THE IMAGE OF LIE POLYNOMIALS ON REAL LIE ALGEBRAS OF DIMENSION UP TO 3

LUCIO CENTRONE AND ŞEHMUS FINDIK

ABSTRACT. Let F_n be the free Lie algebra over $\mathbb{R}$ of rank n generated by $y_1, \ldots, y_n$, and let $f \in F_n'$ be a multilinear Lie polynomial contained in the commutator ideal F_n' of F_n. In this paper, we determine the image

$$\mathrm{Im} f = \{f(w_1, \ldots, w_n) \mid w_i \in L, i = 1, \ldots, n\} \subset L,$$

for Lie algebras L of dimension ≤ 3, and of the Lie algebra of dimension 4 stated in a paper of Baker dating back to 1901. In all the cases studied, the L'vov-Kaplansky Conjecture has a positive answer.

doi.org/10.1016/j.jalgebra.2024.07.006

1. Introduction

Motivated by the problem known as *L'vov-Kaplansky Conjecture*, the study of images of polynomials attracted several authors in the last decade. In few words, given an (possibly not) associative multilinear polynomial $f = f(x_1, \ldots, x_m)$ in m variables, the L'vov-Kaplansky Conjecture asks whether or not the image of f on the full matrix algebra $M_n(K)$ over a field K, seen as a function from $M_n(K)^m$ to $M_n(K)$ itself, is a vector space. In the case of the full matrix algebra, we only have four possibilities, i.e., $M_n(K)$ itself, the set of traceless matrices $sl_n(K)$, the set of scalar matrices (identified with the ground field K), or the set $\{0\}$. Nevertheless, a complete answer to the L'vov-Kaplansky Conjecture is known only for some values of n or of the total degree m of the polynomial. For instance, the case $m = 2$ is a consequence of a well-known result by Shoda [22] and Albert and Muckenhoupt [1] which states that any trace zero matrix is in the image of a commutator. The case $m = 3$ was handled in [11] where only partial results were achieved. If the size of the matrix algebra is $n = 2$, then we have a solution in [10] for K being a quadratically closed field and in [14] for K being the field of real numbers. The case $n = 3$ has a partial solution too and we address the reader to the paper [11]. Indeed, the analogue of the L'vov-Kaplansky Conjecture does not hold if we evaluate the image of a polynomial on a non-simple or not-finite dimensional algebra and, on the purpose, in [21], the authors provide an example of a non-simple finite dimensional algebra whose images on a certain class of polynomials are not subspaces. The L'vov-Kaplansky Conjecture has been studied on several relevant algebras such as the algebra of upper triangular matrices $UT_n(K)$ and that of strictly upper triangular matrices [8, 13, 6] (see [20] for relations with Waring type problems) as

---

2020 *Mathematics Subject Classification.* 17B01; 17B30.

Key words and phrases. images of polynomials, Lie algebras.

L. Centrone was supported by PNRR MUR project PE0000023-NQSTI.

well as the algebra of quaternions [15]. We also have some results in the graded case for the full matrix algebra $M_n(K)$ [5, 7].

In a nonassociative setting, we can cite the results obtained in [17] for some classes of simple Jordan algebras, or for a rock-paper-scissors algebra in [16]. In [2] the authors showed L'vov-Kaplansky Conjecture is true for Lie polynomials of degree 3 and 4 evaluated on the simple Lie algebras $\mathfrak{su}(n,K)$ and $\mathfrak{so}(n,K)$, whereas in [12] all the possible images of a Lie polynomial in $M_2(K)$ where founded and an example of a polynomial whose image is the set of non-nilpotent trace zero matrices, together with 0 was showed. We would also mention the paper [19] by Nehra and Rani about the evaluation of Lie polynomials of degree 2 on nilpotent Lie algebras whose derived algebra has dimension less than or equal to 4.

The present paper fits in the environment of evaluation of Lie polynomials on Lie algebras. More precisely, we consider all real Lie algebras of dimension up to 3 and we show the L'vov-Kaplansky Conjecture is true on the whole set of Lie algebras we have considered. Moreover, we also consider evaluations on multilinear Lie polynomials over a four dimensional Lie algebra which appeared in a paper by Baker (see [4]) and representing key examples of matrices of infinitesimal transformations generating continuous groups. We want to point out among the Lie algebras of dimension three we have $\mathfrak{so}(3,\mathbb{R})$ that means our work improves consistently the results obtained in [2]. Moreover, it is worth mentioning $\mathfrak{so}(3,\mathbb{R})$ is isomorphic to the real three-dimensional space endowed with the cross product.

2. Preliminaries

All algebras and vector spaces throughout the paper will be considered on the ground field $\mathbb{R}$ of real numbers unless stated otherwise.

A *Lie algebra* L is a vector space with a bilinear map (commutator) $[.,.]$ satisfying the following identities.

$$[x,y]+[y,x]=0 \quad \text{(skew-symmetry)}$$
$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0 \quad \text{(Jacobi identity)}$$

The Lie algebra L is called *abelian* if $[x,y]=0$ for all $x,y\in L$; whereas L is called *metabelian* if $[[x,y],[z,t]]=0$ for all $x,y,z,t\in L$. We shall use the left normed commutators throughout the paper; i.e.,

$$[x,y,z]=[[x,y],z].$$

Now consider the free Lie algebra F_n over $\mathbb{R}$ of rank n generated by $y_1,\ldots,y_n$. We denote by $F_n'=[F_n,F_n]$ its commutator ideal generated by all commutators $[x,y]$, where $x,y\in F_n$.

The set $ML(F_n)$ of multilinear polynomials of F_n in the set of variables $\{y_1,\ldots,y_n\}$ is a vector space and an S_n-module, too with the natural left action of the symmetric group on n elements. The following result is well known, see for instance [9, 3].

Proposition 2.1. *The space $ML(F_n)$ has a linear basis of elements of the form*

$$[y_1,y_{\pi(2)},\ldots,y_{\pi(n)}],$$

where $\pi\in S_{n-1}$ acts on the set $\{2,\ldots,n\}$.

The quotient algebra $M_n=F_n/(F_n')'=F_n/F_n''$ is the *free metabelian* Lie algebra of rank n generated by $z_i=y_i+F_n''$, $i=1,\ldots,n$. Let $u\in M_n'$ and $x,y\in M_n$, where M_n' is the commutator ideal of M_n. Then by the fact that $[x,y] \in M_n'$, we have

$$[u,x,y] = -[x,y,u] - [y,u,x] = [u,y,x].$$

Thus by induction

$$u\mathrm{ad}z_{i_1}\cdots\mathrm{ad}z_{i_k} = [u,z_{i_1},\ldots,z_{i_k}] = [u,z_{\pi(i_1)},\ldots,z_{\pi(i_k)}] = u\mathrm{ad}z_{\pi(i_1)}\cdots\mathrm{ad}z_{\pi(i_k)},$$

for each permutation $\pi \in S_k$ of $\{i_1,\ldots,i_k\}$. This yields that M_n' is a right $\mathbb{R}[z_1,\ldots,z_n]$-module by the action

$$u \cdot p(z_1,\ldots,z_n) = up(\mathrm{ad}z_1,\ldots,\mathrm{ad}z_n),$$

where $\mathbb{R}[z_1,\ldots,z_n]$ stands for the commutative associative unital polynomial algebra in n variables.

It is well known (see for instance [3]) that the commutator ideal M_n' has a linear basis consisting of the elements of the form

$$[z_i,z_j,z_{i_1},\ldots,z_{i_k}]\ ,\quad j > i \leq i_1 \leq \cdots \leq i_k.$$

As a consequence, we get the following.

Remark 2.2. Let $f \in M_n'$ be a multihomogeneous Lie polynomial of multidegree

$$(1+a_1,\ldots,1+a_n),\quad a_i \geq 0,\ 1 \leq i \leq n,$$

with respect to the ordered set $\{z_1,\ldots,z_n\}$ of generators of the free metabelian Lie algebra M_n. Then f can be written as

$$\begin{aligned} f(z_1,\ldots,z_n) =& \sum_{j=2}^{n} \alpha_{1j}[z_1,z_j]\cdot(z_1^{a_1}\cdots z_{j-1}^{1+a_{j-1}} z_j^{a_j} z_{j+1}^{1+a_{j+1}}\cdots z_n^{1+a_n})\\ =&[z_1,z_2]\cdot(z_1^{a_1}z_2^{a_2}z_3^{1+a_3}\cdots z_n^{1+a_n}) + [z_1,z_3]\cdot(z_1^{a_1}z_2^{1+a_2}z_3^{a_3}z_4^{1+a_4}\cdots z_n^{1+a_n})\\ &+\cdots+[z_1,z_n]\cdot(z_1^{a_1}z_2^{1+a_2}\cdots z_{n-1}^{1+a_{n-1}}z_n^{a_n}). \end{aligned} \tag{1}$$

For the sake of notation, we used the symbol z^a to denote the adjoint map $\mathrm{ad}z^a$. Similarly, a multihomogeneous Lie polynomial g of multidegree

$$(0,1+a_2,\ldots,1+a_n),\quad a_i \geq 0,\ 2 \leq i \leq n,$$

is of the form

$$\begin{aligned} g(z_1,\ldots,z_n) =& \sum_{j=3}^{n} \alpha_{2j}[z_2,z_j]\cdot(z_2^{a_2}\cdots z_{j-1}^{1+a_{j-1}} z_j^{a_j} z_{j+1}^{1+a_{j+1}}\cdots z_n^{1+a_n})\\ =&[z_2,z_3]\cdot(z_2^{a_2}z_3^{a_3}z_4^{1+a_4}\cdots z_n^{1+a_n}) + [z_2,z_4]\cdot(z_2^{a_2}z_3^{1+a_3}z_4^{a_4}z_5^{1+a_5}\cdots z_n^{1+a_n})\\ &+\cdots+[z_2,z_n]\cdot(z_2^{a_2}z_3^{1+a_3}\cdots z_{n-1}^{1+a_{n-1}}z_n^{a_n}). \end{aligned} \tag{2}$$

Now let $f(y_1,\ldots,y_n)$ be a multihomogeneous Lie polynomial of multidegree

$$(1+a_1,\ldots,1+a_n),\quad a_i \geq 0,\ 1 \leq i \leq n,$$

in the commutator ideal $\in F_n'$ of the free Lie algebra, and consider any metabelian Lie algebra L. If $w_1,\ldots,w_n \in L$, then $f(w_1,\ldots,w_n)$ is of the form (1). Similarly a multihomogeneous Lie polynomial $g(y_1,\ldots,y_n) \in F_n'$ of multidegree

$$(0,1+a_2,\ldots,1+a_n),\quad a_i \geq 0,\ 2 \leq i \leq n,$$

can be expressed as (2) when being evaluated in the metabelian Lie algebra L.

3. Main results

In this section, we consider the Lie algebras of dimension 2 and 3 with respect to the Mubarakzyanov's Classification [18] of real Lie algebras. Furthermore, we handle the 4 dimensional Lie algebra considered in a paper [4] by Baker in 1901. We skip abelian Lie algebras because in this case, $\mathrm{Im}f = 0$ for any $f \in F_n'$. We also do not handle the simple Lie algebra $\mathfrak{g}_{3,6}$ of dimension 3, known as Bianchi VIII or $\mathfrak{sl}(2,\mathbb{R})$, with multiplication table on its basis $\{x_1, x_2, x_3\}$ as follows.

$\mathfrak{g}_{3,6}$	x_1	x_2	x_3
x_1	0	x_1	$2x_2$
x_2	$-x_1$	0	x_3
x_3	$-2x_2$	$-x_3$	0

Because in this case it is known by [9] that $\mathrm{Im}f = \mathfrak{g}_{3,6}$ for $f \in F_n'$ that is not a polynomial identity of $\mathfrak{g}_{3,6}$.

In the following lines, the notation $\mathfrak{g}_1$ stands for the one dimensional abelian Lie algebra.

3.1. The Lie algebra $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1$.

The Lie algebra $\mathfrak{g}_{2,1}$ is the 2 dimensional Lie algebra with the multiplication table on the basis elements x_1, x_2 as below.

$\mathfrak{g}_{2,1}$	x_1	x_2
x_1	0	x_1
x_2	$-x_1$	0

Its commutator ideal $(\mathfrak{g}_{2,1})'$ is contained in the vector space spanned on x_1, and hence $(\mathfrak{g}_{2,1})'' = 0$. Therefore it is metabelian (solvable of class 2).

Theorem 3.1. *Let $f(y_1, \ldots, y_n)$ be a multihomogeneous polynomial in the commutator ideal F_n' of the free Lie algebra F_n that is not a polynomial identity of $\mathfrak{g}_{2,1}$. Then,*

$$\mathrm{Im}f = \{f(w_1, \ldots, w_n) \mid w_i \in \mathfrak{g}_{2,1}, i = 1, \ldots, n\} = \mathrm{Span}_{\mathbb{R}}\{x_1\}.$$

Proof. (i) Let $f \in F_n'$ be a multihomogeneous polynomial of multidegree

$$(1 + a_1, \ldots, 1 + a_n), \quad a_i \geq 0, \ 1 \leq i \leq n.$$

Then by Remark 2.2, we may assume that f is of the form

$$f(y_1, \ldots, y_n) = \sum_{j=2}^{n} \alpha_{1j}[y_1, y_j] \cdot (y_1^{a_1} \cdots y_{j-1}^{1+a_{j-1}} y_j^{a_j} y_{j+1}^{1+a_{j+1}} \cdots y_n^{1+a_n})$$

for some $\alpha_{1j} \in \mathbb{R}$. There exists at least one nonzero coefficient $\alpha_{1j'} \neq 0$, $1 < j'$, since $f \neq 0$. In this case, choosing

$$w_j = x_2, \ w_{j'} = -\frac{\beta}{\alpha_{1j'}} x_1 + x_2, \ j \neq j'$$

gives that

$$\begin{aligned} f(w_1, \ldots, w_n) &= \alpha_{1j'}[w_1, w_{j'}] \cdot (w_1^{a_1} \cdots w_{j'-1}^{1+a_{j'-1}} w_{j'}^{a_{j'}} w_{j'+1}^{1+a_{j'+1}} \cdots w_n^{1+a_n}) \\ &= \alpha_{1j'}[x_2, -\frac{\beta}{\alpha_{1j'}} x_1] \cdot (-\frac{\beta}{\alpha_{1j'}} x_1 + x_2)^{a_{j'}} (x_2^{n-2+\sum_{j \neq j'} a_j}) \\ &= (\beta x_1) \cdot x_2^{n-2+\sum_j a_j} = \beta[x_1, \underbrace{x_2, \ldots, x_2}_{n-2+\sum_j a_j}] = \beta x_1. \end{aligned}$$

(*ii*) Let $f \in F'_n$ be multihomogeneous of multidegree

$$(0, 1 + a_2, \dots, 1 + a_n), \quad a_i \geq 0, \ 2 \leq i \leq n.$$

Then f can be expressed as

$$f(y_1, \dots, y_n) = \sum_{j=3}^{n} \alpha_{2j} [y_2, y_j] \cdot (y_2^{a_2} \cdots y_{j-1}^{1+a_{j-1}} y_j^{a_j} y_{j+1}^{1+a_{j+1}} \cdots y_n^{1+a_n})$$

for some $\alpha_{2j} \in \mathbb{R}$. In a similar way, considering a nonzero coefficient $\alpha_{2j'}$, one may choose

$$w_j = x_2, \ w_{j'} = -\frac{\beta}{\alpha_{2j'}} x_1 + x_2, \ 1 < j \neq j'$$

that implies $f(w_1, \dots, w_n) = \beta x_1$.

The technique used in (*i*) and (*ii*) can be used inductively for all types of multihomogeneous functions f.

□

The proof of Theorem 3.1 also works for the next result.

Corollary 3.2. *Let $f \in F'_n$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{2,1}$. Then,*

$$\mathrm{Im} f = \{f(w_1, \dots, w_n) \mid w_i \in \mathfrak{g}_{2,1} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1\} = \mathrm{Span}_{\mathbb{R}}\{x_1\}.$$

Remark 3.3. Corollary 3.2 covers the result for the 3 dimensional Lie algebra $\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$ that is decomposable and solvable, called Bianchi III, with the multiplication table on the basis $\{x_1, x_2, x_3\}$ given as follows.

$\mathfrak{g}_{2,1} \oplus \mathfrak{g}_1$	x_1	x_2	x_3
x_1	0	x_1	0
x_2	$-x_1$	0	0
x_3	0	0	0

3.2. **The Lie algebra $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1$.** The Lie algebra $\mathfrak{g}_{3,1}$ is the 3 dimensional nilpotent of class 3 Lie algebra, called Heisenberg-Weyl algebra or Bianchi II, with the multiplication table on the basis elements x_1, x_2, x_3 as below.

$\mathfrak{g}_{3,1}$	x_1	x_2	x_3
x_1	0	0	0
x_2	0	0	x_1
x_3	0	$-x_1$	0

Theorem 3.4. *Let $f \in F'_n$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{3,1}$. Then,*

$$\mathrm{Im} f = \{f(w_1, \dots, w_n) \mid w_i \in \mathfrak{g}_{3,1}, i = 1, \dots, n\} = \mathrm{Span}_{\mathbb{R}}\{x_1\}.$$

Proof. Because of the nilpotency index, the only possibility for the total degree of a polynomial $f \in F'_n$ is 2, and when it is of multidegree

$$(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$$

filled with 1's only at i-th and j-th positions, then f is of the form

$$f(y_1, \dots, y_n) = \alpha_{ij} [y_i, y_j]$$

for some $0 \neq \alpha_{ij} \in \mathbb{R}$. Then choosing

$$w_i = \frac{\beta}{\alpha_{ij}} x_2 \ , \quad w_j = x_3 \ , \quad w_k = 0 \ , \quad k \neq i, j,$$

it turns out $f(w_1, \ldots, w_n) = \beta x_1$ and the proof follows.

□

Corollary 3.5. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{3,1}$. Then,*

$$\mathrm{Im} f = \{f(w_1, \ldots, w_n) \mid w_i \in \mathfrak{g}_{3,1} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1\} = \mathrm{Span}_{\mathbb{R}}\{x_1\}.$$

3.3. **The Lie algebra $\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1$.** The Lie algebra $\mathfrak{g}_{3,2}$ is the 3 dimensional metabelian Lie algebra, called Bianchi IV, with the multiplication table on the basis elements x_1, x_2, x_3 as below.

$\mathfrak{g}_{3,2}$	x_1	x_2	x_3
x_1	0	0	x_1
x_2	0	0	$x_1 + x_2$
x_3	$-x_1$	$-x_1 - x_2$	0

Theorem 3.6. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{3,2}$. Then,*

$$\mathrm{Im} f = \{f(w_1, \ldots, w_n) \mid w_i \in \mathfrak{g}_{3,2}\} = \mathrm{Span}_{\mathbb{R}}\{x_1, x_2\}.$$

Proof. The proof is similar to the one of Theorem 3.1. Let $f \in F_n'$ be a multihomogeneous polynomial of multidegree

$$(1 + a_1, \ldots, 1 + a_n), \quad a_i \geq 0, \ 1 \leq i \leq n.$$

By Remark 2.2, we may assume f is of the form

$$f(y_1, \ldots, y_n) = \sum_{j=2}^{n} \alpha_{1j} [y_1, y_j] \cdot (y_1^{a_1} \cdots y_{j-1}^{1+a_{j-1}} y_j^{a_j} y_{j+1}^{1+a_{j+1}} \cdots y_n^{1+a_n})$$

for some $\alpha_{1j} \in \mathbb{R}$ and suppose further $\alpha_{1j'} \neq 0$ for some $1 < j'$. Now choose

$$w_{j'} = -\frac{\beta_1 - (n - 1 + \sum a_j)\beta_2}{\alpha_{1j'}} x_1 - \frac{\beta_2}{\alpha_{1j'}} x_2 + x_3, \quad w_j = x_3, \ j \neq j'.$$

Then $f(w_1, \ldots, w_n) = \beta_1 x_1 + \beta_2 x_2$. Any other case can be solved similarly and we are done.

□

Corollary 3.7. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{3,2}$. Then,*

$$\mathrm{Im} f = \{f(w_1, \ldots, w_n) \mid w_i \in \mathfrak{g}_{3,2} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1\} = \mathrm{Span}_{\mathbb{R}}\{x_1, x_2\}.$$

3.4. **The Lie algebra $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1$.** The Lie algebra $\mathfrak{g}_{3,3}$ is the 3 dimensional metabelian Lie algebra, called Bianchi V, with the multiplication table on the basis elements x_1, x_2, x_3 as below.

$\mathfrak{g}_{3,3}$	x_1	x_2	x_3
x_1	0	0	x_1
x_2	0	0	x_2
x_3	$-x_1$	$-x_2$	0

Theorem 3.8. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{3,3}$. Then,*

$$\mathrm{Im} f = \{f(w_1, \ldots, w_n) \mid w_i \in \mathfrak{g}_{3,3}\} = \mathrm{Span}_{\mathbb{R}}\{x_1, x_2\}.$$

Proof. Similar to the proof of Theorem 3.1, then choose

$$w_{j'} = -\frac{\beta_1}{\alpha_{1j'}} x_1 - \frac{\beta_2}{\alpha_{1j'}} x_2 + x_3, \quad w_j = x_3, \; j \neq j'$$

and the proof follows.

□

Corollary 3.9. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{3,3}$. Then,*

$$\mathrm{Im} f = \{f(w_1, \ldots, w_n) \mid w_i \in \mathfrak{g}_{3,3} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1\} = \mathrm{Span}_{\mathbb{R}}\{x_1, x_2\}.$$

3.5. **The Lie algebra $\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1$.** The Lie algebra $\mathfrak{g}_{3,4}$ is the 3 dimensional metabelian Lie algebra, called Bianchi VI, with the multiplication table on the basis elements x_1, x_2, x_3 as below.

$\mathfrak{g}_{3,4}$	x_1	x_2	x_3
x_1	0	0	x_1
x_2	0	0	γx_2
x_3	$-x_1$	$-\gamma x_2$	0

where , $-1 \leq \gamma < 1$, $\gamma \neq 0$. It is called Poincaré algebra when $\gamma = -1$.

Theorem 3.10. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{3,4}$. Then,*

$$\mathrm{Im} f = \{f(w_1, \ldots, w_n) \mid w_i \in \mathfrak{g}_{3,4}\} = \mathrm{Span}_{\mathbb{R}}\{x_1, x_2\}.$$

Proof. Similar to the proof of Theorem 3.1, choose

$$w_{j'} = -\frac{\beta_1}{\alpha_{1j'}} x_1 - \frac{\gamma^{-n+1-\sum a_j}\beta_2}{\alpha_{1j'}} x_2 + x_3, \quad w_j = x_3, \; j \neq j'.$$

□

Corollary 3.11. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{3,4}$. Then,*

$$\mathrm{Im} f = \{f(w_1, \ldots, w_n) \mid w_i \in \mathfrak{g}_{3,4} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1\} = \mathrm{Span}_{\mathbb{R}}\{x_1, x_2\}.$$

3.6. **The Lie algebra $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1$.** The Lie algebra $\mathfrak{g}_{3,5}$ is the 3 dimensional metabelian Lie algebra, called Bianchi VII, with the multiplication table on the basis elements x_1, x_2, x_3 as below.

$\mathfrak{g}_{3,5}$	x_1	x_2	x_3
x_1	0	0	$\gamma x_1 - x_2$
x_2	0	0	$x_1 + \gamma x_2$
x_3	$-\gamma x_1 + x_2$	$-x_1 - \gamma x_2$	0

where $\gamma \geq 0$.

We need the following technical lemma.

Lemma 3.12. *(i) If* $[x_1,\underbrace{x_3,\ldots,x_3}_{k}]=cx_1+dx_2$*, for some* $c,d\in\mathbb{R}$*, then*

$$[x_2,\underbrace{x_3,\ldots,x_3}_{k}]=-dx_1+cx_2$$

for any $k\geq 1$*.*

(ii) $[x_1,x_3,\underbrace{x_3,\ldots,x_3}_{k}]\neq 0$ *and* $[x_2,x_3,\underbrace{x_3,\ldots,x_3}_{k}]\neq 0$*, for any* $k\geq 0$*.*

Proof. (i) We perform an induction on $k\geq 1$. First, by the multiplication table, we have

$$[x_1,x_3]=\gamma x_1-x_2 \quad\text{and}\quad [x_2,x_3]=x_1+\gamma x_2$$

proving the case $k=1$. Now assume that

$$[x_1,\underbrace{x_3,\ldots,x_3}_{k}]=cx_1+dx_2 \quad\text{and}\quad [x_2,\underbrace{x_3,\ldots,x_3}_{k}]=-dx_1+cx_2\ ,$$

for some $c,d\in\mathbb{R}$. Then

$$[x_1,\underbrace{x_3,\ldots,x_3}_{k},x_3]=[cx_1+dx_2,x_3]=(c\gamma+d)x_1+(-c+d\gamma)x_2$$

and

$$[x_2,\underbrace{x_3,\ldots,x_3}_{k},x_3]=[-dx_1+cx_2,x_3]=(-d\gamma+c)x_1+(d+c\gamma)x_2$$

which completes the proof.

(ii) Assume that $[c_1x_1+c_2x_2,x_3]=0$ for some $c_1,c_2\in\mathbb{R}$. Then,

$$(c_1\gamma+c_2)x_1+(-c_1+c_2\gamma)x_2=0,$$

which gives the homogeneous system

$$\begin{aligned}\gamma c_1+c_2&=0\\ -c_1+\gamma c_2&=0\end{aligned}$$

The determinant

$$\begin{vmatrix}\gamma & 1\\ -1 & \gamma\end{vmatrix}=1+\gamma^2\neq 0.$$

of the coefficients of the system yields $c_1=c_2=0$. However, we have $[x_1,x_3]=\gamma x_1-x_2$ is nonzero and, inductively,

$$[x_1,x_3,\underbrace{x_3,\ldots,x_3}_{k}]=[\gamma x_1-x_2,\underbrace{x_3,\ldots,x_3}_{k}]$$

cannot be zero. The same holds for $[x_2,x_3,\underbrace{x_3,\ldots,x_3}_{k}]$, as well.

□

Theorem 3.13. *Let* $f\in F_n'$ *be a multihomogeneous polynomial that is not a polynomial identity of* $\mathfrak{g}_{3,5}$*. Then,*

$$\mathrm{Im}f=\{f(w_1,\ldots,w_n)\mid w_i\in\mathfrak{g}_{3,5}\}=\mathrm{Span}_{\mathbb{R}}\{x_1,x_2\}.$$

Proof. Let $f \in F_n'$ be a multihomogeneous polynomial of multidegree

$$(1+a_1, \ldots, 1+a_n), \quad a_i \geq 0, \ 1 \leq i \leq n$$

be of the form

$$f(y_1, \ldots, y_n) = \sum_{j=2}^{n} \alpha_{1j}[y_1, y_j] \cdot (y_1^{a_1} \cdots y_{j-1}^{1+a_{j-1}} y_j^{a_j} y_{j+1}^{1+a_{j+1}} \cdots y_n^{1+a_n})$$

for some $\alpha_{1j} \in \mathbb{R}$, and assume $\alpha_{1j'} \neq 0$ for some $1 < j'$. We shall show that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$w_{j'} = c_1 x_1 + c_2 x_2 + x_3, \quad w_j = x_3, \ j \neq j'.$$

with $f(w_1, \ldots, w_n) = \beta_1 x_1 + \beta_2 x_2$ for any given $\beta_1, \beta_2 \in \mathbb{R}$. Assume that

$$[x_1, \underbrace{x_3, \ldots, x_3}_{n-1+\sum a_j}] = cx_1 + dx_2 \ \text{ and } \ [x_1, \underbrace{x_2, \ldots, x_2}_{n-1+\sum a_j}] = -dx_1 + cx_2$$

for some $c, d \in \mathbb{R}$ by (i) of Lemma 3.12. We have the following.

$$\begin{aligned}
f(w_1, \ldots, w_n) &= \alpha_{1j'}[w_1, w_{j'}] \cdot (w_1^{a_1} \cdots w_{j'-1}^{1+a_{j'-1}} w_{j'}^{a_{j'}} w_{j'+1}^{1+a_{j'+1}} \cdots w_n^{1+a_n}) \\
&= \alpha_{1j'}[x_3, c_1x_1 + c_2x_2 + x_3] \cdot (x_3)^{n-2+\sum a_j} \\
&= -\alpha_{1j'}c_1[x_1, \underbrace{x_3, \ldots, x_3}_{n-1+\sum a_j}] - \alpha_{1j'}c_2[x_2, \underbrace{x_3, \ldots, x_3}_{n-1+\sum a_j}] \\
&= -\alpha_{1j'}c_1(cx_1 + dx_2) - \alpha_{1j'}c_2(-dx_1 + cx_2) \\
&= -\alpha_{1j'}(c_1c - c_2d)x_1 - \alpha_{1j'}(c_1d + c_2c)x_2.
\end{aligned}$$

Now $f(w_1, \ldots, w_n) = \beta_1 x_1 + \beta_2 x_2$ gives the system

$$\begin{aligned}
-\alpha_{1j'}(c_1c - c_2d) &= \beta_1 \\
-\alpha_{1j'}(c_1d + c_2c) &= \beta_2
\end{aligned}$$

with the determinant of coefficients with respect to unknowns c_1, c_2 as follows:

$$\begin{vmatrix} -\alpha_{1j'}c & \alpha_{1j'}d \\ -\alpha_{1j'}d & -\alpha_{1j'}c \end{vmatrix} = \alpha_{1j'}^2(c^2 + d^2).$$

Indeed it is nonzero by (ii) of Lemma 3.12, and we are done. □

Corollary 3.14. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{3,5}$. Then,*

$$\text{Im} f = \{f(w_1, \ldots, w_n) \mid w_i \in \mathfrak{g}_{3,5} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1\} = \text{Span}_{\mathbb{R}}\{x_1, x_2\}.$$

3.7. **The Lie algebra $\mathfrak{g}_{3,7} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1$.** The Lie algebra $\mathfrak{g}_{3,7}$ is the 3 dimensional simple Lie algebra, called Bianchi IX or $\mathfrak{so}(3, \mathbb{R})$, with the multiplication table on the basis elements x_1, x_2, x_3 as below.

$\mathfrak{g}_{3,7}$	x_1	x_2	x_3
x_1	0	x_3	$-x_2$
x_2	$-x_3$	0	x_1
x_3	x_2	$-x_1$	0

In the sequel, we shall provide two results: the first one, more constructive, showing what kind of substitutions must be performed on a multilinear monomial in order to show its image is the whole $\mathfrak{so}(3,\mathbb{R})$; the second one, theoretical, showing the image of any multilinear Lie polynomial that is not a polynomial identity is again $\mathfrak{so}(3,\mathbb{R})$.

Theorem 3.15. *Let $f_\pi \in F'_n$ be the multilinear polynomial of the form*

$$f_\pi(y_1,\dots,y_n) = [y_{\pi(1)},\dots,y_{\pi(n)}] \ , \quad \pi \in S_n \ ,$$

that is not a polynomial identity of $\mathfrak{g}_{3,7}$. Then,

$$\mathrm{Im} f_\pi = \{f_\pi(w_1,\dots,w_n) \mid w_i \in \mathfrak{g}_{3,7}\} = \mathfrak{g}_{3,7}.$$

Proof. It is sufficient to prove the statement of the theorem for only

$$f(y_1,\dots,y_n) = [y_1,\dots,y_n].$$

We have to show that there exist $w_i = c_i x_1 + d_i x_2 + e_i x_3 \in \mathfrak{g}_{3,7}$, $i = 1,\dots,n$, such that

$$f(w_1,\dots,w_n) = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

for given real numbers β_1,β_2,β_3.

(i) Initially let us state the following technical straightforward results.

$$[x_1,\underbrace{x_3,\dots,x_3}_{n}] = \begin{cases} -x_2 & \text{if } n \equiv 1 \ (\text{mod} 4) \\ -x_1 & \text{if } n \equiv 2 \ (\text{mod} 4) \\ x_2 & \text{if } n \equiv 3 \ (\text{mod} 4) \\ x_1 & \text{if } n \equiv 0 \ (\text{mod} 4) \end{cases} \qquad [x_2,\underbrace{x_3,\dots,x_3}_{n}] = \begin{cases} x_1 & \text{if } n \equiv 1 \ (\text{mod} 4) \\ -x_2 & \text{if } n \equiv 2 \ (\text{mod} 4) \\ -x_1 & \text{if } n \equiv 3 \ (\text{mod} 4) \\ x_2 & \text{if } n \equiv 0 \ (\text{mod} 4) \end{cases}$$

$$[x_1,\underbrace{x_2,\dots,x_2}_{n}] = \begin{cases} x_3 & \text{if } n \equiv 1 \ (\text{mod} 4) \\ -x_1 & \text{if } n \equiv 2 \ (\text{mod} 4) \\ -x_3 & \text{if } n \equiv 3 \ (\text{mod} 4) \\ x_1 & \text{if } n \equiv 0 \ (\text{mod} 4) \end{cases} \qquad [x_3,\underbrace{x_2,\dots,x_2}_{n}] = \begin{cases} -x_1 & \text{if } n \equiv 1 \ (\text{mod} 4) \\ -x_3 & \text{if } n \equiv 2 \ (\text{mod} 4) \\ x_1 & \text{if } n \equiv 3 \ (\text{mod} 4) \\ x_3 & \text{if } n \equiv 0 \ (\text{mod} 4) \end{cases}$$

(ii) Let $\beta_3 = 0$. Then choosing $w_1 = w_3 = \cdots = w_n$, we get that

$$f(w_1,\dots,w_n) = -[w_2,\underbrace{w_1,\dots,w_1}_{n-1}].$$

Now by (i), choosing $w_j = x_3$, where $j \neq 2$, and

$$w_2 = \begin{cases} -\beta_1 x_1 - \beta_2 x_2 & \text{if } n \equiv 1 \ (\text{mod} 4) \\ \beta_2 x_1 - \beta_1 x_2 & \text{if } n \equiv 2 \ (\text{mod} 4) \\ \beta_1 x_1 + \beta_2 x_2 & \text{if } n \equiv 3 \ (\text{mod} 4) \\ -\beta_2 x_1 + \beta_1 x_2 & \text{if } n \equiv 0 \ (\text{mod} 4) \end{cases}$$

implies that $f(w_1,\dots,w_n) = \beta_1 x_1 + \beta_2 x_2$.

(iii) Let $\beta_3 \neq 0$ and let $n = 2$. The fact

$$\left[c_1 x_1 + d_1 x_2 - \frac{c_1\beta_1 + d_1\beta_2}{\beta_3} x_3, c_2 x_1 + d_2 x_2 - \frac{c_2\beta_1 + d_2\beta_2}{\beta_3} x_3\right] = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

implies that choosing

$$w_1 = \beta_3 x_1 - \beta_1 x_3 \qquad w_2 = x_2 - \frac{\beta_2}{\beta_3} x_3$$

gives $f(w_1, w_2) = [w_1, w_2] = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$, $\beta_3 \neq 0$.

Now let $n > 2$. Similar to (ii), utilizing (i), one may obtain

$$[w_1, \dots, w_{n-1}] = \beta_3 x_1 - \beta_1 x_3$$

by choosing $w_j = x_2$, $j \notin \{2, n\}$, and

$$w_2 = \begin{cases} -\beta_1 x_1 - \beta_3 x_3 & \text{if } n \equiv 0 \text{ (mod4)} \\ \beta_3 x_1 - \beta_1 x_3 & \text{if } n \equiv 1 \text{ (mod4)} \\ \beta_1 x_1 - \beta_3 x_3 & \text{if } n \equiv 2 \text{ (mod4)} \\ -\beta_3 x_1 + \beta_1 x_3 & \text{if } n \equiv 3 \text{ (mod4)} \end{cases}$$

Thus, the choice $w_n = x_2 - \frac{\beta_2}{\beta_3} x_3$ yields that $f(w_1, \dots, w_n) = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$. □

Corollary 3.16. *Let $f \in F'_n$ be a multilinear polynomial of the form*

$$f_\pi(y_1, \dots, y_n) = [y_{\pi(1)}, \dots, y_{\pi(n)}] \ , \quad \pi \in S_n \ ,$$

that is not a polynomial identity of $\mathfrak{g}_{3,7}$. Then,

$$\mathrm{Im} f = \{f(w_1, \dots, w_n) \mid w_i \in \mathfrak{g}_{3,7} \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_1\} = \mathfrak{g}_{3,7}.$$

As stated before, we can generalize the result obtained in Theorem 3.15. Here is the complete statement.

Theorem 3.17. *Let $f \in F'_n$ be a multilinear polynomial that is not a polynomial identity of $\mathfrak{g}_{3,7}$. Then*

$$\mathrm{Im} f = \mathfrak{g}_{3,7}.$$

Proof. We shall make use of the well known fact $\mathfrak{so}(3, \mathbb{R})$ is isomorphic to $\mathbb{R}^3$ endowed with the *cross product* "$\times$". Keeping this in mind, given any $w = w_1 \in \mathbb{R}^3$ of length 1, it is possible to find out, via a small modification of the orthonormalization process of Graham-Schmidt, $w_2, w_3 \in \mathbb{R}^3$ so that $\mathcal{B} = \{w_1, w_2, w_3\}$ is an orthonormal basis of $\mathbb{R}^3$ and $w_1 \times w_2 = w_3$, $w_2 \times w_3 = w_1$ and $w_3 \times w_1 = w_2$. Of course the map μ sending $x_i \mapsto w_i$, where $i = 1, 2, 3$ is an automorphism of Lie algebras. Now, if $f \in F'_n$ is a multilinear polynomial that is not a polynomial identity of $\mathfrak{g}_{3,7}$, then, because μ is an automorphism, we have

$$\mathrm{Im} f = \{f(a_1, \dots, a_n) \mid a_i \in \mathfrak{g}_{3,7}\} = \{f(\mu(a_1), \dots, \mu(a_n)) \mid a_i \in \mathfrak{g}_{3,7}\}.$$

It is easy to see there exist $a_1, \dots, a_n \in \mathfrak{g}_{3,7}$ such that $f(a_1, \dots, a_n) = x_1$. Indeed, we have $w = w_1 = \mu(x_1) = f(\mu(a_1), \dots, \mu(a_n)) \in \mathrm{Im} f$. This shows $\mathrm{Im} f = \mathbb{R}^3 \cong \mathfrak{so}(3, \mathbb{R})$ and the proof follows. □

3.8. **The Lie algebra $\mathfrak{g}_{4,3} \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_1$.** The Lie algebra $\mathfrak{g}_{4,3}$ is the 4 dimensional indecomposable metabelian Lie algebra, stated in Baker's paper [4] in 1901, with the multiplication table on the basis elements x_1, x_2, x_3, x_4 as below.

$\mathfrak{g}_{4,3}$	x_1	x_2	x_3	x_4
x_1	0	x_2	x_4	0
x_2	$-x_2$	0	0	0
x_3	$-x_4$	0	0	0
x_4	0	0	0	0

Theorem 3.18. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{4,3}$.*
(i) If $\deg_f = 2$, then

$$\mathrm{Im} f = \{f(w_1, \dots, w_n) \mid w_i \in \mathfrak{g}_{4,3}\} = \mathrm{Span}_{\mathbb{R}}\{x_2, x_4\}.$$

(ii) If $\deg_f > 2$, then

$$\mathrm{Im} f = \{f(w_1, \dots, w_n) \mid w_i \in \mathfrak{g}_{4,3}\} = \mathrm{Span}_{\mathbb{R}}\{x_2\}.$$

Proof. (i) Let $f \in F_n'$ be of degree 2, and of multidegree

$$(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0).$$

Then, similar to the proof of Theorem 3.4, f is of the form

$$f(y_1, \dots, y_n) = \alpha_{ij}[y_i, y_j]$$

for some $0 \neq \alpha_{ij} \in \mathbb{R}$. Then choosing

$$w_i = \beta_4 x_1 + x_2 \ , \quad w_j = -\beta_2 x_1 + x_3 \ , \quad w_k = 0 \ , \quad k \neq i, j$$

gives $f(w_1, \dots, w_n) = \beta_2 x_2 + \beta_4 x_4$.

(ii) Similarly to the proof of Theorem 3.1, choose

$$w_{j'} = \beta_2(-1)^{n-2+\sum a_j} x_2, \quad w_j = x_1, \ j \neq j'.$$

□

Corollary 3.19. *Let $f \in F_n'$ be a multihomogeneous polynomial that is not a polynomial identity of $\mathfrak{g}_{4,3}$. (i) If $\deg_f = 2$, then*

$$\mathrm{Im} f = \{f(w_1, \dots, w_n) \mid w_i \in \mathfrak{g}_{4,3} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1\} = \mathrm{Span}_{\mathbb{R}}\{x_2, x_4\}.$$

(ii) If $\deg_f > 2$, then

$$\mathrm{Im} f = \{f(w_1, \dots, w_n) \mid w_i \in \mathfrak{g}_{4,3} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_1\} = \mathrm{Span}_{\mathbb{R}}\{x_2\}.$$

Acknowledgements

The second named author is very thankful to Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro for the creative atmosphere and the warm hospitality during his visit as a post-doctoral fellow when this project was carried out.

References

[1] A.A. Albert, B. Muckenhoupt, On matrices of trace zeros. Mich. Math. J. 4 (1957), 1-3. https://doi.org/10.1307/mmj/1028990168
[2] B.E. Anzis, Z.M. Emrich, K.G. valiveti, On the images of Lie polynomials evaluated on Lie algebras. Linear Algebra Appl. 469 (2015), 51-75. https://doi.org/10.1016/j.laa.2014.11.015
[3] Yu. A. Bahturin, Identical Relations in Lie Algebras (Russian), Nauka, Moscow, 1985. Translation: VNU Science Press, Utrecht, 1987.
[4] H.F. Baker, On the exponential theorem for a simply transitive continuous group, and the calculation of the finite equations from the constants of structure. Proc. London Math. Soc. 34 (1901), 91-127. https://doi.org/10.1112/plms/s1-34.1.91
[5] L. Centrone, T.C. de Mello, Images of graded polynomials on matrix algebras. J. Algebra 614 (2023), 650–669. https://doi.org/10.1016/j.jalgebra.2022.09.031
[6] P.S. Fagundes, The images of multilinear polynomials on strictly upper triangular matrices. Linear Algebra Appl. 563 (2019), 287-301. https://doi.org/10.1016/j.laa.2018.11.014
[7] P. Fagundes, P. Koshlukov, Images of multilinear graded polynomials on upper triangular matrix algebras. Canadian J. Math. (2022), 1-26. https://doi.org/10.4153/S0008414X22000438

[8] I.G. Gargate, T.C. de Mello, Images of multilinear polynomials on $n \times n$ upper triangular matrices over infinite fields. Israel J. Math. 252 (1) (2022), 337-354. https://doi.org/10.1007/s11856-022-2350-2
[9] A. Giambruno, M. Zaicev, Polynomial identities and asymptotic methods, Math. Surveys Monogr., 122, American Mathematical Society, Providence, RI, 2005.
[10] A. Kanel-Belov, S. Malev, L. Rowen, The images of non-commutative polynomials evaluated on 2×2 matrices. Proc. Am. Math. Soc. 140 (2) (2012), 465-478.
[11] A. Kanel-Belov, S. Malev, L. Rowen, The images of multilinear polynomials evaluated on matrices. Proc. Am. Math. Soc. 144 (1) (2016), 7-19.
[12] A. Kanel-Belov, S. Malev, L. Rowen, The images of Lie polynomials evaluated on matrices. Comm. Algebra 45 (11) (2017), 4801-4808. https://doi.org/10.1080/00927872.2017.1282959
[13] Y. Luo, Y. Wang, On Fagundes-Mello conjecture. J. Algebra 592 (2022), 118-152. https://doi.org/10.1016/j.jalgebra.2021.11.008
[14] S. Malev, The images of non-commutative polynomials evaluated on 2×2 matrices over an arbitrary field. J. Algebra Appl. 13 (6) (2014), Article 1450004. https://doi.org/10.1142/S0219498814500042
[15] S. Malev, The images of noncommutative polynomials evaluated on the quaternion algebra. J. Algebra Appl. 20 (5) (2021), Article 2150074. https://doi.org/10.1142/S0219498821500742
[16] S. Malev, C. Pines, The images of multilinear non-associative polynomials evaluated on a rock-paper-scissors algebra with unit over an arbitrary field and its subalgebras. Chebyshevskiĭ Sb. 21 (4) (2020), 129-139.
[17] S. Malev, R. Yavich, R. Shayer, Evaluations of multilinear polynomials on low rank Jordan algebras. Commun. Algebra 50 (7) (2022), 2840-2845. https://doi.org/10.1080/00927872.2021.2021221
[18] G. M. Mubarakzyanov, G.M., On solvable Lie algebras, Izv. Vys. Ucheb. Zaved. Matematika (in Russian) 1 (32) (1963), 114–123.
[19] N. Nehra, S. Rani, Image of Lie polynomial of degree 2 evaluated on nilpotent Lie algebra. Comm. Algebra 51 (1) (2023), 46-62. https://doi.org/10.1080/00927872.2022.2087227
[20] S. Panja, S. Prasad, The image of polynomials and Waring type problems on upper triangular matrix algebras. J. Algebra 631 (2023), 148-193. https://doi.org/10.1016/j.jalgebra.2023.04.027
[21] E. Santulo, F. Yukihide Yasumura, On the image of polynomials evaluated on incidence algebras: a counter-example and a solution, arXiv:1902.08116v1 (2019).
[22] K. Shoda, Einige Sätze über Matrizen. Jpn. J. Math., 13 (3) (1937), 361-365.

Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via Edoardo Orabona, 4, 70125 Bari, Italy

IMECC, Universidade Estadual de Campinas, Rua Sergio Buarque de Holanda 651,13083-859, Campinas (SP), Brazil
Email address: `lucio.centrone@uniba.it, centrone@unicamp.br`

Department of Mathematics, Çukurova University, 01330 Balcali, Adana, Turkey

Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via Edoardo Orabona, 4, 70125 Bari, Italy
Email address: `sfindik@cu.edu.tr`