This is the Pre-print of the paper
G.Fragnelli, G.Marinoschi, R.M. Mininni, S.Romanelli
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Journal of Evolution Equations 15 (2015), 27–51
DOI: https://doi.org/10.1007/s00028-014-0247-1

Identification of a diffusion coefficient in strongly degenerate parabolic equations with interior degeneracy

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Abstract. We are concerned with the identification of the diffusion coefficient $u(x)$ in a strongly degenerate parabolic diffusion equation. The strong degeneracy means that $u \in W^{1,\infty}$, u vanishes at an interior point of the space domain and $\frac{1}{u} \notin L^1$. The aim is to identify u from certain observations on the solution, by treating the identification problem as a nonlinear optimal control problem with the control in coefficients. The requirements related to the strong degeneracy of the equation impose to search the control u in $W^{1,\infty}$, restriction which represents a novelty and induces a particular difficulty in the determination of the optimality conditions. We prove the existence of a control and compute the optimality conditions both for homogeneous Dirichlet and Dirichlet-Neumann boundary conditions associated to the state system. In the case with a final time observation and homogeneous Dirichlet-Neumann boundary conditions a very explicit form of the control and its uniqueness are provided by technical arguments.

Keywords: degenerate diffusion equations, interior degeneracy, inverse problems, optimal control, optimality conditions

MSC 2010: 35K65, 35R30, 49N45, 49KXX

1 Introduction

In this article we study two identification problems in relation with an evolution equation with a second order differential degenerate operator, in divergence form, $A_1 y := (uy_x)_x$, when the diffusion coefficient u vanishes at an interior point of an one dimensional space domain. The purpose is to determine u from certain observations on the solution to the evolution equation.

Degenerate parabolic operators naturally arise in many problems: Budyko-Sellers models in climatology (see, e.g., [22]), boundary layer models in physics (see, e.g., [6]), Wright-Fisher and Fleming-Viot models in genetics (see, e.g., [12], [21]), Black-Merton-Scholes models in mathematical finance (see, e.g., [19]). The degenerate operator A_1 has been studied under different boundary conditions, see, for example, [8], [9], [15], [23]. In [11], [15], the authors consider degenerate operators with boundary conditions of Dirichlet, Neumann, periodic, or nonlinear Robin type. In [1] the authors consider the degenerate operator in divergence and in non divergence form with Dirichlet or Neumann boundary conditions, giving more importance to controllability problems of the associated parabolic evolution equations. However, all previous papers deal with a degenerate operator with degeneracy at the boundary of the domain, for example of the form of the double power function $u(x) = x^{k_1}(1-x)^{k_2}$, $x \in [0,1]$, where k_1 and k_2 are positive constants.

To the best of our knowledge, Stahel's paper [20] is the first treating a problem with a degeneracy which may be interior. In [7] there is treated a class of variational degenerate elliptic problems with interior and boundary degeneracy in the case that there exists $k \in (0,2]$ such that u decreases more slowly than $|x-z|^k$ near every point $z \in u^{-1}\{0\}$ in the case of bounded domains. The assumption regarding the interior degeneracy of u is generalized in [13] when $N = 1$. In particular, in [13] the authors analyze in detail degenerate operators in divergence and non-divergence form, under Dirichlet boundary conditions in spaces $L^2(0,1)$ with or without weight and show that under suitable assumptions they generate analytic semigroups. In [14] the authors consider the null controllability of parabolic equations with coefficients having the same interior degeneracy. In [10] a control problem involving a nonlinear nonautonomous operator degenerating on a positive measure interior subset of the space domain is studied.

The interest in this kind of interior degeneracy problems is due to the fact that they govern diffusion of a substance in water, soil or air, heat flow in a material, diffusion of a population in a habitat. The nonhomogeneity of the medium is expressed by the space dependence of the diffusion coefficient with its possible vanishing at some points. For example, a certain composite material can block the heat flow at a certain point, or the migration of small mammal species can degenerate due to environmental heterogeneity and barriers (see, for example, [5], [16] and the references therein). For this reason it is important to study identification and control problems associated to these degenerate equations. In particular, in [5] and [16] the authors consider two optimal problems: the first one is to minimize the damage and trapping costs, the second one is to maximize the difference between harvesting cost and economic revenue. Another application can be related to the study of the design of biological channels (see, e.g., [18]) in the case when the metabolite diffusion coefficient vanishes.

From the mathematical point of view and in connection with the work of Fragnelli et al. (see [13]) we focus on identifying, on the basis of some observations, the diffusion coefficient in the degenerate parabolic equation

$$\frac{\partial y}{\partial t} - \frac{\partial}{\partial x}\left(u(x)\frac{\partial y}{\partial x}\right) = f \text{ in } Q := (0,T) \times (0,L),\ T, L \in (0,\infty), \tag{1}$$

with the initial condition

$$y(0,x)=y_0(x) \text{ in } (0,L), \tag{2}$$

and various boundary conditions, in particular of homogeneous Dirichlet type

$$y(t,0)=y(t,L)=0 \text{ in } (0,T). \tag{3}$$

We consider a diffusion coefficient u vanishing at a point $x_0 \in (0,L)$, such that $u \in W^{1,\infty}(0,L)$ and $\frac{1}{u} \notin L^1(0,L)$, which corresponds to the strongly degenerate case, so-called slow diffusion in [13].

As far as we know, the particular properties related to the behavior of $\frac{1}{u}$ have not been considered in other identification or control papers.

In some practical situations, for instance in a pollutant diffusion process or in a diffusion of a population (bacteria e.g.) in a medium, it is of interest to identify the diffusion coefficient which is suspected to determine high levels of concentration (or density) y in a certain subset of the flow domain due to the possible diffusion stopping at some point x_0. A similar interest is in the design of a composite material for determining the material properties which preserve the temperature at certain high values.

We mainly aim to determine the function $u(x)$ in the system (1)-(3) from the observation of the spatial mean (M_T) of the state at a final time T. However, since other physical quantities, as the mean value M or the mean flux M_f over Q may be measured, we shall identify u by combining these possible observations in a unique mathematical problem of minimization of a cost functional. Namely, first we study the problem

$$\text{Minimize } \left\{ \frac{\lambda_1}{2}\left(\int_Q u(x)y_x^u(t,x)dxdt - M_f\right)^2 + \frac{\lambda_2}{2}\left(\int_0^L y^u(T,x)dx - M_T\right)^2 \right.$$
$$\left. + \frac{\lambda_3}{2}\left(\int_Q y^u(t,x)dxdt - M\right)^2 \right\}$$

subject to (1)-(3), for all u in a set U, which will be specified in Section 2.2. Here, $M_f \in \mathbb{R}$, M_T, M, λ_1, λ_2, λ_3, are nonnegative real numbers, and there exists at least one $i \in \{1,2,3\}$ such that $\lambda_i > 0$.

The notation y^u indicates the solution to (1)-(3) corresponding to u.

The various choices of the constants λ_i, $i = 1,2,3$, enhance a higher or lower importance to the terms in the functional, according to the problem requirements.

We provide results for two identification problems in the divergence form (1), for the strongly degenerate case (see Definition 2.1), both for homogeneous Dirichlet boundary conditions and for homogeneous Dirichlet-Neumann boundary conditions.

We treat the above nonlinear minimization problem by approaching it as an optimal control problem in coefficients. The difficulty we face is given by the restriction $u \in W^{1,\infty}(0,L)$, which is also a novelty of this paper. First, we prove the existence in the state system (1)-(3) by a variational technique, and then we prove the existence of a minimizer constructed as a limit of a minimizing sequence. A necessary condition of optimality is obtained in a generalized form which cannot be more processed due to the insufficient regularity of the state function. In order to get a clear representation of the control we introduce an approximating control problem by regularizing the state equation. A technical proof based on the decomposition of the control in two terms corresponding to each restrictions considered in the admissible set leads to an explicit expression of u. The second identification problem requires to identify u from the final observation, by imposing to

$$\text{Minimize} \quad \left\{ \int_0^L y^u(T,x)dx \right\}$$

for all $u \in U$, subject to (1)-(2), with homogeneous Dirichlet-Neumann boundary conditions. In this case we exploit some particular properties of the derivatives of the state which allow us to get a very explicit and simple form of u and its uniqueness, which is a strong result in a nonlinear optimal control problem.

2 Homogeneous Dirichlet boundary conditions

2.1 Preliminaries and the state system

We begin with some notation, definitions and results given in [13], considering the operator $A_1 y = (uy_x)_x$ in the strongly degenerate case. For simplicity we denote ϕ_t, ϕ_x the partial derivatives of a generic function $\phi(t,x)$ with respect to t and x.

Definition 2.1. $A_1 y = (uy_x)_x$ is called *strongly degenerate* if there exists $x_0 \in (0,L)$ such that $u(x_0) = 0$, $u(x) > 0$ on $[0,L]\backslash\{x_0\}$, $u \in W^{1,\infty}(0,L)$ and $\frac{1}{u} \notin L^1(0,L)$.

As an example we can mention $u(x) = |x - x_0|^k$, $k \geq 1$.

We define the weighted space

$$H_u^1(0,L) = \{y \in L^2(0,L);\ y \text{ locally absolutely continuous in } [0,L]\backslash\{x_0\}, \sqrt{u}y_x \in L^2(0,L),\ y(0) = y(L) = 0\}, \tag{4}$$

with the norm

$$\|y\|_{H_u^1(0,L)} = \left(\|y\|_{L^2(0,L)}^2 + \left\|\sqrt{u}y_x\right\|_{L^2(0,L)}^2 \right)^{1/2}. \tag{5}$$

According to [13], Proposition 2.3, we have that

$$H_u^1(0,L) = \{y \in L^2(0,L);\ y \text{ locally absolutely continuous in } [0,L]\backslash\{x_0\}, \sqrt{u}y_x \in L^2(0,L),\ uy \text{ is continuous at } x_0,\ y(0) = y(L) = (uy)(x_0) = 0\}.$$

We specify that $H_u^1(0,L)$ is a Hilbert space and $H_u^1(0,L) \hookrightarrow L^2(0,L) \hookrightarrow (H_u^1(0,L))'$, where $(H_u^1(0,L))'$ is the dual of $H_u^1(0,L)$ and "$\hookrightarrow$" means a continuous and dense embedding.

For simplicity, we denote

$$H = L^2(0,L),\ V_u = H_u^1(0,L),\ V_u' = (H_u^1(0,L))',$$

where they are indicated as subscripts.

Let us consider

$$H_u^2(0,L) = \{y \in H_u^1(0,L);\ uy_x \in H^1(0,L)\} \tag{6}$$

and define

$$A_1 y := (uy_x)_x,\ A_1 : D(A_1) \subset L^2(0,L) \to L^2(0,L),\ D(A_1) = H_u^2(0,L).$$

According to [13], Proposition 2.4,

$$\begin{aligned} D(A_1) &= \{y \in L^2(0,L);\ y \text{ is locally absolutely continuous in } [0,L]\backslash\{x_0\}, \\ uy &\in H_0^1(0,L),\ uy_x \in H^1(0,L),\ uy(x_0) = uy_x(x_0) = 0\}. \end{aligned} \tag{7}$$

By [13], Theorem 2.7, $A_1 : D(A_1) \to L^2(0,L)$ is self-adjoint, nonpositive on $L^2(0,L)$ and it generates a positivity preserving semigroup. This result is used further to prove that (1)-(3) has a unique mild solution if $y_0 \in L^2(0,L)$ and $f \in L^2(Q)$. This is a strong solution if $y_0 \in H_u^2(0,L)$ and also if $y_0 \in H_u^1(0,L)$ (see Theorem 4.1 and Remark 4.2 in [13]).

We also recall the following result (Lemma 2.6 in [13]):

Lemma 2.2. *For all $(y,z) \in H_u^2(0,L) \times H_u^1(0,L)$ one has*

$$\int_0^L (uy_x)_x z dx = -\int_0^L uy_x z_x dx. \tag{8}$$

For the purposes of our paper we present the existence result for the state system (1)-(3) by a variational way. For convenience, and where no confusion can be made we shall not write the function arguments in the integrands.

Definition 2.3. Let $y_0 \in L^2(0,L)$, $f \in L^2(0,T;(H_u^1(0,L))')$ and u with the properties of Definition 2.1. We call a *solution* to (1)-(3) a function

$$y \in C([0,T];L^2(0,L)) \cap L^2(0,T;H_u^1(0,L)) \cap W^{1,2}([0,T];(H_u^1(0,L))'), \tag{9}$$

which satisfies the equation

$$\int_0^T \left\langle \frac{dy}{dt}(t), \psi(t) \right\rangle_{V_u', V_u} dt + \int_Q uy_x \psi_x dx dt = \int_0^T \langle f(t), \psi(t) \rangle_{V_u', V_u} dt, \tag{10}$$

for any $\psi \in L^2(0,T;H_u^1(0,L))$, and the initial condition $y(0) = y_0$.

Theorem 2.4. *If $y_0 \in L^2(0,L)$, $f \in L^2(0,T;(H_u^1(0,L))')$, then* (1)-(3) *has a unique solution*

$$y \in C([0,T];L^2(0,L)) \cap L^2(0,T;H_u^1(0,L)) \cap W^{1,2}([0,T];(H_u^1(0,L))'),$$

satisfying the estimate

$$\sup_{t\in[0,T]} \|y(t)\|_H^2 + \int_0^T \|y(t)\|_{V_u}^2 \, dt \le C_T(\|y_0\|_H^2 + \|f\|_{L^2(0,T;V_u')}^2). \tag{11}$$

If, in addition $y_0 \in H_u^1(0,L)$ and $f \in L^2(Q)$, then

$$y \in W^{1,2}([0,T];L^2(0,L)) \cap L^2(0,T;H_u^2(0,L)) \cap L^\infty(0,T;H_u^1(0,L)) \tag{12}$$

and it satisfies

$$\sup_{t\in[0,T]} \|y(t)\|_{V_u}^2 + \int_0^T \left(\left\| \frac{dy}{dt}(t) \right\|_H^2 + \|(uy_x)_x(t)\|_H^2 \right) dt \le C_T \left(\|y_0\|_{V_u}^2 + \|f\|_{L^2(Q)}^2 \right), \tag{13}$$

where C_T denotes several positive constants.

If $y_0 \ge 0$ a.e. on $(0,L)$ and $f \ge 0$ a.e. on Q, then $y(t) \ge 0$ a.e. in $(0,L)$ for all $t \in [0,T]$.

Proof. Let us introduce the linear operator

$$A : H_u^1(0,L) \to (H_u^1(0,L))'$$

by

$$\langle Az, \psi \rangle_{V_u', V_u} = \int_0^L u(x) z_x(x) \psi_x(x) dx, \text{ for any } \psi \in H_u^1(0,L). \tag{14}$$

It is continuous and monotone

$$\|Az\|_{V_u'} = \sup_{\psi \in V_u, \|\psi\|_{V_u} \le 1} \left| \langle Az, \psi \rangle_{V_u', V_u} \right| \le \|z\|_{V_u}, \tag{15}$$

$$\langle Az, z \rangle_{V_u', V_u} \ge 0, \tag{16}$$

and has the property

$$\langle Az, z \rangle_{V_u', V_u} = \int_0^L u z_x^2 dx = \|z\|_{V_u}^2 - \|z\|_H^2. \tag{17}$$

We consider the Cauchy problem

$$\begin{aligned} \frac{dy}{dt}(t) + Ay(t) &= f(t), \text{ a.e. } t \in (0,T) \\ y(0) &= y_0. \end{aligned} \tag{18}$$

If

$$y_0 \in L^2(0,L), \; f \in L^2(0,T;(H_u^1(0,L))')$$

then the Cauchy problem has a unique solution belonging to the spaces indicated in (9) (see e.g. [17], p. 162). Obviously, this is a solution to (1)-(3) in the sense of Definition 2.3, so that (10) and (18) are equivalent. The estimate (11) follows by setting in (10) $\psi = y$ and performing a few computations involving the Gronwall's lemma.

Now, we observe that $-A_1$ is the restriction of A on $L^2(0, L)$. Indeed, if $y \in H_u^2(0, L)$, by (8) we see that $-A_1$ coincides with A. Therefore, if $y_0 \in D(A_1) = H_u^2(0, L)$ and $f \in W^{1,1}([0, T]; L^2(0, L))$, problem (18) with A replaced by its restriction $-A_1$ has a more regular solution $y \in W^{1,\infty}([0, T]; L^2(0, L)) \cap L^\infty(0, T; H_u^2(0, L))$. This follows by Theorem 4.9 (see [3], p. 151). If $y_0 \in H_u^1(0, L)$ and $f \in L^2(Q)$, the density of $H_u^2(0, L)$ in $H_u^1(0, L)$, the density of $W^{1,1}([0, T]; L^2(0, L))$ in $L^2(Q)$ and some standard estimates lead to the second part of the theorem.

Finally, the nonnegativity of the solution follows by the positivity preserving property of the semigroup generated by A_1. □

2.2 Existence in (P)

We denote

$$\begin{aligned} J(u) &= \frac{\lambda_1}{2}\left(\int_Q u y_x^u dx dt - M_f\right)^2 + \frac{\lambda_2}{2}\left(\int_0^L y^u(T, x) dx - M_T\right)^2 \\ &+ \frac{\lambda_3}{2}\left(\int_Q y^u dx dt - M\right)^2 \end{aligned} \tag{19}$$

and introduce the minimization problem

$$\text{Minimize } J(u) \text{ for all } u \in U, \tag{P}$$

subject to (1)-(3), where

$$\begin{aligned} U &= \{u \in W^{1,\infty}(0, L);\ u_m(x) \leq u(x) \leq u_M(x), \\ u(0) &= u_0,\ u(L) = u_L,\ |u_x(x)| \leq u_\infty \text{ a.e. } x \in (0, L)\}. \end{aligned} \tag{20}$$

We assume the following hypotheses:

$$\begin{aligned} x_0 &\in (0, L), \text{ fixed and known, } u_\infty \in [0, \infty), \\ u_m,\ u_M &\in C[0, L],\ u_M(x) \leq \alpha(x) u_m(x) \text{ for } x \in [0, L],\ \alpha \in C[0, L], \alpha \geq 1, \\ 0 &< u_m(x) < u_M(x) \text{ for } x \in [0, L]\backslash\{x_0\},\ u_m(x_0) = u_M(x_0) = 0, \end{aligned} \tag{21}$$

and

$$\int_0^L \frac{1}{u_M(x)} dx = +\infty. \tag{22}$$

Then, by (21), for all $u \in U$ we have

$$0 < u_m(0) \leq u_0 \leq u_M(0),\ 0 < u_m(L) \leq u_L \leq u_M(L). \tag{23}$$

These conditions ensure the fact that the operator A_1 is strongly degenerate, because $u \in U$ and (21) imply that $u(x_0) = 0$, $u > 0$ in $[0, L] \backslash \{x_0\}$, and (22) establishes that $\frac{1}{u} \notin L^1(0, L)$.

Moreover, the assumption $u_M(x) \leq \alpha(x) u_m(x)$ implies that if $u, v \in U$ then $\frac{v}{u}(x) \leq \|\alpha\|_{L^\infty(0,L)}$ for $x \in [0, L] \backslash \{x_0\}$ and

$$H_u^1(0, L) = H_v^1(0, L) \text{ for any } u, v \in U. \tag{24}$$

Indeed, if $y \in H_u^1(0, L)$ then y is locally absolutely continuous in $[0, L] \backslash \{x_0\}$, $y \in L^2(0, L)$, $\sqrt{u} y_x \in L^2(0, L)$, $y(0) = y(L) = 0$. Let $v \in U$. Then, by a simple calculation

$$\int_0^L v y_x^2 dx \leq \|\alpha\|_{L^\infty(0,L)} \left\| \sqrt{u} y_x \right\|_{L^2(0,L)}^2 \tag{25}$$

and so $y \in H_v^1(0, L)$. Analogously, we get $H_v^1(0, L) \subset H_u^1(0, L)$.

Theorem 2.5. *Let $y_0 \in L^2(0, L)$, $y_0 \geq 0$ on $(0, L)$, $f \in L^2(Q)$, $f \geq 0$ a.e. on Q. Then (P) has at least one solution u with the corresponding state*

$$y \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H_u^1(0, L)) \cap W^{1,2}([0, T]; (H_u^1(0, L))').$$

If $y_0 \in H_u^1(0, L)$, then the state y is more regular, as in (12).

Proof. Under the specified hypotheses, problem (1)-(3) has a unique nonnegative solution given by Theorem 2.4. Then, $J(u) \geq 0$, its infimum exists and it is nonnegative. Let us denote it by d.

For not overloading the notations we shall drop the superscript ^{u}.

We consider a minimizing sequence $(u_n)_{n \geq 1}$, $u_n \in U$ which satisfies

$$\begin{aligned} d \quad \leq \quad & \frac{\lambda_1}{2} \left(\int_Q u_n y_{nx} dx dt - M_f \right)^2 + \frac{\lambda_2}{2} \left(\int_0^L y_n(T, x) dx - M_T \right)^2 \\ & + \frac{\lambda_3}{2} \left(\int_Q y_n dx dt - M \right)^2 \leq d + \frac{1}{n}, \end{aligned} \tag{26}$$

where the corresponding state y_n is the solution to (1)-(3) (equivalently to (18)) with u replaced by u_n. By Theorem 2.4, it exists for each n, it is unique and satisfies

$$\sup_{t \in [0,T]} \|y_n(t)\|_H^2 + \int_0^T \|y_n(t)\|_{V_{u_n}}^2 dt \leq C, \text{ for any } t \in [0, T], \tag{27}$$

with C a positive constant independent of n, by (13).

Since $u_n \in U$, we deduce that there exist subsequences (denoted still by the subscript $_n$) such that

$$u_n \to u \text{ weak* in } L^\infty(0, L), \text{ as } n \to \infty,$$

$$u_{nx} \to u_x \text{ weak* in } L^\infty(0, L), \text{ as } n \to \infty,$$

$$y_n \to y \text{ weak* in } L^\infty(0,T;L^2(0,L)), \text{ as } n \to \infty,$$

$$y_n(T) \to \zeta \text{ weakly in } L^2(0,L), \text{ as } n \to \infty.$$

By (27), $(\sqrt{u_n} y_{nx})_n$ is bounded in $L^2(Q)$ and there exists $\xi \in L^2(Q)$ such that, on a subsequence (denoted still by the subscript $_n$)

$$\xi_n := \sqrt{u_n} y_{nx} \to \xi \text{ weakly in } L^2(Q), \text{ as } n \to \infty.$$

Let δ be positive, arbitrary. Then $\xi_n \to \xi$ weakly in $L^2(0,T;L^2(\Omega_\delta))$ too, where $\Omega_\delta = (0, x_0 - \delta) \cup (x_0 + \delta, L)$.

The previous first two convergences for u_n imply that

$$u_n \to u \text{ uniformly on } [0,L], \text{ as } n \to \infty,$$

and therefore the sequence $y_{nx} = \frac{\xi_n}{\sqrt{u_n}}$ is bounded in $L^2(0,T;L^2(\Omega_\delta))$ and

$$y_{nx} = \frac{\xi_n}{\sqrt{u_n}} \to \frac{\xi}{\sqrt{u}} \text{ weakly in } L^2(0,T;L^2(\Omega_\delta)), \text{ as } n \to \infty.$$

On the other hand, $y_{nx} \to y_x$ in the sense of distributions. We conclude that $\xi = \sqrt{u} y_x$ a.e. on $(0,T) \times \Omega_\delta$ and since δ is arbitrary we finally get $\xi = \sqrt{u} y_x$ a.e. on Q. It follows that $y \in L^2(0,T;H_u^1(0,L))$ and so

$$y_n \to y \text{ weakly in } L^2(0,T;H_u^1(0,L)), \text{ as } n \to \infty.$$

Next, by the definition of A, see (14), we have that

$$Ay_n \to Ay \text{ weakly in } L^2(0,T;(H_u^1(0,L))'), \text{ as } n \to \infty,$$

and by (18) we deduce that $\left(\frac{dy_n}{dt}\right)_n$ is bounded in $L^2(0,T;(H_u^1(0,L))')$, so that by selecting a subsequence we have

$$\frac{dy_n}{dt} \to \frac{dy}{dt} \text{ weakly in } L^2(0,T;(H_u^1(0,L))'), \text{ as } n \to \infty.$$

By a direct computation we deduce that for all $t \in [0,T]$

$$y_n(t) \to y(t) \text{ weakly in } (H_u^1(0,L))', \text{ as } n \to \infty,$$

whence

$$y_0 = y_n(0) \to y(0),\ y_n(T) \to y(T) \text{ weakly in } (H_u^1(0,L))', \text{ as } n \to \infty.$$

Then, by the limit uniqueness we get $\zeta = y(T)$ and $y(0) = y_0$ a.e. on $(0,L)$.

Now, y_n satisfies (10)

$$\int_0^T \left\langle \frac{dy_n}{dt}(t), \psi(t) \right\rangle_{V_u', V_u} dt + \int_Q u_n y_{nx} \psi_x dx dt = \int_Q f \psi dx dt,$$

for any $\psi \in L^2(0,T;H^1_u(0,L))$ and passing to the limit as $n \to \infty$ we get that y satisfies (10), too. All these assertions prove that y is the solution to (1)-(3) corresponding to u.

Next, if $y_0 \in H^1_u(0,L)$, the solution y previously obtained has the regularity (12) according to Theorem 2.4.

Finally, we pass to the limit in (26) as $n \to \infty$, on the basis of the weakly lower semicontinuity of each convex term in $J(u_n)$, and get that $d = J(u)$.

Since U is closed, then $u(x) \in [u_m(x), u_M(x)]$ which implies by (21) and (22) that $u(x) > 0$ on $[0,L]\backslash\{x_0\}$, $u(x_0) = 0$ and $\frac{1}{u} \notin L^1(0,L)$, so that $u \in U$ and the corresponding operator $A_1 y = (uy_x)_x$ is strongly degenerate. $\square$

2.3 Optimality conditions in the homogeneous Dirichlet case

Proposition 2.6. *Let (u^*, y^*) be a solution to (P). Then u^* satisfies the necessary condition*

$$\int_Q (u^* - u) y^*_x \left[p_x + \lambda_1 \left(\int_Q u^* y^*_x dxdt - M_f \right) \right] dxdt \leq 0 \tag{28}$$

for all $u \in U$, where p is the solution to

$$\begin{aligned} & \frac{\partial p}{\partial t} + (u^* p_x)_x \qquad\qquad (29)\\ = \; & -\lambda_1 u^*_x \left(\int_Q u^* y^*_x dxdt - M_f \right) + \lambda_3 \left(\int_Q y^* dxdt - M \right) \; \text{in } Q, \end{aligned}$$

$$p(T,x) = -\lambda_2 \left(\int_0^L y^*(T,x)dx - M_T \right) \; \text{in } (0,L), \tag{30}$$

$$p(t,0) = p(t,L) = 0 \; \text{in } (0,T). \tag{31}$$

Proof. Let (u^*, y^*) be a solution to (P), $\lambda \in (0,1)$, $u \in U$ and denote

$$u^\lambda(x) = u^*(x) + \lambda v(x),$$

where

$$v(x) = u(x) - u^*(x), \; u \in U. \tag{32}$$

It is obvious that $v \in W^{1,\infty}(0,L)$, $v(x_0) = 0$, $u_m(x) - u_M(x) \leq v(x) \leq u(x)$, $v(0) = v(L) = 0$ and $\frac{1}{v} \notin L^1(0,L)$. We introduce the system

$$\frac{\partial Y}{\partial t} - (u^* Y_x)_x = (v y^*_x)_x \text{ in } Q, \tag{33}$$

$$Y(0,x) = 0 \text{ in } (0,L), \tag{34}$$

$$Y(t,0) = Y(t,L) = 0 \text{ in } (0,T). \tag{35}$$

We note that since $y^* \in L^2(0,T;H^1_{u^*}(0,L))$, after a few calculations it follows that

$$\int_Q v^2(y^*_x)^2 dxdt \le \|u_M\|_{C[0,L]} \|\alpha\|_{L^\infty(0,L)} \int_0^T \left\|\sqrt{u^*}y^*_x(t)\right\|_H^2 dt.$$

Hence, $vy^*_x \in L^2(0,T;L^2(0,L))$ and $g = (vy^*_x)_x \in L^2(0,T;(H^1_{u^*}(0,L))')$, since for any $\psi \in H^1_{u^*}(0,L)$, and a.e. $t \in (0,T)$ we have

$$\left|\langle (vy^*_x)_x(t), \psi\rangle_{V'_{u^*},V_{u^*}}\right| = \left| -\int_0^L vy^*_x(t)\psi_x dx\right| \le \|\alpha\|_{L^\infty(0,L)} \|y^*(t)\|_{V_{u^*}} \|\psi\|_{V_{u^*}}.$$

We state that (33)-(35) has a unique solution

$$Y \in C([0,T];L^2(0,L)) \cap L^2(0,T;V_{u^*}) \cap W^{1,2}([0,T];V'_{u^*}). \tag{36}$$

This follows as in the first part of Theorem 2.4 by defining the operator A from $H^1_{u^*}(0,L)$ to $(H^1_{u^*}(0,L))'$, by

$$\langle Az, \psi\rangle_{V'_{u^*},V_{u^*}} = \int_0^L u^* z_x \psi_x dx, \text{ for any } \psi \in V_{u^*}. \tag{37}$$

Moreover, denoting by $y^\lambda(t,x)$ the solution to (1)-(3) corresponding to $u^\lambda(x)$, one can prove that actually

$$Y(t,x) = \lim_{\lambda\to 0} \frac{y^\lambda(t,x) - y^*(t,x)}{\lambda},$$

so that (33)-(35) is the system of first order variations.

We introduce the dual system (29)-(31). This system has a unique solution

$$p \in C([0,T];H^1_{u^*}(0,L)) \cap L^2(0,T;H^2_{u^*}(0,L)) \cap W^{1,2}([0,T];L^2(0,L)),$$
$$(u^*p_x)_x \in L^2(Q),$$

given still by Theorem 2.4, second part (after making the transformation $t' = T - t$).

Now, we write that (u^*, y^*) is a solution to (P), that is

$$J(u^*) \le J(u), \text{ for all } u \in U,$$

and, in particular, for $u = u^\lambda$. After some algebra we get

$$\lambda_1 I_f \int_Q (-u^*_x Y + vy^*_x) dxdt + \lambda_2 I_T \int_0^L Y(T,x)dx + \lambda_3 I_M \int_Q Y dxdt \ge 0, \tag{38}$$

where

$$I_f = \int_Q u^* y^*_x dxdt - M_f, \; I_T = \int_0^L y^*(T,x)dx - M_T, \; I_M = \int_Q y^* dxdt - M. \tag{39}$$

We test (33) by $p(t)$ and integrate over $(0,T)$. After some calculations we get

$$\begin{aligned}&\int_Q -\left(p_t + (u^*(x)p_x)_x\right) Y dxdt + \int_0^L p(T,x)Y(T,x)dx\\ =\ &\int_0^T \langle (vy_x^*(t))_x, p(t)\rangle_{V'_{u^*},V_{u^*}}\, dt,\end{aligned}$$

which yields, by (29)-(31),

$$\int_Q (-\lambda_1 I_f u_x^* + \lambda_3 I_M) Y dxdt + \lambda_2 I_T \int_0^L Y(T,x)dx = \int_Q vy_x^* p_x dxdt. \tag{40}$$

Comparing with (38) it follows that

$$\int_Q vy_x^* p_x dxdt + \int_Q \lambda_1 I_f v y_x^* dxdt \geq 0$$

with $v = u - u^*$, for all $u \in U$, and this implies (28), as claimed. □

Due to the insufficient regularity of the state we cannot further process (28) in Proposition 2.6 in order to deduce the expression of u^*. Such a result will be obtained in Proposition 2.9 for the approximating control.

2.4 Approximating problem

In order to give a better characterization of the optimality condition (28) we determine an approximating form of it. To this end, for $\varepsilon > 0$, we introduce an approximating problem (P_ε) involving a nondegenerate state equation. The approximating optimality condition may be written more explicitly due to the better regularity of the approximating state and dual variable. Then we show that (P_ε) tends in some sense to (P). Namely, we show that a sequence of solutions to (P_ε) tends to a solution to (P), as $\varepsilon \to 0$.

We introduce

$$\text{Minimize } J(u) \text{ for all } u \in U_\varepsilon, \tag{P_ε}$$

subject to the state system (1)-(3), where

$$\begin{aligned} U_\varepsilon &= \{u \in W^{1,\infty}(0,L);\ u_m(x) + \varepsilon \leq u(x) \leq u_M(x) + 2\varepsilon, \\ u(0) &= u_0^\varepsilon,\ u(L) = u_L^\varepsilon,\ |u_x(x)| \leq u_\infty \text{ a.e. } x \in (0,L)\}. \end{aligned} \tag{41}$$

All hypotheses made in (21)-(23) remain the same and we note that if $u \in U_\varepsilon$, then

$$\varepsilon \leq u(x_0) \leq 2\varepsilon,$$

$$u_m(0) + \varepsilon \leq u_0^\varepsilon \leq u_M(0) + 2\varepsilon, \quad u_m(L) + \varepsilon \leq u_L^\varepsilon \leq u_M(L) + 2\varepsilon.$$

For all $u \in U_\varepsilon$, $u(x) \geq u_m(x) + \varepsilon \geq \varepsilon$, and then system (1)-(3) with $u \in U_\varepsilon$ is nondegenerate.

By the general results concerning nondegenerate evolution equations in Hilbert spaces, if $y_0 \in L^2(0,L)$, and $f \in L^2(Q)$, problem (1)-(3) with $u(x) \geq \varepsilon$ has a unique solution

$$y_\varepsilon \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^1(0,L)) \cap W^{1,2}([0,T]; H^{-1}(0,L)), \quad (42)$$

(see [17], p. 163) and y_ε satisfies the estimate

$$\sup_{t\in[0,T]} \|y_\varepsilon(t)\|_H^2 + \int_0^T \left\|\sqrt{u} y_{\varepsilon x}(t)\right\|_H^2 dt \leq C, \quad (43)$$

with C a positive constant depending on the data and independent of ε.

We denote by $y_{\varepsilon x}$ the derivative of y_ε with respect to x.

Obviously, the control problem (P_ε) has at least a solution $(u_\varepsilon, y_\varepsilon)$, with $u_\varepsilon \in U_\varepsilon$ and y_ε (corresponding to u_ε) satisfying (43). We prove the convergence result $(P_\varepsilon) \to (P)$ as $\varepsilon \to 0$, on the basis of the following lemma.

Lemma 2.7. *Let $y_0 \in L^2(0,L)$, $f \in L^2(Q)$, $y_0 \geq 0$ a.e. on $(0,L)$, $f \geq 0$ a.e. on Q. Let $(u_\varepsilon, y_\varepsilon)$ be a sequence of solutions to (P_ε) such that*

$$u_\varepsilon \to u \text{ uniformly in } [0,L], \text{ as } \varepsilon \to 0, \quad (44)$$

$$u_{\varepsilon x} \to u_x \text{ weak* in } L^\infty(0,L), \text{ as } \varepsilon \to 0. \quad (45)$$

Then, $u \in U$,

$$y_\varepsilon \to y \text{ weakly in } L^2(0,T; H_u^1(0,L)) \cap W^{1,2}([0,T]; (H_u^1(0,L))'), \text{ as } \varepsilon \to 0, \quad (46)$$

$$y_\varepsilon(T) \to y(T) \text{ weakly in } L^2(0,L), \text{ as } \varepsilon \to 0, \quad (47)$$

and y is the solution to (1)-(3) corresponding to u.

Proof. It is easily seen that $\lim_{\varepsilon\to 0} u_\varepsilon = u \in U$ defined by (20). Then, y_ε is the solution to the nondegenerate problem (1)-(3) corresponding to u_ε and y_ε satisfies (43). We get (on subsequences denoted still by the subscript $_\varepsilon$) that

$$y_\varepsilon \to y \text{ weakly in } L^2(0,T; L^2(0,L)), \text{ as } \varepsilon \to 0,$$

$$\sqrt{u_\varepsilon} y_\varepsilon \to \xi \text{ weakly in } L^2(0,T; L^2(0,L)), \text{ as } \varepsilon \to 0,$$

$$y_\varepsilon(T) \to \zeta \text{ weakly in } L^2(0,L), \text{ as } \varepsilon \to 0.$$

We continue the proof in a similar way as in Theorem 2.5 and we get that $\xi = \sqrt{u} y$ a.e. on Q and so $y \in L^2(0,T; H_u^1(0,L))$. By (14) we deduce that

$$Ay_\varepsilon \to Ay \text{ weakly in } L^2(0,T; (H_u^1(0,L))'), \text{ as } \varepsilon \to 0$$

and by (18)

$$\frac{dy_\varepsilon}{dt} \to \frac{dy}{dt} \text{ weakly in } L^2(0,T; (H_u^1(0,L))'), \text{ as } \varepsilon \to 0.$$

Following the arguments in Theorem 2.5 we prove (47) and that y is the solution to (1)-(3) corresponding to u. □

Theorem 2.8. *Let $y_0 \in L^2(0,L)$, $f \in L^2(Q)$, $y_0 \geq 0$ a.e. on $(0,L)$, $f \geq 0$ a.e. on Q. Let $(u_\varepsilon^*, y_\varepsilon^*)_{\varepsilon>0}$ be a sequence of solutions to (P_ε). Then (on subsequences denoted still by ε) we have*

$$u_\varepsilon^* \to u^* \text{ uniformly in } [0,L], \text{ as } \varepsilon \to 0, \tag{48}$$

$$u_{\varepsilon x}^* \to u_x^* \text{ weak* in } L^\infty(0,L), \text{ as } \varepsilon \to 0, \tag{49}$$

$$y_\varepsilon^* \to y^* \text{ weakly in } L^2(0,T;H_{u^*}^1(0,L)) \cap W^{1,2}([0,T];(H_{u^*}^1(0,L))'), \text{ as } \varepsilon \to 0, \tag{50}$$

$$y_\varepsilon(T) \to y(T) \text{ weakly in } L^2(0,L), \text{ as } \varepsilon \to 0. \tag{51}$$

Moreover, y^* is the solution to (1)-(3) corresponding to u^* and (u^*, y^*) is a solution to (P).

Proof. Let $(u_\varepsilon^*, y_\varepsilon^*)$ be a solution to (P_ε), i.e.,

$$J(u_\varepsilon^*) \leq J(u_\varepsilon), \text{ for all } u_\varepsilon \in U_\varepsilon.$$

Under the hypotheses for y_0 it follows that (1)-(3) with $u_\varepsilon \in U_\varepsilon$ has a unique solution $y_\varepsilon \in C([0,T];L^2(0,L)) \cap W^{1,2}([0,T];(H_{u_\varepsilon}^1(0,L))') \cap L^2(0,T;H_{u_\varepsilon}^1(0,L))$ and

$$\begin{aligned}
& \frac{\lambda_1}{2}\left(\int_Q u_\varepsilon^* y_{\varepsilon x}^* dxdt - M_f\right)^2 + \frac{\lambda_2}{2}\left(\int_0^L y_\varepsilon^*(T,x)dx - M_T\right)^2 \\
& + \frac{\lambda_3}{2}\left(\int_Q y_\varepsilon^* dxdt - M\right)^2 \\
\leq\ & \frac{\lambda_1}{2}\left(\int_Q u_\varepsilon y_{\varepsilon x} dxdt - M_f\right)^2 + \frac{\lambda_2}{2}\left(\int_0^L y_\varepsilon(T,x)dx - M_T\right)^2 \\
& + \frac{\lambda_3}{2}\left(\int_Q y_\varepsilon dxdt - M\right)^2
\end{aligned} \tag{52}$$

for all $u_\varepsilon \in U_\varepsilon$. Relations (48)-(49) follow from $u_\varepsilon^* \in U_\varepsilon$ and so the sequence of the corresponding states $(y_\varepsilon^*)_\varepsilon$ converges on a subsequence to y^* the solution to (1)-(3) corresponding to u^*, as established by Lemma 2.7. In particular, we have (46) which implies that

$$u_\varepsilon y_{\varepsilon x} \to u^* y_x^* \text{ weakly in } L^2(Q), \text{ as } \varepsilon \to 0.$$

Similarly, $u_\varepsilon \in U_\varepsilon$ implies that $u_\varepsilon \to u \in U$ uniformly on $[0,L]$ as $\varepsilon \to 0$, and by Lemma 2.7, (46)-(47), we get that $(y_\varepsilon)_\varepsilon$ is convergent to y which is the solution to (1)-(3) corresponding to u.

Passing to the limit in (52) we get

$$\frac{\lambda_1}{2}\left(\int_Q u^* y_x^* dxdt - M_f\right)^2 + \frac{\lambda_2}{2}\left(\int_0^L y^*(T,x)dx - M_T\right)^2 + \frac{\lambda_3}{2}\left(\int_Q y^* dxdt - M\right)^2$$

$$\leq \liminf_{\varepsilon\to 0}\left(\frac{\lambda_1}{2}\left(\int_Q u_\varepsilon^* y_{\varepsilon x}^* dxdt - M_f\right)^2 + \frac{\lambda_2}{2}\left(\int_0^L y_\varepsilon^*(T,x)dx - M_T\right)^2 + \frac{\lambda_3}{2}\left(\int_Q y_\varepsilon^* dxdt - M\right)^2\right)$$

$$\leq \limsup_{\varepsilon\to 0}\left(\frac{\lambda_1}{2}\left(\int_Q u_\varepsilon y_{\varepsilon x} dxdt - M_f\right)^2 + \frac{\lambda_2}{2}\left(\int_0^L y_\varepsilon(T,x)dx - M_T\right)^2 + \frac{\lambda_3}{2}\left(\int_Q y_\varepsilon dxdt - M\right)^2\right)$$

$$\leq \frac{\lambda_1}{2}\left(\int_Q u y_x dxdt - M_f\right)^2 + \frac{\lambda_2}{2}\left(\int_0^L y(T,x)dx - M_T\right)^2 + \frac{\lambda_3}{2}\left(\int_Q y dxdt - M\right)^2,$$

for all $u \in U$. This implies that (u^*, y^*) is a solution to (P). □

2.5 Approximating optimality conditions

Let K be a closed convex subset of a Banach space X having the dual X'. We recall (see [2], p. 4-5) that the indicator function of K is

$$I_K(\xi) = \begin{cases} 0, & \text{if } \xi \in K, \\ +\infty, & \text{if } \xi \notin K, \end{cases}$$

and the subdifferential of I_K coincides with the normal cone to K at ξ,

$$\partial I_K(\xi) = N_K(\xi) = \{\xi^* \in X';\ \langle \xi^*, \xi\rangle_{X',X} \geq 0\}.$$

For $\theta : \mathbb{R} \to [-1,1]$, let us denote by sign θ the graph

$$\text{sign } \theta = \begin{cases} 1, & \text{on } \{x;\ \theta(x) > 0\}, \\ [-1,1], & \text{on } \{x;\ \theta(x) = 0\}, \\ -1, & \text{on } \{x;\ \theta(x) < 0\}. \end{cases}$$

Let $(u_\varepsilon^*, y_\varepsilon^*)$ be a solution to (P_ε) and let us denote by p_ε the solution to (29)-(31) in which u^* and y^* are replaced by u_ε^* and y_ε^*, respectively.

Proposition 2.9. *Let us assume the hypotheses as in* Theorem 2.8. *Then, the approximating optimality condition reads as*

$$\int_0^L (u_\varepsilon^* - u_\varepsilon)(x)\Phi(x)dx \geq 0, \text{ for all } u_\varepsilon \in U_\varepsilon, \tag{53}$$

where

$$\Phi(x) := -\int_0^T y^*_{\varepsilon x}(t,x)\left\{p_{\varepsilon x}(t,x)+\lambda_1\left(\int_Q u^*_\varepsilon y^*_{\varepsilon x}dxdt - M_f\right)\right\}dt. \tag{54}$$

Moreover, Φ has the representation

$$\Phi(x) = -\rho'(x) + \mu(x) \ \textit{a.e.}\ x \in (0,L), \tag{55}$$

$$u^*_{\varepsilon x}(x) \in u_\infty \mathrm{sign}(\rho(x)) \ \textit{a.e.}\ x \in (0,L), \tag{56}$$

where μ, $\rho \in L^1(0,L)$,

$$\begin{cases} \mu(x) \le 0 \ \ \textit{a.e. in}\ \{x \in (0,L);\ u^*_\varepsilon(x) = u_m(x)+\varepsilon\} \\ \mu(x) = 0 \ \ \textit{a.e. in}\ \{x \in (0,L);\ u^*_\varepsilon(x) \in (u_m(x)+\varepsilon, u_M(x)+2\varepsilon)\} \\ \mu(x) \ge 0 \ \ \textit{a.e. in}\ \{x \in (0,L);\ u^*_\varepsilon(x) = u_M(x)+2\varepsilon\}, \end{cases} \tag{57}$$

and

$$\rho(x) = \begin{cases} 0 & \textit{a.e. in}\ \{x \in (0,L); |u^*_{\varepsilon x}(x)| < u_\infty\ \textit{a.e.}\} \\ \nu(x)u^*_{\varepsilon x}(x) & \textit{a.e. in}\ \{x \in (0,L); |u^*_{\varepsilon x}(x)| = u_\infty\ \textit{a.e.}\} \end{cases} \tag{58}$$

with $\nu \in L^1(0,L)$, $\nu \ge 0$ a.e. $x \in (0,L)$.

Proof. The system in variations, the dual system and the optimality conditions are similarly obtained as those for (P). Namely, if $(u^*_\varepsilon, y^*_\varepsilon)$ is a solution to (P_ε) we introduce

$$\begin{aligned} &\frac{\partial p_\varepsilon}{\partial t} + (u^*_\varepsilon p_{\varepsilon x})_x \\ = \ & -\lambda_1 u^*_{\varepsilon x}\left(\int_Q u^*_\varepsilon y^*_{\varepsilon x}dxdt - M_f\right) + \lambda_3\left(\int_Q y^*_\varepsilon dxdt - M\right) \ \text{in}\ Q, \end{aligned} \tag{59}$$

$$p_\varepsilon(T,x) = -\lambda_2\left(\int_0^L y^*_\varepsilon(T,x)dx - M_T\right) \ \text{in}\ (0,L), \tag{60}$$

$$p_\varepsilon(t,0) = p_\varepsilon(t,L) = 0 \ \text{in}\ (0,T), \tag{61}$$

and deduce that u^*_ε, y^*_ε and the dual variable p_ε should satisfy a similar relation to (28), i.e.,

$$\int_Q (u^*_\varepsilon - u_\varepsilon)y^*_{\varepsilon x}\left[p_{\varepsilon x} + \lambda_1\left(\int_Q u^*_\varepsilon y^*_{\varepsilon x}dxdt - M_f\right)\right]dxdt \le 0 \tag{62}$$

for all $u_\varepsilon \in U_\varepsilon$. The solution p_ε is regular, since $u^*_\varepsilon \ge \varepsilon$, and

$$p_\varepsilon \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1_0(0,L)) \cap W^{1,2}([0,T];H^{-1}(0,L)),$$

by the same arguments as for y_ε (see (42)).

The supplementary regularity of y_ε and p_ε implies that (62) can be still written

$$\int_0^L (u_\varepsilon^* - u_\varepsilon)\left(\int_0^T y_{\varepsilon x}^*\left[p_{\varepsilon x} + \lambda_1\left(\int_Q u_\varepsilon^* y_\varepsilon^* dxdt - M_f\right)\right]dt\right)dx \leq 0, \qquad (63)$$

for all $u_\varepsilon \in U_\varepsilon$. We see that $\Phi \in L^1(0,L)$. Then, (53) and (54) imply that

$$\Phi(x) \in N_{U_\varepsilon}(u_\varepsilon^*). \qquad (64)$$

Here, $N_{U_\varepsilon}(u_\varepsilon^*)$ means the normal cone in $L^1(0,L)$ to U_ε at $u_\varepsilon^* \in U_\varepsilon$, that is

$$N_{U_\varepsilon}(u_\varepsilon^*) = \{w \in L^1(0,L);\ \int_0^L (u_\varepsilon^* - u_\varepsilon)w dx \geq 0, \text{ for all } u_\varepsilon \in U_\varepsilon\}.$$

We give further a representation of $N_{U_\varepsilon}(u_\varepsilon^*)$, more exactly we prove that any $w \in N_{U_\varepsilon}(u_\varepsilon^*)$ can be written in the form

$$w = -\rho'(x) + \mu(x) \text{ a.e. } x \in (0,L) \qquad (65)$$

with ρ and μ previously defined.

Indeed, if w is given by (65) then

$$\int_0^L w(x)(u_\varepsilon^* - u_\varepsilon)dx = \int_0^L \mu(x)(u_\varepsilon^* - u_\varepsilon)dx - \int_0^L \rho'(x)(u_\varepsilon^* - u_\varepsilon)dx$$
$$= \int_0^L \mu(x)(u_\varepsilon^* - u_\varepsilon)dx - \rho(u_\varepsilon^* - u_\varepsilon)|_0^L + \int_0^L \rho(x)(u_{\varepsilon x}^* - u_{\varepsilon x})dx \geq 0$$

for any $u_\varepsilon \in U_\varepsilon$, by (57) for the first term and (58) for the last term. This means that $w \in N_{U_\varepsilon}(u_\varepsilon^*)$.

For the inverse implication we take $w \in N_{U_\varepsilon}(u_\varepsilon^*)$ and we note first that U_ε can be written $U_\varepsilon = U_{1\varepsilon} + U_{2\varepsilon}$, where

$$U_{1\varepsilon} = \{v \in W^{1,\infty}(0,L);\ |v_x| \leq u_\infty \text{ a.e. } x \in (0,L),\ v(0) = u_0^\varepsilon,\ v(L) = u_L^\varepsilon\},$$

$$U_{2\varepsilon} = \{v \in L^\infty(0,L);\ u_m(x) + \varepsilon \leq v(x) \leq u_M(x) + 2\varepsilon \text{ a.e. } x \in (0,L)\}.$$

We also remark that $U_{1\varepsilon} \cap \text{int } U_{2\varepsilon} \neq \varnothing$, and so

$$\partial(I_1 + I_2) = \partial I_1 + \partial I_2, \qquad (66)$$

(see [2], p. 7), where I_i are the indicator functions of $U_{i\varepsilon}$ $(i = 1,2)$ and ∂I_i denote their subdifferentials.

Therefore, $w \in N_{U_\varepsilon}(u_\varepsilon^*) = \partial I_{U_\varepsilon}(u_\varepsilon^*)$ is given by

$$w = \xi + \mu,\ \xi \in \partial I_1(u_\varepsilon^*),\ \mu \in \partial I_2(u_\varepsilon^*). \qquad (67)$$

It is obvious that $\mu(x) \in \partial I_2(u_\varepsilon^*) = N_{U_{2\varepsilon}}(u_\varepsilon^*)$ if and only if $\mu(x)$ satisfies (57).

The form of $\xi = -\rho'$ (with ρ given by (58)) follows by some arguments based on general results given in [2], p. 11-15. For the reader's convenience we give a few details adapted to our case.

Let $\xi \in \partial I_1(u_\varepsilon^*)$ and $\gamma \in W^{1,1}(0,L)$ such that $\xi = -\gamma'$ a.e. in $(0,L)$. This means that

$$\xi = -\gamma' \in \partial I_1(u_\varepsilon^*) \tag{68}$$

and $\gamma \in \partial I_F(u_{\varepsilon x}^*)$, where I_F is the indicator function of the set

$$F = \{\zeta \in L^\infty(0,L);\ \zeta = v' \text{ a.e. in } (0,L), v \in U_{1\varepsilon}\}.$$

Indeed, for all $v \in U_{1\varepsilon}$, we have

$$0 \leq \int_0^L \xi(u_\varepsilon^* - v)dx = \int_0^L \gamma(u_{\varepsilon x}^* - v_x)dx = \int_0^L \gamma(u_{\varepsilon x}^* - \zeta)dx, \text{ for all } \zeta \in F.$$

The set F can be decomposed as $F_1 \cap F_2$ where

$$F_1 = \{\zeta \in L^\infty(0,L);\ \zeta = v',\ v \in W^{1,\infty}(0,L),\ v(0) = u_0^\varepsilon,\ v(L) = u_L^\varepsilon\},$$

$$F_2 = \{\zeta \in L^\infty(0,L);\ |\zeta(x)| \leq u_\infty \text{ a.e. } x \in (0,L)\}.$$

Let us assume that there exists $w_0 \in W^{1,\infty}(0,L)$ such that

$$\|w_0\|_{L^\infty(0,L)} < u_\infty,\ w_0(0) = u_0^\varepsilon,\ w_0(L) = u_L^\varepsilon. \tag{69}$$

We note that by (69), $w_0 \in F_1 \cap$ int F_2 and so

$$\partial I_F = \partial I_{F_1} + \partial I_{F_2}$$

(see again [2], p. 7). Therefore, $\gamma \in \partial I_F(u_{\varepsilon x}^*)$ can be written $\gamma = \gamma_1 + \gamma_2$ with $\gamma_i \in \partial I_{F_i}(u_{\varepsilon x}^*)$. The subdifferential ∂I_F is an application from $L^\infty(0,L)$ to $(L^\infty(0,L))'$ and so γ_i are seen as elements belonging to $(L^\infty(0,L))'$, being represented as the sum of a continuous part $\gamma_{ia} \in L^1(0,L)$ and a singular part γ_{is} (see [2], p. 15). Then, $\gamma_{2a} \in \partial I_{F_2}(u_{\varepsilon x}^*)$ a.e. $x \in (0,L)$ and by Proposition 1.9 in [2], p. 11-13, we have

$$\gamma_{2a}(x) = \begin{cases} 0 & \text{a.e. in } \{x \in (0,L); |u_{\varepsilon x}^*(x)| < u_\infty\} \\ \nu(x) u_{\varepsilon x}^*(x) & \text{a.e. in } \{x \in (0,L); |u_{\varepsilon x}^*(x)| = u_\infty\}, \end{cases} \tag{70}$$

where $\nu \in L^1(0,L)$, $\nu \geq 0$ a.e. in $(0,L)$.

Now, $\gamma_{1a} \in \partial I_{F_1}(u_{\varepsilon x}^*)$ and so

$$\int_0^L \gamma_{1a}(x)(u_{\varepsilon x}^* - v_x)(x)dx = -\int_0^L \gamma_{1a}'(x)(u_\varepsilon^* - v)(x)dx \geq 0, \text{ for any } v \in F_1.$$

In particular setting $v := u_\varepsilon^* + l\phi$ with $\phi \in C_0^\infty(0,L)$ and $l > 0$, we get

$$\int_0^L \gamma_{1a}'(x)\phi(x)dx \geq 0 \text{ for any } \phi \in C_0^\infty(0,L).$$

Setting $v := u_\varepsilon^* - l\phi$ we get the inverse inequality and so it follows that

$$\int_0^L \gamma_{1a}'(x)\phi(x)dx = 0 \text{ for any } \phi \in C_0^\infty(0,L),$$

which implies that $\gamma_{1a}'(x) = 0$.

In conclusion, $\gamma = \gamma_{1a} + \gamma_{2a}$, where $(\gamma_{1a})' = 0$ a.e. in $(0,L)$ and $\gamma_{2a} := \rho$ satisfies (70), so that by (68) we have $\xi = -\gamma' = -\rho'$. We note that

$$-\rho' \in \partial I_1(u_\varepsilon^*) \text{ iff } \rho \in \partial I_F(u_{\varepsilon x}^*). \tag{71}$$

By (67) we get (65) and thus relation (55) follows from $\Phi(x) \in N_{U_\varepsilon}(u_\varepsilon^*)$.

On the subset $\{x \in (0,L);\ |u_{\varepsilon x}^*(x)| = u_\infty \text{ a.e.}\}$ we have $\rho(x) = \nu(x)u_{\varepsilon x}^*(x)$ and we observe that there are two cases for $\nu > 0$:

$$u_{\varepsilon x}^*(x) = \begin{cases} u_\infty & \text{if } u_{\varepsilon x}^*(x) = \frac{\rho(x)}{\nu(x)} > 0 \\ -u_\infty & \text{if } u_{\varepsilon x}^*(x) = \frac{\rho(x)}{\nu(x)} < 0. \end{cases}$$

On the subset $\{x \in (0,L);\ |u_{\varepsilon x}^*(x)| < u_\infty \text{ a.e.}\}$ the function $\rho(x) = 0$. Therefore, we deduce (56), as claimed. □

3 Homogeneous Dirichlet-Neumann boundary conditions

In this section we consider the final time minimization problem

$$\text{Minimize } \left(\int_0^L y^u(T,x)dx\right) \text{ for all } u \in U, \tag{P_1}$$

subject to

$$\frac{\partial y}{\partial t} - (uy_x)_x = f \text{ in } Q, \tag{72}$$

$$y(0,x) = y_0(x) \text{ in } (0,L), \tag{73}$$

$$y(t,0) = 0,\ y_x(t,L) = 0 \text{ in } (0,T). \tag{74}$$

The set U and all hypotheses made for the functions occurring in U are the same as in Section 2.2.

The existence of the solution to the new state system is treated by the variational technique, as in the case of the previous state system.

We introduce the space

$$\widetilde{H_u^1}(0,L) = \{y \in L^2(0,L);\ y \text{ locally absolutely continuous in } [0,L]\backslash\{x_0\},$$
$$\sqrt{u}y_x \in L^2(0,L),\ y(0) = 0\},$$

equipped with the scalar product

$$(\theta, \overline{\theta})_{\widetilde{H^1_u}(0,L)} = \int_0^L \theta\overline{\theta}dx + \int_0^L u\theta_x\overline{\theta}_x dx, \text{ for any } \theta, \overline{\theta} \in \widetilde{H^1_u}(0,L).$$

It is a Hilbert space with the norm given by (5) and with the dual denoted by $(\widetilde{H^1_u}(0,L))'$. We have the continuous and dense embeddings $\widetilde{H^1_u}(0,L) \hookrightarrow L^2(0,L) \hookrightarrow (\widetilde{H^1_u}(0,L))'$.

For the subscripts we denote $\widetilde{V_u} = \widetilde{H^1_u}(0,L)$, $\widetilde{V_u'} = (\widetilde{H^1_u}(0,L))'$ and $H = L^2(0,L)$.

The solution to (72)-(74) is defined according to Definition 2.3 by replacing $H^1_u(0,L)$ by $\widetilde{H^1_u}(0,L)$.

Let $u \in U$. We define $\widetilde{A} : \widetilde{H^1_u}(0,L) \to (\widetilde{H^1_u}(0,L))'$,

$$\left\langle \widetilde{A}y, \psi \right\rangle_{\widetilde{V_u'}, \widetilde{V_u}} = \int_0^L u y_x \psi_x dx, \text{ for any } \psi \in \widetilde{H^1_u}(0,L).$$

Then, problem (72)-(74) has a unique solution

$$y \in C([0,T]; L^2(0,L)) \cap L^2(0,T; \widetilde{H^1_u}(0,L)) \cap W^{1,2}([0,T]; (\widetilde{H^1_u}(0,L))') \quad (75)$$

obtained by the first part of Theorem 2.4.

Theorem 3.1. *Let $y_0 \in L^2(0,L)$ and $f \in L^2(Q)$, $y_0 \geq 0$ a.e. on $(0,L)$, $f \geq 0$ a.e. on Q. Then (P_1) has at least one solution.*

The proof is led as in Theorem 2.5.

3.1 Optimality conditions in the homogeneous Dirichlet-Neumann case

Proposition 3.2. *Let the assumptions of* Proposition 2.9 *hold. Assume in addition that*

$$y_0 \in H^2(0,L),\ f \in C([0,T]; H^1(0,L)),\ y_{0x} > 0 \text{ on } [0,L],\ f_x > 0 \text{ on } Q, \quad (76)$$

$$u_M \in W^{1,\infty}(0,L),\ \|u_M'\|_{L^\infty(0,L)} \leq u_\infty. \quad (77)$$

Then, u^*, a solution to (P_1) has the form

$$u^*(x) = \begin{cases} u_0 + u_\infty x & \text{for } x \in [0, x_1) \\ u_M(x) & \text{for } x \in [x_1, x_2) \\ -u_\infty(x - L) + u_L & \text{for } x \in [x_2, L], \end{cases} \quad (78)$$

where x_1 and x_2 are the solutions to

$$u_M(x_1) = u_\infty x_1 + u_0,\ u_M(x_2) = -u_\infty(x_2 - L) + u_L. \quad (79)$$

The function u^* is unique for fixed u_M, u_m, u_0, u_L, u_∞.

Proof. Since, in particular, $y_0 \in L^2(0,L)$ and $f \in C(\overline{Q})$ it follows by Theorem 3.1 that (P_1) has at least a solution u^*. We shall deduce the optimality condition for (P_1) by passing to the limit in the approximating problem

$$\text{Minimize } \left(\int_0^L y^u(T,x)dx \right) \text{ for all } u \in U_\varepsilon, \tag{$P_{1\varepsilon}$}$$

subject to the state system (72)-(74) with U_ε given by (41).

All existence results and computations for the optimality condition in $(P_{1\varepsilon})$ are led similarly as for (P_ε), but the dual system and the optimality condition are slightly modified due to the new cost functional and boundary conditions.

Let $(u_\varepsilon^*, y_\varepsilon^*)$ be a solution to $(P_{1\varepsilon})$. The new dual system is

$$\frac{\partial p_\varepsilon}{\partial t} + (u_\varepsilon^*(x)p_{\varepsilon x})_x = 0 \text{ in } Q, \tag{80}$$

$$p_\varepsilon(T,x) = -1 \text{ in } (0,L), \tag{81}$$

$$p_\varepsilon(t,0) = p_{\varepsilon x}(t,L) = 0 \text{ in } (0,T), \tag{82}$$

(here $\lambda_1 = \lambda_3 = 0$, $\lambda_2 = 1$), and the optimality condition

$$\int_0^T \int_0^L (u_\varepsilon^* - u_\varepsilon) y_{\varepsilon x}^* p_{\varepsilon x} dx dt \le 0, \text{ for all } u_\varepsilon \in U_\varepsilon,$$

is obtained by similar computations as for (62).
Since $y_0 \in H^2(0,L)$, y_ε^* and $p_{\varepsilon x}$, solutions to nondegenerate equations, are more regular

$$y_\varepsilon^*,\ p_\varepsilon \in C^1([0,T];L^2(0,L)) \cap C([0,T];H^2(0,L)),$$

and in particular $y_{\varepsilon x}^*$, $p_{\varepsilon x} \in C([0,T]\times[0,L])$, $y_{\varepsilon xx}^*$, $p_{\varepsilon xx} \in C([0,T];L^2(0,L))$. Then, we can write

$$\int_0^L (u_\varepsilon^* - u_\varepsilon) \left(\int_0^T y_{\varepsilon x}^* p_{\varepsilon x} dt \right) dx \le 0, \text{ for all } u_\varepsilon \in U_\varepsilon. \tag{83}$$

In addition, the convergence of $(P_{1\varepsilon})$ to (P_1) as $\varepsilon \to 0$ follows as in Theorem 2.8.

Moreover, according to Proposition 2.9 we have

$$\widetilde{\Phi}(x) = -\rho'(x) + \mu(x) \text{ a.e. } x \in (0,L), \tag{84}$$

$$u_{\varepsilon x}^*(x) \in u_\infty \text{sign}(\rho(x)) \text{ a.e. } x \in (0,L), \tag{85}$$

where

$$\widetilde{\Phi}(x) := -\int_0^T y_{\varepsilon x}^* p_{\varepsilon x} dt.$$

We continue by establishing the sign of $\widetilde{\Phi}(x)$. The solution y_ε^* is continuous and nonnegative on $[0,T]\times[0,L]$ and the minimum is attained on the boundary.

Then by the strong maximum principle it follows that $y_\varepsilon^* > 0$ on $(0,T) \times (0,L)$ and $-y_{\varepsilon x}^*(t,0) < 0$. Let $z = u_\varepsilon^* y_{\varepsilon x}^*$. Then, it satisfies

$$\begin{aligned} z_t - u_\varepsilon^* z_{xx}^* &= u_\varepsilon^* f_x \text{ in } Q, \\ z(0,x) &= u_\varepsilon^* y_{0x} \text{ in } (0,L), \\ z_x(t,0) &> 0,\ z_x(t,L) = 0 \text{ in } (0,T). \end{aligned}$$

This nondegenerate problem has a unique solution, continuous and positive on $(0,T) \times (0,L)$, by the same strong maximum principle, hence $y_{\varepsilon x}^*(t,x) > 0$, for any $(t,x) \in Q$.

From the dual problem we get $p_\varepsilon \leq 0$ on $[0,T] \times [0,L]$ and by a similar argument $-p_{\varepsilon x}(t,0) > 0$. If $\omega = u_\varepsilon^* p_{\varepsilon x}$ then ω satisfies

$$\begin{aligned} \omega_t + u_\varepsilon^* \omega_{xx}^* &= 0 \text{ in } Q, \\ \omega(0,x) &= 0 \text{ in } (0,L), \\ \omega_x(t,0) &< 0,\ \omega_x(t,L) = 0 \text{ in } (0,T) \end{aligned}$$

and it follows that $p_{\varepsilon x}(t,x) < 0$ for any $(t,x) \in Q$.

In conclusion, $\widetilde{\Phi}(x) > 0$ for all $x \in (0,L)$ and by (84) we note that

$$\rho'(x) = \mu(x) - \widetilde{\Phi}(x)$$

preserves a negative sign only for $\mu(x) \leq 0$, i.e., on the subset

$$U_- = \{x \in (0,L); u_\varepsilon^*(x) < u_M(x) + 2\varepsilon\}.$$

It means that on this set $\rho(x) \neq 0$ except at most one point, and so we get by (85) that

$$|u_{\varepsilon x}^*(x)| = u_\infty \text{ a.e. in } \{x \in (0,L);\ u_\varepsilon^*(x) < u_M(x) + 2\varepsilon\},$$

which is the eikonal equation. This solution cannot actually be uniquely determined by the conditions $u_\varepsilon^*(0) = u_0^\varepsilon$, $u_\varepsilon^*(L) = u_L^\varepsilon$, unless one observes, following a similar argument as in [4], that the function u_ε^* is the maximal element of the set

$$D = \{z \in W^{1,\infty}(0,L);\ |z'(x)| \leq u_\infty \text{ a.e. } x \in (0,L),\ z(x) \leq u_\varepsilon^* \ \forall x \in \partial U_-\}.$$

Here, ∂U_- is the boundary of U_-. To this end, let $z \in D$, and use (84) to get on the right-hand that

$$\begin{aligned} &\int_{U_-} \rho'(x)(u_\varepsilon^*(x) - z(x))^- dx \\ = &\int_{U_-} \mu(x)(u_\varepsilon^*(x) - z(x))^- dx - \int_{U_-} \widetilde{\Phi}(x)(u_\varepsilon^*(x) - z(x))^- dx \leq 0. \end{aligned}$$

On the other hand

$$\int_{U_-}\rho'(x)(u_\varepsilon^*(x)-z(x))^-dx$$
$$= \quad \rho(x)(u_\varepsilon^*(x)-z(x))^-\big|_{\partial U_-}+\int_{U_-}\rho(x)(u_{\varepsilon x}^*(x)-z_x(x))dx\geq 0.$$

Since $\rho'<0$ on U_-, we necessarily obtain that $(u_\varepsilon^*(x)-z(x))^-=0$, meaning that $z(x)\leq u_\varepsilon^*(x)$ for any $z\in D$.

In conclusion, $u_\varepsilon^*(x)$ must have the slope equal to either u_∞ or $-u_\infty$ in the subset where $u_\varepsilon^*(x)<u_M+2\varepsilon$, and being the maximal element of D, it is unique. In the subset where $u_\varepsilon^*(x)=u_M(x)+2\varepsilon$ its slope is $u_M'\in(-u_\infty,u_\infty)$. Therefore, for given data u_M, u_m, u_0^ε, u_L^ε, u_∞, the possible representation of u_ε^* is

$$u_\varepsilon^*(x)=\begin{cases} u_0^\varepsilon+u_\infty x & \text{for } x\in[0,x_1^\varepsilon) \\ u_M(x)+2\varepsilon & \text{for } x\in[x_1^\varepsilon,x_2^\varepsilon) \\ -u_\infty(x-L)+u_L^\varepsilon & \text{for } x\in[x_2^\varepsilon,L], \end{cases} \tag{86}$$

where x_1^ε is the abscissa of the intersection of the graphic of the function $x\to u_M(x)+2\varepsilon$ with the line of slope u_∞ passing through u_0^ε, and x_2^ε is the abscissa of the intersection of the graphic of the function $x\to u_M(x)+2\varepsilon$ with the line of slope u_∞ passing through u_L^ε, i.e.,

$$u_M(x_1^\varepsilon)+2\varepsilon=u_\infty x_1^\varepsilon+u_0^\varepsilon,\ u_M(x_2^\varepsilon)+2\varepsilon=-u_\infty(x_2^\varepsilon-L)+u_L^\varepsilon. \tag{87}$$

Passing to the limit in (86) as $\varepsilon\to 0$, on the basis of the convergence of $(P_{1\varepsilon})$ to (P_1) we get (78). □

As an example, the feature of a possible u^* given by (78) is seen in Fig. 1, by a solid line. It was computed for $L=1$, $T=2$, $x_0=0.7$, $u_M(x)=20(x-x_0)^2$, $u_m(x)=(x-x_0)^2$, $u_0=3, u_L=1, u_\infty=10$.

The graphic of u_M is drawn by a dashed line.

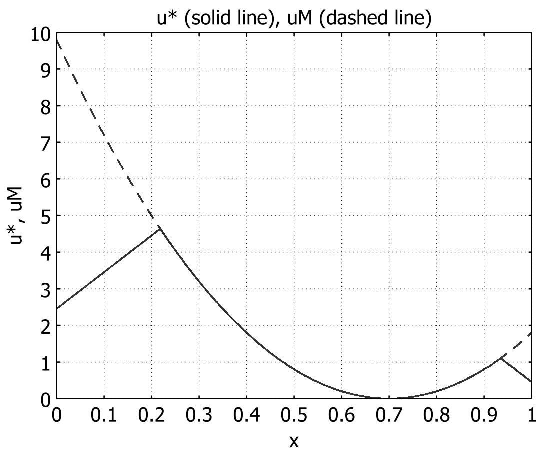

Fig. 1. $u^*(x)$ given by (78) and $u_M(x)$

Remark 3.3. The interest in this study was motivated by the applications to certain real-world models of diffusion, with vanishing diffusion coefficients in the interior of the domain, as specified by the examples and references given in the Introduction. However the results remain true even in the case of boundary degeneracy (namely when $x_0 = 0$ or $x_0 = L$). The proofs follow in a similar way, by specifying that $u_0 = 0$, or $u_L = 0$ in the set U given by (20) and $u_m(0) = u_M(0) = 0$ or $u_m(L) = u_M(L) = 0$ in (23).

As a matter of fact, we also observe that the construction of the optimal control in this case can be reduced to the case with interior degeneracy, by considering the problem on the interval $(-\lambda, L)$. Thus, we set

$$\begin{aligned} U^\lambda &= \{u \in W^{1,\infty}(-\lambda, L);\ u_m^\lambda(x) \le u(x) \le u_M^\lambda(x), \\ u(-\lambda) &= u_0^\lambda > 0,\ u(L) = u_L,\ |u_x(x)| \le u_\infty \text{ a.e. } x \in (-\lambda, L)\}, \end{aligned}$$

with $x_0 = 0 \in (-\lambda, L)$, $u_\infty \in [0, \infty)$, where u_m^λ and u_M^λ are extensions of the functions u_m and u_M on $(-\lambda, 0)$, satisfying conditions (21)-(23). Relation (78) becomes

$$u_\lambda^*(x) = \begin{cases} u_0^\lambda + u_\infty x, & \text{for } x \in [-\lambda, x_1^\lambda) \\ u_M(x), & \text{for } x \in [x_1^\lambda, x_2) \\ -u_\infty(x - L) + u_L, & \text{for } x \in [x_2, L], \end{cases}$$

where x_1^λ is given by $u_M^\lambda(x_1^\lambda) = u_\infty x_1^\lambda + u_0^\lambda$ and $u_0^\lambda \to u_0$ and $x_1^\lambda \to x_1$ as $\lambda \to 0$, where u_0 and x_1 are those previously considered. Then, the solution corresponding to the limit case $x_0 = 0$ is obtained as the limit of u_λ^* as $\lambda \to 0$.

Acknowledgments. The authors thank the anonymous reviewer for the observations on an earlier form of this paper.

G.M. acknowledges the support of INDAM-GNAMPA, Italy, for May-June 2013 and of the grant CNCS–UEFISCDI, project number PN-II-ID-PCE-2011-3-0027. G.F., R.M.M., S.R. acknowledge the support of the GNAMPA project "Equazioni di evoluzione degeneri e singolari: controllo e applicazioni", 2013.

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