : Canonical forms of quadratic Boson fields, *Proceedings of the Conference QBIC 2014*, dedicated to Prof. Masanori Ohya, Eds. L.Accardi, W.Freudenberg, N.Watanabe (2020), http://www.worldscientific.com/worldscibooks/
4. Accardi, L., Boukas, A., Lu, Y. G. : The Vacuum distributions of the truncated Virasoro fields are products of Gamma distributions, *Open Syst. Inf. Dyn.* **24**(1) (2017) 1–26.
5. Accardi, L., Boukas, A., Lu, Y. G., Teretenkov A.: The non–linear and quadratic quantization programs, in *Infinite Dimensional Analysis, Quantum Probability and Applications*: QP41 Conference, Al Ain, UAE, March 28 – April 1, 2021, L. Accardi, A. A. Rawashdeh and F. Mukhamedov eds., Springer 2022

6. Broadbridge, P.: Canonical forms for quadratic Hamiltonians, *Physica A: Statistical Mechanics and its Applications*, 1981.
7. Chebotarev, A. M., Teretenkov, A. E.: Singular value decomposition for the Takagi factorization of symmetric matrices, *Applied Mathematics and Computation* **234**, 380–384 (2014).
8. Colpa, J. H. P.: Diagonalization of the quadratic Boson Hamiltonian, *Physica* **93A**, 327–353 (1978).
9. Derezinski, J.: Bosonic quadratic Hamiltonians, *Journal of Mathematical Physics* **58**, 12, 121101, (2017).
10. Derezinski, J., Gerard, C.: *Mathematics of Quantization and Quantum Fields*, Cambridge University Press, 2013.
11. Horn, R. A., Johnson, C. R.: *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.
12. Kustura, K., Rusconi, C. C., Romero-Isart, O.: Quadratic quantum Hamiltonians: General canonical transformation to a normal form, *Phys. Rev. A* **99**, 022130 (2019).
13. Louck, J. D.: *Theory of Angular Momentum in N-Dimensional Space*, Los Alamos Scientific Laboratory (1960), https://www.osti.gov/biblio/4121939.
14. Lucacs, E.: *Characteristic Functions*, 2nd ed., Griffin, London, 1970.
15. Nielsen, E. B., Rask, O.: Exponentiability of quadratic hamiltonians, *Reports on Mathematical Physics* **52**, 2, 177–186 (2003).
16. Thanh, P. , Napiorkowski, N. M., Solovej, J. P.: Diagonalization of bosonic quadratic Hamiltonians by Bogoliubov transformations, *Journal of Functional Analysis*, **270**, 11, 4340–4368 (2016).
17. Tikochinsky, Y.: On the diagonalization of the general quadratic Hamiltonian for coupled harmonic oscillators, *J. Math. Phys.*, **20**, 406 (1979).
18. Trifonov, D. A.: *Diagonalization of Hamiltonians, Uncertainty Matrices and Robertson Inequality*, in Second International Conference on Geometry, Integrability and Quantization, June 7-15, 2000, Varna, Bulgaria, Ivailo M. Mladenov and Gregory L. Naber, Editors, Coral Press, Sofia 2001.
19. Williamson, J. : On the algebraic problem concerning the normal forms of linear dynamical systems, *American J. of Math.* **58**, 1, 141–163 (1936).
20. Williamson, J. : On the normal forms of linear canonical transformations in dynamics, *American J. of Math.* **59**, 3, 599–617 (1937).
21. Xiao, M. W. : Theory of transformation for the diagonalization of quadratic Hamiltonians, arXiv:0908.0787 (2009).

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