J. Fixed Point Theory Appl. (2022) 24:65
https://doi.org/10.1007/s11784-022-00982-5

Journal of Fixed Point Theory and Applications

Positive solutions for a class of nonlocal problems with possibly singular nonlinearity

Leszek Gasiński, João R. Santos Junior and Gaetano Siciliano

Abstract. We study a class of elliptic problems with homogeneous Dirichlet boundary condition and a nonlinear reaction term f which is nonlocal depending on the L^p-norm of the unknown function. The nonlinearity f can make the problem degenerate since it may even have multiple singularities in the nonlocal variable. We use fixed point arguments for an appropriately defined solution map, to produce multiplicity of classical positive solutions with ordered norms.

Mathematics Subject Classification. 35J50, 35J57, 35J70.

Keywords. Problems with nonlocal terms, multiple asymptotic behaviour, multiplicity result, fixed point methods.

Contents

João R. Santos was partially supported by CNPq 306503/2018-7, Brazil. Gaetano Siciliano was partially supported by Fapesp 2019/27491-0, Capes and CNPq 304660/2018-3, Brazil.

Published online: 14 September 2022

1. Introduction

In this paper we consider the existence of multiple positive solutions for the following class of nonlocal problems:

$$\begin{cases} -\Delta u = f\left(u, \int_\Omega |u|^p \mathrm{d}x\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_I}$$

where $p \geq 1$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and f is a continuous function with a suitable behaviour in the second variable.

In the last decades partial differential equations involving nonlocal terms have attracted a great deal of attention of the mathematical community for different reasons. Indeed, equations involving nonlocal terms are usually more realistic to model different situations in nature, see Furter–Grinfeld [11] for a comparison between local and nonlocal models in population dynamics, or Kirchhoff [15] for an improvement (based on the insertion of a nonlocal term) of the classical D'Alembert's wave equation in string deformation theory. From a mathematical point of view nonlocal equations are challenging since, in general, the presence of a nonlocal term makes the equation much more complicated. In many cases the known techniques cannot be applied in a straightforward way, so the development of alternative approaches is required.

Regarding stationary partial differential equations in very recent years two classes of problems involving nonlocal terms in the diffusion operator have been a quite active research field, namely the Kirchhoff and the generalised Carrier problems. A basic prototype of the first one is

$$\begin{cases} -a\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right) \Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{KP}$$

where a is usually a continuous real function bounded away from zero. Problem (KP) is the stationary version of a hyperbolic model to small transversal vibrations of elastic membranes, see Kirchhoff [15] for details. On the other hand, generalised Carrier problems are of type

$$\begin{cases} -a\left(\int_\Omega |u|^p \mathrm{d}x\right) \Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{GCP}$$

where a is again a continuous real function. When $p = 2$, problem (GCP) is the stationary case of another hyperbolic model of transversal vibrations (not necessarily small) of elastic membranes which was introduced by Carrier in [7], in the unidimensional case. As $p = 1$, (GCP) also appears as the stationary version of parabolic models in the study of the dispersion of biological populations where the diffusion rate of the specie takes into account the whole population. See, for instance, Chipot–Rodrigues [8]. For papers involving operators as in (GCP) with $p > 1$ arbitrary, we refer the reader to Figueiredo–Sousa–Morales-Rodrigo–Suárez [10] and references therein.

Recently, the above problems have been studied also in the degenerate case, that is, when a vanishes in some points. Concerning (KP), this line of research started with the papers of Ambrosetti–Arcoya [2,3]. See also Santos Júnior–Siciliano [16] who establish a multiplicity result of positive solutions depending on the number of degeneration points of the function a. For problem (GCP), some interesting results can be found in the recent papers of Delgado–Morales-Rodrigo–Santos Júnior–Suárez [9] and Gasiński–Santos Júnior [12,13].

Starting with the above considerations, our aim here is to study existence of positive solutions for (possibly) degenerate problems for which problem (GCP) is just a particular case. The main goal of this paper is to prove that when the reaction term involved in the equation depends on the L^p-norm of the unknown function and it allows singularities in some points of its domain, the same sort of multiplicity result (as those observed in degenerate Kirchhoff and Carrier problems, see Santos Júnior–Siciliano [16] and Gasiński–Santos Júnior [12]) holds.

It is worth to point out that such kind of results are much more general than those obtained for degenerate

Carrier problems. In fact, if we choose, the nonlinearity f in (P_I) of the form

$$f\left(u, \int_\Omega |u|^p \mathrm{d}x\right) = \frac{g(u)}{a\left(\int_\Omega |u|^p \mathrm{d}x\right)}, \tag{1.1}$$

with a being a continuous real function which vanishes in some points, then $f(u, \cdot)$ exhibits a diverging behaviour and we fall in the degenerate Carrier problems (GCP).

1.1. Statement of the main result—Theorem 1.1

First let us introduce some notations. Along the paper,

- $\|\cdot\|$ denotes the $H_0^1(\Omega)$-norm,
- $|\cdot|_r$ is the $L^r(\Omega)$-norm,
- λ_1 is the first eigenvalue of the minus Laplacian operator in Ω with zero Dirichlet boundary condition,
- φ_1 is the positive eigenfunction associated to λ_1 normalized in the $H_0^1(\Omega)$-norm,
- e_1 is the positive eigenfunction associated to λ_1 normalized in the $L^\infty(\Omega)$-norm,
- $C_1 > 0$ stands for best constant of the Sobolev embedding of $H_0^1(\Omega)$ into $L^1(\Omega)$;
- $|\Omega|$ is the Lebesgue measure of Ω.

We assume that f is a function having the following behaviour:

(f_0) there exist positive numbers $0 =: t_0 < t_1 < t_2 < \cdots < t_K$ ($K \geq 1$) such that $f : \mathbb{R} \times \mathcal{A} \to \mathbb{R}$ is continuous, where $\mathcal{A} = (0, \infty) \backslash \{t_1, \ldots, t_K\}$. For each $k \in \{1, \ldots, K\}$ and fixed $\alpha \in (t_{k-1}, t_k)$, $f(\cdot, \alpha) \in C^1(\mathbb{R})$, there exists $s_\alpha > 0$ such that $f(s_\alpha, \alpha) = 0$ and $f(s, \alpha) > 0$ for $s \in (0, s_\alpha)$. Moreover $\sup_{\alpha \in Z} s_\alpha < \infty$, for each compact set $Z \subset (t_{k-1}, t_k)$ and $k \in \{1, \ldots, K\}$;

(f_1) for each $k \in \{0, \ldots, K\}$ and any $s \in (0, s_\alpha)$, $\lim_{\alpha \to t_k} f(s, \alpha)/s \in [\lambda_1, +\infty]$;

(f_2) for each $k \in \{1, \ldots, K\}$ and fixed $\alpha \in (t_{k-1}, t_k)$, the map $(0, s_\alpha) \ni s \mapsto f(s, \alpha)/s$ is decreasing;

(f_3) $\inf_{\alpha\in\mathcal{A}} \gamma_\alpha > \lambda_1$, where $\gamma_\alpha := \lim_{s\to 0^+} f(s, \alpha)/s$.

Remark 1. (a) Hypothesis (f_1) makes the problems (P_I) singular in the nonlocal term whenever $\lim_{\alpha\to t_k} f(s, \alpha) = +\infty$.
(b) Hypotheses (f_2) and (f_3) describe the behaviour of the function $\psi_\alpha : (0, s_\alpha) \to (0, \gamma_\alpha)$ defined by

$$\psi_\alpha(s) = \frac{f(s, \alpha)}{s}, \ \forall s \in (0, s_\alpha). \tag{1.2}$$

Hypothesis (f_2) guarantees its monotonicity, which will be crucial in the proof of the uniqueness of solutions for the auxiliary problem ($P_{k,\alpha}$) (see Proposition 2.1) and for other technical issues (see Lemmas 2.2, 2.3, and 2.4). Hypothesis (f_3) says that the limits of the functions ψ_α at zero are separated from λ_1 (uniformly in α).

Moreover we assume that:

(f_4) $t_K < \left(\inf_{\alpha\in\mathcal{A}} s_\alpha\right)^p \int_\Omega e_1^p \mathrm{d}x$;

(f_5) $\inf_{\alpha\in(t_{k-1},t_k)} \max_{s\in[0,s_\alpha]} f(s, \alpha)(s_\alpha^{p-1}/\alpha) < \lambda_1^{1/2}/C_1|\Omega|^{1/2}$, for all $k \in \{1, \ldots, K\}$.

Remark 2. (a) Hypotheses (f_4) gives some bounds on the "edges" of the curvilinear trapezoid $\{s_\alpha : \ \alpha \in \ \mathcal{A}\} \times \mathcal{A}$, in which the interesting domain of f is contained.
(b) Hypotheses (f_5) is quite technical and says that for some particular choice of $\alpha_0 \in (t_{k-1}, t_k)$ the maximum of the quotient $f(s, \alpha_0)/\alpha_0$ over $s \in [0, s_\alpha]$ is bounded by some constant.

Our main result is the following.

Theorem 1.1. *If conditions* (f_0)–(f_5) *hold, then problem* (P_I) *has at least* $2K$ *classical positive solutions with ordered* L^p*-norms, namely*

$$0 < \int_\Omega u_{1,1}^p \mathrm{d}x < \int_\Omega u_{1,2}^p \mathrm{d}x < t_1 < \cdots < t_{K-1} < \int_\Omega u_{K,1}^p \mathrm{d}x < \int_\Omega u_{K,2}^p \mathrm{d}x < t_K.$$

With the notation in (1.1), the Theorem gives existence of multiple and ordered solutions for the degenerate Carrier problem

$$\begin{cases} -a\left(\displaystyle\int_\Omega |u|^p \mathrm{d}x\right) \Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

since a may vanish in some points.

The proof of the above theorem will follow the following general steps:

Step 1. We first introduce an auxiliary problem with a truncated nonlinearity and fixed nonlocal term;

Step 2. By variational methods we show that the auxiliary problem has a unique solution;

Step 3. We define a suitable map which gives the L^p-norm of the solution;

Step 4. Finally, we show that this map has fixed points, which are indeed solutions of the considered problem.

The structure of the paper is the following. In Sect. 2 we develop Step 1 and Step 2 above. In Sect. 3 we address Step 3 and Step 4 above and implement them to the proof of Theorem 1.1. In the final Sect. 4 we give examples of nonlinearities f satisfying our assumptions.

2. Auxiliary problem

2.1. Statement of the auxiliary problem ($P_{k,\alpha}$)

In order to prove our results, the following auxiliary problem will play an important role: for each $k \in \{1, \dots, K\}$ and any $\alpha \in (t_{k-1}, t_k)$ fixed, consider

$$\begin{cases} -\Delta u = \widehat{f}_\alpha(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{k,\alpha}$}$$

where

$$\widehat{f}_\alpha(s) = \begin{cases} f(0,\alpha) & \text{if } s \le 0, \\ f(s,\alpha) & \text{if } 0 < s \le s_\alpha, \\ 0 & \text{if } s_\alpha < s. \end{cases}$$

Remark 3. Since f is continuous, it is an immediate consequence of the definition of $\widehat{f}_\alpha$ that if $\{\alpha_n\} \subset (t_{k-1}, t_k)$ converges to $\alpha \in (t_{k-1}, t_k)$ and $s_n \in (0, s_{\alpha_n})$ is such that $s_n \to s$, for some $s \in (0, s_\alpha)$, then $\widehat{f}_{\alpha_n}(s_n) \to \widehat{f}_\alpha(s)$.

2.2. Existence of the solutions of ($P_{k,\alpha}$)

Since we have "removed" the nonlocal term of (P_I), we can treat problem ($P_{k,\alpha}$) using a variational approach. Define the energy functional corresponding to problem ($P_{k,\alpha}$)

$$I_{k,\alpha}(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \widehat{F}_\alpha(u)\mathrm{d}x, \tag{2.1}$$

where as usual

$$\widehat{F}_\alpha(s) = \int_0^s \widehat{f}_\alpha(\sigma)\mathrm{d}\sigma.$$

It is clearly well defined and of class C^1.

Proposition 2.1. *If conditions (f_0), (f_2) and (f_3) hold, then for each $k \in \{1, \dots, K\}$ and $\alpha \in (t_{k-1}, t_k)$ fixed, problem ($P_{k,\alpha}$) has a unique classical solution $0 < u_\alpha \le s_\alpha$.*

Proof. Since, for each $k \in \{1, \dots, K\}$ and $\alpha \in (t_{k-1}, t_k)$ fixed, $\widehat{f}_\alpha$ is bounded and continuous, the energy functional (2.1) is coercive and weakly lower semicontinuous. Therefore $I_{k,\alpha}$ has a minimum point u_α which is a weak solution of ($P_{k,\alpha}$). Moreover, it follows from conditions (f_2) and (f_3) that

$$\frac{I_{k,\alpha}(s\varphi_1)}{s^2} = \frac{1}{2} - \int_\Omega \frac{\widehat{F}_\alpha(s\varphi_1)}{(s\varphi_1)^2}\varphi_1^2\mathrm{d}x \to \frac{1}{2}\left(1 - \frac{\gamma_\alpha}{\lambda_1}\right) < 0 \quad \text{as } s \to 0^+.$$

The last inequality implies that the minimum point u_α of $I_{k,\alpha}$ is nontrivial, because for $s > 0$ small enough

$$I_{k,\alpha}(u_\alpha) \le I_{k,\alpha}(s\varphi_1) = (I_{k,\alpha}(s\varphi_1)/s^2)s^2 < 0.$$

Then $u_\alpha \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \nabla u_\alpha \nabla v \mathrm{d}x = \int_\Omega \widehat{f}_\alpha(u_\alpha) v \mathrm{d}x \quad \forall\, v \in H_0^1(\Omega). \tag{2.2}$$

Then it is easy to see that any nontrivial weak solution u_α of problem $(P_{k,\alpha})$ satisfies $0 \le u_\alpha \le s_\alpha$. Indeed by choosing $v = (u_\alpha - s_\alpha)^+$ in (2.2) we arrive (by the definition of $\widehat{f}_\alpha$) at

$$\int_\Omega \nabla u_\alpha \nabla (u_\alpha - s_\alpha)^+ \mathrm{d}x = \int_\Omega \widehat{f}_\alpha(u_\alpha)(u_\alpha - s_\alpha)^+ \mathrm{d}x = 0$$

which redly implies $u_\alpha \le s_\alpha$. Analogously one deduces that $0 \le u_\alpha$ by taking $v = u_\alpha^-$ in (2.2). Moreover the weak solution of problem $(P_{k,\alpha})$ is unique by condition (f_2), see Brezis–Oswald [6]. Since $\widehat{f}_\alpha(u_\alpha) = f(u_\alpha, \alpha)$ is bounded and $f(\cdot, \alpha) \in C^1(\mathbb{R})$, it follows from Agmon [1] that u_α is a classical solution. Finally, the maximum principle completes the proof (see Gilbarg–Trudinger [14, Theorem 3.1]). □

2.3. Properties of the solutions of $(P_{k,\alpha})$

Due to Proposition 2.1, we can set

$$c_\alpha := I_{k,\alpha}(u_\alpha) = \min_{u \in H_0^1(\Omega)} I_{k,\alpha}(u). \tag{2.3}$$

Since, by condition (f_2), the map $(0, s_\alpha) \ni s \mapsto \psi_\alpha(s) = \widehat{f}_\alpha(s)/s$ is decreasing (see (1.2) in Remark 1), there exists the inverse $\psi_\alpha^{-1}\colon (0, \gamma_\alpha) \to (0, s_\alpha)$. Thereby, by condition (f_3), for each $\varepsilon \in (0, \inf_{\alpha \in \mathcal{A}} \gamma_\alpha - \lambda_1)$, it makes sense to consider the function

$$y_\alpha := \psi_\alpha^{-1}(\lambda_1 + \varepsilon) e_1.$$

Remark 4. Let $\{\alpha_n\} \subset (t_{k-1}, t_k)$ be a sequence such that $\alpha_n \to \alpha_* \in (t_{k-1}, t_k)$.

Then

$$\psi_{\alpha_n}^{-1}(\lambda_1 + \varepsilon) \to \psi_{\alpha_*}^{-1}(\lambda_1 + \varepsilon).$$

Indeed, if we set $z_n := \psi_{\alpha_n}^{-1}(\lambda_1 + \varepsilon) \in (0, s_{\alpha_n})$, by (f_0) the sequence $\{z_n\}$ is bounded, hence we can assume that $z_n \to z$. Then, by Remark 3

$$\lambda_1 + \varepsilon = \frac{\widehat{f}_{\alpha_n}(z_n)}{z_n} \to \frac{\widehat{f}_{\alpha_*}(z)}{z} \quad \text{as } n \to +\infty$$

and so $z = \psi_{\alpha_*}^{-1}(\lambda_1 + \varepsilon)$.

Remark 5. We observe also that, if $\alpha_n \to \alpha_*$, and $\{u_n\}$ is such that $0 < u_n \le s_{\alpha_n} \le M$ and $u_n \to u_*$ a.e. in Ω, then

$$\widehat{F}_{\alpha_n}(u_n(x)) \to \widehat{F}_{\alpha_*}(u_*(x)).$$

Indeed by definition

$$\begin{aligned}\left|\widehat{F}_{\alpha_n}(u_n(x)) - \widehat{F}_{\alpha_*}(u_*(x))\right| &= \left|\int_0^{u_n(x)} f(\sigma,\alpha_n)\mathrm{d}\sigma - \int_0^{u_*(x)} f(\sigma,\alpha_*)\mathrm{d}\sigma\right| \\ &\le \left|\int_0^{u_n(x)} f(\sigma,\alpha_n)\mathrm{d}\sigma - \int_0^{u_*(x)} f(\sigma,\alpha_n)\mathrm{d}\sigma\right| \\ &\quad + \int_0^{u_*(x)} |f(\sigma,\alpha_*) - f(\sigma,\alpha_n)|\mathrm{d}\sigma \\ &= \int_{u_*(x)}^{u_n(x)} |f(\sigma,\alpha_n)|\mathrm{d}\sigma + o_n(1) = o_n(1).\end{aligned}$$

Lemma 2.2. *If conditions* (f_0), (f_2) *and* (f_3) *hold,* $\alpha \in (t_{k-1}, t_k)$, *then for each* $\varepsilon \in (0, \inf_{\alpha\in\mathcal{A}} \gamma_\alpha - \lambda_1)$, *we have*

$$c_\alpha \le -\frac{1}{2}\varepsilon\psi_\alpha^{-1}(\lambda_1+\varepsilon)^2 \int_\Omega e_1^2 \mathrm{d}x. \tag{2.4}$$

where c_α *is given in* (2.3).

Proof. Note that, by condition (f_2), we have

$$\widehat{F}_\alpha(t) \ge \frac{1}{2}\widehat{f}_\alpha(s)s \quad \forall\ s \ge 0.$$

Hence,

$$\frac{I_{k,\alpha}(y_\alpha)}{\psi_\alpha^{-1}(\lambda_1+\varepsilon)^2} \le \frac{1}{2}\left[\|e_1\|^2 - \int_\Omega \frac{\widehat{f}_\alpha(y_\alpha)}{\psi_\alpha^{-1}(\lambda_1+\varepsilon)^2} y_\alpha \mathrm{d}x\right],$$

or equivalently

$$\frac{I_{k,\alpha}(y_\alpha)}{\psi_\alpha^{-1}(\lambda_1+\varepsilon)^2} \le \frac{1}{2}\left[\|e_1\|^2 - \int_\Omega \frac{\widehat{f}_\alpha(y_\alpha)}{y_\alpha} e_1^2 \mathrm{d}x\right].$$

Using the definition of e_1 and condition (f_2), we get

$$\frac{I_{k,\alpha}(y_\alpha)}{\psi_\alpha^{-1}(\lambda_1+\varepsilon)^2} \le \frac{1}{2}\left[\|e_1\|^2 - \int_\Omega \frac{\widehat{f}_\alpha(\psi_\alpha^{-1}(\lambda_1+\varepsilon))}{\psi_\alpha^{-1}(\lambda_1+\varepsilon)} e_1^2 \mathrm{d}x\right].$$

Now, using the definition of ψ_α^{-1}, we deduce

$$\frac{I_{k,\alpha}(y_\alpha)}{\psi_\alpha^{-1}(\lambda_1+\varepsilon)^2} \le \frac{1}{2}\left[\|e_1\|^2 - (\lambda_1+\varepsilon)\int_\Omega e_1^2 \mathrm{d}x\right] = -\frac{1}{2}\varepsilon\int_\Omega e_1^2\mathrm{d}x.$$

Therefore,

$$c_\alpha \le I_{k,\alpha}(y_\alpha) \le -\frac{1}{2}\varepsilon\psi_\alpha^{-1}(\lambda_1+\varepsilon)^2 \int_\Omega e_1^2\mathrm{d}x,$$

which concludes the proof. □

The next two technical results will be helpful in what follows.

Lemma 2.3. *If conditions* (f_0), (f_2) *and* (f_3) *hold, then*

$$u_\alpha \ge z_\alpha := \psi_\alpha^{-1}(\lambda_1)e_1 \quad \forall\ \alpha \in (t_{k-1}, t_k). \tag{2.5}$$

Proof. From condition (f_2) and definition of ψ_α^{-1} it follows that

$$\lambda_1 = \frac{\widehat{f}_\alpha(\psi_\alpha^{-1}(\lambda_1))}{\psi_\alpha^{-1}(\lambda_1)} \leq \frac{\widehat{f}_\alpha(z_\alpha)}{z_\alpha}.$$

Thus

$$-\Delta(z_\alpha) = \lambda_1 z_\alpha \leq \widehat{f}_\alpha(z_\alpha) \text{ in } \Omega.$$

Therefore z_α is a subsolution of $(P_{k,\alpha})$. Inequality (2.5) follows now from condition (f_2) and Ambrosetti–Brezis–Cerami [4, Lemma 3.3]. □

Lemma 2.4. *If conditions* (f_0)*–*(f_2) *hold, then*

$$\liminf_{\alpha \to t_{k-1}^+} \psi_\alpha^{-1}(\lambda_1) \geq \inf_{\alpha \in (t_{k-1}, t_k)} s_\alpha. \tag{2.6}$$

The same holds for $\alpha \to t_k^-$.

Proof. Assuming the contrary, there would exist a sequence $\{\alpha_n\} \subseteq (t_{k-1}, t_k)$ such that $\alpha_n \to t_{k-1}^+$ (or alternatively $\alpha_n \to t_k^-$) and

$$\psi_{\alpha_n}^{-1}(\lambda_1) < T < \inf_{\alpha \in (t_{k-1}, t_k)} s_\alpha,$$

for some $T \in \mathbb{R}$. Since $T \in [0, s_{\alpha_n}]$ for all $n \geq 1$ and ψ_{α_n} is decreasing on $[0, s_{\alpha_n}]$, we would have

$$\lambda_1 > \psi_{\alpha_n}(T) = \frac{f(T, \alpha_n)}{T}.$$

But this contradicts hypothesis (f_1). So the proof is done. □

3. Proof of the main result

3.1. Auxiliary map $\mathcal{P}_k$

In virtue of Proposition 2.1, given $p \geq 1$, for any $k \in \{1, \ldots, K\}$ we can define the map

$$\mathcal{P}_k : \alpha \in (t_{k-1}, t_k) \longmapsto \int_\Omega u_\alpha^p \mathrm{d}x \in \mathbb{R} \tag{3.1}$$

where u_α is the unique solution of problem $(P_{k,\alpha})$. The strategy of proving Theorem 1.1 will be to show that $\mathcal{P}_k$ is continuous and has two fixed points. Indeed any fixed point, let us say $\overline{\alpha}$, of $\mathcal{P}_k$ satisfies by definition

$$\begin{cases} -\Delta u_{\overline{\alpha}} = \widehat{f}\left(u, \int_\Omega u_{\overline{\alpha}}^p \mathrm{d}x\right) & \text{in } \Omega, \\ 0 < u_{\overline{\alpha}} \leq s_{\overline{\alpha}} & \text{in } \Omega, \\ u_{\overline{\alpha}} = 0 & \text{on } \partial\Omega, \end{cases}$$

and hence $u_{\overline{\alpha}}$ is a solution of (P_I) with L^p-norm in (t_{k-1}, t_k).

3.2. Continuity of $\mathcal{P}_k$

Proposition 3.1. *If conditions* (f_0), (f_2) *and* (f_3) *hold, then for each* $k \in \{1, 2, \ldots, K\}$, *the map* $\mathcal{P}_k$ *defined in* (3.1) *is continuous.*

Proof. Let $\{\alpha_n\} \subset (t_{k-1}, t_k)$ be a sequence such that $\alpha_n \to \alpha_*$, for some $\alpha_* \in (t_{k-1}, t_k)$. Denote by u_n the positive solution of ($P_{k,\alpha}$) with $\alpha = \alpha_n$. Since,

$$\frac{1}{2}\|u_n\|^2 - \int_\Omega \widehat{F}_{\alpha_n}(u_n)\mathrm{d}x = I_{k,\alpha_n}(u_n) < 0, \tag{3.2}$$

we get

$$\|u_n\| \leq \left[2\widehat{F}_{\alpha_n}(s_{\alpha_n})|\Omega|\right]^{1/2}, \quad \forall\, n \in \mathbb{N}. \tag{3.3}$$

Note that by assumption (f_0), $\{s_{\alpha_n}\} \subset (0, M)$, for some $M > 0$, then

$$\widehat{F}_{\alpha_n}(s_{\alpha_n}) = \int_0^{s_{\alpha_n}} \widehat{f}_{\alpha_n}(\sigma)\mathrm{d}\sigma \leq \int_0^M \widehat{f}_{\alpha_n}(\sigma)\mathrm{d}\sigma$$

and this last integral is bounded uniformly in n by Remark 3. Consequently the right hand side in (3.3) is bounded and so is $\{u_n\}$ in $H_0^1(\Omega)$. Then, up to a subsequence, there exists $u_* \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u_* \;\text{ in }\; H_0^1(\Omega). \tag{3.4}$$

Thus, from Remark 3, passing to the limit as $n \to \infty$ in

$$\int_\Omega \nabla u_n \nabla v \mathrm{d}x = \int_\Omega \widehat{f}_{\alpha_n}(u_n) v \mathrm{d}x \quad \forall\, v \in H_0^1(\Omega),$$

we get

$$\int_\Omega \nabla u_* \nabla v \mathrm{d}x = \int_\Omega \widehat{f}_{\alpha_*}(u_*) v \mathrm{d}x \quad \forall\, v \in H_0^1(\Omega).$$

So, u_* is a nonnegative weak solution of ($P_{k,\alpha}$) with $\alpha = \alpha_*$. We are going to show that $u_* \neq 0$. In fact, passing to the limit as $n \to \infty$ in

$$\int_\Omega \nabla u_n \nabla u_* \mathrm{d}x = \int_\Omega \widehat{f}_{\alpha_n}(u_n) u_* \mathrm{d}x$$

and

$$\|u_n\|^2 = \int_\Omega \widehat{f}_{\alpha_n}(u_n) u_n \mathrm{d}x,$$

we conclude that

$$\|u_n\| \to \|u_*\|. \tag{3.5}$$

From (3.4) and (3.5), we have

$$u_n \to u_* \;\text{ in }\; H_0^1(\Omega). \tag{3.6}$$

As a consequence, by Remark 5 and the Dominated Convergence Theorem (being $0 < u_n \leq M$) we infer

$$\int_\Omega \widehat{F}_{\alpha_n}(u_n)\mathrm{d}x \to \int_\Omega \widehat{F}_{\alpha_*}(u_*)\mathrm{d}x. \tag{3.7}$$

By Lemma 2.2, there exists $\varepsilon > 0$, small enough, such that

$$I_{k,\alpha_n}(u_n) \leq -\frac{1}{2}\varepsilon\psi_{\alpha_n}^{-1}(\lambda_1+\varepsilon)^2 \int_\Omega e_1^2 \mathrm{d}x \quad \forall\, n \in \mathbb{N}.$$

So, passing to the limit as $n \to \infty$ and using (3.6), we obtain by Remark 4 and (3.7) ,

$$I_{k,\alpha_*}(u_*) \leq -\frac{1}{2}\varepsilon\psi_{\alpha_*}^{-1}(\lambda_1+\varepsilon)^2 \int_\Omega e_1^2 \mathrm{d}x < 0.$$

Therefore $u_* \neq 0$. Arguing as in the proof of Proposition 2.1 we can show that u_* is a positive classical solution of ($P_{k,\alpha}$) with $\alpha = \alpha_*$. Since such a solution is unique, we conclude that

$$u_* = u_{\alpha_*}. \tag{3.8}$$

Using that $0 \leq u_n \leq s_{\alpha_n} \leq M$ we infer $\sup_n\{|u_n|_\infty, |u_*|_\infty\} \leq M$ and finally

$$\int_\Omega |u_n - u_*|^p \mathrm{d}x \leq |u_n - u_*|_\infty^{p-1} \int_\Omega |u_n - u_*| \mathrm{d}x \to 0$$

which implies

$$\mathcal{P}_k(\alpha_n) \to \mathcal{P}_k(\alpha_*)$$

and concludes the proof. □

Remark 6. It is worth noticing that, with the notation of the previous proof,

$$u_n \to u_* \quad \text{in } C^1(\overline{\Omega}). \tag{3.9}$$

In fact, by (3.8) we infer

$$-\Delta(u_n - u_*) = \widehat{f}_{\alpha_n}(u_n) - \widehat{f}_{\alpha_*}(u_*) =: g_n(x) \quad \forall\, n \in \mathbb{N}. \tag{3.10}$$

From (f_0), the continuity of f and inequalities $0 \leq u_n \leq s_{\alpha_n}, 0 \leq u_* \leq s_{\alpha_*}$, there exists a constant $C > 0$, such that

$$|g_n|_\infty \leq C_2 \quad \forall\, n \in \mathbb{N}. \tag{3.11}$$

It follows from (3.10), (3.11) and Theorem 0.5 of Ambrosetti–Prodi [5] that there exists $\beta \in (0,1)$ such that

$$\|u_n - u_*\|_{C^{1,\beta}(\overline{\Omega})} \leq C' \quad \forall\, n \in \mathbb{N},$$

for some $C' > 0$. By the compactness of embedding from $C^{1,\beta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and (3.6), up to a subsequence, we have (3.9).

3.3. Existence of fixed points of $\mathcal{P}_k$

Proposition 3.2. *If conditions (f_0)–(f_5) hold, then the map $\mathcal{P}_k$ defined by* (3.1) *has at least two fixed points $t_{k-1} < \alpha_{1,k} < \alpha_{2,k} < t_k$.*

Proof. We start with two claims describing the behaviour of $\mathcal{P}_k$.
Claim 1 $\lim_{\alpha \to t_{k-1}^+} \mathcal{P}_k(\alpha) > t_{k-1}$ and $\lim_{\alpha \to t_k^-} \mathcal{P}_k(\alpha) > t_k$.

From Lemma 2.3, we have

$$\mathcal{P}_k(\alpha) \geq (\psi_\alpha^{-1}(\lambda_1))^p \int_\Omega e_1^p \mathrm{d}x \quad \forall\, \alpha \in (t_{k-1}, t_k).$$

Hence, by condition (f_4) and Lemma 2.4, we get

$$\liminf_{\alpha\to t_{k-1}^+} \mathcal{P}_k(\alpha) \geq \left(\inf_{\alpha\in(t_{k-1},t_k)} s_\alpha\right)^p \int_\Omega e_1^p \mathrm{d}x > t_K > t_{k-1},$$
$$\liminf_{\alpha\to t_k^-} \mathcal{P}_k(\alpha) \geq \left(\inf_{\alpha\in(t_{k-1},t_k)} s_\alpha\right)^p \int_\Omega e_1^p \mathrm{d}x > t_K > t_k.$$

Claim 2 There exists $\alpha_0 \in (t_{k-1}, t_k)$ such that $\mathcal{P}_k(\alpha_0) < \alpha_0$.

For each $\alpha \in (t_{k-1}, t_k)$, let w_α be the unique solution (which is positive) of the problem

$$\begin{cases} -\Delta u = u_\alpha^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where u_α is the unique positive solution of ($P_{k,\alpha}$). Hence, multiplying by u_α and integrating by parts, we have

$$\int_\Omega \nabla w_\alpha \nabla u_\alpha \mathrm{d}x = \int_\Omega u_\alpha^p \mathrm{d}x = \mathcal{P}_k(\alpha).$$

On the other hand, by the definition of u_α, we get

$$\mathcal{P}_k(\alpha) = \int_\Omega \widehat{f}_\alpha(u_\alpha) w_\alpha \mathrm{d}x \tag{3.12}$$

thus,

$$\mathcal{P}_k(\alpha) \leq \left(\max_{s\in[0,s_\alpha]} \widehat{f}_\alpha(s)\right) C_1 \|w_\alpha\|, \tag{3.13}$$

(where $C_1 > 0$ is the best constant of the Sobolev embedding from $H_0^1(\Omega)$ into $L^1(\Omega)$).

From the definition of w_α, the fact that $0 < u_\alpha \leq s_\alpha$ and Hölder's inequality, we obtain

$$\|w_\alpha\| \leq \frac{1}{\sqrt{\lambda_1}} \left(\int_\Omega u_\alpha^{2(p-1)} \mathrm{d}x\right)^{1/2} \leq \frac{1}{\sqrt{\lambda_1}} s_\alpha^{p-1} |\Omega|^{1/2}. \tag{3.14}$$

Applying (3.14) in (3.13), we obtain

$$\mathcal{P}_k(\alpha) \leq \left(\max_{s\in[0,s_\alpha]} \widehat{f}_\alpha(s)\right) \frac{C_1}{\sqrt{\lambda_1}} s_\alpha^{p-1} |\Omega|^{1/2} \quad \forall\, \alpha \in (t_{k-1}, t_k).$$

Using condition (f_5) we get the conclusion of Claim 2.

From the continuity of $\mathcal{P}_k$ (see Proposition 3.1), Claims 1 and 2 and the intermediate value theorem for continuous real functions, we conclude that $\mathcal{P}_k$ has at least two fixed points $\alpha_{1,k}$ and $\alpha_{2,k}$ in the interval (t_{k-1}, t_k). The conclusions of Claims 1 and 2 are illustrated in Figure 1.

□

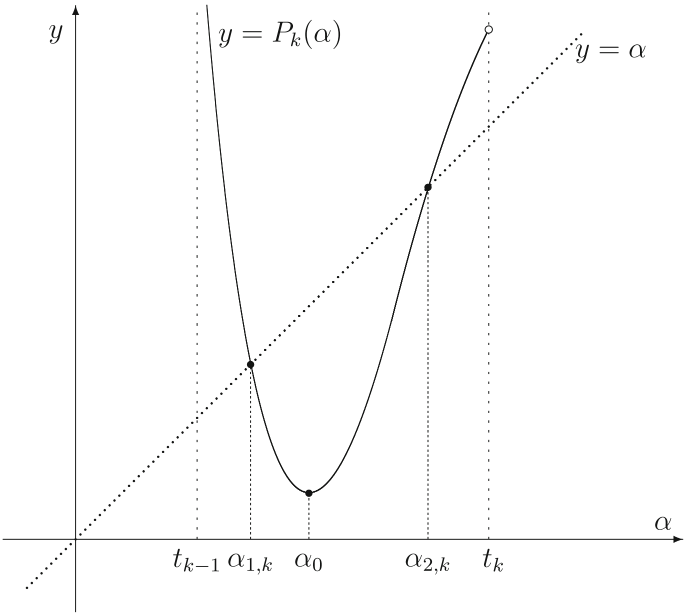

Figure 1. Properties of $\mathcal{P}_k$ in (t_{k-1}, t_k) due to Claims 1 and 2

3.4. Conclusion of the proof of Theorem 1.1

From Propositions 2.1 and 3.2 it follows that problem (P_I) has two classical positive solutions $u_{k,1}$ and $u_{k,2}$ such that

$$t_{k-1} < \int_\Omega u_{k,1}^p \mathrm{d}x < \int_\Omega u_{k,2}^p \mathrm{d}x < t_k,$$

for any fixed $k \in \{1, \ldots, K\}$. This finishes the proof. $\square$

4. Examples of nonlinearities *f*

Let us provide some examples of function f satisfying hypotheses of Theorem 1.1

(a) Let us take two integers $K \geq 1$, $u \geq (K/\int_\Omega e_1^p \mathrm{d}x)^{1/p}$ and let $S\colon [0, K] \to (u, +\infty)$ be a C^1-function. We put

$$\overline{s} := \max_{t\in[0,K]} S(t), \quad \underline{s} := \min_{t\in[0,K]} S(t) > u, \quad \text{and} \quad M := \lambda_1^{1/2}/(2C_1|\Omega|^{1/2}).$$

Let $L\colon \mathbb{R} \to [0, \infty)$ be a continuous function such that $L(w) = 0$ for $w \leq 0$, L is increasing and C^1 on $(0, \infty)$, $\lim_{w\to\infty} L(w) = +\infty$, $L(\underline{s}) > \lambda_1$, and $\max_{s\in[0,\overline{s}]} s \cdot L(\overline{s} - s) < M/\overline{s}^{p-1}$.

Now, defining

$$f(s, t) := s \cdot L\left(\frac{S(t) - s}{|\sin \pi t|}\right), \tag{4.1}$$

we will show that f satisfies hypothesis (f_0)–(f_5), with $t_i = i$ for $i \in \{0, 1, \ldots, K\}$ and $s_\alpha = S(\alpha)$.

Indeed, $f\colon \mathbb{R}\times\mathcal{A} \to \mathbb{R}$, where $\mathcal{A} = (0,\infty)\setminus\{1,\ldots,K\}$, $f(\cdot,\alpha) \in C^1(\mathbb{R})$ for all $\alpha \in \mathcal{A}$, $f(s_\alpha,\alpha) = 0$, $f(s,\alpha) > 0$ for $s \in (0, s_\alpha)$. Thus hypothesis (f_0) holds. By the assumptions on L, it is also obvious that (f_1) is valid. Also, the map $(0, s_\alpha) \ni s \mapsto f(s,\alpha)/s$ is decreasing (as L is increasing), hence (f_2) holds. For any $k \in \{1,\ldots,K\}$ and $\alpha \in (t_{k-1}, t_k)$, we have

$$\gamma_\alpha = \lim_{s\to 0^+} \frac{f(s,\alpha)}{s} = \lim_{s\to 0^+} L\left(\frac{s_\alpha - s}{|\sin \pi\alpha|}\right) = L\left(\frac{s_\alpha}{|\sin \pi\alpha|}\right) \geq L(s_\alpha) \geq L(\underline{s}) > \lambda_1,$$

thus hypothesis (f_3) holds. Hypothesis (f_4) is clear, by the assumptions on the map S (see the definition of u). Finally, for a fixed $k \in \{1,\ldots,K\}$, we have

$$\inf_{\alpha\in(t_{k-1},t_k)} \max_{s\in[0,s_\alpha]} \frac{f(s,\alpha)s_\alpha^{p-1}}{\alpha} \leq \max_{s\in[0,\overline{s}]} \frac{s\overline{s}^{p-1}}{k-\frac{1}{2}} L\left(\frac{S(k-\frac{1}{2}) - s}{|\sin \pi/2|}\right)$$

$$\leq 2\overline{s}^{p-1} \max_{s\in[0,\overline{s}]} sL(\overline{s} - s) < 2\overline{s}^{p-1} M/\overline{s}^{p-1} = \frac{\lambda_1^{1/2}}{C_1|\Omega|^{1/2}},$$

thus hypothesis (f_5) holds.

(b) Let us now provide some particular examples of functions S and L satisfying the above assumptions.

Take K, u, and M as defined in (a) and put $a := u+1$, $n > \frac{(\lambda_1+1)(a+1)^p}{M} - 1$, and $\beta := \frac{\lambda_1+1}{a^n}$. Next choose any $b \in \left(a, \min\left\{a\sqrt[n]{\frac{(n+1)M}{(\lambda_1+1)(a+1)^p}},\ a+1\right\}\right)$. Now we define functions $S\colon [0,K] \to (0,+\infty)$ and $L\colon \mathbb{R} \to [0,\infty)$, by

$$S(t) := \frac{b-a}{K}t + a \quad \forall t \in [0,K], \qquad L(w) := \begin{cases} \beta w^n & \text{for } w \geq 0, \\ 0 & \text{for } w < 0. \end{cases}$$

It is easy to check, that $\underline{s} = a > u$ and $\overline{s} = b$. Moreover the function L is continuous, increasing and C^1 on $(0,\infty)$, $\lim_{w\to\infty} L(w) = +\infty$, $L(\underline{s}) = L(a) = \beta a^n = \frac{\lambda_1+1}{a^n}a^n = \lambda_1 + 1 > \lambda_1$, and finally the maximum of the function $[0,\overline{s}] \ni s \longmapsto s \cdot L(\overline{s} - s) = \beta s(b-s)^n$ is attained at $s_{max} = \frac{b}{1+n}$ and so

$$\max_{s\in[0,\overline{s}]} s\cdot L(\overline{s}-s) = \frac{\lambda_1+1}{1+n}\left(\frac{b}{a}\right)^n b\left(\frac{n}{1+n}\right)^n < \frac{\lambda_1+1}{1+n}\frac{(n+1)M}{(\lambda_1+1)(a+1)^p}(a+1)$$
$$= M/(a+1)^{p-1} < M/b^{p-1} = M/\overline{s}^{p-1},$$

by the definition of b. Thus both functions S and L fit into the framework of example (a), and so the function f defined by (4.1) satisfies hypotheses (f_0)–(f_5).

(c) For another choice of functions S and L satisfying the assumptions of (a), take again K, u, and M as in (a) and put $a := u + 1$. Next fix two numbers b and d such that $a < b < d < \min\left\{a + \frac{Ma}{(\lambda_1+1)(a+1)^p},\ a+1\right\}$ and define $c := \frac{(\lambda_1+1)(d-a)}{a}$. Let $S\colon [0,K] \to [a,b]$ be any C^1-function. Then $\underline{s} \geq a$ and $\overline{s} \leq b$.

On the interval $[0,b]$ we defined an increasing C^1-function $L(w)=\frac{-cw}{w-d}$ and extend it outside $[0,b]$ in such a way that it is still C^1 and increasing. We have $L(\underline{s})\geq L(a)=\lambda_1+1>\lambda_1$ and

$$\max_{s\in[0,\overline{s}]} s\cdot L(\overline{s}-s)\leq \max_{s\in[0,b]} s\cdot L(b-s)=\max_{s\in[0,b]} -cs\frac{b-s}{b-s-d}=\max_{s\in[0,b]}\frac{cs(b-s)}{s+(d-b)},$$

where last maximum is attained at $s_{max}=\sqrt{d(d-b)}-(d-b)\in(0,b)$, with the value

$$\begin{aligned}\max_{s\in[0,b]}\frac{cs(b-s)}{s+(d-b)} &= \frac{(\lambda_1+1)(d-a)}{a}\big(1-\sqrt{(d-b)/d}\big)\big(d-\sqrt{d(d-b)}\big)\\ &\leq \frac{(\lambda_1+1)}{a}(d-a)d<\frac{(\lambda_1+1)}{a}\frac{Ma}{(\lambda_1+1)(a+1)^p}(a+1)\\ &=\frac{M}{(a+1)^{p-1}}\leq\frac{M}{b^{p-1}}\leq\frac{M}{\overline{s}^{p-1}},\end{aligned}$$

thus both functions S and L fit into the framework of example (a).

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no conflict of interests of any type.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Agmon, S.: The L_p approach to the Dirichlet problem. Ann. Scuola Norm. Sup. Pisa **13**, 405–448 (1959)

[2] Ambrosetti, A., Arcoya, D.: Positive solutions of elliptic Kirchhoff equations. Adv. Nonlinear Stud. **17**(1), 3–16 (2017)

[3] Ambrosetti, A., Arcoya, D.: Remarks on non homogeneous elliptic Kirchhoff equations. Nonlinear Differ. Equ. Appl. **23**, Art. 57 (2016)

[4] Ambrosetti, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)

[5] Ambrosetti, A., Prodi, G.: A Primer of Nonlinear Analysis. Cambridge University Press, Cambridge (1993)

[6] Brezis, H., Oswald, L.: Remarks on sublinear elliptic equations. Nonlinear Anal. **10**(1), 55–64 (1986)

[7] Carrier, G.F.: On the non-linear vibration problem of the elastic string. Q. J. Appl. Math. **3**, 151–165 (1945)

[8] Chipot, M., Rodrigues, J.F.: On a class of nonlocal nonlinear elliptic problems. RAIRO Modélisation mathématique et analyse numérique **26**(3), 447–467 (1992)

[9] Delgado, M., Morales-Rodrigo, C., Santos Júnior, J.R., Suárez, A.: Non-local degenerate diffusion coefficients break down the components of positive solutions. Adv. Nonlinear Stud. **20**(1), 19–30 (2019)

[10] Figueiredo-Sousa, T., Morales-Rodrigo, C., Suárez, A.: A non-local non-autonomous diffusion problem: linear and sublinear cases. Z. Angew. Math. Phys. **68**(5), Art. 108 (2017)

[11] Furter, J., Grinfeld, M.: Local vs. nonlocal interactions in population dynamics. J. Math. Biol. **27**, 65–80 (1989)

[12] Gasiński, L., Santos Júnior, J.R.: Multiplicity of positive solutions for an equations with degenerate nonlocal diffusion. Comput. Math. Appl. **78**, 136–143 (2019)

[13] Gasiński, L., Santos Júnior, J.R.: Nonexistence and multiplicity of positive solutions for an equation with degenerate nonlocal diffusion. Bull. Lond. Math. Soc. **52**, 489–497 (2020)

[14] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1983)

[15] Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)

[16] Santos Júnior, J.R., Siciliano, G.: Positive solutions for a Kirchhoff problem with vanishing nonlocal term. J. Differ. Equ. **265**, 2034–2043 (2018)

Leszek Gasiński
Department of Mathematics
Pedagogical University of Cracow
Podchorazych 2
30-084 Cracow,
Poland
e-mail: `leszek.gasinski@up.krakow.pl`

João R. Santos Junior
Faculdade de Matemática Instituto de Ciências Exatas e Naturais
Universidade Federal do Pará
Avenida Augusto corrêa 01
Belém 66075-110 PA
Brazil
e-mail: `joaojunior@ufpa.br`

Gaetano Siciliano
Departamento de Matemática Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010
São Paulo SP 05508-090
Brazil
e-mail: `sicilian@ime.usp.br`

Accepted: August 30, 2022.