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Calculus of Variations

On the equivalence between an Onofri-type inequality by Del Pino–Dolbeault and the sharp logarithmic Moser–Trudinger inequality

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Abstract

In this paper we consider the N-dimensional Euclidean Onofri inequality proved by del Pino and Dolbeault (Int Math Res Not IMRN 15:3600–3611, 2013) for smooth compactly supported functions in $\mathbb{R}^N$, $N \geq 2$. We extend the inequality to a suitable weighted Sobolev space, although no clear connection with standard Sobolev spaces on $\mathbb{S}^N$ through stereographic projection is present, except for the planar case. Moreover, in any dimension $N \geq 2$, we show that the Euclidean Onofri inequality is equivalent to the logarithmic Moser–Trudinger inequality with sharp constant proved by Carleson and Chang (Bull Sci Math 110(2):113–127, 1986) for balls in $\mathbb{R}^N$.

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Contents

1 Introduction

In this paper we discuss connections between Onofri-type inequalities on the Euclidean space and the logarithmic Moser–Trudinger inequality on balls in $\mathbb{R}^N$. Let $N \geq 2$ be an integer and let ω_{N-1} and V_N denote respectively the $N-1$ dimensional measure of the unit sphere

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and the volume of the unit ball in $\mathbb{R}^N$. In [9] del Pino and Dolbeault obtained the following N-dimensional Euclidean Onofri inequality for any $u \in C_0^\infty(\mathbb{R}^N)$:

$$\ln\left(\int_{\mathbb{R}^N} e^u \, d\mu_N\right) \leq \frac{1}{\widetilde{\omega_N}} \int_{\mathbb{R}^N} H_N(u, \mu_N)\, dx + \int_{\mathbb{R}^N} u \, d\mu_N, \tag{1.1}$$

where

$$\widetilde{\omega_N} := N^N \left(\frac{N}{N-1}\right)^{N-1} \omega_{N-1}, \tag{1.2}$$

$$\mu_N(x) := \frac{1}{V_N \left(1 + |x|^{\frac{N}{N-1}}\right)^N}, \quad d\mu_N(x) := \mu_N(x)\, dx, \tag{1.3}$$

and

$$H_N(u, \mu_N) := |\nabla v_N + \nabla u|^N - |\nabla v_N|^N - N|\nabla v_N|^{N-2} \nabla v_N \cdot \nabla u, \tag{1.4}$$

with

$$v_N(x) := \ln \mu_N(x). \tag{1.5}$$

They were able to obtain (1.1) by considering the endpoint of a family of optimal Gagliardo-Nirenberg interpolation inequalities, discovered in [7] and extended in [8]. We point out that the Euclidean Onofri inequality (1.1) was obtained with different techniques also by Agueh et al. [2], for any $u \in W^{1,N}(\mathbb{R}^N)$. We also refer to [15] for an improved version of (1.1) involving singular weights (see also [10–12]).

In dimension $N = 2$, (1.1) becomes

$$\ln\left(\int_{\mathbb{R}^2} e^u \, d\mu_2\right) \leq \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} u \, d\mu_2 \tag{1.6}$$

for any $u \in C_0^\infty(\mathbb{R}^N)$. Actually, the 2-dimensional Euclidean Onofri inequality (1.6) holds for any function $u \in L^1(\mathbb{R}^2, d\mu_2)$ such that $|\nabla u| \in L^2(\mathbb{R}^2, dx)$. Indeed, by using the stereographic projection from $\mathbb{S}^2$ onto $\mathbb{R}^2$ with respect to the north pole, (1.6) is equivalent to the following classical Onofri inequality

$$\ln\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^u \, dv_{g_0}\right) \leq \frac{1}{16\pi} \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 \, dv_{g_0} + \frac{1}{4\pi} \int_{\mathbb{S}^2} u \, dv_{g_0}, \tag{1.7}$$

for any $u \in H^1(\mathbb{S}^2, g_0)$, where g_0 is the standard metric on $\mathbb{S}^2$. Inequality (1.7) plays an important role in spectral analysis for the Laplace-Beltrami operator thanks to Polyakov's formula (see [18, 19, 21, 22]) and it was proved in 1982 by Onofri [17] using conformal invariance and an earlier result by Aubin [1].

Inequality (1.7) is also strictly related to the Moser–Trudinger inequality [16, 20, 24, 25]. Specifically, in [16, Theorem 2] Moser proved via symmetrization techniques that there exists a constant $S > 1$ such that

$$\frac{1}{4\pi} \int_{\mathbb{S}^2} e^{4\pi \left(\frac{u - \frac{1}{4\pi}\int_{\mathbb{S}^2} u \, dv_{g_0}}{\int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 \, dv_{g_0}}\right)^2} dv_{g_0} \leq S, \tag{1.8}$$

for any $u \in H^1(\mathbb{S}^2, g_0)$. As a straightforward consequence of (1.8) and Young's inequality, one can prove that

$$\ln\left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^u \, dv_{g_0}\right) \leq \frac{1}{16\pi} \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 \, dv_{g_0} + \frac{1}{4\pi} \int_{\mathbb{S}^2} u \, dv_{g_0} + \ln S, \tag{1.9}$$

which is a non-optimal version of (1.7) due to the constant $\ln S > 0$. For this reason (1.7) is sometimes called Moser–Onofri or Moser–Trudinger–Onofri inequality in the literature.

An extension of (1.7) to $\mathbb{S}^N$ in higher dimensions was stated in [4, 5], however no clear connection with (1.1) through stereographic projection is present. Taking into account this, we want to extend (1.1) to a suitable weighted Sobolev space. More precisely, we will consider the space

$$W_{\mu_N}(\mathbb{R}^N) := \{u \in L^1(\mathbb{R}^N, d\mu_N) : \ |\nabla u| \in L^N(\mathbb{R}^N, dx), \\ |\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)\},$$

endowed with the norm

$$\|u\|_{\mu_N} := \int_{\mathbb{R}^N} |u|\, d\mu_N + \|\nabla u\|_N + \left(\int_{\mathbb{R}^N} |\nabla u|^2 |\nabla v_N|^{N-2}\, dx \right)^{\frac{1}{2}}. \tag{1.10}$$

We observe that, if $N = 2$, $W_{\mu_2}(\mathbb{R}^2)$ coincides with the set of functions $u \in L^1(\mathbb{R}^2, d\mu_2)$ such that $|\nabla u| \in L^2(\mathbb{R}^2, dx)$, in which (1.6) holds.

To extend (1.1) to $W_{\mu_N}(\mathbb{R}^N)$, we show that smooth compactly supported functions of $\mathbb{R}^N$ are dense in $W_{\mu_N}(\mathbb{R}^N)$ with respect to the norm (1.10).

Theorem 1.1 *Assume $N \geq 2$. Then*

$$W_{\mu_N}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}.$$

Therefore, by using the previous density result, we are able to prove the following Theorem.

Theorem 1.2 *Assume $N \geq 2$. Then, for any $u \in W_{\mu_N}(\mathbb{R}^N)$, we have*

$$\ln \left(\int_{\mathbb{R}^N} e^u\, d\mu_N \right) \leq \frac{1}{\widetilde{\omega_N}} \int_{\mathbb{R}^N} H_N(u, \mu_N)\, dx + \int_{\mathbb{R}^N} u\, d\mu_N. \tag{1.11}$$

Now, let us recall the Logarithmic Moser–Trudinger inequality on domains of $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Similarly to (1.9), by means of the Moser–Trudinger inequality for the Sobolev space $W_0^{1,N}(\Omega)$ [16, Theorem 1], it is simple to prove that there exists a constant $c = c(N)$ such that

$$\ln \left(\frac{1}{|\Omega|} \int_\Omega e^u\, dx \right) \leq \frac{1}{\widetilde{\omega_N}} \int_\Omega |\nabla u|^N\, dx + \ln c, \tag{1.12}$$

for any $u \in W_0^{1,N}(\Omega)$.

In [6], Carleson and Chang improved inequality (1.12) when Ω is the unit ball B_1 of $\mathbb{R}^N$. Indeed they prove that for any $u \in W_0^{1,N}(B_1)$:

$$\ln \left(\frac{1}{V_N} \int_{B_1} e^u\, dx \right) < \frac{1}{\widetilde{\omega_N}} \int_{B_1} |\nabla u|^N\, dx + \sum_{k=1}^{N-1} \frac{1}{k}. \tag{1.13}$$

Actually, by (1.13) one can easily show that for any open ball $B \subset \mathbb{R}^N$, and for any $u \in W_0^{1,N}(B)$, we have

$$\ln \left(\frac{1}{|B|} \int_B e^u\, dx \right) < \frac{1}{\widetilde{\omega_N}} \int_B |\nabla u|^N\, dx + \sum_{k=1}^{N-1} \frac{1}{k}.$$

Inequality (1.13) is sharp, in the sense that the functional $J : W_0^{1,N}(B_1) \to \mathbb{R}$ defined as

$$J(u) := \frac{1}{\widetilde{\omega_N}} \int_{B_1} |\nabla u|^N \, dx - \ln \left(\frac{1}{V_N} \int_{B_1} e^u \, dx \right), \tag{1.14}$$

satisfies

$$\inf_{W_0^{1,N}(B_1)} J = -\sum_{k=1}^{N-1} \frac{1}{k}.$$

Our goal is to establish a natural connection between the sharp inequality (1.13) and the inequality (1.11). We can consider the functional $I : W_{\mu_N}(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$I(u) := \frac{1}{\widetilde{\omega_N}} \int_{\mathbb{R}^N} H_N(u, \mu_N) \, dx + \int_{\mathbb{R}^N} u \, d\mu_N - \ln \left(\int_{\mathbb{R}^N} e^u \, d\mu_N \right). \tag{1.15}$$

In our main result, without using (1.1) or (1.13), we prove the following identity between $\inf_{W_0^{1,N}(B_1)} I$ and $\inf_{W_{\mu_N}(\mathbb{R}^N)} J$.

Theorem 1.3 *For any $N \geq 2$, we have*

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I = \inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k}.$$

Theorem 1.3 shows that the sharp inequalities (1.11) and (1.13) are equivalent. In particular, we can give a simple alternative proof of (1.11). On the one hand, it follows from (1.13) that, for any $u \in W_0^{1,N}(B_1)$:

$$J(u) > -\sum_{k=1}^{N-1} \frac{1}{k} = \inf_{W_0^{1,N}(B_1)} J.$$

On the other hand, since $I(0) = 0$, Theorem 1.3 yields

$$0 = \inf_{W_{\mu_N}(\mathbb{R}^N)} I = \min_{W_{\mu_N}(\mathbb{R}^N)} I.$$

Theorem 1.3 is interesting even in the simplest case $N = 2$. Indeed, as announced in [3] (see also [13]), it shows that Onofri's inequality (1.7) on $\mathbb{S}^2$ is equivalent to the sharp Logarithmic Moser–Trudinger inequality (1.13) by Carleson and Chang.

As discussed above the inequalities with non-optimal constants (1.9) and (1.12) are direct corollaries of the Moser–Trudinger inequality for the sphere [16, Theorem 2] and its analogue for bounded domains [16, Theorem 1]. In [16], Moser proved these theorems indipendently, although their proofs are based on similar arguments relying on symmetrization techniques. He explicitly states that it seems impossible to deduce one Theorem from the other. Although the equivalence between Moser's theorems is still an open problem, here we have proved the equivalence between the corresponding logarithmic inequalities with sharp constants (1.13) (with $N = 2$) and (1.7).

2 Preliminaries and function spaces

In this section we establish some preliminary estimates and discuss the optimal weighted Sobolev space in which the N-Euclidean Onofri inequality holds. Throughout the paper we let $\widetilde{\omega}_N, \mu_N, d\mu_N$ and v_N be defined as in (1.2), (1.3), (1.4) and (1.5). A straightforward computation in polar coordinates shows that $d\mu_N$ is a probability measure on $\mathbb{R}^N$, namely

$$\int_{\mathbb{R}^N} d\mu_N = 1.$$

Moreover, we have

$$\nabla v_N(x) = -\frac{N^2}{N-1}\,\frac{|x|^{\frac{1}{N-1}}}{1+|x|^{\frac{N}{N-1}}}\,\frac{x}{|x|},$$

and

$$\Delta_N v_N(x) = -N^N\left(\frac{N}{N-1}\right)^{N-1} V_N\mu_N(x), \tag{2.1}$$

where Δ_N denotes the N-Laplacian operator, defined by

$$\Delta_N u := \operatorname{div}\left(|\nabla u|^{N-2}\nabla u\right).$$

Let us denote

$$R_N(X,Y) := |X+Y|^N - |X|^N - N|X|^{N-2}X\cdot Y \qquad (X,Y)\in\mathbb{R}^N\times\mathbb{R}^N,$$

and

$$\begin{aligned} H_N(u(x),\mu_N(x)) &:= R_N(\nabla v_N(x),\nabla u(x)) \\ &= |\nabla v_N(x)+\nabla u(x)|^N - |\nabla v_N(x)|^N - N|\nabla v_N(x)|^{N-2}\nabla v_N(x)\cdot\nabla u(x). \end{aligned}$$

Lemma 2.1 *For any $N\in\mathbb{N}$, $N\geq 2$, there exists a constant c_N such that*

$$0\leq R_N(X,Y)\leq c_N(|Y|^N+|Y|^2|X|^{N-2}),\quad \forall\, X,Y\in\mathbb{R}^N.$$

Proof If $N=2$, $R_2(X,Y)=|Y|^2$ does not depend on X and the thesis is trivial. Let $N\geq 3$ and let $X,Y\in\mathbb{R}^N$. We consider the C^2 function $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(t) = |tY+X|^N.$$

By a direct computation, we have

$$f'(t) = N\,|tY+X|^{N-2}\,(tY+X)\cdot Y,$$

and

$$f''(t) = N(N-2)\,|tY+X|^{N-4}\,((tY+X)\cdot Y)^2 + N\,|tY+X|^{N-2}\,|Y|^2$$

if $tY+X\neq 0$, and $f''(t)=0$ otherwise.

Now we observe that, if $t\in[0,1]$, we have

$$0\leq f''(t)\leq 2c_N\left(|Y|^N+|Y|^2|X|^{N-2}\right), \tag{2.2}$$

where $c_N = N(N-1)2^{N-4}$. By Taylor's theorem with Lagrange remainder, there exists $\bar{t} \in]0,1[$ such that

$$\frac{1}{2} f''(\bar{t}) = f(1) - f(0) - f'(0) = |Y + X|^N - |X|^N - N\,|X|^{N-2}\, X \cdot Y.$$

Taking into account (2.2), we deduce that

$$0 \le R_N(X,Y) \le c_N \left(|Y|^N + |Y|^2 |X|^{N-2}\right).$$

□

We define

$$W_{\mu_N}(\mathbb{R}^N) := \{u \in L^1(\mathbb{R}^N, d\mu_N) : \ |\nabla u| \in L^N(\mathbb{R}^N, dx), \\ |\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)\}.$$

In this space we introduce the following norm:

$$\|u\|_{\mu_N} := \int_{\mathbb{R}^N} |u|\, d\mu_N + \|\nabla u\|_N + \left(\int_{\mathbb{R}^N} |\nabla u|^2 |\nabla v_N|^{N-2}\, dx\right)^{\frac{1}{2}}.$$

An immediate consequence of Lemma 2.1 is the following integrability condition for H_N.

Corollary 2.2 *Let $u \in W_{\mu_N}(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} H_N(u, \mu_N)\, dx < +\infty.$$

It is interesting to compare W_{μ_N} with $W^{1,N}$.

Lemma 2.3 *If $u \in W^{1,N}(\mathbb{R}^N)$ is such that $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$, then $u \in W_{\mu_N}(\mathbb{R}^N)$. Moreover, we have $W_{\mu_N}(\mathbb{R}^N) \setminus W^{1,N}(\mathbb{R}^N) \neq \emptyset$.*

Proof Let $u \in W^{1,N}(\mathbb{R}^N)$ such that $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$.

$$\int_{\mathbb{R}^N} |u(x)|\, d\mu_N = \int_{\mathbb{R}^N} |u(x)| \frac{1}{V_N \left(1 + |x|^{\frac{N}{N-1}}\right)^N}\, dx.$$

By assumption $\|u\|_N < +\infty$, while

$$\gamma_N := \int_{\mathbb{R}^N} \frac{1}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{\frac{N^2}{N-1}}}\, dx < +\infty.$$

By Hölder inequality we have

$$\int_{\mathbb{R}^N} |u(x)|\, d\mu_N \le \frac{\gamma_N^{\frac{N-1}{N}}}{V_N} \|u\|_N < +\infty.$$

Therefore $u \in W_{\mu_N}(\mathbb{R}^N)$.

Finally, the constant $u \equiv 1$ is such that $u \in W_{\mu_N}(\mathbb{R}^N) \setminus W^{1,N}(\mathbb{R}^N)$. □

The next lemma shows that the condition $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$ is necessary for the integrability of $H(u, \mu_N)$, at least in even dimension.

Lemma 2.4 *If $N \geq 4$ is even, then*

$$H(u, \mu_N) \geq \frac{N}{2}|\nabla u|^2 |\nabla v_N|^{N-2}.$$

Proof The proof is based on the following inequalities, which hold for $a, b \in \mathbb{R}$ and $k \in \mathbb{N}$, with $a \geq 0$ and $a + b \geq 0$:

(1) if $k \geq 2$, then $(a+b)^k \geq a^k + ka^{k-1}b + b^k$.
(2) if $k \geq 3$, then $(a+b)^k \geq a^k + ka^{k-1}b + kab^{k-1} + b^k$.

Both (1) and (2) can be easily proved by induction on k. Let $X, Y \in \mathbb{R}^N$. If $\frac{N}{2}$ is even, using inequality (1) with $k = \frac{N}{2}$, $a = |X|^2$ and $b = 2X \cdot Y + |Y|^2$ we get that

$$\begin{aligned} R_N(X, Y) &= \left(|X|^2 + 2X \cdot Y + |Y|^2\right)^{\frac{N}{2}} - |X|^N - N|X|^{N-2} X \cdot Y \\ &\geq \frac{N}{2}|X|^{N-2}|Y|^2 + \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}}. \end{aligned}$$

Since $\frac{N}{2}$ is even, we get

$$R_N(X, Y) \geq \frac{N}{2}|X|^{N-2}|Y|^2. \tag{2.3}$$

Similarly, if $\frac{N}{2}$ is odd (hence $N \geq 6$), we can apply inequality (2) (with k, a and b as above) to get:

$$\begin{aligned} R_N(X, Y) &\geq \frac{N}{2}|X|^{N-2}|Y|^2 + \frac{N}{2}|X|^2 \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}-1} + \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}} \\ &= \frac{N}{2}|X|^{N-2}|Y|^2 + \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}-1} \left(\frac{N}{2}|X|^2 + 2X \cdot Y + |Y|^2\right) \\ &\geq \frac{N}{2}|X|^{N-2}|Y|^2 + \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}-1} |X + Y|^2. \end{aligned}$$

Since $\frac{N}{2} - 1$ is even, we get again (2.3). Hence (2.3) holds whenever $\frac{N}{2}$ is integer and the conclusion follows by taking $X = \nabla v_N$ and $Y = \nabla u$. $\square$

Remark 2.5 If $N \geq 3$ and $u \in W^{1,N}(\mathbb{R}^N)$, in general the integrability condition $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$ of Lemma 2.3 is not satisfied. Indeed, let

$$u(x) := \sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}} w(2(|x| - k)),$$

where

$$w(r) := \begin{cases} 1 + r & \text{if } r \in [-1, 0[, \\ 1 - r & \text{if } r \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

A direct computation shows that the function u satisfies

$$u \in W^{1,N}(\mathbb{R}^N) \quad \text{and} \quad |\nabla u|^2 |\nabla v_N|^{N-2} \notin L^1(\mathbb{R}^N, dx). \tag{2.4}$$

In particular, if N is even, Lemma 2.4 yields

$$\int_{\mathbb{R}^N} H(u, \mu_N)\, dx = +\infty.$$

To prove (2.4) we observe that the functions $x \to w(2(|x|-k))$ have disjoint support as k varies in $\mathbb{N}$ with $k \geq 2$. Then

$$|u(x)|^N = \sum_{k=2}^{\infty} \frac{1}{k^N (\ln k)^{\frac{N}{2}}} (w(2(|x|-k)))^N.$$

Integrating in polar coordinates and using that $0 \leq w \leq 1$, we find

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^N \, dx &\leq \omega_{N-1} \sum_{k=2}^{\infty} \frac{1}{k^N (\ln k)^{\frac{N}{2}}} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \rho^{N-1} \, d\rho \\ &= \frac{\omega_{N-1}}{N} \sum_{k=2}^{\infty} \frac{1}{k^N (\ln k)^{\frac{N}{2}}} \left[\left(k+\frac{1}{2}\right)^N - \left(k-\frac{1}{2}\right)^N \right] \\ &\leq c_1(N) \sum_{k=2}^{\infty} \frac{1}{k (\ln k)^{\frac{N}{2}}} < +\infty\end{aligned}$$

since $\frac{N}{2} > 1$. Similarly, we have

$$|\nabla u(x)| = 2 \sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}} |w'(2(|x|-k))|,$$

and

$$\int_{\mathbb{R}^N} |\nabla u(x)|^N \, dx = c_2(N) \sum_{k=2}^{\infty} \frac{1}{k^N (\ln k)^{\frac{N}{2}}} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \rho^{N-1} \, d\rho < +\infty.$$

Now, we observe that $|\nabla v_N(x)| \sim \frac{N^2}{N-1} \frac{1}{|x|}$ if $|x| \gg 1$, hence, taking k_0 sufficiently large, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2 |\nabla v_N|^{N-2} \, dx &\geq c_3(N) + c_4(N) \sum_{k=k_0}^{\infty} \frac{1}{k^2 \ln k} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{1}{\rho^{N-2}} \rho^{N-1} \, d\rho \\ &\geq c_5(N) \left(1 + \sum_{k=k_0}^{\infty} \frac{1}{k \ln k} \right) = +\infty.\end{aligned}$$

Now, we want to prove the density result of Theorem 1.1. Let us define

$$\mathcal{X} := \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}.$$

Clearly, $C_0^\infty(\mathbb{R}^N) \subset W_{\mu_N}(\mathbb{R}^N)$, so that $\mathcal{X} \subseteq W_{\mu_N}(\mathbb{R}^N)$. Our aim is to prove that the two spaces coincide. We will prove this in two steps. Firstly, by using a similar idea of Tintarev and Fieseler in [23, Remark 2.2], we show that bounded functions in $W_{\mu_N}(\mathbb{R}^N)$ belong to $\mathcal{X}$.

Lemma 2.6 *Let $u \in W_{\mu_N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then $u \in \mathcal{X}$.*

Proof For any $k \in \mathbb{N}$, $k \geq 2$, let us denote

$$k^* := \frac{1}{\left(1 - \frac{1}{\sqrt[k]{k}}\right)^k}.$$

We define

$$\eta_k(x) := \begin{cases} \frac{1}{\sqrt[k]{k}} & \text{if } |x| \leq 1, \\ \dfrac{1}{\sqrt[k]{|x|}} + \dfrac{1}{\sqrt[k]{k}} - 1 & \text{if } 1 < |x| \leq k^*, \\ 0 & \text{if } k^* < |x|. \end{cases}$$

Let $u \in W_{\mu_N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We define $u_k := u\,\eta_k$. Observe that

$$\nabla u_k(x) := \begin{cases} \dfrac{1}{\sqrt[k]{k}} \nabla u(x) & \text{if } |x| < 1, \\ \left(\dfrac{1}{\sqrt[k]{k}} + \dfrac{1}{\sqrt[k]{|x|}} - 1\right) \nabla u(x) - \dfrac{1}{k} \dfrac{u(x)}{|x|^{\frac{1}{k}+1}} \dfrac{x}{|x|} & \text{if } 1 < |x| < k^*, \\ 0 & \text{if } k^* < |x|. \end{cases}$$

First of all, since $u \in L^\infty(\mathbb{R}^N)$ and $\eta_k(x) \to 1$ for any $x \in \mathbb{R}^N$ as $k \to +\infty$, we can apply dominated convergence theorem to say that

$$\int_{\mathbb{R}^N} |u - u_k(x)|\, d\mu_N \to 0 \quad \text{as } k \to +\infty.$$

Now we want to prove

$$\|\nabla(u - u_k)\|_N \to 0 \quad \text{as } k \to +\infty.$$

We will denote with $o(1)$ any quantity that goes to 0 as $k \to +\infty$.

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u - \nabla u_k|^N \, dx \\ &\quad = \int_{|x|<1} |\nabla u - \nabla u_k|^N \, dx + \int_{1<|x|<k^*} |\nabla u - \nabla u_k|^N \, dx + \int_{|x|>k^*} |\nabla u|^N \, dx. \end{aligned}$$

We have

$$\int_{|x|<1} |\nabla u - \nabla u_k|^N \, dx = \left(1 - \frac{1}{\sqrt[k]{k}}\right)^N \int_{|x|<1} |\nabla u|^N \, dx = o(1).$$

Since $\|\nabla u\|_N < +\infty$ and $k^* \to +\infty$ as $k \to +\infty$, we easily obtain

$$\int_{|x|>k^*} |\nabla u - \nabla u_k|^N \, dx = \int_{|x|>k^*} |\nabla u|^N \, dx = o(1).$$

Since $u \in L^\infty(\mathbb{R}^N)$, there exists a constant M_u such that $|u(x)| \leq M_u$ for almost every $x \in \mathbb{R}^N$.

$$\begin{aligned}
&\int_{1<|x|<k^*} |\nabla u - \nabla u_k|^N \, dx \\
&\quad = \int_{1<|x|<k^*} \left| \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}} \right) \nabla u(x) + \frac{1}{k} \frac{u(x)}{|x|^{\frac{1}{k}+1}} \frac{x}{|x|} \right|^N \, dx \\
&\quad \le C \int_{1<|x|<k^*} \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}} \right)^N |\nabla u(x)|^N \, dx \\
&\quad + C \frac{M_u^N}{k^N} \int_{1<|x|<k^*} \frac{1}{|x|^{\frac{N}{k}+N}} \, dx.
\end{aligned}$$

Now we observe that

$$\frac{M_u^N}{k^N} \int_{1<|x|<k^*} \frac{1}{|x|^{\frac{N}{k}+N}} \, dx < \frac{c_1}{k^N} \int_1^{+\infty} \frac{1}{\rho^{\frac{N}{k}+1}} \, d\rho = \frac{c_2}{k^{N-1}} = o(1).$$

Moreover, we can apply dominated convergence theorem to say that

$$\int_{1<|x|<k^*} \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}} \right)^N |\nabla u(x)|^N \, dx \to 0 \quad \text{as } k \to +\infty.$$

Hence

$$\|\nabla(u - u_k)\|_N \to 0 \quad \text{as } k \to +\infty.$$

It remains to prove, in the case $N \ge 3$, that

$$\left(\int_{\mathbb{R}^N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} \, dx \right)^{\frac{1}{2}} \to 0 \quad \text{as } k \to +\infty.$$

We observe that $|\nabla v_N(x)| \sim \dfrac{1}{|x|}$ as $|x| \to +\infty$, by which there exists a constant $r_N > 1$ such that $|\nabla v_N(x)| \le \dfrac{2}{|x|}$ if $|x| \ge r_N$. Since $k \to +\infty$ implies $k^* \to +\infty$, we can assume from the beginning that k is such that $k^* > r_N$. Moreover, we recall that $|\nabla v_N|$ is bounded from above.

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} \, dx \\
&\quad = \int_{|x|<1} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} \, dx + \int_{1<|x|<r_N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} \, dx \\
&\quad + \int_{r_N<|x|<k^*} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} \, dx + \int_{k^*<|x|} |\nabla u|^2 |\nabla v_N|^{N-2} \, dx.
\end{aligned}$$

Reasoning in a similar way to the previous case, we obtain

$$\int_{|x|<1} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} \, dx = o(1),$$

$$\int_{1<|x|<r_N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} \, dx = o(1),$$

and

$$\int_{k^*<|x|} |\nabla u|^2 |\nabla v_N|^{N-2}\, dx = o(1).$$

Now we observe that

$$\begin{aligned}
&\int_{r_N<|x|<k^*} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2}\, dx \\
&= \int_{r_N<|x|<k^*} \left| \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}}\right) \nabla u(x) + \frac{1}{k} \frac{u(x)}{|x|^{\frac{1}{k}+1}} \frac{x}{|x|} \right|^2 |\nabla v_N|^{N-2}\, dx \\
&\le C \int_{r_N<|x|<k^*} \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}}\right)^2 |\nabla u(x)|^2\, |\nabla v_N|^{N-2}\, dx + C \frac{1}{k^2} \int_{r_N<|x|} \frac{1}{|x|^{\frac{2}{k}+N}}\, dx.
\end{aligned}$$

Clearly,

$$\frac{1}{k^2} \int_{r_N<|x|} \frac{1}{|x|^{\frac{2}{k}+N}}\, dx = o(1).$$

Moreover, taking into account that $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$, we can apply dominated convergence theorem to say that

$$\int_{r_N<|x|<k^*} \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}}\right)^2 |\nabla u(x)|^2\, |\nabla v_N|^{N-2}\, dx = o(1).$$

Finally, we have

$$\left(\int_{\mathbb{R}^N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2}\, dx \right)^{\frac{1}{2}} \to 0 \quad \text{as } k \to +\infty.$$

We have proved that $\{u_k\}_k$ is a sequence of compactly supported functions such that $\|u - u_k\|_{\mu_N} \to 0$ as $k \to +\infty$.
We want to take a sequence $\{\tilde{u}_k\}_k \subset C_0^\infty(\mathbb{R}^N)$ such that $\|u - \tilde{u}_k\|_{\mu_N} \to 0$ as $k \to +\infty$.
Let us denote with B_{k^*} the open ball $B_{k^*}(O)$.
Since $u_{k|_{B_{k^*}}} \in W_0^{1,N}(B_{k^*})$, there exists $\tilde{u}_k \in C_0^\infty(B_{k^*})$ such that

$$\left\| u_{k|_{B_{k^*}}} - \tilde{u}_k \right\|_{L^N(B_{k^*})} + \left\| \nabla \left(u_{k|_{B_{k^*}}} - \tilde{u}_k \right) \right\|_{L^N(B_{k^*})} < \frac{1}{k}.$$

We can extend to 0 in $\mathbb{R}^N \setminus B_{k^*}$ and consider $\tilde{u}_k \in C_0^\infty(\mathbb{R}^N)$. Hence

$$\|u_k - \tilde{u}_k\|_N + \|\nabla (u_k - \tilde{u}_k)\|_N < \frac{1}{k}.$$

Since

$$\gamma_N := \int_{\mathbb{R}^N} \frac{1}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{\frac{N^2}{N-1}}}\, dx < +\infty,$$

we can apply Hölder inequality and say that

$$\int_{\mathbb{R}^N} |u_k(x) - \tilde{u}_k(x)|\, d\mu_N < \gamma_N^{\frac{N-1}{N}} \|u_k - \tilde{u}_k\|_N < \frac{\gamma_N^{\frac{N-1}{N}}}{k} = o(1).$$

Now let k such that $k^* > r_N$. Since

$$\int_{0<|x|\le k^*} |\nabla v_N|^N \, dx \le C \left(\ln k^* + 1\right),$$

we have

$$\int_{\mathbb{R}^N} |\nabla(u_k - \tilde{u}_k)|^2 |\nabla v_N|^{N-2} \, dx < C \frac{(\ln k^* + 1)^{\frac{N-2}{N}}}{k^2}$$

Now we observe that, as $k \to +\infty$

$$\left(\ln k^* + 1\right)^{\frac{N-2}{N}} \sim k^{\frac{N-2}{N}} \left(-\ln\left(1 - \frac{1}{\sqrt[k]{k}}\right)\right)^{\frac{N-2}{N}},$$

by which

$$\frac{(\ln k^* + 1)^{\frac{N-2}{N}}}{k^2} \sim \frac{\left(-\ln\left(1 - \frac{1}{\sqrt[k]{k}}\right)\right)^{\frac{N-2}{N}}}{k^{1+\frac{2}{N}}} \to 0.$$

Hence

$$\int_{\mathbb{R}^N} |\nabla(u_k - \tilde{u}_k)|^2 |\nabla v_N|^{N-2} \, dx \to 0.$$

Finally, we have

$$\|u - \tilde{u}_k\|_{\mu_N} \le \|u - u_k\|_{\mu_N} + \|u_k - \tilde{u}_k\|_{\mu_N} \to 0 \quad \text{as } k \to +\infty.$$

□

Thanks to previous lemma, it remains to prove that any function $u \in W_{\mu_N}(\mathbb{R}^N)$ can be approximated by a sequence of bounded functions in $W_{\mu_N}(\mathbb{R}^N)$.

Lemma 2.7
Let $u \in W_{\mu_N}(\mathbb{R}^N)$. There exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W_{\mu_N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $\|u - u_k\|_{\mu_N} \to 0$ as $k \to +\infty$.

Proof Let $u \in W_{\mu_N}(\mathbb{R}^N)$. Taking into account [14, Lemma A.4, Chapter 2], we have $|u| \in W_{\mu_N}(\mathbb{R}^N)$ with

$$\nabla|u| = \begin{cases} \nabla u & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) > 0\}, \\ 0 & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) = 0\}, \\ -\nabla u & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) < 0\}. \end{cases}$$

For any $\lambda > 0$, we can define the *truncated function of u of parameter λ* as $u_\lambda := \max\{-\lambda, \min\{u, \lambda\}\}$, that is

$$u_\lambda(x) = \begin{cases} \lambda & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) \ge \lambda\}, \\ u & \text{a.e. in } \{x \in \mathbb{R}^N \mid -\lambda < u(x) < \lambda\}, \\ -\lambda & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) \le -\lambda\}. \end{cases}$$

Let $k \in \mathbb{N}$, $k \ge 1$, and let $\{u_k\}_{k\in\mathbb{N}}$ a sequence of truncated function of u of parameter k. Clearly, $\{u_k\}_{k\in\mathbb{N}} \subset W_{\mu_N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Let us denote with $E_k := \{x \in \mathbb{R}^N \mid |u(x)| \geq k\}$. Hence

$$\int_{\mathbb{R}^N} |u - u_k|\, d\mu_N = \int_{E_k} |u - u_k|\, d\mu_N \leq \int_{E_k} |u|\, d\mu_N.$$

Since $u \in L^1(\mathbb{R}^N, d\mu_N)$, by Chebyshev's inequality, we know that

$$|E_k|_{\mu_N} \to 0 \qquad \text{as } k \to +\infty.$$

Moreover, we can apply absolute continuity of the integral and say that

$$\int_{E_k} |u|\, d\mu_N \to 0 \qquad \text{as } k \to +\infty.$$

Therefore, we deduce $u_k \to u$ in $L^1(\mathbb{R}^N, d\mu_N)$.

Let now $\varepsilon > 0$ and let us consider the set

$$\left\{ x \in \mathbb{R}^N \ : \ \left| \frac{\partial u_k}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right| \geq \varepsilon \right\}.$$

Since

$$\left\{ x \in \mathbb{R}^N \ : \ \left| \frac{\partial u_k}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right| \geq \varepsilon \right\} \subset E_k,$$

we deduce that

$$\frac{\partial u_k}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{in measure with respect to } \mu_N.$$

Hence there exists a subsequence $\{u_{k_j}\}_{j \in \mathbb{N}}$ such that $\nabla u_{k_j} \to \nabla u$ almost everywhere in $\mathbb{R}^N$ with respect to the measure μ_N. Since the Lebesgue measure dx is absolutely continuous with respect to the measure $d\mu_N$, we deduce that $\nabla u_{k_j} \to \nabla u$ almost everywhere in $\mathbb{R}^N$ also with respect to the Lebesgue measure.

Finally, since

$$\left|\nabla u(x) - \nabla u_{k_j}(x)\right|^N \leq |\nabla u(x)|^N \quad \text{a.e. } x \in \mathbb{R}^N,$$

and

$$|\nabla u(x) - \nabla u_{k_j}(x)|^2 |\nabla v_N(x)|^{N-2} \leq |\nabla u(x)|^2 |\nabla v_N(x)|^{N-2} \quad \text{a.e. } x \in \mathbb{R}^N,$$

we can apply dominated convergence theorem to say that

$$\|\nabla u - \nabla u_{k_j}\|_N \to 0 \quad \text{as } j \to +\infty,$$

and

$$\left(\int_{\mathbb{R}^N} |\nabla u - \nabla u_{k_j}|^2 |\nabla v_N|^{N-2}\, dx \right)^{\frac{1}{2}} \to 0 \quad \text{as } j \to +\infty.$$

□

Taking into account Lemmas 2.6 and 2.7, we have proved Theorem 1.1, that is

$$W_{\mu_N}(\mathbb{R}^N) = \mathcal{X} = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}.$$

Now, we can extend the N-dimensional Euclidean Onofri inequality to the weighted Sobolev space $W_{\mu_N}(\mathbb{R}^N)$.

Proof of Theorem 1.2 Let $u \in W_{\mu_N}(\mathbb{R}^N)$. Since $W_{\mu_N}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}$, there exists a sequence $\{u_m\}_{m\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^N)$ such that $\|u - u_m\|_{\mu_N} \to 0$. By (1.1), we have

$$\ln\left(\int_{\mathbb{R}^N} e^{u_m}\,d\mu_N\right) \le \frac{1}{\widetilde{\omega_N}}\int_{\mathbb{R}^N} H_N(u_m,\mu_N)\,dx + \int_{\mathbb{R}^N} u_m\,d\mu_N$$

for any $m \in \mathbb{N}$.

The condition $\|u - u_m\|_{\mu_N} \to 0$ implies that there exists a subsequence, that we recall $\{u_m\}_{m\in\mathbb{N}}$, and a nonnegative function $w \in L^N(\mathbb{R}^N, dx)$ with $w^2|\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$, such that

- $u_m(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $m \to +\infty$;
- $\nabla u_m(x) \to \nabla u(x)$ a.e. in $\mathbb{R}^N$ as $m \to +\infty$;
- for any m, $|\nabla u_m(x)| \le w(x)$ a.e. in $\mathbb{R}^N$.

We easily obtain that

$$\int_{\mathbb{R}^N} u_m\,d\mu_N \to \int_{\mathbb{R}^N} u\,d\mu_N \quad \text{as } m \to +\infty.$$

It's easy to observe that

$$H_N(u_m(x),\mu_N(x)) \to H_N(u(x),\mu_N(x)) \quad \text{a.e. in } \mathbb{R}^N \quad \text{as } \ m \to +\infty.$$

Taking into account Lemma 2.1, we have

$$\begin{aligned} 0 \le H_N(u_m(x),\mu_N(x)) &\le \frac{c_N}{2}\left(|\nabla u_m(x)|^N + |\nabla u_m(x)|^2|\nabla v_N(x)|^{N-2}\right) \\ &\le \frac{c_N}{2}\left(w(x)^N + w(x)^2|\nabla v_N(x)|^{N-2}\right), \end{aligned}$$

almost everywhere in $\mathbb{R}^N$ and for any m.

Since $w^N + w^2|\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$, we can apply dominated convergence theorem to say that

$$\int_{\mathbb{R}^N} H_N(u_m,\mu_N)\,dx \to \int_{\mathbb{R}^N} H_N(u,\mu_N)\,dx.$$

Finally, by Fatou's Lemma, we have

$$\int_{\mathbb{R}^N} e^u\,d\mu_N \le \liminf_{m\to+\infty}\left(\int_{\mathbb{R}^N} e^{u_m}\,d\mu_N\right) \le e^{\frac{1}{\widetilde{\omega_N}}\int_{\mathbb{R}^N} H_N(u,\mu_N)\,dx + \int_{\mathbb{R}^N} u\,d\mu_N},$$

and passing to logarithm we can conclude that

$$\ln\left(\int_{\mathbb{R}^N} e^u\,d\mu_N\right) \le \frac{1}{\widetilde{\omega_N}}\int_{\mathbb{R}^N} H_N(u,\mu_N)\,dx + \int_{\mathbb{R}^N} u\,d\mu_N.$$

□

3 Equivalence between Logarithmic Moser–Trudinger inequality and Euclidean Onofri inequality

This section is devoted to the proof Theorem 1.3, that is to the equivalence between the sharp inequalities (1.11) and (1.13).

Proof of Theorem 1.3 Let I and J be the functionals defined in (1.15) and (1.14). We need to prove that

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I = \inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k}.$$

We begin with the proof of " $\geq$ ".

Let $u \in W_{\mu_N}(\mathbb{R}^N)$. Since $W_{\mu_N}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}$, we can assume $u \in C_0^\infty(\mathbb{R}^N)$ and the thesis will follow by density.

Let us consider the following cutoff function $\Psi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \Psi \leq 1$, $\Psi \equiv 1$ if $|x| \leq \frac{1}{2}$, and $\Psi \equiv 0$ if $|x| \geq 1$.

Let $r > 0$ and let us define $\Psi_r(x) := \Psi\left(\frac{x}{r}\right)$, so that $\Psi_r \equiv 1$ if $|x| \leq \frac{r}{2}$, and $\Psi_r \equiv 0$ if $|x| \geq r$. Obviously, $\Psi_r(x) \to 1$ as $r \to +\infty$, for any $x \in \mathbb{R}^N$.

For simplicity of notation, let's say $B_r := B_r(O)$; moreover, we will denote with $o(1)$ any function that goes to zero as $r \to +\infty$.

Using the characteristic function χ, we can write

$$\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)}\,dx = \int_{\mathbb{R}^N} e^{u(x)\Psi_r(x)+v_N(x)} \chi_{B_r}(x)\,dx.$$

Now we observe that $|u|$ is bounded by a constant M_u, so for any $x \in \mathbb{R}^N$ and for any $r > 0$,

$$\left| e^{u(x)\Psi_r(x)+v_N(x)} \chi_{B_r}(x) \right| \leq e^{M_u+v_N(x)} = e^{M_u} \mu_N(x) \in L^1(\mathbb{R}^N).$$

Moreover, for any $x \in \mathbb{R}^N$,

$$e^{u(x)\Psi_r(x)+v_N(x)} \chi_{B_r}(x) \to e^{u(x)+v_N(x)} \quad \text{as } r \to +\infty.$$

Therefore we can apply dominated convergence theorem and say that, as $r \to +\infty$,

$$\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)}\,dx \to \int_{\mathbb{R}^N} e^{u(x)+v_N(x)}\,dx = \int_{\mathbb{R}^N} e^u\,d\mu_N.$$

We can write

$$\ln\left(\int_{\mathbb{R}^N} e^u\,d\mu_N\right) = \lim_{r\to+\infty} \ln\left(\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)}\,dx\right),$$

therefore we can estimate

$$\ln\left(\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)}\,dx\right).$$

In general the function $u\Psi_r + v_N \notin W_0^{1,N}(B_r)$.

If we define $w := u\Psi_r + v_N - v_N(r)$, (where with $v_N(r)$ we make a slight abuse of notation to denote a constant function on B_r whose value is $v_N(x)$ as $|x| = r$), we have $w \in W_0^{1,N}(B_r)$.

$$\begin{aligned}
\ln\left(\int_{B_r} e^{u(x)\Psi_r(x)}\,d\mu_N\right) &= \ln\left(\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)}\,dx\right) \\
&= N \ln\left(\frac{r}{1+r^{\frac{N}{N-1}}}\right) + \ln\left(\frac{1}{|B_r|}\int_{B_r} e^{w(x)}\,dx\right)
\end{aligned}$$

$$= -\frac{N}{N-1}\ln r + \ln\left(\frac{1}{|B_r|}\int_{B_r} e^{w(x)}\,dx\right) + o(1).$$

Since $w \in W_0^{1,N}(B_r)$, we have

$$\inf_{W_0^{1,N}(B_1)} J < \frac{1}{\widetilde{\omega_N}}\int_{B_r} |\nabla w|^N\,dx - \ln\left(\frac{1}{|B_r|}\int_{B_r} e^{w(x)}\,dx\right).$$

Therefore

$$\ln\left(\int_{B_r} e^{u\Psi_r}\,d\mu_N\right) < \frac{1}{\widetilde{\omega_N}}\int_{B_r} |\nabla w|^N\,dx - \frac{N}{N-1}\ln r - \inf_{W_0^{1,N}(B_1)} J + o(1).$$

We need to compute $\displaystyle\frac{1}{\widetilde{\omega_N}}\int_{B_r} |\nabla w|^N\,dx$.

Since

$$\nabla w = \nabla(u\Psi_r) + \nabla v_N,$$

and

$$H_N(u\Psi_r, \mu_N) = |\nabla w|^N - |\nabla v_N|^N - N|\nabla v_N|^{N-2}\nabla v_N \cdot \nabla(u\Psi_r),$$

we have

$$|\nabla w|^N = H_N(u\Psi_r, \mu_N) + |\nabla v_N|^N + N|\nabla v_N|^{N-2}\nabla v_N \cdot \nabla(u\Psi_r),$$

by which

$$\int_{B_r} |\nabla w|^N\,dx = \int_{B_r} H_N(u\Psi_r, \mu_N)\,dx + \int_{B_r} |\nabla v_N|^N\,dx + \int_{B_r} |\nabla v_N|^{N-2}\nabla v_N \cdot \nabla(u\Psi_r)\,dx.$$

By Green's formula and taking into account (2.1), we obtain

$$\frac{N}{\widetilde{\omega_N}}\int_{B_r} |\nabla v_N|^{N-2}\nabla v_N \cdot \nabla(u\Psi_r)\,dx = \int_{B_r} u\Psi_r\mu_N\,dx.$$

Now, we need to compute $\displaystyle\frac{1}{\widetilde{\omega_N}}\int_{B_r} |\nabla v_N|^N\,dx$.

$$\begin{aligned}
\frac{1}{\widetilde{\omega_N}}\int_{B_r} |\nabla v_N|^N\,dx &= \frac{N^{2N}}{(N-1)^N\widetilde{\omega_N}}\int_{B_r} \frac{|x|^{\frac{N}{N-1}}}{\left(1+|x|^{\frac{N}{N-1}}\right)^N}\,dx \\
&= \int_0^{r^{\frac{N}{N-1}}} \frac{t^{N-1}}{(1+t)^N}\,dt \\
&= \int_0^{r^{\frac{N}{N-1}}} \frac{1}{1+t}\,dt - \sum_{k=0}^{N-2}\binom{N-1}{k}\int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N}\,dt \qquad (3.1) \\
&= \frac{N}{N-1}\ln r - \sum_{k=0}^{N-2}\binom{N-1}{k}\int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N}\,dt + o(1).
\end{aligned}$$

By induction, we prove that

$$\binom{N-1}{k}\int_0^{r^{\frac{N}{N-1}}}\frac{t^k}{(1+t)^N}\,dt=\frac{1}{N-k-1}+o(1) \tag{3.2}$$

for any $k=0,\dots,N-2$.
If $k=0$, we have

$$\binom{N-1}{k}\int_0^{r^{\frac{N}{N-1}}}\frac{t^k}{(1+t)^N}\,dt=\int_0^{r^{\frac{N}{N-1}}}\frac{1}{(1+t)^N}\,dt=\frac{1}{N-1}+o(1).$$

Now, let $N\geq 3$, $k\geq 1$, and let assume (3.2) to hold for $k-1$.

$$\begin{aligned}
&\binom{N-1}{k}\int_0^{r^{\frac{N}{N-1}}}\frac{t^k}{(1+t)^N}\,dt\\
&=\binom{N-1}{k}\left[o(1)+\frac{k}{N-1}\int_0^{r^{\frac{N}{N-1}}}\frac{t^{k-1}(1+t)}{(1+t)^N}\,dt\right]\\
&=\binom{N-1}{k}\left[o(1)+\frac{\frac{k}{N-1}}{\binom{N-1}{k-1}}\left(\frac{1}{N-k}+o(1)\right)+\frac{k}{N-1}\int_0^{r^{\frac{N}{N-1}}}\frac{t^k}{(1+t)^N}\,dt\right]\\
&=\frac{1}{N-1}+\frac{k}{N-1}\binom{N-1}{k}\int_0^{r^{\frac{N}{N-1}}}\frac{t^k}{(1+t)^N}\,dt+o(1).
\end{aligned}$$

Therefore

$$\binom{N-1}{k}\int_0^{r^{\frac{N}{N-1}}}\frac{t^k}{(1+t)^N}\,dt=\frac{1}{N-k-1}+o(1).$$

By (3.2) we deduce that

$$\sum_{k=0}^{N-2}\binom{N-1}{k}\int_0^{r^{\frac{N}{N-1}}}\frac{t^k}{(1+t)^N}\,dt=\sum_{k=1}^{N-1}\frac{1}{k}+o(1). \tag{3.3}$$

Finally,

$$\frac{1}{\widetilde{\omega_N}}\int_{B_r}|\nabla v_N|^N\,dx=\frac{N}{N-1}\ln r-\sum_{k=1}^{N-1}\frac{1}{k}+o(1). \tag{3.4}$$

Summing up, we have shown that

$$\inf_{W_0^{1,N}(B_1)}J+\sum_{k=1}^{N-1}\frac{1}{k}<I(u\Psi_r)+o(1).$$

Now, we pass to the limit as $r\to+\infty$.
We observe that $|u|$ is bounded by a constant M_u, so for any $x\in\mathbb{R}^N$ and for any $r>0$,

$$|u(x)\Psi_r(x)\mu_N(x)|\leq M_u\mu_N(x)\in L^1(\mathbb{R}^N).$$

Moreover, for any $x \in \mathbb{R}^N$,

$$u(x)\Psi_r(x)\mu_N(x) \to u(x)\mu_N(x) \quad \text{as } r \to +\infty.$$

We can apply dominated convergence theorem to say that

$$\int_{\mathbb{R}^N} u\Psi_r \, d\mu_N \to \int_{\mathbb{R}^N} u d\mu_N.$$

Now we observe that, for any $x \in \mathbb{R}^N$,

$$H_N(u(x)\Psi_r(x), \mu_N(x)) \to H_N(u(x), \mu_N(x)) \qquad \text{as } r \to +\infty.$$

Moreover, taking into account Lemma 2.1 and $u \in C_0^\infty(\mathbb{R}^N)$, there exists a function $v \in L^1(\mathbb{R}^N, dx)$ such that, for any $x \in \mathbb{R}^N$ and for any $r > 1$, we have

$$0 \le H_N(u(x)\Psi_r(x), \mu_N(x)) \le v(x).$$

Therefore we can apply dominated convergence theorem and say that

$$\frac{1}{\widetilde{\omega_N}} \int_{\mathbb{R}^N} H_N(u\Psi_r, \mu_N) \, dx \to \frac{1}{\widetilde{\omega_N}} \int_{\mathbb{R}^N} H_N(u, \mu_N) \, dx.$$

Finally,

$$\inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k} \le I(u),$$

by which

$$\inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k} \le \inf_{C_0^\infty(\mathbb{R}^N)} I.$$

By density, that is by the proof of Theorem 1.2, we obtain

$$\inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k} \le \inf_{W_{\mu_N}(\mathbb{R}^N)} I.$$

Now, we prove "$\le$". Let $u \in W_0^{1,N}(B_1)$ and let $r > 0$. We define the function $u_r : \mathbb{R}^N \to \mathbb{R}$ as

$$u_r(x) = \begin{cases} u\left(\dfrac{x}{r}\right) - v_N(x) + v_N(r) & \text{if } |x| \le r, \\ 0 & \text{if } |x| > r. \end{cases}$$

By Lemma 2.3, we have $u_r \in W_{\mu_N}(\mathbb{R}^N)$, by which

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I \le I(u_r),$$

where

$$I(u_r) = \frac{1}{\widetilde{\omega_N}} \int_{\mathbb{R}^N} H_N(u_r, \mu_N) \, dx + \int_{\mathbb{R}^N} u_r \, d\mu_N - \ln\left(\int_{\mathbb{R}^N} e^{u_r} \, d\mu_N\right). \tag{3.5}$$

We need to compute the terms in (3.5).

$$\ln\left(\int_{\mathbb{R}^N} e^{u_r}\, d\mu_N\right) = \ln\left(\mu_N(r)\int_{B_r} e^{u(\frac{x}{r})}\, dx + \int_{\mathbb{R}^N\setminus B_r} \mu_N(x)\, dx\right);$$

$$\mu_N(r)\int_{B_r} e^{u(\frac{x}{r})}\, dx = \left(\frac{r}{1+r^{\frac{N}{N-1}}}\right)^N \frac{1}{V_N}\int_{B_1} e^{u(y)}\, dy;$$

$$\int_{\mathbb{R}^N\setminus B_r} \mu_N(x)\, dx = \left[\left(\frac{t}{1+t}\right)^{N-1}\right]_{r^{\frac{N}{N-1}}}^{+\infty} = 1 - \frac{r^N}{\left(1+r^{\frac{N}{N-1}}\right)^{N-1}}.$$

Therefore

$$\begin{aligned}
&\ln\left(\int_{\mathbb{R}^N} e^{u_r}\, d\mu_N\right)\\
&= \ln\left(\left(\frac{r}{1+r^{\frac{N}{N-1}}}\right)^N \frac{1}{V_N}\int_{B_1} e^{u}\, dx + 1 - \frac{r^N}{\left(1+r^{\frac{N}{N-1}}\right)^{N-1}}\right)\\
&= -\frac{N}{N-1}\ln r + \ln\left(\frac{1}{V_N}\int_{B_1} e^{u}\, dx + N - 1 + o(1)\right) + o(1).
\end{aligned}$$

Now, we compute $\frac{1}{\widetilde{\omega_N}}\int_{\mathbb{R}^N} H_N(u_r,\mu_N)\, dx$.

We have

$$\begin{aligned}
\frac{1}{\widetilde{\omega_N}}\int_{\mathbb{R}^N} H_N(u_r,\mu_N)\, dx &= \frac{1}{\widetilde{\omega_N}}\int_{B_1} |\nabla u|^N\, dx + \frac{N-1}{\widetilde{\omega_N}}\int_{B_r} |\nabla v_N|^N\, dx\\
&\quad - \frac{N}{\widetilde{\omega_N}}\int_{B_r} |\nabla v_N|^{N-2}\nabla v_N\cdot\nabla_x u\left(\frac{x}{r}\right)\, dx.
\end{aligned}$$

Taking into account (3.4), we have

$$\frac{N-1}{\widetilde{\omega_N}}\int_{B_r} |\nabla v_N|^N\, dx = N\ln r - (N-1)\sum_{k=1}^{N-1}\frac{1}{k} + o(1).$$

Moreover, by Green's formula and taking into account (2.1), we obtain

$$-\frac{N}{\widetilde{\omega_N}}\int_{B_r} |\nabla v_N|^{N-2}\nabla v_N\cdot\nabla_x u\left(\frac{x}{r}\right)\, dx = -\int_{B_r} u\left(\frac{x}{r}\right)\, d\mu_N.$$

Therefore

$$\begin{aligned}
&\frac{1}{\widetilde{\omega_N}}\int_{\mathbb{R}^N} H_N(u_r,\mu_N)\, dx\\
&= \frac{1}{\widetilde{\omega_N}}\int_{B_1} |\nabla u|^N dx - \int_{B_r} u\left(\frac{x}{r}\right)\, d\mu_N + N\ln r - (N-1)\sum_{k=1}^{N-1}\frac{1}{k} + o(1).
\end{aligned}$$

Now, we compute $\int_{\mathbb{R}^N} u_r \, d\mu_N$.

$$\int_{\mathbb{R}^N} u_r \, d\mu_N = \int_{B_r} u\left(\frac{x}{r}\right) d\mu_N - \int_{B_r} v_N(x) \, d\mu_N + v_N(r) \int_{B_r} d\mu_N.$$

$$v_N(r) \int_{B_r} d\mu_N = -\ln V_N - \frac{N^2}{N-1} \ln r + o(1).$$

$$-\int_{B_r} v_N(x) \, d\mu_N = \ln V_N + N^2 \int_0^r \frac{\rho^{N-1} \ln\left(1+\rho^{\frac{N}{N-1}}\right)}{\left(1+\rho^{\frac{N}{N-1}}\right)^N} d\rho + o(1).$$

Taking into account (3.1) and (3.3), we have

$$\begin{aligned}
&N^2 \int_0^r \frac{\rho^{N-1} \ln\left(1+\rho^{\frac{N}{N-1}}\right)}{\left(1+\rho^{\frac{N}{N-1}}\right)^N} d\rho \\
&\quad = N(N-1) \int_0^{r^{\frac{N}{N-1}}} \frac{1}{(1+t)^2} \left(\frac{t}{1+t}\right)^{N-2} \ln(1+t) \, dt \\
&\quad = N \left(\left[\left(\frac{t}{1+t}\right)^{N-1} \ln(1+t) \right]_0^{r^{\frac{N}{N-1}}} - \int_0^{r^{\frac{N}{N-1}}} \frac{t^{N-1}}{(1+t)^N} dt \right) \\
&\quad = N \left(\frac{r^N - \left(1+r^{\frac{N}{N-1}}\right)^{N-1}}{\left(1+r^{\frac{N}{N-1}}\right)^{N-1}} \ln\left(1+r^{\frac{N}{N-1}}\right) + \sum_{k=1}^{N-1} \frac{1}{k} + o(1) \right) \\
&\quad = N \sum_{k=1}^{N-1} \frac{1}{k} + o(1).
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} u_r \, d\mu_N = \int_{B_r} u\left(\frac{x}{r}\right) d\mu_N - \frac{N^2}{N-1} \ln r + N \sum_{k=1}^{N-1} \frac{1}{k} + o(1).$$

Summing up, we have shown that

$$\begin{aligned}
&\inf_{W_{\mu_N}(\mathbb{R}^N)} I \\
&\leq \frac{1}{\widetilde{\omega_N}} \int_{B_1} |\nabla u|^N dx + \sum_{k=1}^{N-1} \frac{1}{k} - \ln\left(\frac{1}{V_N} \int_{B_1} e^u dx + N - 1 + o(1)\right) + o(1).
\end{aligned}$$

Passing to the limit as $r \to +\infty$, we obtain

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I \leq \frac{1}{\widetilde{\omega_N}} \int_{B_1} |\nabla u|^N \, dx + \sum_{k=1}^{N-1} \frac{1}{k} - \ln\left(\frac{1}{V_N} \int_{B_1} e^u \, dx + N - 1\right),$$

by which

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I < J(u) + \sum_{k=1}^{N-1} \frac{1}{k}.$$

Finally,

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I \leq \inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k}.$$

□

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