Controlling and measuring a superposition of position and momentum

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We present an observation of quantum interference between position and momentum with photons. We prepared a superposition of position and momentum states by using slits, lenses, and an interferometer. The precise control and measuring of propagation time of the photon results in a clear quantum interference of 85% visibility. The generated quantum state is then used to verify the violation of particle propagation inequality proposed by Hofmann [Phys. Rev. A **96**, 020101(R) (2017)], resulting in 5.9% below the statistically predicted lower bound. We believe that this result is a major step towards further developments in quantum theory, such as fundamental modifications to the interpretation of quantum mechanics.

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I. INTRODUCTION

The precise control and measurement of physical quantities is at the heart of physics, although the uncertainty principle prohibits the simultaneous measurement of noncommutative observables in quantum mechanics [1]. It is therefore of great interest to investigate a particle trajectory, which is described by noncommuting observables of position and momentum for both theory and experiment [2–11]. In this context, there have been lots of efforts on identifying the particle trajectories [12,13] such as the direct measurement of the wave function [14–16], the identification of particle trajectories using weak measurements [17,18], and the Feynman path integral form [19–21]. These findings significantly advance the fundamental comprehension of quantum mechanics. However, the explication of these trajectories and their physical significance remains a subject of debate when quantum interference arises. Therefore, it is worthwhile to explore an alternative approach that measures the degree to which the probability flux predicted by quantum mechanics deviates from classical mechanical assumptions, while preserving the meaning of physical quantities such as position and momentum. Indeed, it has been shown that in free space, even when the momentum component of the wave function is defined in the positive interval, probability flow goes in the negative direction when quantum interference arises [22].

Hofmann established a stringent inequality for particle propagation in this framework by utilizing the motion dynamics outlined in Newton's first law to describe the probability flux. He showed that the interference effect in quantum mechanics can greatly enhance the violation of this inequality, particularly when the overlap between position and momentum is optimized [23]. Specifically, probability distributions are assigned to the noncommutative positions and momenta in Heisenberg's equations of motion at a given initial time and a lower bound on the probability of finding a particle at some later time has been derived using the assumption that particles travel in straight lines. An inequality then arises when the initial state consists of a superposition of two states, one with a well-defined initial position and one with a well-defined range of momenta. When these two interfere, this reduces the probability of finding the particle within the range predicted by straight line paths at intermediate times probability. Studies have shown that by optimizing the widths of position and momentum, the defect probabilities can exceed 7% [23]. This suggests that there may be a need to revise the relationships between physical properties at distinct points in time.

In this article, we present an experimental verification of the particle propagation inequality with photons, as proposed by Hofmann [23,24]. We prepared the superposition of position and momentum states by using slits, lenses, and an interferometer [25–29] and then measured the quantum state at about 470 ps after the initial time. Our experimental data showed clear quantum interference between position and momentum with a high visibility of 85%. The obtained probability distribution was then used to test the rigorous particle propagation inequality which is derived from the statistics

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in terms of position and momentum. Our results confirmed that the inequality was violated by approximately 5.9% in the probability of finding a particle. Our statistical investigation showed that the uncertainty principle is not sufficient to explain the particle propagation of a nonclassical superposition state even for a single particle. Therefore, we believe that this result is a major step towards further developments in quantum theory, such as fundamental modifications to the interpretation of quantum theory [30].

II. THEORETICAL OVERVIEW

Let us first summarize the propagation inequality of a particle in free space discussed in Ref. [23]. In quantum mechanics, the time evolution of the position operator $\hat{x}(t)$ for a single particle is described by the Heisenberg's formulation. In terms of initial position $\hat{x}(0)$ and momentum $\hat{p}_x$, the time evolution of the position operator in free space is given by

$$\hat{x}(t) = \hat{x}(0) + \frac{\hat{p}_x}{m}t. \tag{1}$$

If we replace these operators with concrete values of x and p_x, this equation corresponds to Newton's first law, which states that a particle moves in a straight line in time. However, due to the uncertainty relation in quantum mechanics, the concrete values of the noncommutative observables ($\hat{x}$ and $\hat{p}_x$) cannot be identified simultaneously. In Ref. [23] a lower bound on the probability of finding a particle at an intermediate time was derived by assigning a separate probability distribution to the noncommutative observables of position and momentum at an initial time. Specifically, if a particle is assumed to be moving in a straight line, then a particle existing in an interval of width $-L/2 < x < L/2$ with a probability of $P(L)$ and also in an interval $-B/2 < p_x < B/2$ at $t = 0$ with a probability of $P(B)$ must pass through the position interval $M = L + Bt/m$ with a probability of $P(M, t)$. This constraint requires $P(M, t) \geqslant P(L \text{ AND B})$. The lowest possible value of $P(L \text{ AND B})$ can be expressed using the experimentally observable probabilities of finding the particle in L and B, resulting in the statistical limit of finding the particle at an intermediate time as follows [23],

$$P(M, t) \geqslant P(L) + P(B) - 1. \tag{2}$$

Because $P(M, t)$, $P(L)$, and $P(B)$ are obtained from the actual statistics of position and momentum in a different time, this equation can thus be confirmed in an experiment.

Many quantum states violate the inequality in the equation, but in this work we have studied the originally considered states in Ref. [23], which is a superposition of a position state $|L\rangle$ localized in the spatial interval L and a momentum state $|B\rangle$ localized in the momentum interval B which is given by

$$|\psi(t = 0)\rangle = \frac{1}{\sqrt{2(1 + \langle L|B\rangle)}}(|L\rangle + |B\rangle). \tag{3}$$

Note that the constructive interference between a position state $|L\rangle$ and a momentum state $|B\rangle$ enhances the total probability $P(L) + P(B)$ at $t = 0$ in Eq. (2). With time, the shape of the position wave function $\langle x|\hat{U}|L\rangle$ broadens and approaches the sinc function, while the shape of the momentum wave function $\langle x|\hat{U}|B\rangle$ does not change with time. The exact forms of the wave functions for $t > mL^2/(2\pi\hbar)$ are given by

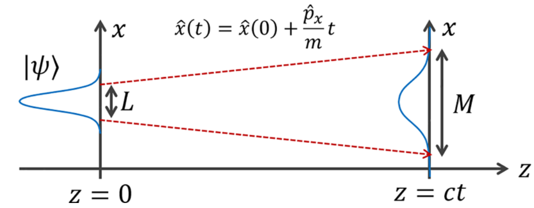

FIG. 1. Conceptual illustration of the propagation inequality. The blue line shows a quantum state. The particle moves along the z axis. $\hat{p}_x$ corresponds to the transverse momentum of the photon. The red dotted line shows the boundary of the trajectory where photons in the initial position interval L and momentum B move through the interval M.

$$\langle x|\hat{U}|L\rangle = \sqrt{\frac{mL}{2\pi\hbar t}}\left[\frac{2\hbar t}{mLx}\sin\left(\frac{mL}{2\hbar t}x\right)\right] \times \exp\left(i\frac{m}{2\hbar t}x^2 - i\frac{\pi}{4}\right), \tag{4}$$

$$\langle x|\hat{U}|B\rangle = \sqrt{\frac{B}{2\pi\hbar}}\left[\frac{2\hbar}{Bx}\sin\left(\frac{B}{2\hbar}x\right)\right]. \tag{5}$$

A strong violation of the inequality can be observed at around $t_M = mL/B$ when the shapes of the wave function of position and momentum match perfectly, where strong quantum interference between the position and momentum states reduces the probability of $P(M, t)$ in Eq. (2).

III. EXPERIMENTAL SETUP

In this context, we have experimentally investigated the inequality of Eq. (2) with photons. We utilized a transverse position x and momentum p_x of photons propagating at velocity c along the z axis, where the position $z = ct$ is used to measure the time t (Fig. 1). Using the paraxial approximation of $p_z \approx p = h/\lambda$, the transverse position can be identified as $\hat{x}(t) = \hat{x}(0) + (\hat{p}_x/p)ct$, so that the effective mass in Eq. (1) can be translated into $m = p/c = h/(c\lambda)$ where λ is the wavelength of photons. In this configuration, the initial quantum state $|\psi\rangle$ at $t = 0$ corresponds to the state at $z = 0$, and the state at t corresponds to the state at $z = ct$.

Figure 2(a) shows the experimental setup for creating the superposition state of Eq. (3), which is based on a Mach-Zehnder interferometer equipped with a slit and a lens. The photon is input to the interferometer and divided by the first beam splitter, resulting in a superposition of the photons in one path and in the other path. To produce a position state $|L\rangle$, we put a slit of width L in one arm of the interferometer [Fig. 2(b)], so that the state after the slit approximately corresponds to an image of the slit. Likewise, to produce a momentum state, we used an effect on the Fourier transform of the position by using a lens. We put a slit of width L' in the other arm of the interferometer and the lens was placed on the position after the slit at the distance of focal length f of the lens [Fig. 2(c)]. Under the paraxial approximation, the momentum information before the lens is mapped onto the

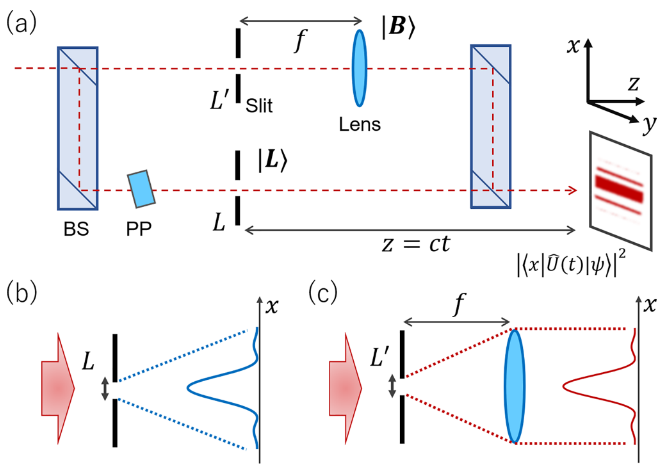

FIG. 2. (a) Experimental setup for producing the superposition of position and momentum. BS: beam splitter; PP: phase plate. (b) The position state $|L\rangle$ was prepared by using slit L in the bottom arm of the interferometer and (c) the momentum state $|B\rangle$ was prepared by using slit L' and the lens f in the upper arm of the interferometer. The lens f' was used to create an image of the slit L at the position of $2f'$ from the lens f'.

position scaled by $h/(f\lambda)$ after the lens. The corresponding momentum B is therefore given by $B = hL'/(f\lambda)$. The state then corresponds to a superposition of position $|L\rangle$ and momentum $|B\rangle$.

In order for the inequality in Eq. (2) to be strongly violated, t, L, and B must be chosen carefully. To quantify the violation of Eq. (2) at different times, we define the defect probability as

$$P_{\text{defect}}(t) = P(L) + P(B) - 1 - P(M, t). \tag{6}$$

The condition for the inequality to be maximally violated is obtained by maximizing this probability. As shown in Ref. [23], the maximum value of this probability is obtained when $LB = 0.024(2\pi\hbar)$ with $t_M = mL/B$ where the shapes of the wave functions of position and momentum match perfectly. In the experiment, we chose the slit widths of the position and momentum state for $L = 47$ μm and $L' = 37$ μm, respectively, and we chose a lens of $f = 10$ cm. LB was approximately calculated as $0.022(2\pi\hbar)$. The position $z_M = ct_M$ where the shapes of the wave functions of position and momentum match perfectly is calculated as 10 cm.

Figure 3(a) shows the dependence of the defect probability on the position scaled by z_M. As shown in the figure, the maximum value of P_{defect} is obtained at around $z = z_M$. More precisely, the maximum value of P_{defect} is obtained at a time $z = 1.1z_M$, which is slightly different from $z = z_M$ in our choice of L and B. Note that for our experimental parameters of the slit widths and lens we have chosen, the optimal condition is different from the original ones in Ref. [23]. Note also that the violation of the inequality can be observed over a fairly wide time range from about $z \sim z_M$ to $z \sim 8z_M$, which correspond to the position from $z = 10$ cm to $z = 80$ cm.

In this proof-of-principle demonstration, we used a weak coherent laser light with a wavelength of 800 nm. The laser was attenuated by using a neutral density filter at a single-photon level. The average photon number existing in the interferometer was estimated as 0.6, which ensures that the contribution of more than two photons from the laser was negligible. The interferometer was stable enough during the experiment due to the displaced common path design shown in Fig. 2(a). The obtained visibility of our interferometer was about 85% as discussed later. We used a single-photon sensitivity charge coupled device (CCD) to measure the interference fringe which ensures that the detected interference fringe is constructed from a pseudo-single-photon source of weak coherent light.

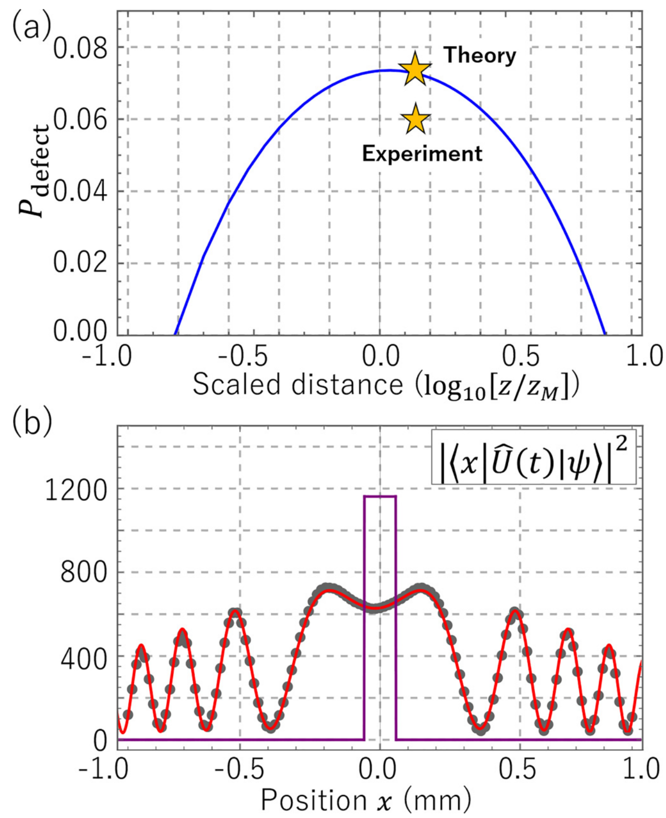

FIG. 3. (a) Dependence of the defect probability on the normalized position of the photon scaled by z_M. The star-shaped points represent the conditions of the present experiment. The degradation of the experimental values with respect to the theoretical values is due to the visibility of the interference fringes. (b) Experimentally observed interference fringe between position and momentum at $z = 1.4z_M$. The gray dot shows the experimental data. The red line shows the fitted curve. The purple line shows the probability densities corresponding to the minimal value of $P(M)$ which is expected from Eq. (2). $|\psi(t)\rangle = \hat{U}(t)|\psi\rangle$, where $\hat{U}(t)$ is the unitary operator of time evolution in free space.

IV. RESULTS

Figure 3(b) shows the interference fringe at an intermediate position $z = 1.4z_M$ (approximately 470 ps in time) which is slightly longer than its optimal time of $1.1z_M$. We have clearly observed an interference fringe between the position and momentum state. From the fitting, the measured visibility

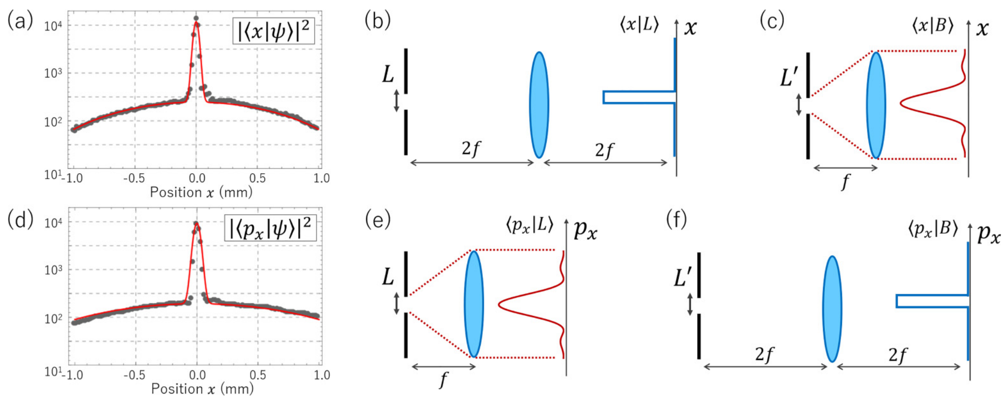

FIG. 4. (a) Interference fringe of position and momentum obtained experimentally in position space for the initial state of $|\psi\rangle$. Red is the fitting curve. The fitting was done using the Gaussian function instead of the slit function. (b) is the measurement system for the position state in position space, and (c) is the measurement system for the momentum state in position space. (d) Interference fringe of position and momentum obtained experimentally in momentum space for the initial state of $|\psi\rangle$. (e) is the measurement system for the position state in the momentum space, and (f) is the measurement system for the momentum state in the momentum space.

was roughly 85%. Figures 4(a) and 4(d) show the measured probability densities at $z = 0$ in the position basis and momentum basis, respectively. For the measurement in the position basis, the interference fringe was taken on the image plane of the position slit L outside of the interferometer as shown in Figs. 4(b) and 4(c). The probability of $P(L)$ is then calculated by integrating the probability density at $z = 0$ over the position x from $-L/2$ to $L/2$. From the fitted curve, the probability was calculated as $P(L) = 56.5\%$. For the measurement in the momentum basis, we used the effect on the Fourier transform by using the lens, where the state of Eq. (3) is mapped into the momentum space of $\hat{p}_x$. For the position state of $\langle p_x|L\rangle$, we put the lens at the position after the slit at the distance of focal length f of the lens, where the momentum information before the lens is mapped onto the position scaled by $h/(f\lambda)$ after the lens. For the momentum state of $\langle p_x|B\rangle$, we put the lens at the position after the slit at a distance of $2f$. The interference fringe was then taken on the image plane of the position slit L' outside of the interferometer as shown in Figs. 4(e) and 4(f). The probability of $P(B)$ was then calculated by integrating the probability density at $t = 0$ over the momentum from $-B/2$ to $B/2$. From the fitted curve, the probability was calculated as $P(B) = 56.5\%$.

Finally, we analyze the data. At an initial position of $t = 0$, we obtained the probabilities of $P(L)$ and $P(B)$ as 56.5%. We also measured the probability at the intermediate position. From the propagation inequality of Eq. (2), the minimum value of the probability of finding the photon within an interval M at $t = 1.4t_M$ was predicted as $P(L) + P(B) - 1 \approx 13.1\%$, which corresponds to the area under the purple line as shown in Fig. 3(b). The figure clearly shows that the actual measured probability of $P(M, t)$ is smaller than the value. The value of $P(M = L + 1.4\,Bt_M/m)$ is calculated by integrating the probability density at $t = 1.4t_M$ over the position x from $-M/2$ to $M/2$. From the fitting, which takes into account the visibility of the fringe, the probability was calculated as $P(M, t = 1.4t_M) = 7.2\%$. From the measured data, we obtained $P_{\text{defect}}(t = 1.4t_M) \approx 5.9\%$, which means that the 5.9% probability cannot be explained by the statistical limit which assumes that the particles move along a straight line.

V. DISCUSSION AND CONCLUSIONS

To demonstrate the precise control of position and momentum in quantum mechanics, it is interesting to compare our experimental results with the quantum uncertainty limit for the widths of the position and momentum distributions. This limit is expressed quantitatively as $P(L) + P(B) \leqslant 1 + \sqrt{U}$, where $U = LB/(2\pi\hbar)$ [31,32]. Notably, this inequality presents a more quantitative measure of statistical uncertainty, as opposed to the variance of the distributions, particularly when the product of L and B is smaller than Planck's constant. In our experiment, with $U = LB/(2\pi\hbar) \approx 0.022$, we obtained $1 + \sqrt{U} \approx 1.15$ and $P(L) + P(B) = 1.13$. These results indicate that the deviation between our experimental data and the lower limit is approximately 2%. Therefore, we conducted our experiment under conditions that were close to the limits of quantum uncertainty. This was achieved by implementing precise control over both the position and momentum of the system. While this study adopted the argument proposed in Ref. [23], it is crucial to emphasize that the violation of the particle propagation inequality is primarily an interference effect between position and momentum. Therefore, despite the superposition state by a rectangular function being the state that is closest to violating the particle propagation inequality, it is interesting to explore other quantum states that are theoretically and experimentally tractable for a deeper comprehension of quantum mechanics through particle propagation inequalities [32].

In conclusion, we have experimentally demonstrated the superposition of position and momentum states by precisely adjusting the slit width, lens, and the propagation distance of the photons. The strong interference of position and momentum was obtained at about 470 ps after the initial time, when the two states are almost overlapped. We then investigated how much the quantum coherent process between position and momentum can deviate from the lower bound on the probability of finding the particle, which is derived from the statistical analysis of the photon. Note that our experimental result does not give a new interpretation for quantum particle trajectories. Instead, on the basis of the observed statistics, we have quantitatively showed that at least Newton's first law falls about 5.9% short of explaining quantum mechanical probability flows, due to quantum interference effects. Therefore, we believe that this result is a major step towards further developments in quantum theory, such as fundamental modifications to the interpretation of quantum theory.

In this proof-of principle experiment, we employed a single-photon state [33] that was obtained through postselection via the use of attenuated weak coherent (laser) light. While our current measurement system does not fully capture the multiphoton interference patterns that arise from more than two photons, the contributions of more than two photons will obscure the single-photon pattern. This analysis can be performed by observing the correlation of counts between different positions in the interference pattern [34]. Therefore, it would be worthwhile to explore the inequality with multiparticle quantum states, including the entangled state.

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