DOI: 10.1142/S021902572250028X

DE FINETTI-TYPE THEOREMS ON QUASI-LOCAL ALGEBRAS AND INFINITE FERMI TENSOR PRODUCTS

VITONOFRIO CRISMALE, STEFANO ROSSI, AND PAOLA ZURLO

Abstract. Local actions of $\mathbb{P}_{\mathbb{N}}$, the group of finite permutations on $\mathbb{N}$, on quasi-local algebras are defined and proved to be $\mathbb{P}_{\mathbb{N}}$-abelian. It turns out that invariant states under local actions are automatically even, and extreme invariant states are strongly clustering. Tail algebras of invariant states are shown to obey a form of the Hewitt and Savage theorem, in that they coincide with the fixed-point von Neumann algebra. Infinite graded tensor products of C^*-algebras, which include the CAR algebra, are then addressed as particular examples of quasi-local algebras acted upon $\mathbb{P}_{\mathbb{N}}$ in a natural way. Extreme invariant states are characterized as infinite products of a single even state, and a de Finetti theorem is established. Finally, infinite products of factorial even states are shown to be factorial by applying a twisted version of the tensor product commutation theorem, which is also derived here.

Mathematics Subject Classification: 46L06, 60G09, 60F20, 46L53.
Key words: quasi-local algebras, infinite $\mathbb{Z}_2$-graded tensor products, tail algebras, de Finetti's theorems, product states.

1. Introduction

Distributional symmetries for families of random variables concern invariance of any finite joint distribution of them under some measurable transformations. For their importance in probability theory, invariance under shifts, finite permutations or rotations are certainly worth mentioning. In these cases the random variables are respectively named stationary, exchangeable or rotatable, and the reader is referred to [15] for an extensive account of the subject in the setting of commutative probability spaces. The investigation of distributional symmetries was initiated by de Finetti's celebrated theorem, which shows that sequences of two-point valued exchangeable random variables are conditionally independent and identically distributed. Phrased differently, any finite joint distribution of them is obtained by randomization of the binomial distribution. This result has since found several generalizations. To name but one of these, the probability measures on the Tychonov product of compact Hausdorff spaces which are invariant under the action of finite permutations are in fact mixtures of product measures, as proved by Hewitt and Savage in [14].

Now the C^*-algebraic counterpart of Tychonov products is provided by the theory of tensor products of C^*-algebras. Therefore, it is no wonder that the earliest non-commutative settings for the generalizations of de Finetti's theorem came from the infinite tensor products of a given unital C^*-algebra. In [20], Størmer carried out a thorough analysis of all permutation-invariant states of the (minimal) infinite tensor product $\otimes^{\mathbb{N}}\mathfrak{A}$ of an assigned C^*-algebra $\mathfrak{A}$. Among the main results obtained in that paper, it is worthwhile to mention that the extreme points of the (weakly-$*$) compact convex set of such states may be identified with infinite product states of a single state on $\mathfrak{A}$. Furthermore, the convex set in question is actually a Choquet simplex, which allows for a decomposition of any invariant state into an integral of extreme invariant states with respect to a unique barycentric measure. To our knowledge, though, it was not until the early 90s that this line of research got a new lease of life, when far more emphasis was laid on the probabilistic interpretation. In this respect, Accardi and Lu proved a general non-commutative version of the Hewitt and Savage theorem, [2]. In a later paper, [1], connections between exchangeability and singleton conditions were also established. Not long after, Köstler obtained a non-commutative de Finetti theorem within the formalism of von Neumann algebras in [16], where exchangeability is seen to imply independence with respect to the tail algebra, although the converse may fail to hold, as remarked by the author himself. Finally, also motivated by the key role played in physics by the canonical anti-commutation rules, Crismale and Fidaleo provided a version of the theorem for states right on the CAR algebra, [8]. Although the CAR algebra is isomorphic with the UHF algebra of type 2^∞ and is thus an infinite tensor product of $\mathbb{M}_2(\mathbb{C})$ with itself, the de Finetti theorem proved in the last mentioned paper cannot be reached by an application of the results in [20], not least because the action of the permutations is not the same as the one considered by Størmer. In fact, the results obtained there take into account the canonical $\mathbb{Z}_2$-grading of the CAR algebra as well. In particular, any symmetric state turns out to be even, namely grading-invariant. Furthermore, extreme symmetric states feature the same properties as in the work of Stormer. The novelty, however, is that the product must be intended in the sense of Araki and Moriya, [4], and the factor state must be an even state on $\mathbb{M}_2(\mathbb{C})$, thought of as a graded C^*-algebra with even (odd) part given by diagonal (anti-diagonal) matrices, for the product state to even make sense. Unlike what happens with usual tensor products, the action of $\mathbb{P}_\mathbb{N}$, the group of finite permutations on $\mathbb{N}$, on the CAR algebra is no longer asymptotically abelian. Nevertheless, the corresponding C^*-dynamical system is $\mathbb{P}_\mathbb{N}$-abelian, see [18] for the definition. It is ultimately this circumstance which guarantees that the set of symmetric states is still a Choquet simplex.

This paper in part aims to resume the analysis carried out in [8] in order to frame it in the broader scope of quasi-local algebras, the interest in which is undoubtedly justified by the many appearences they make in quantum field theory and statistical mechanics. In the present work, however, quasi-local algebras are mainly thought of as a source of examples of $\mathbb{Z}_2$-graded C^*-algebras. In particular, in Section 2 we first single out actions of $\mathbb{P}_\mathbb{N}$ which are fully compatible with the local structure of the algebras addressed, see Definition 2.3. More in detail, Proposition 2.4 shows that any such action is $\mathbb{P}_\mathbb{N}$-abelian. Moreover, its invariant states are automatically even, with extreme states being weakly clustering. These are then shown to be strongly clustering in Theorem 2.6. Tail algebras of invariant states are then given a good deal of attention. In Proposition 2.9 we show that the tail algebra of an extreme invariant state is always trivial. Tail algebras corresponding to non-extreme invariant states, too, can be analized in full detail. In the first place, their structure is disciplined by a form of the Hewitt and Savage theorem, in so far as they coincide with the $\mathbb{P}_\mathbb{N}$-invariant part of the center of the von Neumann algebra generated by the given state. As a consequence, they are always abelian and decompose into a direct integral of ergodic components, as proved in Proposition 2.10. The section ends with Proposition 2.12 and Proposition 2.13, which provide de Finetti-type theorems for nets of local algebras. In particular, under the assumption of additivity of the net, Proposition 2.13 characterizes symmetry of states in terms of a condition reminiscent of identical distribution, and conditional independence of the local algebras with respect to the conditional expectation onto the tail algebra. Section 3 is devoted to infinite $\mathbb{Z}_2$-graded tensor products as distinguished and particularly well-behaved instances of quasi-local algebras. After providing a quick exposition of infinite graded tensor products, we show in Example 3.2 how the CAR algebra can be recovered as a suitable infinite product of this type. The group of finite permutations acts in a natural way on infinite graded tensors products. Invariant states for this action lend themselves to a more accomplished description as opposed to the case of quasi-local algebras. In particular, extreme states can be identified with infinite products of a single even state, Proposition 3.5. Moreover, as shown in Proposition 3.6, the action also turns out to be weakly ergodic when it is the minimal product to be dealt with. Finally, infinite graded tensor products offer quite a natural setting to state a fully-fledged version of de Finetti's theorem, for in this case invariant states correspond to exchangeable quantum stochastic processes, see also [9, 10]. This is done in Theorem 3.8, where such processes are characterized in terms of identical distribution and conditional independence.

In Section 4 we further develop the analysis of infinite product states by showing pureness (factoriality) when each factor is pure (factorial), Proposition 4.6 (Proposition 4.10). The proof of these results heavily relies on a twisted version of the well-known tensor product commutation theorem, which is obtained in Theorem 4.4 for the tensor product of two graded von Neumann algebras, and in Theorem 4.5 for the tensor product of infinitely many von Neumann algebras. Finally, the analysis by Størmer in [20] on the type of factor one can obtain from the GNS representation of product states applies to the present framework, and the relative results are gathered in Proposition 4.12.

2. Symmetric states on quasi-local algebras

By a $\mathbb{Z}_2$-graded C^*-algebra we mean a pair $(\mathfrak{A}, \theta)$ made up of a (unital) C^*-algebra and a (unital) $*$-automorphism θ which is involutive, namely $\theta^2 = \mathrm{id}_{\mathfrak{A}}$. Setting $\mathfrak{A}_1 := \{a \in \mathfrak{A} : \theta(a) = a\}$ and $\mathfrak{A}_{-1} := \{a \in \mathfrak{A} : \theta(a) = -a\}$, one easily sees that $\mathfrak{A}$ decomposes as

$$\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_{-1} \,,$$

where the direct sum is topological, and

$$(\mathfrak{A}_i)^* = (\mathfrak{A}^*)_i \,,\; \mathfrak{A}_i\mathfrak{A}_j \subset \mathfrak{A}_{ij} \,, \quad i, j = 1, -1 \,.$$

Note that $\mathfrak{A}_1$ is a (unital) C^*-subalgebra of $\mathfrak{A}$, while $\mathfrak{A}_{-1}$ is only an involutive closed subspace of $\mathfrak{A}$. The subspaces $\mathfrak{A}_i$, $i = 1, -1$ are often referred to as the homogeneous components of $\mathfrak{A}$, and correspondingly any element of $\mathfrak{A}_i$ is called a homogeneous element of $\mathfrak{A}$. For any homogeneous element $x \in \mathfrak{A}_{\pm 1}$ we denote its *grade* by

$$\partial(x) = \pm 1.$$

It is easy to see that considering an involutive $*$-automorphism θ on $\mathfrak{A}$ amounts to assigning a decomposition of $\mathfrak{A}$ into a topological direct sum as above. Indeed, if one is given such a decomposition, then the corresponding automorphism θ can be defined as

$$\theta\lceil_{\mathfrak{A}_1} := \mathrm{id}_{\mathfrak{A}_1} \,, \quad \theta\lceil_{\mathfrak{A}_{-1}} := -\mathrm{id}_{\mathfrak{A}_{-1}} \,.$$

Note that

$$\varepsilon_\theta := \frac{1}{2}(\mathrm{id}_{\mathfrak{A}} + \theta) \,,$$

defines a faithful conditional expectation onto $\mathfrak{A}_1$. When there is no risk of confusion, we will suppress the underscript from ε_θ and simply write ε. The $*$-subalgebra $\mathfrak{A}_+ := \mathfrak{A}_1$ and the subspace $\mathfrak{A}_- := \mathfrak{A}_{-1}$ are commonly referred to as the *even part* and the *odd part* of $\mathfrak{A}$, respectively. Clearly, any $a \in \mathfrak{A}$ can be written as a sum $a = a_+ + a_-$, with $a_+ \in \mathfrak{A}_+$, $a_- \in \mathfrak{A}_-$, and this decomposition is unique. Taking $\theta = \mathrm{id}_{\mathfrak{A}}$, one sees that any $*$-algebra $\mathfrak{A}$ is equipped with a $\mathbb{Z}_2$ trivial grading. Here, $\mathfrak{A}_+ = \mathfrak{A}$ and $\mathfrak{A}_- = \{0\}$.

A simple example of $\mathbb{Z}_2$-graded $*$-algebra is obtained by taking a Hilbert space $\mathcal{H}$, and a bounded self-adjoint unitary U on $\mathcal{H}$. The adjoint action $\mathrm{ad}_U(\cdot) := U \cdot U^*$ is an involutive $*$-automorphism which induces a $\mathbb{Z}_2$-grading on $\mathcal{B}(\mathcal{H})$.

Let $\big(\mathfrak{A}_i, \theta_i\big)$, $i = 1, 2$, be two $\mathbb{Z}_2$-graded $*$-algebras. The map $T : \mathfrak{A}_1 \to \mathfrak{A}_2$ is said to be *even* if it is grading-equivariant, *i.e.*

$$T \circ \theta_1 = \theta_2 \circ T \,.$$

When $\theta_2 = \mathrm{id}_{\mathfrak{A}_2}$, the map $T : \mathfrak{A}_1 \to \mathfrak{A}_2$ is even if and only if it is grading-invariant, that is $T \circ \theta_1 = T$. If T is $\mathbb{Z}_2$-linear, then it is even if and only if $T\lceil_{\mathfrak{A}_{1,-}} = 0$. When $\big(\mathfrak{A}_2, \theta_2\big) = \big(\mathbb{C}, \mathrm{id}_{\mathbb{C}}\big)$, a functional $f : \mathfrak{A}_1 \to \mathbb{C}$ is even if and only if $f \circ \theta = f$.

In the sequel, we will denote by $\mathcal{S}_+(\mathfrak{A})$ the weakly-$*$ compact convex subset of all even states. Even states play a role in giving a $\mathbb{Z}_2$-grading to their GNS structures. More in detail, suppose that $(\mathfrak{A}, \theta)$ is a $\mathbb{Z}_2$-graded C^*-algebra, and $\varphi \in \mathcal{S}_+(\mathfrak{A})$. Let $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi, V_{\theta,\varphi})$ be the GNS covariant representation of φ, where the unitary self-adjoint $V_{\theta,\varphi}$ fixes ξ_φ and verifies

$$\pi_\varphi(\theta(a)) = V_{\theta,\varphi} \pi_\varphi(a) V_{\theta,\varphi}\,, \quad a \in \mathfrak{A}\,.$$

Then, $(\mathcal{B}(\mathcal{H}), \mathrm{ad}_{V_{\theta,\varphi}})$ is a $\mathbb{Z}_2$-graded C^*-algebra. If φ is a pure state, though, evenness is no longer necessary for a unitary on $\mathcal{H}_\varphi$ implementing the grading to exist. In fact, all is needed is that π_φ and $\pi_{\varphi\circ\theta}$ are not disjoint representations[1]. More precisely, one has the following.

Proposition 2.1. *Let $\varphi \in \mathcal{S}_+(\mathfrak{A})$ be a pure state such that π_φ and $\pi_{\varphi\circ\theta}$ are not disjoint. Then there exists a self-adjoint unitary $U \in \pi_\varphi(\mathfrak{A}_+)''$ such that*

$$U\pi_\varphi(a)U^* = \pi_\varphi(\theta(a)),\ a \in \mathfrak{A} \quad \text{and } \langle U\xi_\varphi, \xi_\varphi\rangle \geq 0\,.$$

Proof. Same proof as Lemma 3.1 in [4]. □

We can now move on to consider quasi-local algebras as notable examples of $\mathbb{Z}_2$-graded C^*-algebra. To this aim, denote by $\mathcal{P}_0(\mathbb{N})$ the set of all finite subsets of $\mathbb{N}$.

Definition 2.2. By a quasi-local algebra over $\mathcal{P}_0(\mathbb{N})$ we mean a unital $\mathbb{Z}_2$-graded C^*-algebra $(\mathfrak{A}, \theta)$, where $\mathfrak{A}$ is the inductive limit of a net $\{\mathfrak{A}(I) : I \in \mathcal{P}_0(\mathbb{N})\}$ of local unital C^*-subalgebras $\mathfrak{A}(I) \subset \mathfrak{A}$ such that:

(i) for every $I, J \in \mathcal{P}_0(\mathbb{N})$ with $I \subset J$, one has $\mathfrak{A}(I) \subset \mathfrak{A}(J)$;
(ii) for every $I \in \mathcal{P}_0(\mathbb{N})$, one has $\theta(\mathfrak{A}(I)) = \mathfrak{A}(I)$;
(iii) for every $I, J \in \mathcal{P}_0(\mathbb{N})$ with $I \cap J = \emptyset$, and homogeneous $x \in \mathfrak{A}(I)$ and $y \in \mathfrak{A}(J)$, x and y commute when one of them is even, and anticommute when they are both odd.

[1]This means that there exists a non-null intertwining operator T, *i.e.* a $0 \neq T \in \mathcal{B}(\mathcal{H}_\varphi, \mathcal{H}_{\varphi\circ\theta})$ such that $T\pi_\varphi(a) = \pi_{\varphi\circ\theta}(a)T$ for all $a \in \mathfrak{A}$.

We should mention that the net of local algebras can of course be indexed by more general sets than $\mathcal{P}_0(\mathbb{N})$, see *e.g.* [6]. However, the choice of $\mathcal{P}_0(\mathbb{N})$ made here is the most appropriate insofar as we want our quasi local-algebra to be acted upon by $\mathbb{P}_\mathbb{N}$, the group of finite permutations of $\mathbb{N}$. More precisely, throughout this section we will be focusing on local actions of $\mathbb{P}_\mathbb{N}$ on $\mathfrak{A}$, as defined below.

Definition 2.3. A local action of $\mathbb{P}_\mathbb{N}$ on a quasi-local C^*-algebra $\mathfrak{A}$ is a group homomorphism $\alpha : \mathbb{P}_\mathbb{N} \to \mathrm{Aut}(\mathfrak{A})$ such that

(i) the action is grading-equivariant, that is $\alpha_\sigma \circ \theta = \theta \circ \alpha_\sigma$, for every permutation $\sigma \in \mathbb{P}_\mathbb{N}$;
(ii) for every finite subset $I \subset \mathbb{N}$ and $\sigma \in \mathbb{P}_\mathbb{N}$, one has $\alpha_\sigma(\mathfrak{A}(I)) = \mathfrak{A}(\sigma(I))$.

We next show that the states of $\mathfrak{A}$ which are invariant under such an action of $\mathbb{P}_\mathbb{N}$ enjoy good properties. First, they are automatically even. Second, they are weakly (and in fact strongly) clustering as soon as they are extreme. These properties are proved in the propositions below. Before stating them, though, some notation and definitions need to be set first.
A state ω on $\mathfrak{A}$ is invariant under α, or equivalently α-invariant, if $\omega \circ \alpha_\sigma = \omega$, for every $\sigma \in \mathbb{P}_\mathbb{N}$. The set of all α-invariant states, which we denote by $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A})$, is weakly-$*$ compact and convex. Its extreme states are called the *ergodic* states for the action of $\mathbb{P}_\mathbb{N}$. The set of all invariant extreme states will be denoted by $\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))$. We will be using the terms invariant states and symmetric states interchangeably throught the paper.
If now $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$ is the GNS triple associated with a given state ω in $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A})$, the action of every α_σ can be implemented on the Hilbert space $\mathcal{H}_\omega$ by a unitary U^ω_σ uniquely determined by

$$U^\omega_\sigma \pi_\omega(a)\xi_\omega := \pi_\omega(\alpha_\sigma(a))\xi_\omega, \quad a \in \mathfrak{A}\,.$$

We denote by $\mathcal{H}^{\mathbb{P}_\mathbb{N}}_\omega \subset \mathcal{H}_\omega$ the closed subspace of all invariant vectors under the action of the unitaries U^ω_σ, namely

$$\mathcal{H}^{\mathbb{P}_\mathbb{N}}_\omega := \{\xi \in \mathcal{H}_\omega : U^\omega_\sigma \xi = \xi, \text{ for all}\, \sigma \in \mathbb{P}_\mathbb{N}\}\,.$$

The orthogonal projection onto $\mathcal{H}^{\mathbb{P}_\mathbb{N}}_\omega$ is denoted by E_ω. As is clear, the one-dimensional subspace $\mathbb{C}\xi_\omega$ is contained in $\mathcal{H}^{\mathbb{P}_\mathbb{N}}_\omega$, which means E_ω is never 0. As is known from the general theory of group actions through automorphisms on C^*-algebras, the condition that $\mathcal{H}^{\mathbb{P}_\mathbb{N}}_\omega$ reduces to $\mathbb{C}\xi_\omega$ implies that ω is extreme in $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A})$, see *e.g.* [18, Proposition 3.1.10]. The reverse implication may well fail to hold for a given action of a given group G on a general C^*-algebra $\mathfrak{A}$. However, it does hold provided that the system $(\mathfrak{A}, G, \alpha)$ is what is known as a G-abelian dynamical system. This is by definition the case when, for every G-invariant state ω, the set $E_\omega \pi_\omega(\mathfrak{A})E_\omega$ is an abelian family of operators acting on $\mathcal{H}_\omega$.

Among other things, we next show that any local action of $\mathbb{P}_\mathbb{N}$ on a quasi-local C^*-algebra is always $\mathbb{P}_\mathbb{N}$-abelian.

For every natural n, denote by $\mathbb{P}_n \subset \mathbb{P}_\mathbb{N}$ the finite subgroup of permutations which act trivially from $n+1$ onwards. Note that $\mathbb{P}_\mathbb{N} = \bigcup_n \mathbb{P}_n$. We adopt the notation in [8] and define the Cesàro average of an arbitrary operator-valued function $f : \mathbb{P}_\mathbb{N} \to \mathcal{B}(\mathcal{H})$ as

$$M(f(\sigma)) := \lim_{n\to\infty} \frac{1}{n!} \sum_{\sigma\in\mathbb{P}_n} f(\sigma)$$

as long as the limit exists in a suitable sense (for instance in the strong/weak operator topology). We recall that for any ω in $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A})$ and $\sigma \in \mathbb{P}_\mathbb{N}$ one has $M(U^\omega_\sigma) = E_\omega$, where the equality is understood in the strong operator topology, see [8, Proposition 3.1].

Proposition 2.4. *Let α be a local action of $\mathbb{P}_\mathbb{N}$ on a quasi-local algebra $\mathfrak{A}$. If $\omega \in \mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A})$, then*

1. *ω is even;*
2. *$E_\omega\pi_\omega(\mathfrak{A})E_\omega$ is a commuting family of operators, hence the dynamical system is $\mathbb{P}_\mathbb{N}$-abelian;*
3. *$\omega \in \mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))$ if and only if $\dim\mathcal{H}^{\mathbb{P}_\mathbb{N}}_\omega = 1$.*

Proof. As for (1), we need to show that any symmetric state ω vanishes on all odd elements of $\mathfrak{A}$. By density, it is enough to prove that $\omega(a) = 0$ for every a which is a localized odd element, say $a \in \mathfrak{A}(I)$ for some finite subset $I \subset \mathbb{N}$. Denoting by $\{\cdot,\cdot\}$ the anticommutator, for an a as before we have

$$\begin{aligned}
&\{E_\omega\pi_\omega(a)E_\omega, E_\omega\pi_\omega(a^*)E_\omega\}\\
=&M(E_\omega\pi_\omega(a)U^\omega_\sigma\pi_\omega(a^*)E_\omega + E_\omega\pi_\omega(a^*)U^\omega_\sigma\pi_\omega(a)E_\omega)\\
=&M(E_\omega\pi_\omega(\{a, \alpha_\sigma(a^*)\})E_\omega)\\
=&\lim_{n\to\infty} \frac{1}{n!} \sum_{\sigma\in\mathbb{P}_n} E_\omega\pi_\omega(\{a, \alpha_\sigma(a^*)\})E_\omega = 0
\end{aligned}$$

where the last equality holds because for every n such that $I \subset \{1,\dots,n\}$ one has $|\{\sigma \in \mathbb{P}_n : \sigma(I)\cap I \neq \emptyset\}| \leq C(n-1)!$, with C being a constant that does not depend on n, see [8, Lemma 3.3], whereas if σ is such that $\sigma(I) \cap I = \emptyset$ then $\{a, \alpha_\sigma(a^*)\} = 0$ by virtue of (iii) of Definition 2.2. This readily implies that

$$E_\omega\pi_\omega(a)E_\omega = 0 \quad \text{for any odd } a \in \mathfrak{A}\,. \tag{2.1}$$

In particular, for such an a one has

$$\omega(a) = \langle\pi_\omega(a)\xi_\omega, \xi_\omega\rangle = \langle E_\omega\pi_\omega(a)E_\omega\xi_\omega, \xi_\omega\rangle = 0\,,$$

and so (1) is proved.

As for (2), thanks to Equality (2.1) it is enough to verify that the commutator $[E_\omega\pi_\omega(a)E_\omega, E_\omega\pi_\omega(a)E_\omega]$ is 0 for even $a, b \in \mathfrak{A}$, which can

be seen with similar computations to those in (1).
Property (3) holds thanks to Proposition 3.1.12 in [18]. □

For any fixed integer $n \geq 1$, we denote by σ_n the permutation acting on $\mathbb{N}$ as

$$\sigma_n(k) = \begin{cases} k + 2^{n-1}, & 1 \leq k \leq 2^{n-1} \\ k - 2^{n-1}, & 2^{n-1} < k \leq 2^n \\ k, & k > 2^n \end{cases} \tag{2.2}$$

The next result is key to further characterize extreme symmetric states.

Lemma 2.5. *If $\omega \in \mathcal{S}(\mathfrak{A})$ is an extreme symmetric state with respect to a local action α of $\mathbb{P}_\mathbb{N}$ on a quasi-local algebra $\mathfrak{A}$, then for every $a \in \mathfrak{A}$ one has*

$$\lim_{n\to\infty} \pi_\omega(\alpha_{\sigma_n}(a))\xi_\omega = \omega(a)\xi_\omega,$$

in the weak operator topology.

Proof. The proof is the same as in Lemma 5.2 in [8]. The only thing that needs to be taken care of is that if $a \in \mathfrak{A}(I)$ for some finite subset $I \subset \mathbb{N}$ and $\sigma \in \mathbb{P}_\mathbb{N}$, there exists $N_{\sigma,a} \in \mathbb{N}$ such that $\alpha_{\sigma\sigma_n}(a) = \alpha_{\sigma_n}(a)$ for every $n \geq N_{A,\sigma}$. To this end, let $r, s \in \mathbb{N}$ such that $I \subset \{1, \dots, r\}$ and the restriction of σ to $\{n \in \mathbb{N} : n \geq s\}$ is the identity. Set $N_{\sigma,a} := \max\{r, s\}$. For $n \geq N_{\sigma,a}$, we have $\alpha_{\sigma\sigma_n}(a) = \alpha_\sigma(\alpha_{\sigma_n}(a)) = \alpha_{\sigma_n}(a)$ because α_σ acts as the identity on each $\mathfrak{A}(J)$ if $J \cap \{1, \dots s-1\} = \emptyset$. □

Before stating the announced characterization, we recall that a symmetric state ω is strongly clustering (or mixing) if for every $a, b \in \mathfrak{A}$ one has $\lim_n \omega(\alpha_{\sigma_n}(a)b) = \omega(a)\omega(b)$, *cf.* [20].

Theorem 2.6. *For a symmetric state ω on a quasi-local algebra $\mathfrak{A}$ acted upon $\mathbb{P}_\mathbb{N}$ through a local action α the following conditions are equivalent:*

1. *ω is extreme;*
2. *ω is strongly clustering;*
3. *$\omega(ab) = \omega(a)\omega(b)$ for every $a \in \mathfrak{A}(I)$ and $b \in \mathfrak{A}(J)$ and finite subsets $I, J \subset \mathbb{N}$ such that $I \cap J = \emptyset$.*

Proof. The equivalence (1) ⇔ (2) can be proved exactly as is done in [8, Theorem 5.3]. The implication (3) ⇒ (2) is obvious, so it remains to show that (2) ⇒ (3). To this aim, consider $\sigma \in \mathbb{P}_\mathbb{N}$ such that σ is the identity on I and coincides with σ_m on J, where σ_m is the permutation defined in (2.2). We have

$$\omega(ab) = \omega(\alpha_\sigma(ab)) = \omega(a\alpha_{\sigma_m}(b)) = \lim_m \omega(a\alpha_{\sigma_m}(b)) = \omega(a)\omega(b)\,,$$

where in the second-last equality we have used that $\{\omega(a\alpha_{\sigma_m}(b))\}_{m\in\mathbb{N}}$ is actually a constant sequence. □

The next result shows that $\mathbb{P}_\mathbb{N}$ is represented by a large group of automorphisms in the sense of [19] whenever it acts on a quasi-local C^*-algebra as in Definition 2.3. This means that for any invariant state $\omega \in \mathcal{S}(\mathfrak{A})$ and any self-adjoint $a \in \mathfrak{A}$ one has

$$\overline{\mathrm{conv}}\{\pi_\omega(\alpha_\sigma(a)) : \sigma \in \mathbb{P}_\mathbb{N}\} \cap \pi_\omega(\mathfrak{A})' \neq \emptyset\,.$$

Proposition 2.7. *Any local action α of $\mathbb{P}_\mathbb{N}$ on a quasi local C^*-algebra $\mathfrak{A}$ is a large group of automorphisms.*

Proof. The proof can be done as in [8, Theorem 4.2] once we have first established asymptotic abelianness in average of any symmetric state. More explicitly, we need to show that if ω is a symmetric state on $\mathfrak{A}$, then $M\{\omega(c[\alpha_\sigma(a), b]d)\} = 0$ for every $a, b, c, d \in \mathfrak{A}$.
We start by observing that

$$\begin{aligned} M\{\omega(c\alpha_\sigma(a)bd)\} =& M\{\omega(\alpha_\sigma(a_+)cbd)\} \\ &+ M\{\omega(\alpha_\sigma(a_-)(c_+ - c_-)bd)\} \end{aligned}$$

as follows by applying [8, Lemma 3.3]. Now the second summand in the right-hand side of the equality above is 0 since $E_\omega\pi_\omega(a_-)E_\omega = 0$ thanks to (2.1). By $\mathbb{P}_\mathbb{N}$-abelianness we then have

$$\begin{aligned} M\{\omega(c\alpha_\sigma(a)bd)\} &= M\{\omega(\alpha_\sigma(a_+)cbd)\} \\ &= \langle \pi_\omega(a_+)E_\omega\pi_\omega(cbd)\xi_\omega, \xi_\omega\rangle \\ &= \langle \pi_\omega(cbd)E_\omega\pi_\omega(a_+)\xi_\omega, \xi_\omega\rangle \\ &= M\{\omega(cbd\alpha_\sigma(a_+))\} \\ &= M\{\omega(cb\alpha_\sigma(a_+)d)\}\,, \end{aligned}$$

where the last equality is due again to [8, Lemma 3.3]. Now, arguing as above, one easily sees that $M\{\omega(cb\alpha_\sigma(a_-)d)\} = 0$, which ends the proof. □

As the dynamical system $(\mathfrak{A}, \mathbb{P}_\mathbb{N})$ is $\mathbb{P}_\mathbb{N}$-abelian, by [18, Theorem 3.1.14], one has that the set of symmetric states is indeed a Choquet simplex. This means that any $\mathbb{P}_\mathbb{N}$-invariant state is the barycenter of a unique probability measure which is pseudo-supported on the set of extreme states, see [6], page 322. More in detail, we have

Proposition 2.8. *Let α be a local action of $\mathbb{P}_\mathbb{N}$ on a quasi-local C^*-algebra $\mathfrak{A}$. If $\omega \in \mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A})$, then there exists a unique probability measure μ pseudo-supported on $\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))$ such that*

$$\omega(a) = \int_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \psi(a)\mathrm{d}\mu(\psi)\,, \quad a \in \mathfrak{A}\,. \tag{2.3}$$

We recall that with any state ω on a quasi-local algebra $\mathfrak{A}$ it is possible to associate a von Neumann algebra $\mathfrak{Z}_\omega^\perp \subset \mathcal{B}(\mathcal{H}_\omega)$ defined as

$$\mathfrak{Z}_\omega^\perp = \bigcap_{n=1}^\infty \bigvee_{I \in \mathcal{F}_n} \pi_\omega(\mathfrak{A}(I))''\,,$$

where $\mathcal{F}_n$ collects all the finite subsets $I \subset \mathbb{N}$ such that $I \subset \{n, n+1, \dots\}$. This algebra is commonly known as the tail algebra of the state ω, although in quantum statistical mechanics is typically referred to as the algebra at infinity, see also [6, Definition 2.6.4]. The tail algebra of an ergodic symmetric state is shown to be trivial below.

Proposition 2.9. *Let α be a local action of $\mathbb{P}_\mathbb{N}$ on a quasi-local C^*-algebra $\mathfrak{A}$. The tail algebra $\mathfrak{Z}_\omega^\perp$ of any ω in $\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))$ is trivial.*

Proof. We first show that $\mathfrak{Z}_\omega^\perp$ is contained in the fixed-point von Neumann algebra $\{T \in \mathcal{B}(\mathcal{H}_\omega) : U_\sigma^\omega T = T U_\sigma^\omega,\ \sigma \in \mathbb{P}_\mathbb{N}\}$, where U_σ^ω is the unitary implementator of α_σ in $\mathcal{H}_\omega$, *i.e.* $U_\sigma^\omega \pi_\omega(a)\xi_\omega = \pi_\omega(\alpha_\sigma(a))\xi_\omega$, $a \in \mathfrak{A}$. Let T be in $\mathfrak{Z}_\omega^\perp$ and $\sigma \in \mathbb{P}_\mathbb{N}$. Then there exists n_o such that σ acts trivially on $\{n_o, n_o+1, \dots,\}$. In particular, $\mathrm{ad}_{U_\sigma^\omega}$ acts trivially on $\pi_\omega(\mathfrak{A}(I))$ for every finite subset I contained in $\{n_o, n_o+1, \dots\}$. As a result, $\mathrm{ad}_{U_\sigma^\omega}$ still acts trivially on $\bigvee_{I\in\mathcal{F}_{n_o}} \pi_\omega(\mathfrak{A}(I))''$. Now since T sits in particular in $\bigvee_{I\in\mathcal{F}_{n_o}} \pi_\omega(\mathfrak{A}(I))''$, we must have $U_\sigma^\omega T = T U_\sigma^\omega$.
Furthermore, by Theorem 2.6.5 in [6] we also have that $\mathfrak{Z}_\omega^\perp$ is contained in in $\pi_\omega(\mathfrak{A})' \cap \pi_\omega(\mathfrak{A})''$. In particular, ξ_ω is separating for $\mathfrak{Z}_\omega^\perp$. Indeed, from $\mathfrak{Z}_\omega^\perp \subset \pi_\omega(\mathfrak{A})'$ we see $\pi_\omega(\mathfrak{A})'' \subset (\mathfrak{Z}_\omega^\perp)'$, hence ξ_ω is cyclic for $(\mathfrak{Z}_\omega^\perp)'$. We are ready to reach the conclusion. Indeed, if T lies in $\mathfrak{Z}_\omega^\perp$, then $T\xi_\omega$ is an invariant vector. By extremality of ω and $\mathbb{P}_\mathbb{N}$-abelianness, we then have $T\xi_\omega = \lambda\xi_\omega$ for some $\lambda \in \mathbb{C}$, which means $T = \lambda 1$ since ξ_ω is separating for such a T. □

The next proposition provides a quantum analogue of the well-known Hewitt and Savage theorem that the tail and the symmetric σ-algebras of an exchangeable sequence of random variables actually coincide, see [14]. For the reader's convenience we recall that a sequence of random variables is exchangeable if the joint distribution of any finite subset of variables is invariant under permutations.

Given ω in $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A})$, we set $Z_{\mathbb{P}_\mathbb{N}}(\omega) := \mathcal{Z}(\pi_\omega(\mathfrak{A})'') \cap U_\omega(\mathbb{P}_\mathbb{N})'$, where

$$U_\omega(\mathbb{P}_\mathbb{N})' := \{T \in \mathcal{B}(\mathcal{H}_\omega) : U_\sigma^\omega T = T U_\sigma^\omega,\ \sigma \in \mathbb{P}_\mathbb{N}\}$$

and $\mathcal{Z}(\pi_\omega(\mathfrak{A})'') := \pi_\omega(\mathfrak{A})'' \cap \pi_\omega(\mathfrak{A})'$ is the center of $\pi_\omega(\mathfrak{A})''$. The vector state $\langle \cdot \xi_\omega, \xi_\omega \rangle$ on $\pi_\omega(\mathfrak{A})''$ will be denoted by φ_{ξ_ω}.

Proposition 2.10. *The tail algebra $\mathfrak{Z}_\omega^\perp$ of a symmetric state $\omega \in \mathcal{S}(\mathfrak{A})$ coincides with $Z_{\mathbb{P}_\mathbb{N}}(\omega)$.*
Moreover, there exists a unique conditional expectation $E_\omega : \pi_\omega(\mathfrak{A})'' \to \mathfrak{Z}_\omega^\perp$. This is given by

$$E_\omega(X) = \int_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))}^{\oplus} \langle X_\psi \xi_\psi, \xi_\psi \rangle \mathrm{d}\mu(\psi), \quad X \in \pi_\omega(\mathfrak{A})'',$$

where μ is the measure appearing in (2.3)*, and* $X = \int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} X_\psi \mathrm{d}\mu(\psi)$*. In addition, E_ω preserves the vector state φ_{ξ_ω}.*

Proof. By applying Theorem 4.4.3 in [6] and Proposition 3.1.10 in [18], we see that the abelian von Neumann algebra $Z_{\mathbb{P}_\mathbb{N}}(\omega)$ decomposes into a direct integral as

$$Z_{\mathbb{P}_\mathbb{N}}(\omega) = \int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \mathbb{C}1_{\mathcal{H}_\psi} \mathrm{d}\mu(\psi) \cong L^\infty(\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A})), \mu)\,.$$

Because the diagonal operators of $\int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \mathcal{H}_\psi \mathrm{d}\mu(\psi)$ are contained in $\pi_\omega(\mathfrak{A})''$, we can apply Lemma 8.4.1 in [12] to find that

$$\pi_\omega(\mathfrak{A})'' = \int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \pi_\psi(\mathfrak{A})'' \mathrm{d}\mu(\psi)\,.$$

The above decomposition enables us to identify the tail algebra. Indeed, by Theorem 4.4.6 in [6] and Proposition 2.9 one has

$$\begin{aligned}
\mathfrak{Z}^\perp_\omega &= \bigcap_{n=1}^{\infty} \bigvee_{I\in\mathcal{F}_n} \pi_\omega(\mathfrak{A}(I))'' = \bigcap_{n=1}^{\infty} \bigvee_{I\in\mathcal{F}_n} \int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \pi_\psi(\mathfrak{A}(I))'' \mathrm{d}\mu(\psi) \\
&= \int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \bigcap_{n=1}^{\infty} \bigvee_{I\in\mathcal{F}_n} \pi_\psi(\mathfrak{A}(I))'' \mathrm{d}\mu(\psi) \\
&= \int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \mathbb{C}1_{\mathcal{H}_\psi} \mathrm{d}\mu(\psi) = Z_{\mathbb{P}_\mathbb{N}}(\omega)\,.
\end{aligned}$$

Since by Proposition 2.7 $\mathbb{P}_\mathbb{N}$ acts as a large group of automorphisms on $\mathfrak{A}$, Theorem 3.1 in [19] applies yielding the existence of a unique conditional expectation, E_ω, from $\pi_\omega(\mathfrak{A})''$ onto $Z_{\mathbb{P}_\mathbb{N}}(\omega) = \mathfrak{Z}^\perp_\omega$.

All is left to do is prove the formula for E_ω. To this end, note that $F(X) := \int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \langle X_\psi \xi_\psi, \xi_\psi\rangle \mathrm{d}\mu(\psi)$, X in $\pi_\omega(\mathfrak{A})''$, defines a conditional expectation of $\pi_\omega(\mathfrak{A})''$ onto $\mathfrak{Z}^\perp_\omega$ as it is the direct integral of states. By uniqueness one sees that $F = E_\omega$. Finally E_ω is seen to preserve the vector state $\langle \cdot\xi_\omega, \xi_\omega\rangle$ by means of simple computations, see also [9, Theorem 5.3]. □

Before we can state our version of de Finetti's theorem tailored to the present context, we need to recall what should be meant by conditional independence for a net of local algebras $\{\mathfrak{A}(I) : I \in \mathcal{P}_0(\mathbb{N})\}$ with respect to a given state ω of the quasi-local algebra $\mathfrak{A}$. We start by recalling that for any such state ω the tail algebra $\mathfrak{Z}^\perp_\omega$ will always be commutative, see [6, Theorem 2.6.5], and thus expected. In other words, there will always exist a conditional expectation $F_\omega : \pi_\omega(\mathfrak{A})'' \to \mathfrak{Z}^\perp_\omega$. As is customary, we will need to work under the hypothesis that such conditional expectation is normal and φ_{ξ_ω}-preserving, that is $\langle F_\omega[X]\xi_\omega, \xi_\omega\rangle = \langle X\xi_\omega, \xi_\omega\rangle$ for any $X \in \pi_\omega(\mathfrak{A})''$.

Definition 2.11. The net $\{\mathfrak{A}(I) : I \in \mathcal{P}_0(\mathbb{N})\}$ of the local algebras is conditionally independent with respect to a conditional expectation F_ω as above if for any $I, J \in \mathcal{P}_0(\mathbb{N})$ with $I \cap J = \emptyset$ we have

$$F_\omega[XY] = F_\omega[X]F_\omega[Y]$$

for every $X \in \pi_\omega(\mathfrak{A}(I))'' \bigvee \mathfrak{Z}_\omega^\perp$ and $Y \in \pi_\omega(\mathfrak{A}(J))'' \bigvee \mathfrak{Z}_\omega^\perp$;

We are now ready to state our result.

Proposition 2.12. *Let α be a local action of $\mathbb{P}_\mathbb{N}$ on a net of local C^*-algebras with quasi-local algebra $\mathfrak{A}$. If $\omega \in \mathcal{S}(\mathfrak{A})$ is symmetric, then $E_\omega \circ \mathrm{ad}_{U_\sigma^\omega} = E_\omega$ for every $\sigma \in \mathbb{P}_\mathbb{N}$. Conversely, $\omega \in \mathcal{S}(\mathfrak{A})$ is symmetric if $F_\omega \circ \mathrm{ad}_{U_\sigma^\omega} = F_\omega$, $\sigma \in \mathbb{P}_\mathbb{N}$, for some normal φ_{ξ_ω}-preserving conditional expectation $F_\omega : \pi_\omega(\mathfrak{A})'' \to \mathfrak{Z}_\omega^\perp$.*
Moreover, in this case the net is conditionally independent with respect to E_ω.

Proof. Suppose that ω is a symmetric state. Then by Proposition 2.10, the tail algebra is given by $\mathfrak{Z}_\omega^\perp = \int^\oplus_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \mathbb{C}1_{\mathcal{H}_\psi} \mathrm{d}\mu(\psi)$ and the unique conditional expectation $E_\omega : \pi_\omega(\mathfrak{A})'' \to \mathfrak{Z}_\omega^\perp$ decomposes as $E_\omega(X) = \int^\oplus_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} \langle X_\psi \xi_\psi, \xi_\psi\rangle \mathrm{d}\mu(\psi)$ for every $X \in \pi_\omega(\mathfrak{A})''$. We observe that for any $\sigma \in \mathbb{P}_\mathbb{N}$ the unitary U_σ^ω decomposes into a direct integral as well. More precisely, $U_\sigma^\omega = \int^\oplus_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\mathfrak{A}))} U_\sigma^\psi \mathrm{d}\mu(\psi)$, where U_σ^ψ is the unitary acting on $\mathcal{H}_\psi$ as $U_\sigma^\psi \pi_\psi(x)\xi_\psi = \pi_\psi(\alpha_\sigma(x))\xi_\psi$, $x \in \mathfrak{A}$. Using this decomposition of U_σ^ω, it is now straightforward to check that $E_\omega \circ \mathrm{ad}_{U_\sigma^\omega} = E_\omega$, for every $\sigma \in \mathbb{P}_\mathbb{N}$.
The converse implication follows by direct computation. Indeed, let $F_\omega : \pi_\omega(\mathfrak{A})'' \to \mathfrak{Z}_\omega^\perp$ be a conditional expectation such that $F_\omega \circ \mathrm{ad}_{U_\sigma^\omega} = F_\omega$ for every $\sigma \in \mathbb{P}_\mathbb{N}$. For $a \in \mathfrak{A}$ and $\sigma \in \mathbb{P}_\mathbb{N}$ one then has

$$\begin{aligned}\omega(\alpha_\sigma(a)) =&\langle \pi_\omega(\alpha_\sigma(a))\xi_\omega, \xi_\omega\rangle = \langle F_\omega \circ \mathrm{ad}_{U_\sigma^\omega}(\pi_\omega(a))\xi_\omega, \xi_\omega\rangle \\ =&\langle F_\omega(\pi_\omega(a))\xi_\omega, \xi_\omega\rangle = \omega(a)\,,\end{aligned}$$

which shows that ω is symmetric, and thus $F_\omega = E_\omega$ thanks to Proposition 2.10.

As for conditional independence, fix now $I_1, I_2 \subset \mathbb{N}$ finite subsets with $I_1 \cap I_2 = \emptyset$, and for $i = 1, 2$ take $X_i \in \pi_\omega(\mathfrak{A}(I_i))'' \bigvee \mathfrak{Z}_\omega^\perp$. We need to show that $E_\omega(X_1X_2) = E_\omega(X_1)E_\omega(X_2)$. To this end, we start by considering two localized elements, that is $X_i \in \pi_\omega(\mathfrak{A}(I_i))$, $i = 1, 2$, with $I_1 \cap I_2 = \emptyset$. In this case, by using Proposition 2.10 and (3) in the statement of Theorem 2.6 we have

$$
\begin{aligned}
&E_\omega(X_1X_2)\\
&=\int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_{\mathbb{N}}}(\mathfrak{A}))}\langle X_{1,\psi}X_{2,\psi}\,\xi_\psi,\xi_\psi\rangle 1_{\mathcal{H}_\psi}\mathrm{d}\mu(\psi)\\
&=\int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_{\mathbb{N}}}(\mathfrak{A}))}\langle X_{1,\psi}\,\xi_\psi,\xi_\psi\rangle\langle X_{2,\psi}\,\xi_\psi,\xi_\psi\rangle 1_{\mathcal{H}_\omega}\mathrm{d}\mu(\psi)\\
&=\int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_{\mathbb{N}}}(\mathfrak{A}))}\langle X_{1,\psi}\,\xi_\psi,\xi_\psi\rangle 1_{\mathcal{H}_\omega}\mathrm{d}\mu(\psi)\int^{\oplus}_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_{\mathbb{N}}}(\mathfrak{A}))}\langle X_{2,\psi}\,\xi_\psi,\xi_\psi\rangle 1_{\mathcal{H}_\omega}\mathrm{d}\mu(\psi)\\
&=E_\omega(X_1)E_\omega(X_2)\,.
\end{aligned}
$$

Since E_ω is a normal conditional expectation, by density the above equality still holds for $X_i\in\pi_\omega(\mathfrak{A}(I_i))''$, $i=1,2$.
We are now ready to deal with the general case. As $\mathfrak{Z}^\perp_\omega$ is contained in the center of $\pi_\omega(\mathfrak{A})''$, we may assume that X_i, $i=1,2$, is of the form $X_i=\sum_{j\in F}T^i_jC^i_j$, where F is a finite set, $\{T^i_j : j\in F\}\subset\pi_\omega(\mathfrak{A}(I_i))''$ and $\{C^i_j : j\in F\}\subset\mathfrak{Z}^\perp_\omega$ for $i=1,2$. Since $X_1X_2=\sum_{j,l\in F}T^1_jT^2_lC^1_jC^2_l$, we find

$$
\begin{aligned}
E_\omega(X_1X_2)&=\sum_{j,l\in F}E_\omega(T^1_jT^2_l)C^1_jC^2_l=\sum_{j,l\in F}E_\omega(T^1_j)E_\omega(T^2_l)C^1_jC^2_l\\
&=\sum_{j\in F}E_\omega(T^1_j)C^1_j\sum_{l\in F}E_\omega(T^2_l)C^2_l=E_\omega(X_1)E_\omega(X_2)
\end{aligned}
$$

and the proof is complete. □

If more assumptions are made on the structure of the net of the local algebras, the global invariance condition $E_\omega\circ\mathrm{ad}_{U^\omega_\sigma}=E_\omega$ can be recast in a seemingly weaker way. To this end, from now on we will assume that the net $\{\mathfrak{A}(I) : I\in\mathcal{P}_0(\mathbb{N})\}$ is *additive*[2], namely that $\mathfrak{A}(I\cup J)=C^*(\mathfrak{A}(I),\mathfrak{A}(J))$ for every $I,J\in\mathcal{P}_0(\mathbb{N})$. In particular, for any finite subset $I\subset\mathbb{N}$ we have $\mathfrak{A}(I)=C^*(\mathfrak{A}(\{i\}) : i\in I)$.
In the present context Theorem 2.12 can be stated as follows.

Proposition 2.13. *Let α be a local action of $\mathbb{P}_\mathbb{N}$ on an additive net of local C^*-algebras. A state $\omega\in\mathcal{S}(\mathfrak{A})$ on the quasi-local algebra $\mathfrak{A}$ is symmetric if and only if:*

(i) *the local algebras are conditionally independent w.r.t. E_ω;*
(ii) *for every $i\in\mathbb{N}$, $E_\omega[\pi_\omega(\alpha_\sigma(a))]=E_\omega[\pi_\omega(a)]$, $a\in\mathfrak{A}(\{i\}),\sigma\in\mathbb{P}_\mathbb{N}$.*

Proof. By virtue of Theorem 2.12 we need only show that (i) and (ii) imply that $E_\omega\circ\mathrm{ad}_{U^\omega_\sigma}=E_\omega$.
Since E_ω is a normal conditional expectation, by density of $\pi_\omega(\mathfrak{A})$ in

[2]The terminology is borrowed from algebraic quantum field theory, see Definition 4.13 in [3].

the bicommutant $\pi_\omega(\mathfrak{A})''$, it is enough to verify the equality on $\pi_\omega(\mathfrak{A})$. As $\mathfrak{A}$ is in turn the inductive limit of the local algebras, the thesis will be achieved if we show that for any fixed finite subset $I \subset \mathbb{N}$ one has $E_\omega[\alpha_\sigma(\pi_\omega(a))] = E_\omega[\pi_\omega(a)]$ for every $a \in \mathfrak{A}(I)$. By additivity there is no loss of generality to assume that a factors into a product as $a = a_{i_1}a_{i_2}\cdots a_{i_n}$, with $i_l \neq i_k$ (possible repetitions of the same index at different places are dealt with by means of (iii) in Definition 2.2). For any $\sigma \in \mathbb{P}_\mathbb{N}$ we then have:

$$\begin{aligned}
&E_\omega[\pi_\omega(a_{i_1}a_{i_2}\cdots a_{i_n})] = E_\omega[\pi_\omega(a_{i_1})]E_\omega[\pi_\omega(a_{i_2})]\cdots E_\omega[\pi_\omega(a_{i_n})]\\
&= E_\omega[\pi_\omega(\alpha_\sigma(a_{i_1}))]E_\omega[\pi_\omega(\alpha_\sigma(a_{i_2}))]\cdots E_\omega[\pi_\omega(\alpha_\sigma(a_{i_n}))]\\
&= E_\omega[\pi_\omega(\alpha_\sigma(a_{i_1}a_{i_2}\cdots a_{i_n})]\,,
\end{aligned}$$

which ends the proof. □

It is worth noting that when the quasi-local algebra arises as a quotient of the infinite free product $*_\mathbb{N}\mathfrak{B}$ of a sample C^*-algebra $\mathfrak{B}$, the conditions (i) and (ii) in the statement above return the usual notion for a (quantum) stochastic process to be conditionally independent and identically distributed with respect to the tail algebra. Indeed, in this case the states of the quotient are in a one-to-one correspondence with the stochastic processes on the sample algebra $\mathfrak{B}$, see *e.g.* Theorem 3.4 and Definition 4.1 in [9] or Theorem 2.3 in [10].

3. Processes on infinite graded tensor products

In this section, we collect some results on $\mathbb{Z}_2$-graded algebraic structures obtained as tensor products of graded $*$-algebras. Consider the C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and denote by $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ the algebraic tensor product $\mathfrak{A}_1 \odot \mathfrak{A}_2$ with the product and involution given by

$$(a_1\otimes a_2)(a_1'\otimes a_2') := a_1a_1'\otimes a_2a_2'\,, \quad (a_1\otimes a_2)^* := a_1^*\otimes a_2^*\,,$$

for all $a_1, a_1' \in \mathfrak{A}_1$, $a_2, a_2' \in \mathfrak{A}_2$. Let us denote by $\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2$ and $\mathfrak{A}_1 \otimes_{\min} \mathfrak{A}_2$ the completions of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ with respect to the maximal and minimal C^*-cross norm, respectively, see [21].
If one takes $\omega_1 \in \mathcal{S}(\mathfrak{A}_1)$ and $\omega_2 \in \mathcal{S}(\mathfrak{A}_2)$, their product state $\psi_{\omega_1,\omega_2} \in \mathcal{S}(\mathfrak{A}_1 \otimes_{\min} \mathfrak{A}_2)$ is well defined also on $\mathfrak{A}_1 \otimes_{\max} \mathfrak{A}_2$, and consequently the notation $\psi_{\omega_1,\omega_2} \in \mathcal{S}(\mathfrak{A}_1 \otimes \mathfrak{A}_2)$ will be used in the sequel.

Suppose that $(\mathfrak{A}_1, \theta_1)$ and $(\mathfrak{A}_2, \theta_2)$ are $\mathbb{Z}_2$-graded $*$-algebras, and consider the linear space $\mathfrak{A}_1 \odot \mathfrak{A}_2$. In what follows, we recall the definition of the involutive $\mathbb{Z}_2$-graded tensor product, which will be henceforth denoted by $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2$. For homogeneous elements $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$ and

$i, j \in \mathbb{Z}_2$, we set

$$\varepsilon(a_1, a_2) := \begin{cases} -1 & \text{if } \partial(a_1) = \partial(a_2) = -1\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

$$\epsilon(i, j) := \begin{cases} -1 & \text{if } i = j = -1\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Given $x, y \in \mathfrak{A}_1 \odot \mathfrak{A}_2$ with

$$x := \oplus_{i,j\in\mathbb{Z}_2} x_{i,j} \in \oplus_{i,j\in\mathbb{Z}_2} (\mathfrak{A}_{1,i} \odot \mathfrak{A}_{2,j})\,,$$
$$y := \oplus_{i,j\in\mathbb{Z}_2} y_{i,j} \in \oplus_{i,j\in\mathbb{Z}_2} (\mathfrak{A}_{1,i} \odot \mathfrak{A}_{2,j})\,,$$

the involution, which by a minor abuse of notation we continue to denote by *, and the multiplication on $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2$ are defined as (see also *e.g.* [7])

$$x^* := \sum_{i,j\in\mathbb{Z}_2} \epsilon(i, j) x^*_{i,j}\,,$$
$$xy := \sum_{i,j,k,l\in\mathbb{Z}_2} \epsilon(j, k) x_{i,j} y_{k,l}\,.$$

The $*$-algebra thus obtained, in [7] referred to as the Fermi tensor product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, also carries a $\mathbb{Z}_2$-grading. This is induced by the $*$-automorphism $\theta = \theta_1 \hat{\otimes} \theta_2$, whose action on simple tensors is given by

$$\theta_1 \hat{\otimes} \theta_2 (a_1 \hat{\otimes} a_2) := \theta_1(a_1) \hat{\otimes} \theta_2(a_2)\,, \quad a_1 \in \mathfrak{A}_1\,,\ a_2 \in \mathfrak{A}_2\,, \tag{3.1}$$

where $a_1 \hat{\otimes} a_2$ is nothing but $a_1 \otimes a_2$ thought of as an element of the $\mathbb{Z}_2$-graded $*$-algebra $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2$, since $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2 = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ as linear spaces. As of now, we will use $a_1 \otimes a_2$ and $a_1 \hat{\otimes} a_2$ interchangeably when no confusion can occur.

The even and odd part of the Fermi product are respectively

$$\big(\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2\big)_+ = \big(\mathfrak{A}_{1,+} \odot \mathfrak{A}_{2,+}\big) \oplus \big(\mathfrak{A}_{1,-} \odot \mathfrak{A}_{2,-}\big)\,,$$
$$\big(\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2\big)_- = \big(\mathfrak{A}_{1,+} \odot \mathfrak{A}_{2,-}\big) \oplus \big(\mathfrak{A}_{1,-} \odot \mathfrak{A}_{2,+}\big)\,.$$

The construction of the algebraic Fermi tensor product can of course be performed with an arbitrary number n of C^*-algebras $\mathfrak{A}_i$, $i = 1, 2, \dots, n$. As usual, as a linear space $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{A}_n$ is given by the algebraic tensor product $\mathfrak{A}_1 \odot \mathfrak{A}_2 \odot \cdots \odot \mathfrak{A}_n$. Product and involution can be defined by carefully exploiting the associativity of the usual tensor product. The $*$-algebra $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{A}_n$ can be turned into a $\mathbb{Z}_2$-graded algebra by the grading $\theta^{(n)} := \theta_1 \hat{\otimes} \theta_2 \hat{\otimes} \cdots \hat{\otimes} \theta_n$ defined as in (3.1).

For $\omega_i \in \mathcal{S}(\mathfrak{A}_i)$, $i = 1, 2$, the state ψ_{ω_1,ω_2} has a counterpart in $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2$ by means of the product functional $\omega_1 \times \omega_2$, defined as usual by

$$\omega_1 \times \omega_2 \bigg(\sum_{j=1}^n a_{1,j} \hat{\otimes} a_{2,j} \bigg) := \sum_{j=1}^n \omega_1(a_{1,j}) \omega_2(a_{2,j})\,,$$

for all $\sum_{j=1}^{n} a_{1,j}\hat{\otimes}a_{2,j} \in \mathfrak{A}_1\hat{\otimes}\mathfrak{A}_2$. Contrary to the case of a trivial grading, the functional defined above is not necessarily positive, unless at least one between ω_1 and ω_2 is even, see [7, Proposition 7.1]. More in general, given $\omega_i \in \mathcal{S}(\mathfrak{A}_i)$, we denote by $\omega_1 \times \omega_2 \times \cdots \times \omega_n$ the linear functional on $\mathfrak{A}_1\hat{\otimes}\mathfrak{A}_2\hat{\otimes}\cdots\hat{\otimes}\mathfrak{A}_n$ defined on simple tensors as

$$\omega_1 \times \omega_2 \times \cdots \times \omega_n(a_1\hat{\otimes}a_2\hat{\otimes}\cdots\hat{\otimes}a_n) := \omega_1(a_1)\omega_2(a_2)\cdots\omega_n(a_n)$$

for every $a_i \in \mathfrak{A}_i$. The following proposition is a straightforward generalization of [11, Proposition 2.6].

Proposition 3.1. *Let $(\mathfrak{A}_i, \theta_i)$ be graded C^*-algebras, $i = 1, 2, \ldots n$. Given $\omega_i \in \mathcal{S}(\mathfrak{A}_i)$, then their product state $\omega_1 \times \omega_2 \times \cdots \times \omega_n$ is positive if and only if at least $n-1$ of them are even. Moreover, $\omega_1\times\omega_2\times\cdots\times\omega_n$ is even if and only if all states $\omega_1, \omega_2, \ldots, \omega_n$ are even.*

Proof. A simple induction on n. □

As in the case with only two factors, which has been addressed in [11], the product $\mathfrak{A}_1\hat{\otimes}\mathfrak{A}_2\hat{\otimes}\cdots\hat{\otimes}\mathfrak{A}_n$ will in general admit many C^*-completions. We first consider the minimal completion, namely the one obtained by completing with respect to the norm

$$\|x\|_{\min} := \sup\{\|\pi_\omega(x)\| : \omega = \omega_1\times\omega_2\cdots\times\omega_n,\, \omega_i \in \mathcal{S}_+(\mathfrak{A}_i),\, i = 1, \ldots, n\}\,,$$

which we denote by $\mathfrak{A}_1\hat{\otimes}_{\min}\mathfrak{A}_2\hat{\otimes}_{\min}\cdots\hat{\otimes}_{\min}\mathfrak{A}_n$. It is still a $\mathbb{Z}_2$-graded C^*-algebra, with the grading obtained by extending $\theta^{(n)}$ to the minimal completion, *cf.* [11, Proposition 4.7].

Minimal infinite tensor Fermi products can be defined through inductive limits. More precisely, if $\{(\mathfrak{A}_i, \theta_i) : i \in \mathbb{N}\}$ is a countable family of unital $\mathbb{Z}_2$-graded C^*-algebras, then for each $n \in \mathbb{N}$ we can consider the injective homomorphism

$$\Phi_n : \mathfrak{A}_1\hat{\otimes}_{\min}\mathfrak{A}_2\hat{\otimes}_{\min}\cdots\hat{\otimes}_{\min}\mathfrak{A}_n \to \mathfrak{A}_1\hat{\otimes}_{\min}\mathfrak{A}_2\hat{\otimes}_{\min}\cdots\hat{\otimes}_{\min}\mathfrak{A}_n\hat{\otimes}_{\min}\mathfrak{A}_{n+1}$$

completely determined by

$$\Phi_n(a_1\hat{\otimes}a_2\hat{\otimes}\cdots\hat{\otimes}a_n) = a_1\hat{\otimes}a_2\hat{\otimes}\cdots\hat{\otimes}a_n\hat{\otimes}1\,,$$

for every $a_i \in \mathfrak{A}_i$ and $i = 1, \ldots, n$, where by a slight abuse of notation 1 denotes the unity of $\mathfrak{A}_i$ for every $i \in \mathbb{N}$.
Clearly, $\{\mathfrak{A}_1\hat{\otimes}_{\min}\mathfrak{A}_2\hat{\otimes}_{\min}\cdots\hat{\otimes}_{\min}\mathfrak{A}_n, \Phi_n\}$ is an inductive system of C^*-algebras, whose limit we denote by $\hat{\otimes}_{\min}^{i\in\mathbb{N}}\mathfrak{A}_i$ and call the (minimal) infinite Fermi tensor product.
We denote by ι_n the embedding of $\mathfrak{A}_1\hat{\otimes}_{\min}\mathfrak{A}_2\hat{\otimes}_{\min}\cdots\hat{\otimes}_{\min}\mathfrak{A}_n$ into $\hat{\otimes}_{\min}^{i\in\mathbb{N}}\mathfrak{A}_i$. Henceforth, we will often write $a_1\hat{\otimes}a_2\hat{\otimes}\cdots\hat{\otimes}a_n\hat{\otimes}1\,\hat{\otimes}1\,\cdots$ rather than write $\iota_n(a_1\hat{\otimes}a_2\hat{\otimes}\cdots\hat{\otimes}a_n)$, as is commonly done in the literature.

Infinite Fermi tensor products provide examples of quasi-local algebras. Here, the net of local subalgebras is as follows. For every

$I = \{i_1, \dots, i_{|I|}\}$ finite subset of $\mathbb{N}$, we denote by $\mathfrak{A}(I) \subset \hat{\otimes}_{\min}^{i\in\mathbb{N}}\mathfrak{A}_i$ the unital C^*-subalgebra generated by simple tensors of the type

$$1\hat{\otimes}\cdots\hat{\otimes}a_{i_1}\hat{\otimes}\cdots\hat{\otimes}a_{i_2}\hat{\otimes}\cdots\hat{\otimes}a_{i_{|I|}}\hat{\otimes}1\hat{\otimes}1\cdots$$

when the a_{i_j}'s vary in $\mathfrak{A}_{i_j}$, $j = 1, \dots, |I|$.

If now $\mathfrak{A}$ is a fixed unital C^*-algebra, we denote by $\mathfrak{A}^{(n)}$ the minimal Fermi tensor product of $\mathfrak{A}$ with itself n times and by $\hat{\otimes}_{\min}^{\mathbb{N}}\mathfrak{A}$ the corresponding infinite graded tensor product. We still denote by $\iota_n : \mathfrak{A}^{(n)} \to \hat{\otimes}_{\min}^{\mathbb{N}}\mathfrak{A}$ the embeddings of $\mathfrak{A}^{(n)}$ into $\hat{\otimes}_{\min}^{\mathbb{N}}\mathfrak{A}$.
Finally, note that $\hat{\otimes}_{\min}^{\mathbb{N}}\mathfrak{A}$ is still a $\mathbb{Z}_2$-graded C^*-algebra, whose grading, which we denote by $\hat{\otimes}^{\mathbb{N}}\theta$, is obtained as the inductive limit ot the $\theta^{(n)}$'s.

In the following example we show how the CAR algebra can be reobtained as an infinite Fermi tensor product.

Example 3.2. Our starting data is the $\mathbb{Z}_2$-graded C^*-algebra $(\mathfrak{A}, \theta) = (\mathbb{M}_2(\mathbb{C}), \mathrm{ad}(U))$, where U is the (Pauli) unitary matrix $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Note that, given $B \in \mathbb{M}_2(\mathbb{C})$, one has $UBU^* = B$ if and only if B is a diagonal matrix and $UBU^* = -B$ if and only if B is anti-diagonal.
We next show that $\hat{\otimes}^{\mathbb{N}}\mathbb{M}_2(\mathbb{C})$ is $*$-isomorphic with the CAR algebra and $\hat{\otimes}^{\mathbb{N}}\mathrm{ad}(U)$ is its usual grading. To this end, set $A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
Note that A is odd and $A^2 = 0$, $A^*A + AA^* = I$. For every $j \in \mathbb{N}$, denote by $i_j : \mathbb{M}_2(\mathbb{C}) \to \hat{\otimes}^{\mathbb{N}}\mathbb{M}_2(\mathbb{C})$ the injective *-homomorphism given by

$$i_j(B) = 1\hat{\otimes}1\hat{\otimes}\cdots\hat{\otimes}\underbrace{B}_{\text{j-th place}}\hat{\otimes}1\hat{\otimes}1\cdots,\ B \in \mathbb{M}_2(\mathbb{C}).$$

and define $a_j := i_j(A)$, $j \in \mathbb{N}$. Since A generates $\mathbb{M}_2(\mathbb{C})$, $\{a_j : j \in \mathbb{N}\}$ is a set of generators of the infinite Fermi tensor product of $\mathbb{M}_2(\mathbb{C})$ with itself. Now the relations $a_j a_k + a_k a_j = 0$ and $a_j a_k^* + a_k^* a_j = \delta_{j,k} I$, $j, k \in \mathbb{N}$, are a straighforward consequence of the equalities $A^2 = 0$, $A^*A + AA^* = I$ and of the fact that A is odd. As the CAR algebra, $\mathrm{CAR}(\mathbb{N})$, is the universal C^*-algebra generated by b_j's satisfying the above relations, we find that there must exist a surjective $*$-homomorphism $\Psi : \mathrm{CAR}(\mathbb{N}) \to \hat{\otimes}^{\mathbb{N}}\mathbb{M}_2(\mathbb{C})$ such that $\Psi(b_j) = a_j$, for every $j \in \mathbb{N}$. By simplicity of $\mathrm{CAR}(\mathbb{N})$, Ψ is also injective and so $\mathrm{CAR}(\mathbb{N}) \cong \hat{\otimes}^{\mathbb{N}}\mathbb{M}_2(\mathbb{C})$. Finally, as for the grading, it is enough to observe that each a_j is odd w.r.t. $\hat{\otimes}^{\mathbb{N}}\mathrm{ad}(U)$.

If $\omega_i \in \mathcal{S}_+(\mathfrak{A}_i)$ are even states for every $i \in \mathbb{N}$, then their infinite product, $\underset{i\in\mathbb{N}}{\times}\omega_i$, is the state on $\hat{\otimes}_{\min}^{i\in\mathbb{N}}\mathfrak{A}_i$ uniquely determined by

$$\underset{i\in\mathbb{N}}{\times}\omega_i(x_1 \hat{\otimes} \cdots \hat{\otimes} x_n \hat{\otimes} 1 \hat{\otimes} 1 \hat{\otimes} \cdots) = \omega_1(x_1)\omega_2(x_2)\cdots\omega_n(x_n)$$

for every $x_i \in \mathfrak{A}_i$, $i = 1, 2, \ldots, n$, and every $n \in \mathbb{N}$.
If ω_i is a fixed even state ω on $\mathfrak{A}$ for each $i \in \mathbb{N}$, we then simply denote by $\times^{\mathbb{N}}\omega$ the product state $\underset{i\in\mathbb{N}}{\times}\omega_i$ on $\hat{\otimes}_{\min}^{\mathbb{N}}\mathfrak{A}$.

Similarly to what we have seen for C^*-algebras, the definition of the $\mathbb{Z}_2$-graded tensor product of two Hilbert spaces, as given in [11], can easily be extended to an arbitrary number of spaces. We start by recalling that a $\mathbb{Z}_2$-graded Hilbert space is a pair $(\mathcal{H}, U)$, where $\mathcal{H}$ is a (complex) Hilbert space and U a self-adjoint unitary acting on $\mathcal{H}$. In such a situation, $\mathcal{H}$ decomposes into an orthogonal direct sum of the type

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \,,$$

where $\mathcal{H}_+ := \mathrm{Ker}(I - U)$ and $\mathcal{H}_- := \mathrm{Ker}(I + U)$. As usual, vectors in $\mathcal{H}_+$ ($\mathcal{H}_-$) are called *even* (*odd*) vectors. Even or odd vectors are collectively referred to as homogeneous vectors.
The Hilbert tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two $\mathbb{Z}_2$-graded Hilbert spaces $(\mathcal{H}_1, U_1)$ and $(\mathcal{H}_2, U_2)$ will always be conceived of as a graded Hilbert space, with the natural grading associated with $U_1 \otimes U_2$.

We also recall that infinite tensor products of Hilbert spaces can be defined as direct limits of finite products following a construction due to von Neumann, which we rather quickly sketch for convenience. We first observe that, given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, for any unit vector $\xi \in \mathcal{H}_2$ the map $\mathcal{H}_1 \ni x \to x \otimes \xi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is isometric. Given a sequence $\{(\mathcal{H}_i, \xi_i) : i \in \mathbb{N}\}$ of Hilbert spaces, where for each $i \in \mathbb{N}$ $\xi_i \in \mathcal{H}_i$ is a unit vector, we can consider the isometries $\Phi_n : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \to \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_{n+1}$ given by

$$\Phi_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) := x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes \xi_{n+1}.$$

The infinite tensor product of the Hilbert spaces $\mathcal{H}_i$ with respect to the sequence $\boldsymbol{\xi} = \{\xi_i\}_{i\in\mathbb{N}}$ is by definition the inductive limit of the direct system $\{(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n, \Phi_n) : n \in \mathbb{N}\}$, and will be denoted by $\underset{\boldsymbol{\xi}}{\otimes}\mathcal{H}_i$. For each integer n, we denote by ι_n the isometric embedding of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ into $\underset{\boldsymbol{\xi}}{\otimes}\mathcal{H}_i$. Note that $\iota_{n+1} \circ \Phi_n = \iota_n$ for every n by definition of inductive limit. Instead of $\iota_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n)$ we will every so often write $x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes \xi_{n+1} \otimes \xi_{n+2} \otimes \cdots$. When all the Hilbert spaces $\mathcal{H}_i$ are graded, say by self-adjoint unitaries $U_i \in \mathcal{B}(\mathcal{H}_i)$, then the infinite product $\underset{\boldsymbol{\xi}}{\otimes}\mathcal{H}_i$ can be equipped with a $\mathbb{Z}_2$-grading through the self-adjoint unitary $\otimes_{n\in\mathbb{N}}U_n$, whose definition is deferred to Section 4.

In [11] the Fermi product of grading-equivariant representations was defined for two representations. Obviously, the construction given there continues to work with an arbitrary number n of representations. In other terms, if $\pi_i : \mathfrak{A}_i \to \mathcal{B}(\mathcal{H}_i)$, with $i = 1, 2, \ldots, n$, are grading-equivariant representations acting on the $\mathbb{Z}_2$-graded Hilbert spaces $(\mathcal{H}_i, U_i)$, then it is possible to define a representation π of the Fermi product $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{A}_n$ acting on the Hilbert space $\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ as

$$\pi(a_1 \hat{\otimes} a_2 \hat{\otimes} \cdots \hat{\otimes} a_n) := \pi_1(a_1) \hat{\otimes} \pi_2(a_2) \hat{\otimes} \cdots \hat{\otimes} \pi_n(a_n)$$

for every $a_i \in \mathfrak{A}_i$, $i = 1, 2, \ldots, n$. Note that in the formula above the symbol $\hat{\otimes}$ is actually used to denote the Fermi tensor product of two or more operators acting on (possibly different) Hilbert spaces, as defined in [11]. For convenience, we recall the definition with two (homogeneous) operators $T_i \in \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$:

$$T_1 \hat{\otimes} T_2(\xi_1 \otimes \xi_2) := \; T_1\xi_1 \otimes T_2\xi_2$$

for homogeneous vectors $\xi_i \in \mathcal{H}_i$, where, as usual, the sign $\varepsilon(T_2, \xi_1)$ is -1 if T_2 and ξ_1 are both odd. Notice that if T_1 and T_2 are both even, then $T_1 \hat{\otimes} T_2$ is nothing but the usual tensor product $T_1 \otimes T_2$ of operators.

We will say that π is the Fermi tensor product of the representations π_i and write $\pi = \pi_1 \hat{\otimes} \pi_2 \hat{\otimes} \cdots \hat{\otimes} \pi_n$. It is easy to verify that π can be extended to a representation of the completion $\mathfrak{A}_1 \hat{\otimes}_{\min} \mathfrak{A}_2 \hat{\otimes}_{\min} \cdots \hat{\otimes}_{\min} \mathfrak{A}_n$. With a slight abuse of notation, we continue to denote this extension by $\pi_1 \hat{\otimes} \pi_2 \hat{\otimes} \cdots \hat{\otimes} \pi_n$. Once finitely many representations have been dealt with, infinitely many representations can be handled easily. For the sake of simplicity, we only consider cyclic representations.

Indeed, if $\pi_{\omega_i} : \mathfrak{A}_i \to \mathcal{B}(\mathcal{H}_{\omega_i})$, with $i \in \mathbb{N}$, are the GNS representations of the even states $\omega_i \in \mathcal{S}_+(\mathfrak{A}_i)$, then their Fermi product is the representation $\hat{\bigotimes}_{i\in\mathbb{N}} \pi_{\omega_i}$ of the Fermi product C^*-algebra $\hat{\otimes}_{\min} \mathfrak{A}_i$ acting on the Hilbert space $\bigotimes_{\boldsymbol{\xi}} \mathcal{H}_{\omega_i}$, with $\boldsymbol{\xi} := \{\xi_{\omega_i}\}_{i\in\mathbb{N}}$, uniquely determined by

$$\hat{\bigotimes_{i\in\mathbb{N}}} \pi_{\omega_i}(a_1 \hat{\otimes} \cdots \hat{\otimes} a_n \hat{\otimes} 1 \hat{\otimes} 1 \hat{\otimes} \cdots) = \pi_{\omega_1}(a_1) \hat{\otimes} \cdots \hat{\otimes} \pi_{\omega_n}(a_n) \hat{\otimes} 1 \hat{\otimes} 1 \hat{\otimes} \cdots .$$

Remark 3.3. The representation $\hat{\bigotimes}_{i\in\mathbb{N}} \pi_{\omega_i}$ is still cyclic and coincides (up to unitary equivalence) with the GNS representation of the product state $\omega = \times_{i\in\mathbb{N}} \omega_i$. Indeed, the unit vector ξ defined as

$$\xi := \iota_n(\xi_{\omega_1} \otimes \cdots \otimes \xi_{\omega_n}) =: \bigotimes_{i\in\mathbb{N}} \xi_{\omega_i}$$

(note that the definition does not depend on n) is cyclic, and the equality $\omega(\cdot) = \langle \hat{\bigotimes}_{i\in\mathbb{N}} \pi_{\omega_i}(\cdot)\xi, \xi \rangle$ is easily checked.

Remark 3.4. Note that if we start with a faithful even state ω on $\mathfrak{A}$, then its product $\times^{\mathbb{N}}\omega$ will be a faithful state on $\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}$. This can actually be seen much in the same way as in the proof of Proposition 5 in [13], page 22.

Finally, we will simply denote by $\otimes^{\mathbb{N}}\mathcal{H}_\omega$ the infinite product $\underset{\boldsymbol{\xi}}{\otimes}\mathcal{H}_{\omega_i}$, with $\mathcal{H}_{\omega_i}=\mathcal{H}_\omega$ and $\boldsymbol{\xi}$ being sequence constantly equal to ξ_ω.

As already remarked, the infinite Fermi tensor product $\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}$ of a given C^*-algebra $\mathfrak{A}$ is a quasi-local algebra. In addition, $\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}$ is acted upon by $\mathbb{P}_{\mathbb{N}}$ in a natural way. Indeed, associated with any $\sigma\in\mathbb{P}_{\mathbb{N}}$ there is a $*$-automorphism $\alpha_\sigma\in\mathrm{Aut}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})$, which is completely determined by

$$\alpha_\sigma(\iota_n(a_1\hat{\otimes}a_2\hat{\otimes}\cdots\hat{\otimes}a_n))=\iota_n(a_{\sigma(1)}\hat{\otimes}a_{\sigma(2)}\hat{\otimes}\cdots\hat{\otimes}a_{\sigma(n)})\,,$$

for $n\in\mathbb{N}$, $a_i\in\mathfrak{A}$, $i=1,2,\dots,n$. Clearly, this is a local action of $\mathbb{P}_{\mathbb{N}}$ in the sense of Definition 2.3. In particular, Theorem 2.6 applies to the present context in a strengthened fashion. More precisely, the extreme symmetric states can now be characterized as infinite products of a given state on $\mathfrak{A}$.

Proposition 3.5. *Let $(\mathfrak{A},\theta)$ be a $\mathbb{Z}_2$-graded C^*-algebra. If ω is a symmetric state on $\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}$, then the following are equivalent:*

1. *ω is extreme;*
2. *ω is strongly clustering;*
3. *there exists an even state $\rho\in\mathcal{S}(\mathfrak{A})$ such that $\omega=\times^{\mathbb{N}}\rho$.*

Proof. In light of Theorem 2.6 we need only show that (2) and (3) are equivalent, which can be done exactly as in [8, Theorem 5.3]. □

As a consequence of the above result, we find that the extreme symmetric states of $\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}$ are sufficiently many to separate its points. This circumstance plays an instrumental role in proving weak ergodicity of the permutation action on an infinite product, as shown below.

Proposition 3.6. *For any given $\mathbb{Z}_2$-graded C^*-algebra $(\mathfrak{A},\theta)$, the C^*-dynamical system $(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A},\mathbb{P}_{\mathbb{N}},\{\alpha_\sigma:\sigma\in\mathbb{P}_{\mathbb{N}}\})$ is weakly ergodic, i.e. $x\in\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}$ with $\alpha_\sigma(x)=x$ for every $\sigma\in\mathbb{P}_{\mathbb{N}}$ implies $x=\lambda 1$ for some $\lambda\in\mathbb{C}$.*

Proof. We start by showing that for any given x in $\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}$ there exists a separable $\mathbb{Z}_2$-graded subalgebra $\widetilde{\mathfrak{A}}\subset\mathfrak{A}$ such that x belongs to $\hat{\otimes}^{\mathbb{N}}_{\min}\widetilde{\mathfrak{A}}$. Indeed, by definition x is the limit in norm of a sequence $\{x_n\}_{n\in\mathbb{N}}$, where the x_n's are elements of the form

$$x_n=\sum_{k\leq K_n}a^{(n)}_{1,k}\hat{\otimes}a^{(n)}_{2,k}\hat{\otimes}\cdots\hat{\otimes}a^{(n)}_{L_n,k}\hat{\otimes}1\hat{\otimes}1\cdots$$

where K_n, L_n are suitable integers and $a^{(n)}_{i,k}$ belongs to $\mathfrak{A}$ for every integers $k \leq K_n$ and $i \leq L_n$. The countably generated C^*-subalgebra

$$\widetilde{\mathfrak{A}} := C^*\{a^{(n)}_{i,k}, \theta(a^{(n)}_{i,k}) : i \leq L_n, k \leq K_n, n \in \mathbb{N}\} \subset \mathfrak{A}$$

clearly does the job. Let now x in $\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}$ be a fixed point, that is $\alpha_\sigma(x) = x$ for every $\sigma \in \mathbb{P}_\mathbb{N}$, and let ρ be a faithful even state on the separable C^*-algebra $\widetilde{\mathfrak{A}}$ considered above. Note that the action of $\mathbb{P}_\mathbb{N}$ leaves $\hat{\otimes}^{\mathbb{N}}_{\min}\widetilde{\mathfrak{A}}$ invariant. Define ω as the infinite product of ρ with itself. By Remark 3.4 ω is still faithful. Since x is invariant under the action of all σ's in $\mathbb{P}_\mathbb{N}$, we have that $\pi_\omega(x)\xi_\omega$ lies in $\mathcal{H}^{\mathbb{P}_\mathbb{N}}_\omega$. By virtue of Propositions 2.4 and 3.5 there exists $\lambda \in \mathbb{C}$ such that $\pi_\omega(x)\xi_\omega = \lambda\xi_\omega$. By faithfulness of ω we find $x = \lambda 1$. □

The compact convex set $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})$ is again a Choquet simplex. Moreover, its extreme points make up a closed set since the bijection $\mathcal{S}_+(\mathfrak{A}) \ni \rho \overset{T}{\mapsto} \times^{\mathbb{N}}\rho \in \mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}))$ establishes a homeomorphism between the two topological spaces. In particular, for any $\omega \in \mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})$, there exists a unique probability measure μ which is now genuinely supported on $\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}))$ such that

$$\omega = \int_{\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}))} \psi \, \mathrm{d}\mu(\psi)\,.$$

Because $\mathcal{S}_+(\mathfrak{A})$ and $\mathcal{E}(\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}))$ are homeomorphic compact spaces, the above equality can also be rewritten as

$$\omega = \int_{\mathcal{S}_+(\mathfrak{A})} \times^{\mathbb{N}}\rho \, \mathrm{d}\mu^*(\rho)\,,$$

where μ^* is the probability measure on $\mathcal{S}_+(\mathfrak{A})$ induced by μ through T, *i.e.* $\mu^*(B) = \mu(T(B))$, for any Borel set $B \subset \mathcal{S}_+(\mathfrak{A})$.

The structure of our Choquet simplex can be further analyzed by spotting its faces. This was done in [20] in the case of usual infinite tensor products and can be easily adapted to the present situation. More explicitly, Theorem 2.9 and Corollary 2.10 in [20] admit a straightforward extension to the graded case. In order to the state it, we keep the same notation as in the above mentioned paper and denote by X the generic type of a given von Neumann algebra. In other words, X can be I, II_1, II_∞, III.

Proposition 3.7. *For any X as above, define the convex subsets*

$$\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})_X := \{\omega \in \mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A}) : \pi_\omega(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})'' \textit{ is of type}\, X\}\,.$$

Then $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})_X$ is a face of $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})$. Moreover, $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})$ is the closed convex hull of the faces $\mathcal{S}^{\mathbb{P}_\mathbb{N}}(\hat{\otimes}^{\mathbb{N}}_{\min}\mathfrak{A})_X$.

We are next going to draw our attention to maximal completions of infinite graded tensor products. As we will recall, these can also be obtained as quotients of a universal C^*-algebra, the infinite free product of a given C^*-algebra. It is this very property that makes maximal completions particularly suited to establishing a correspondence between their symmetric states and quantum stochastic processes of a particular form. Notably, this allows us to come to a version of de Finetti's theorem for the processes thus obtained. With this in mind, we start by quickly outlining how maximal completions of infinite products can be got to.

For finitely many factors $\mathfrak{A}_i$, $i = 1, 2, \ldots, n$, the maximal Fermi tensor product $\mathfrak{A}_1 \hat{\otimes}_{\max} \mathfrak{A}_2 \hat{\otimes}_{\max} \cdots \hat{\otimes}_{\max} \mathfrak{A}_n$ is nothing but the completion of the algebraic product $\mathfrak{A}_1 \hat{\otimes} \mathfrak{A}_2 \hat{\otimes} \cdots \hat{\otimes} \mathfrak{A}_n$ with respect to the maximal C^*-norm, see [7] for the details. Infinite products, as usual, are dealt with by taking inductive limits. Henceforth we will be focusing on the maximal infinite tensor product $\hat{\otimes}^{\mathbb{N}}_{\max} \mathfrak{A}$ of a given $\mathbb{Z}_2$-graded C^*-algebra $(\mathfrak{A}, \theta)$.

First, we observe that $\hat{\otimes}^{\mathbb{N}}_{\max} \mathfrak{A}$ can also be recovered as a suitable quotient of the infinite free product $*^{\mathbb{N}} \mathfrak{A}$ of $\mathfrak{A}$ with itself, see [5] for a thorough account of free products. Note that $*^{\mathbb{N}} \mathfrak{A}$ is a $\mathbb{Z}_2$-graded C^*-algebra with grading given by $\theta^* := *^{\mathbb{N}} \theta$. For every $j \in \mathbb{N}$, we will denote by $i_j : \mathfrak{A} \to *^{\mathbb{N}} \mathfrak{A}$ the j-th embedding of $\mathfrak{A}$ into its infinite free product, *cf.* [5]. Consider now the closed two-sided ideal I of $*^{\mathbb{N}} \mathfrak{A}$ generated by elements of the form $[i_j(a), i_k(b)]_{\theta^*}$ as a, b vary in $\mathfrak{A}$ and $j \neq k$, where for homogeneous x, y in $*^{\mathbb{N}} \mathfrak{A}$ the symbol $[x, y]_{\theta^*}$ is the commutator of x and y if at least one of them is even, or the anti-commutator when x and y are both odd.

An easy application of [7, Theorem 8.4] shows that the quotient C^*-algebra $*^{\mathbb{N}} \mathfrak{A} / I$ is $*$-isomorphic with $\hat{\otimes}^{\mathbb{N}}_{\max} \mathfrak{A}$. We will denote by $\Psi : *^{\mathbb{N}} \mathfrak{A} \to \hat{\otimes}^{\mathbb{N}}_{\max} \mathfrak{A}$ the canonical projection onto the quotient.

Following [9], by a quantum stochastic process we mean a quadruple $(\mathfrak{A}, \{\iota_j : j \in \mathbb{N}\}, \mathcal{H}, \xi)$, where $\mathfrak{A}$ is a unital C^*-algebra, $\mathcal{H}$ is a Hilbert space, $\iota_j : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a $*$-representation for every $j \in \mathbb{N}$, and $\xi \in \mathcal{H}$ is a cyclic vector for the von Neumann algebra $\bigvee_{j \in \mathbb{N}} \iota_j(\mathfrak{A})$. As shown in [9], there is a one-to-one correspondence between stochastic processes on $\mathfrak{A}$ and states of the infinite free product $*^{\mathbb{N}} \mathfrak{A}$. This is realized as follows. Starting from a state ω on $*^{\mathbb{N}} \mathfrak{A}$, the corresponding process is obtained as

$$\iota_j := \pi_\omega \circ i_j, \quad j \in \mathbb{N}. \tag{3.2}$$

Note that the GNS vector ξ_ω is certainly cyclic for $\bigvee_{j \in \mathbb{N}} \iota_j(\mathfrak{A})$. Now $\mathbb{P}_{\mathbb{N}}$ acts naturally on the infinite free product $*^{\mathbb{N}} \mathfrak{A}$. Indeed for any $\sigma \in \mathbb{P}_{\mathbb{N}}$ there is a unique automorphism α_σ of $*^{\mathbb{N}} \mathfrak{A}$ determined by

$$\alpha_\sigma(i_{j_1}(a_1) i_{j_2}(a_2) \cdots i_{j_n}(a_n)) = i_{\sigma(j_1)}(a_1) i_{\sigma(j_2)}(a_2) \cdots i_{\sigma(j_n)}(a_n)$$

for $j_1 \neq j_2 \neq \cdots \neq j_n \in \mathbb{N}$, $a_1, a_2, \ldots, a_n \in \mathfrak{A}$, $n \in \mathbb{N}$. Invariant states under this action of $\mathbb{P}_\mathbb{N}$ are again referred to as symmetric states and correspond to so-called exchangeable processes. We recall that a process $(\mathfrak{A}, \{\iota_j : j \in \mathbb{N}\}, \mathcal{H}, \xi)$ is said to be exchangeable if for every $j_1 \neq j_2 \neq \cdots \neq j_n \in \mathbb{N}$, $n \in \mathbb{N}$, $a_1, a_2, \ldots, a_n \in \mathfrak{A}$, and $\sigma \in \mathbb{P}_\mathbb{N}$ one has

$$\langle \iota_{j_1}(a_1)\iota_{j_2}(a_2)\cdots\iota_{j_n}(a_n)\xi, \xi\rangle = \langle \iota_{\sigma(j_1)}(a_1)\iota_{\sigma(j_2)}(a_2)\cdots\iota_{\sigma(j_n)}(a_n)\xi, \xi\rangle .$$

We are actually interested in processes $(\mathfrak{A}, \{\iota_j : j \in \mathbb{N}\}, \mathcal{H}, \xi)$ where the sample algebra $\mathfrak{A}$ is in fact a $\mathbb{Z}_2$-graded C^*-algebra, and such that for homogeneous $a, b \in \mathfrak{A}$ and $j \neq k$ $\iota_j(a)$ and $\iota_k(b)$ commute if at least one between a or b is even and anti-commute otherwise. Clearly, processes of this type arise from states of the quotient $*^\mathbb{N}\mathfrak{A}/I \cong \hat{\otimes}^\mathbb{N}_{\max}\mathfrak{A}$, and therefore they will be referred to as $\mathbb{Z}_2$-graded processes on the sample algebra $\mathfrak{A}$. Like minimal graded infinite products, maximal ones are seen at once to be quasi-local algebras. In addition, the natural action of $\mathbb{P}_\mathbb{N}$ on them is of course local. As a consequence, Proposition 2.10 applies, so if ω is a symmetric state on $\hat{\otimes}^\mathbb{N}_{\max}\mathfrak{A}$, we denote by $E_\omega : \pi_\omega(\hat{\otimes}^\mathbb{N}_{\max}\mathfrak{A})'' \to \mathfrak{Z}^\perp_\omega$ the unique conditional expectation onto the (commutative) tail algebra. That said, we are now ready to state a de Finetti-type theorem for graded processes.

Theorem 3.8. *A $\mathbb{Z}_2$-graded process $(\mathfrak{A}, \{\iota_j : j \in \mathbb{N}\}, \mathcal{H}, \xi)$, with corresponding $\omega \in \mathcal{S}(\hat{\otimes}^\mathbb{N}_{\max}\mathfrak{A})$, is exchangeable if and only if:*

(i) *the process is conditionally independent w.r.t. E_ω, namely*

$$E_\omega[XY] = E_\omega[X]E_\omega[Y]$$

for every $X \in \big(\bigvee_{i\in I}\iota_i(\mathfrak{A})\big)\bigvee\mathfrak{Z}^\perp_\omega$ and $Y \in \big(\bigvee_{j\in J}\iota_j(\mathfrak{A})\big)\bigvee\mathfrak{Z}^\perp_\omega$, and $I, J \subset \mathbb{N}$ finite disjoint subsets;

(ii) *the process is identically distributed w.r.t. E_ω, namely*

$$E_\omega[\iota_j(a)] = E_\omega[\iota_k(a)]$$

for every $j, k \in \mathbb{N}$ and $a \in \mathfrak{A}$.

Proof. It is an application of Proposition 2.13. Indeed, $\hat{\otimes}^\mathbb{N}_{\max}\mathfrak{A}$ is a quasi-local C^*-algebra coming from the additive net of local algebras $\{\mathfrak{A}(I) : I \in \mathcal{P}_0(\mathbb{N})\}$, where $\mathfrak{A}(I)$ is the unital C^*-subalgebra generated by simple tensors of the type

$$1\hat{\otimes}\cdots\hat{\otimes}a_{i_1}\hat{\otimes}\cdots\hat{\otimes}a_{i_2}\hat{\otimes}\cdots\hat{\otimes}a_{i_{|I|}}\hat{\otimes}1\hat{\otimes}1\cdots$$

when the a_{i_j}'s vary in $\mathfrak{A}$, $j = 1, \ldots, |I|$.
In order to apply the aforementioned proposition, though, we first need to ascertain that the equality $\bigvee_{i\in I}\iota_i(\mathfrak{A}) = \pi_\omega(\mathfrak{A}(I))''$ holds for any finite subset I. This follows by additivity and (3.2), for we have

$$\bigvee_{j\in I}\iota_j(\mathfrak{A}) = \bigvee_{j\in I}\pi_\omega(i_j(\mathfrak{A})) = \pi_\omega\bigg(C^*(i_j(\mathfrak{A}) : j \in I)\bigg)'' = \pi_\omega(\mathfrak{A}(I))'' ,$$

where, by a slight abuse of notation, $i_j : \mathfrak{A} \to \hat{\otimes}^{\mathbb{N}}_{\max}\mathfrak{A}$ denotes the map $\Psi \circ i_j$. □

4. The twisted commutant of a Fermi product and product states

The main goal of this section is to prove that an infinite product of even factorial states is still factorial. This task will be accomplished by making use of the so-called twisted commutant, see [7] and the references therein. For the reader's convenience, though, we recall some basic definitions. By a $\mathbb{Z}_2$-graded von Neumann algebra we mean a pair $(\mathcal{M}, U)$, where $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $U \in \mathcal{U}(\mathcal{H})$ is a self-adjoint unitary such that $U\mathcal{M}U = \mathcal{M}$. With such a U it is possible to associate a $*$-automorphism of $(\mathcal{B}(\mathcal{H}), \mathrm{ad}_U)$, commonly known as *twist automorphism*, see *e.g.* [7] and references therein, which is defined as

$$\eta_U(T_+ + T_-) := T_+ + iUT_-$$

for $T = T_+ + T_-$ in $\mathcal{B}(\mathcal{H})$. The *twisted commutant* of $\mathcal{M}$ is $\mathcal{M}^\wr := \eta_U(\mathcal{M}') = \eta_U(\mathcal{M})'$. Obviously, the definition makes sense with any subset of $\mathcal{B}(\mathcal{H})$. Again, more details are found in [7]. Here, we will limit ourselves to observing that $\eta_U^2 = \mathrm{ad}_U$. We start with a preliminary lemma.

Lemma 4.1. *Let $(\mathcal{H}, U)$ be a $\mathbb{Z}_2$-graded Hilbert space and let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $*$-algebra such that $U\mathcal{A}U = \mathcal{A}$. A vector $\xi \in \mathcal{H}$ with $U\xi = \xi$ is cyclic for $\mathcal{A}$ if and only if it is cyclic for $\eta_U(\mathcal{A})$.*

Proof. We start by observing that if $T \in \mathcal{A}$, then both $T_+ := \frac{T+UTU}{2}$ and $T_- := \frac{T-UTU}{2}$ are still in $\mathcal{A}$. The thesis is reached thanks to the following computation

$$\eta_U(T)\xi = T_+\xi + iUT_-\xi = (T_+ - iT_-)\xi$$

which shows that the map $\mathcal{A}\xi \ni T\xi \mapsto \eta_U(T)\xi \in \mathcal{A}\xi$ is a linear bijection. □

Here follows a twisted version of Theorem 2 in [17]. We denote by $\mathcal{A}_s$ the set of all self-adjoint elements of a given $*$-algebra $\mathcal{A}$. Following [17], for any subspace $K \subset \mathcal{H}$ we denote by $K^\perp$ the *real* orthogonal complement, namely $K^\perp = \{x \in \mathcal{H} : \Re\langle x, k\rangle = 0, k \in K\}$, where $\Re$ is the real part of a complex number.

Lemma 4.2. *Let $(\mathcal{H}, U)$ be a $\mathbb{Z}_2$-graded Hilbert space and let $\mathcal{A}, \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be unital $*$-subalgebras such that $U\mathcal{A}U = \mathcal{A}$ and and $U\mathcal{B}U = \mathcal{B}$. If $\xi \in \mathcal{H}$ is an even cyclic vector for $\mathcal{A}$ and $\mathcal{A} \subset \mathcal{B}^\wr$, then the following conditions are equivalent:*

1. $\mathcal{A}^\wr = \mathcal{B}^{\wr\wr}$;
2. $\eta_U(\mathcal{A}_s)\xi + i\mathcal{B}_s\xi$ *is dense in* $\mathcal{H}$;
3. $[\eta_U(\mathcal{A}_s)\xi]^\perp = i\overline{\mathcal{B}_s\xi}$.

Proof. Throughout the proof η_U will be simply written as η to ease the notation. We start by showing that (2) and (3) are equivalent. First, observe $i\mathcal{B}_s\xi \subset [\eta(\mathcal{A}_s)\xi]^\perp$. Indeed, for $B \in \mathcal{B}_s$ and $A \in \mathcal{A}_s$, we have that $\eta(A)B$ is self-adjoint because $\eta(A)$ and B commute since $\mathcal{A} \subset \mathcal{B}^\wr$ (that is $\eta(\mathcal{A}) \subset \mathcal{B}'$), but then $\Re\langle iB\xi, \eta(A)\xi\rangle = \Re\langle i\eta(A)B\xi, \xi\rangle = 0$. From the observation above (2) and (3) are seen to be equivalent by a straightforward application of the following general fact: $X + X^\perp$ is dense in $\mathcal{H}$ for any real subspace $X \subset \mathcal{H}$.

We next show that either (2) or (3) implies (1). First, note that (2) or (3) implies $(\mathcal{A}^\wr)_s\xi \subset \overline{\mathcal{B}_s\xi}$. Indeed, the same computation as above shows that in general $(\mathcal{A}^\wr)_s\xi \subset [i\eta(\mathcal{A}_s)]^\perp$. Obviously, we only have to prove the inclusion $\mathcal{A}^\wr \subset \mathcal{B}^{\wr\wr} = \mathcal{B}''$. To this aim, fix T in $(\mathcal{A}^\wr)_s$ and $R \in \mathcal{B}'_s$. We need to show that $RT = TR$. Since ξ is cyclic for $\mathcal{A}$, by Lemma 4.1 it is also cyclic for $\eta(\mathcal{A})$, which means it suffices to verify that

$$\langle RTA\xi, C\xi\rangle = \langle TRA\xi, C\xi\rangle$$

for every $A, C \in \eta(\mathcal{A})$. Now there exists a sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_s$ such that $\|T\xi - B_n\xi\| \to 0$, and we have

$$\langle RTA\xi, C\xi\rangle = \langle RAT\xi, C\xi\rangle = \lim_n \langle RAB_n\xi, C\xi\rangle = \lim_n \langle B_nRA\xi, C\xi\rangle$$

where in the last equality we have used that by hypothesis the inclusion $\eta(\mathcal{A}) \subset \mathcal{B}'$ holds. Then

$$\begin{aligned}\lim_n \langle B_nRA\xi, C\xi\rangle &= \lim_n \langle RA\xi, B_nC\xi\rangle = \lim_n \langle RA\xi, CB_n\xi\rangle \\ &= \langle RA\xi, CT\xi\rangle = \langle RA\xi, TC\xi\rangle = \langle TRA\xi, C\xi\rangle\,,\end{aligned}$$

and we are done.

That (1) implies (2) can be seen in the exact same way as in the proof of Theorem 2 in [17] provided that $\mathcal{A}$ is replaced with $\eta(\mathcal{A})$. □

Remark 4.3. Taking $\mathcal{A} = \mathcal{B}^\wr$ in the above result, one finds that $\mathcal{A}_s\xi + i(\mathcal{A}^\wr)_s\xi$ is a dense subspace of $\mathcal{H}$ and $[\eta_U(\mathcal{A}_s)\xi]^\perp = i\overline{(\mathcal{A}^\wr)_s\xi}$.

Our aim now is to use Lemma 4.2 to come to a twisted version of the tensor product commutation theorem. For completeness' sake, we recall that this states that the commutant of the tensor product of two (or infinitely many) von Neumann algebras equals the tensor product of their commutants. The first general proof was obtained in [22] and later simplified in [17].

We first need to introduce graded (or Fermi) products of von Neumann algebras. We directly discuss infinite products. If $\{(\mathcal{M}_n, U_n) : n \in \mathbb{N}\}$ is a family of $\mathbb{Z}_2$-graded von Neumann algebras on the Hilbert spaces $\mathcal{H}_n$ and $\boldsymbol{\xi} := \{\xi_n : n \in \mathbb{N}\}$ is a sequence of unit vectors $\xi_n \in \mathcal{H}_n$ such that $U_n\xi_n = \xi_n$ for every $n \in \mathbb{N}$, the infinite graded product $\hat{\otimes}_{\boldsymbol{\xi}}\mathcal{M}_n$ is the von Neumann algebra on the Hilbert space $\otimes_{\boldsymbol{\xi}}\mathcal{H}_n$ generated by

operators $T_1\hat{\otimes}T_2\hat{\otimes}\cdots\hat{\otimes}T_k\hat{\otimes}1\hat{\otimes}1\cdots$ with $T_i \in \mathcal{M}_i$ for $i = 1, 2, \ldots, k$ and $k \in \mathbb{N}$.
The condition $U_n\xi_n = \xi_n$, $n \in \mathbb{N}$, comes in useful to define a self-adjoint unitary on $\underset{\boldsymbol{\xi}}{\otimes}\mathcal{H}_n$ as the infinite product $\otimes_{n\in\mathbb{N}}U_n$. This is understood as the strong limit of the sequence given by finite products of the type

$$\otimes_{i=1}^n U_i\otimes 1\otimes 1\otimes\cdots$$

which is easily verified to be Cauchy in the strong operator topology. The operator $\otimes_{n\in\mathbb{N}}U_n$ thus obtained is a self-adjoint unitary as it is the limit of self-adjoint unitaries. Moreover, $\underset{\boldsymbol{\xi}}{\hat{\otimes}}\mathcal{M}_n$ is invariant under the adjoint action of $\otimes_{n\in\mathbb{N}}U_n$. Phrased differently, $(\underset{\boldsymbol{\xi}}{\hat{\otimes}}\mathcal{M}_n, \otimes_{n\in\mathbb{N}}U_n)$ is a $\mathbb{Z}_2$-graded von Neumann algebra.

In order to arrive at the general form of our product commutation theorem, we start by attacking the case of a product of two von Neumann algebras.

Theorem 4.4. *If $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras on $\mathbb{Z}_2$-graded Hilbert spaces $(\mathcal{H}, U)$ and $(\mathcal{K}, V)$ such that $U\mathcal{M}U = \mathcal{M}$ and $V\mathcal{N}V = \mathcal{N}$, then*

$$(\mathcal{M}\,\hat{\otimes}\mathcal{N})^\wr = \mathcal{M}^\wr\,\hat{\otimes}\mathcal{N}^\wr\,.$$

Proof. We start with the inclusion $\mathcal{M}^\wr\,\hat{\otimes}\mathcal{N}^\wr \subset (\mathcal{M}\,\hat{\otimes}\mathcal{N})^\wr$, which can be checked by direct computation as follows. Since for any von Neumann algebra $\mathcal{L}$ by definition one has $\mathcal{L}^\wr = \eta(\mathcal{L}')$, we need to show that

$$[\eta_U(M')\hat{\otimes}\eta_V(N'), \eta_{U\otimes V}(M\hat{\otimes}N)] = 0$$

for every homogeneous $M \in \mathcal{M}, N \in \mathcal{N}, M' \in \mathcal{M}', N' \in \mathcal{N}'$. This requires an easy but tedious inspection of the signs, which we leave out.

The converse implication is obtained as an application of Lemma 4.2 with $\mathcal{A} = \mathcal{M}\,\hat{\otimes}\mathcal{N}$ and $\mathcal{B} = \mathcal{M}^\wr\,\hat{\otimes}\mathcal{N}^\wr$. First note that without loss of generality we may assume that $\mathcal{M}$ has an even cyclic vector $\xi_1 \in \mathcal{H}$ and $\mathcal{N}$ has an even cyclic vector $\xi_2 \in \mathcal{K}$. This can be seen as in [17] and references therein because even normal states on a graded von Neumann algebra separate its points. In particular, $\xi := \xi_1\otimes\xi_2$ is an even cyclic vector for $\mathcal{M}\hat{\otimes}\mathcal{N}$.
In order to apply Lemma 4.2, we need to make sure that $\eta(\mathcal{M}\,\hat{\otimes}\mathcal{N})_s\xi + i(\mathcal{M}^\wr\,\hat{\otimes}\mathcal{N}^\wr)_s\xi$ is dense in $\mathcal{H}\otimes\mathcal{K}$, where $\eta := \eta_U\hat{\otimes}\eta_V$. Now, as is easily checked, $\eta(\mathcal{M}\,\hat{\otimes}\mathcal{N})_s\xi \supset \eta_U(\mathcal{M}_s)\xi_1\otimes\eta_V(\mathcal{N}_s)\xi_2$ and $(\mathcal{M}^\wr\,\hat{\otimes}\mathcal{N}^\wr)_s\xi \supset (\mathcal{M}^\wr)_s\xi_1\otimes(\mathcal{N}^\wr)_s\xi_2$, which means it is enough to verify that

$$\eta_U(\mathcal{M}_s)\xi_1\otimes\eta_V(\mathcal{N}_s)\xi_2 + i(\mathcal{M}^\wr)_s\xi_1\otimes(\mathcal{N}^\wr)_s\xi_2$$

is dense in $\mathcal{H} \otimes \mathcal{K}$. In light of Remark 4.3, we are reconducted to verifying that

$$\eta_U(\mathcal{M}_s)\xi_1 \otimes \eta_V(\mathcal{N}_s)\xi_2 + i\big([\eta_U(\mathcal{M}_s)\xi_1]^\perp \otimes [\eta_V(\mathcal{N}_s)\xi_2]^\perp\big)$$

is dense, which follows from the final lemma in [17]. □

We can finally state the general version.

Theorem 4.5. *If $\{(\mathcal{H}_i, U_i) : i \in \mathbb{N}\}$ is a family of $\mathbb{Z}_2$-graded Hilbert spaces, and $\mathcal{N}_i \subset \mathcal{B}(\mathcal{H}_i)$ are von Neumann algebras such that $U_i\mathcal{N}_iU_i = \mathcal{N}_i$, $\imath \in \mathbb{N}$, then*

$$\big(\hat{\otimes}_{\boldsymbol{\xi}}\mathcal{N}_i\big)^\wr = \hat{\otimes}_{\boldsymbol{\xi}}\mathcal{N}_i^\wr$$

for any sequence $\boldsymbol{\xi} := \{\xi_i : i \in \mathbb{N}\}$ of unit vectors $\xi_i \in \mathcal{H}_i$ with $U_i\xi_i = \xi_i$, $i \in \mathbb{N}$.

Proof. First note that a straightforward induction shows that Theorem 4.4 holds for any finite Fermi tensor product. Again, the inclusion $\hat{\otimes}_{\boldsymbol{\xi}}\mathcal{N}_i^\wr \subset \big(\hat{\otimes}_{\boldsymbol{\xi}}\mathcal{N}_i\big)^\wr$ is trivially satisfied.

For the converse inclusion, take T in $\big(\hat{\otimes}_{\boldsymbol{\xi}}\mathcal{N}_i\big)^\wr$. We will show that T sits in the weak closure of $\hat{\otimes}_{\boldsymbol{\xi}}\mathcal{N}_i^\wr$. Set $\mathcal{H} := \otimes_{\boldsymbol{\xi}}\mathcal{H}_i$. Now a neighborhood of T for the weak operator topology is of the form

$$\mathcal{G} = \{S \in \mathcal{B}(\mathcal{H}) : |\langle (T-S)x_i, y_i\rangle| < \varepsilon, i = 1, 2, \dots, n\},$$

for some $x_i, y_i \in \mathcal{H}$, $i = 1, 2, \dots, n$, and $\varepsilon > 0$. By definition of $\mathcal{H}$, there exists $N \in \mathbb{N}$ such that

$$\|Px_i - x_i\| \le \varepsilon \text{ and } \|Py_i - y_i\| \le \varepsilon, \text{ for every } i = 1, 2, \dots, n,$$

where P is the projection uniquely determined on simple tensors $\otimes_{i\in\mathbb{N}}u_i$ in $\mathcal{H}$ by

$$P(\otimes u_i) = \otimes_{i=1}^N u_i \otimes (\otimes_{i\ge N+1}\langle u_i, \xi_i\rangle \xi_i)\,.$$

The same calculations as in the proof of Proposition 9 on page 34 of [13] show that PT lies in $(\hat{\otimes}_{i=1}^N\mathcal{N}_i)^\wr \hat{\otimes} \mathbb{C} \hat{\otimes} \mathbb{C} \cdots$. Since we have

$$(\hat{\otimes}_{i=1}^N\mathcal{N}_i)^\wr \hat{\otimes} \mathbb{C} \hat{\otimes} \mathbb{C} \cdots = \hat{\otimes}_{i=1}^N\mathcal{N}_i^\wr \hat{\otimes} \mathbb{C} \hat{\otimes} \mathbb{C} \cdots \subset \hat{\otimes}_{\boldsymbol{\xi}}\mathcal{N}_i^\wr$$

the thesis will be arrived at as long as we make sure that $PT \in \mathcal{G}$. This follows exactly as in the above reference. □

As an easy application of the theorem above, we provide the following result, where pureness of product states is addressed.

Proposition 4.6. *Let $(\mathfrak{A}_i, \theta_i)$ be $\mathbb{Z}_2$-graded C^*-algebras, and let $\omega_i \in \mathcal{S}(\mathfrak{A}_i)$ be pure states, $i \in \mathbb{N}$. Suppose all of these states are even but one, say ω_1. If π_{ω_1} and $\pi_{\omega_1\circ\theta}$ are unitarily equivalent, then the product state $\times_i\omega_i$ is pure as well.*

Proof. It suffices to note that under the above hypotheses $\pi_{\times_i \omega_i}$ is still (unitarily equivalent to) $\hat{\bigotimes}_{\boldsymbol{\xi}} \pi_{\omega_i}$ with $\boldsymbol{\xi} := \{\xi_{\omega_i}\}_{i\in\mathbb{N}}$, see Proposition 2.1, which means Theorem 4.5 applies. □

We are going to further apply Theorem 4.5 to infer factoriality of an infinite product of even factorial states. To do so, we first establish a couple of related results.

Lemma 4.7. *Let $(\mathcal{H}, U)$ be a $\mathbb{Z}_2$-graded Hilbert space and let $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{B}(\mathcal{H})$ be von Neumann algebras with $U\mathcal{N}_iU = \mathcal{N}_i$, $i = 1, 2$, then*

$$(\mathcal{N}_1 \cap \mathcal{N}_2)^{\wr} = \mathcal{N}_1^{\wr} \vee \mathcal{N}_2^{\wr} \quad \text{and} \quad (\mathcal{N}_1 \vee \mathcal{N}_2)^{\wr} = \mathcal{N}_1^{\wr} \cap \mathcal{N}_2^{\wr}.$$

Proof. The first equality is arrived at through the chain of equalities below

$$\begin{aligned}(\mathcal{N}_1 \cap \mathcal{N}_2)^{\wr} &= \eta_U((\mathcal{N}_1 \cap \mathcal{N}_2)') = \eta_U(\mathcal{N}_1' \vee \mathcal{N}_2') \\ &= \eta_U(\mathcal{N}_1') \vee \eta_U(\mathcal{N}_2') = \mathcal{N}_1^{\wr} \vee \mathcal{N}_2^{\wr}.\end{aligned}$$

The second follows analogously. □

Proposition 4.8. *Let $(\mathcal{H}, U)$ and $(\mathcal{K}, V)$ be $\mathbb{Z}_2$-graded Hilbert spaces. If $\mathcal{M}_i \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{N}_i \subset \mathcal{B}(\mathcal{K})$, $i = 1, 2$, are von Neumann algebras such that $U\mathcal{M}_iU = \mathcal{M}_i$ and $V\mathcal{N}_iV = \mathcal{N}_i$, $i = 1, 2$, then*

$$(\mathcal{M}_1 \hat{\otimes} \mathcal{N}_1) \cap (\mathcal{M}_2 \hat{\otimes} \mathcal{N}_2) = (\mathcal{M}_1 \cap \mathcal{M}_2) \hat{\otimes} (\mathcal{N}_1 \cap \mathcal{N}_2)$$

and

$$(\mathcal{M}_1 \hat{\otimes} \mathcal{N}_1) \vee (\mathcal{M}_2 \hat{\otimes} \mathcal{N}_2) = (\mathcal{M}_1 \vee \mathcal{M}_2) \hat{\otimes} (\mathcal{N}_1 \vee \mathcal{N}_2).$$

Proof. As for the first equality, only the inclusion

$$(\mathcal{M}_1 \hat{\otimes} \mathcal{N}_1) \cap (\mathcal{M}_2 \hat{\otimes} \mathcal{N}_2) \subset (\mathcal{M}_1 \cap \mathcal{M}_2) \hat{\otimes} (\mathcal{N}_1 \cap \mathcal{N}_2)$$

needs to be dealt with, for the converse inclusion is trivially verified. To this aim, we show that

$$\big((\mathcal{M}_1 \cap \mathcal{M}_2) \hat{\otimes} (\mathcal{N}_1 \cap \mathcal{N}_2)\big)^{\wr} \subset \big((\mathcal{M}_1 \hat{\otimes} \mathcal{N}_1) \cap (\mathcal{M}_2 \hat{\otimes} \mathcal{N}_2)\big)^{\wr}.$$

Now by Theorem 4.4 and Lemma 4.7 we have

$$\begin{aligned}\big((\mathcal{M}_1 \cap \mathcal{M}_2) \hat{\otimes} (\mathcal{N}_1 \cap \mathcal{N}_2)\big)^{\wr} &= (\mathcal{M}_1 \cap \mathcal{M}_2)^{\wr} \hat{\otimes} (\mathcal{N}_1 \cap \mathcal{N}_2)^{\wr} \\ &= (\mathcal{M}_1^{\wr} \vee \mathcal{M}_2^{\wr}) \hat{\otimes} (\mathcal{N}_1^{\wr} \vee \mathcal{N}_2^{\wr}) \\ &\subset (\mathcal{M}_1^{\wr} \hat{\otimes} \mathcal{N}_1^{\wr}) \vee (\mathcal{M}_2^{\wr} \hat{\otimes} \mathcal{N}_2^{\wr}) \\ &= (\mathcal{M}_1 \hat{\otimes} \mathcal{N}_1)^{\wr} \vee (\mathcal{M}_2 \hat{\otimes} \mathcal{N}_2)^{\wr} \\ &= \big((\mathcal{M}_1 \hat{\otimes} \mathcal{N}_1) \cap (\mathcal{M}_2 \hat{\otimes} \mathcal{N}_2)\big)^{\wr}.\end{aligned}$$

In the second equality both inclusions can be verified directly. □

Remark 4.9. By using Theorem 4.5 one sees that the first equality of the above result holds with infinite graded tensor products as well.

Before stating our next result, we recall that a factor is a von Neumann algebra with trivial center.

Proposition 4.10. *Under the same hypotheses as in Theorem 4.5, an infinite Fermi tensor product is a factor if and only if each component is a factor.*

Proof. Set $\mathcal{R} := \hat{\otimes}_{\boldsymbol{\xi}} \mathcal{R}_i$, where the $\mathcal{R}_i$'s are all factors. With $U = \otimes_{i\in\mathbb{N}} U_i$, thanks to Theorem 4.5 and Remark 4.9 we have

$$\eta_U(\mathcal{R}) \cap \mathcal{R}^{\wr} = \hat{\otimes}_{\boldsymbol{\xi}} \eta_{U_i}(\mathcal{R}_i) \cap \hat{\otimes}_{\boldsymbol{\xi}} \mathcal{R}_i^{\wr} = \hat{\otimes}_{\boldsymbol{\xi}} \left(\eta_{U_i}(\mathcal{R}_i) \cap \mathcal{R}_i^{\wr}\right) = \mathbb{C}\,,$$

which shows that $\mathcal{R}$ is still a factor. The converse implication is obvious. □

A representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ of a given C^*-algebra is said to be factorial if $\pi(\mathfrak{A})''$ is a factor, *i.e.* $\pi(\mathfrak{A})'' \cap \pi(\mathfrak{A})' = \mathbb{C}1$. A state φ of a C^*-algebra is factorial if its GNS representation is. Moreover, the type of a factorial state is by definition the same as the type of the factor generated by its GNS representation.

Proposition 4.11. *Let $(\mathfrak{A}_i, \theta_i)$ be $\mathbb{Z}_2$-graded C^*-algebras, and let ω_i be in $\mathcal{S}_+(\mathfrak{A}_i)$, $i \in \mathbb{N}$. The product state $\omega = \times_i \omega_i$ is factorial if and only if each ω_i is.*

Proof. A straightforward application of Remark 3.3 and Proposition 4.10. □

Actually, far more can be said about the type of factor one can obtain from a GNS representation as above. In fact, the analysis conducted in [20] for tensor products carries over almost *verbatim* to the graded case. More precisely, we can provide a graded version of Theorem 2.2 in [20]. We limit ourselves to stating the result since the proof is exactly the same as the original by Størmer.

Proposition 4.12. *If ω is an even factorial state on a $\mathbb{Z}$-graded C^*-algebra $(\mathfrak{A}, \theta)$, then*

(i) $\times^{\mathbb{N}}\omega$ *is of type I_1 if and only if ω is mutiplicative;*
(ii) $\times^{\mathbb{N}}\omega$ *is of type I_∞ if and only if ω is pure but is not multiplicative;*
(iii) $\times^{\mathbb{N}}\omega$ *is of type II_1 if and only if ω is a trace but is not multiplicative.*
(iv) $\times^{\mathbb{N}}\omega$ *is of type II_∞ if and only if the restriction of the vector state φ_{ξ_ω} to $\pi_\omega(\mathfrak{A})'$ is a trace, and ω is neither pure nor a trace.*
(v) $\times^{\mathbb{N}}\omega$ *is of type III if and only if the restriction of the vector state φ_{ξ_ω} to $\pi_\omega(\mathfrak{A})'$ is not a trace.*

Acknowledgments

The authors acknowledge Italian INDAM-GNAMPA.

References

[1] Accardi L., Ben Ghorbal A., Crismale V., Lu Y.G. *Singleton conditions and quantum de Finetti's theorems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **11** (2008), 639–660.

[2] Accardi L., Lu Y. G. *A continuous version of de Finetti's theorem*, Ann. Probab. **21** (1993), 1478–1493.

[3] Araki H. *Mathematical Theory of Quantum Fields*, International Series of Monographs on Physics, 101, Oxford University Press, 2009.

[4] Araki H., Moriya H. *Joint extension of states of subsystems for a CAR system*, Commun. Math. Phys. **237** (2003), 105–122.

[5] Avitzour D. *Free products of C^*-algebras*, Trans. Am. Math. Soc. **271** (1982), 423–435.

[6] Bratteli O., Robinson D.W. *Operator algebras and quantum statistical mechanics 1*, 2nd edition, Springer-Verlag, New York, 1997.

[7] Crismale V., Duvenhage R., Fidaleo F. *C^*-fermi systems and detailed balance*, Anal. Math. Phys. **11** (2021), Paper No. 11.

[8] Crismale V., Fidaleo F. *de Finetti theorem on the CAR algebra*, Commun. Math. Phys. **315** (2012), 135–152.

[9] Crismale V., Fidaleo F. *Exchangeable stochastic processes and symmetric states in quantum probability*, Ann. Mat. Pura Appl., **194** (2015), 969–993.

[10] Crismale V., Fidaleo F. *Symmetries and ergodic properties in quantum probability*, Colloq. Math. **149** (2017), 1–20.

[11] Crismale V., Rossi S., Zurlo P. *On C^*-norms on Fermi tensor products*, to appear in Banach J. Math. Anal., arXiv preprint 2112.03988.

[12] Dixmier J. *C^*-algebras*, North Holland, Amsterdam 1977.

[13] Guichardet A., *Tensor products of C^*-algebras Part II. Infinite tensor products*, Lecture Notes Series No. 13, Aahrus Universitet.

[14] Hewitt E., Savage L.F. *Symmetric measures on Cartesian products* Trans. Am. Math. Soc. **80**, 470–501. (1955)

[15] Kallenberg O. *Probabilistic Symmetries and Invariance principles*, Springer, Berlin 2005.

[16] Köstler C. *A noncommutative extended De Finetti theorem* J. Funct. Anal. **258** (2010), 1073–1120.

[17] Rieffel M., Van Daele A., *The commutation theorem for tensor products of von Neumann algebras*, Bull. London. Math. Soc. **7** (1975), 257–260.

[18] Sakai S. *C^*-Algebras and W^*-Algebras*, Springer, Berlin 1971.

[19] Størmer E. *Large Groups of Automorphims of C^*-Algebras*, Commun. Math. Phys. **5** (1967), 1–22.

[20] Størmer E. *Symmetric States of infinite Tensor Products of C*-algebras.* J. Funct. Anal. **3** (1969), 48–68.

[21] Takesaki M. *Theory of operator algebras I*, Springer, Berlin–Heidelberg–New York 1979.

[22] Tomita M. *Standard forms of von Neumann algebras*, Vth functional analysis simposium of the mathematical society of Japan, Sendai 1967.

Vitonofrio Crismale, Dipartimento di Matematica, Università degli studi di Bari, Via E. Orabona, 4, 70125 Bari, Italy
Email address: vitonofrio.crismale@uniba.it

Stefano Rossi, Dipartimento di Matematica, Università degli studi di Bari, Via della E. Orabona, 4, 70125 Bari, Italy
Email address: stefano.rossi@uniba.it

Paola Zurlo, Dipartimento di Matematica, Università degli studi di Bari, Via E. Orabona, 4, 70125 Bari, Italy
Email address: paola.zurlo@uniba.it