



On holomorphic distributions on Fano threefolds

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ABSTRACT

This paper is devoted to the study of holomorphic distributions of dimension and codimension one on smooth weighted projective complete intersection Fano manifolds X which is three-dimensional and with Picard number equal to one. We study the relations between algebro-geometric properties of the singular set of singular holomorphic distributions and their associated sheaves. We characterize either distributions whose tangent sheaf or conormal sheaf are arithmetically Cohen Macaulay (aCM) on X . We also prove that a codimension one locally free distribution with trivial canonical bundle on any Fano threefold, with Picard number equal to one, has a tangent sheaf which either splits or it is stable.

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1. Introduction

In complex manifolds, holomorphic distributions and foliations have been much studied (see [2], [5], [8], [16]). An important problem is to analyse when the tangent and conormal sheaves split, as well as the properties of singular schemes of distributions. With these properties make it possible to classify of codimension one distributions and study the stability of the tangent sheaf.

The relation between the tangent sheaf of a holomorphic distribution and its singular locus has been recently of great interest. Indeed, L. Giraldo and A. J. Pan-Collantes showed in [16] that the tangent sheaf of a foliation of codimension one on \mathbb{P}^3 splits if and only if its singular scheme Z is aCM, and more recently, M. Corrêa, M. Jardim and R. Vidal Martins extended this result in [8], showing that the tangent sheaf of

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a codimension one locally free distribution on \mathbb{P}^n splits as a sum of line bundles if and only if its singular scheme is aCM.

Section 3 is devoted precisely to extend this result for the smooth weighted projective complete intersection Fano threefolds with Picard number one. Such varieties have been classified by Iskovskih [19,20] and Mukai [24]. The *index* of X is a basic invariant of these manifolds. This is the largest integer ι_X such that the canonical line bundle K_X is divisible by ι_X in $\text{Pic}(X)$.

In [21], Kobayashi and Ochiai showed that the index ι_X is at most $\dim(X) + 1$ and $\iota_X = \dim(X) + 1$, if and only if $X \simeq \mathbb{P}^n$. Moreover, $\iota_X = \dim(X)$, if and only if $X \simeq Q^n \subset \mathbb{P}^{n+1}$, where Q^n is a smooth quadric. In particular, when X is a Fano threefold, it can have index $\iota_X = 4, 3, 2, 1$. Separating the varieties by the index, we prove the following result, where we denote by r is an integer such that $r = c_1(TX) - c_1(T_{\mathcal{F}})$, i.e. the difference of the first Chern classes of the tangent bundle of the variety with the tangent sheaf of the distribution.

Theorem 1.1. *Let \mathcal{F} be a distribution of codimension one on a smooth weighted projective complete intersection Fano threefold X , such that the tangent sheaf $T_{\mathcal{F}}$ is locally free. If $Z = \text{Sing}(\mathcal{F})$ is the singular scheme of \mathcal{F} , then:*

- (1) *If $\iota_X = 3$ and $T_{\mathcal{F}}$ either splits as a sum of line bundles or is a spinor bundle, then Z is arithmetically Buchsbaum, with $h^1(Q^3, I_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$. Conversely, if Z is arithmetically Buchsbaum with $h^1(Q^3, I_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$ and $h^2(T_{\mathcal{F}}(-2)) = h^2(T_{\mathcal{F}}(-1 - c_1(T_{\mathcal{F}}))) = 0$, then $T_{\mathcal{F}}$ either split or is a spinor bundle.*
- (2) *If $\iota_X = 2$ and $T_{\mathcal{F}}$ has no intermediate cohomology, then $H^1(X, I_Z(r+t)) = 0$ for $t < -6$ and $t > 8$. Conversely, if $H^1(X, I_Z(r+t)) = 0$ for $t < -6$ and $t > 8$, and $H^2(X, T_{\mathcal{F}}(t)) = 0$ for $t \leq 8$ and $H^1(X, T_{\mathcal{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathcal{F}})$, then $T_{\mathcal{F}}$ has no intermediate cohomology.*
- (3) *If $\iota_X = 1$ and $T_{\mathcal{F}}$ has no intermediate cohomology, then $H^1(X, I_Z(r+t)) = 0$ for $t < -4$ and $t > 4$. Conversely, if $H^1(X, I_Z(r+t)) = 0$ for $t < -4$ and $t > 4$, and $H^2(X, T_{\mathcal{F}}(t)) = 0$ for $t \leq 4$ and $H^1(X, T_{\mathcal{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathcal{F}})$, then $T_{\mathcal{F}}$ has no intermediate cohomology.*

In section 4 we will focus on the conormal sheaf $N_{\mathcal{F}}^*$ of a foliation of dimension one on Fano threefold. M. Corrêa, M. Jardim and R. Vidal Martins showed in [8] that the conormal sheaf $N_{\mathcal{F}}^*$ of a foliation of dimension one on \mathbb{P}^n splits if and only if its singular scheme Z is arithmetically Buchsbaum with $h^1(\mathcal{I}_Z(d-1)) = 1$ being the only nonzero intermediate cohomology. We manage to extend the mentioned result for any Fano threefolds and obtain the following result.

Theorem 1.2. *Let \mathcal{F} be a distribution of dimension one on a smooth weighted projective complete intersection Fano threefold X , with index ι_X and Picard number one. If $Z = \text{Sing}(\mathcal{F})$ is the singular scheme of \mathcal{F} , then:*

- (1) *If $N_{\mathcal{F}}^*$ is arithmetically Cohen Macaulay, then Z is arithmetically Buchsbaum, with $h^1(X, I_Z(r)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$.*
- (2) *If Z is arithmetically Buchsbaum with $h^1(X, I_Z(r)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$ and $h^2(N_{\mathcal{F}}^*) = h^2(N_{\mathcal{F}}^*(-c_1(N_{\mathcal{F}}^*) - \iota_X)) = 0$ and $\iota_X \in \{1, 2, 3\}$, then $N_{\mathcal{F}}^*$ is arithmetically Cohen Macaulay.*

We address the properties of their singular schemes of codimension one holomorphic distributions on a threefold in section 5. D. Cerveau and A. Lins Neto in [7] have classified codimension one foliations on \mathbb{P}^3 with trivial canonical bundle. In [5], O. Calvo-Andrade, M. Corrêa and M. Jardim have studied codimension one distributions by analyzing the properties of their singular schemes and tangent sheaves. In [22], F. Loray,

J. Pereira and F. Touzet have studied codimension one foliation, with numerically trivial canonical bundle, on Fano threefolds with Picard number one.

In [5], O. Calvo-Andrade, M. Corrêa and M. Jardim showed a cohomological criterion for the connectedness of the singular scheme of codimension one distributions on \mathbb{P}^3 [5, Theorem 3.7]. We extend this result for the other Fano threefolds with Picard number one. More precisely, we prove the following result.

Theorem 1.3. *Let \mathcal{F} be a distribution of dimension one on a smooth weighted projective complete intersection Fano threefold X , with index ι_X and Picard number one. Denote by C the component of pure dimension one of $Z = \text{Sing}(\mathcal{F})$. If $h^2(T_{\mathcal{F}}(-r)) = 0$ and $C \subset X$, $C \neq \emptyset$, then $Z = C$ is connected and $T_{\mathcal{F}}$ is locally free. Conversely, for $r \neq \iota_X$, if $Z = C$ and it is connected, then $T_{\mathcal{F}}$ is locally free and $h^2(T_{\mathcal{F}}(-r)) = 0$.*

In section 6 we will prove that if \mathcal{F} is a codimension one distribution on Fano threefold with tangent sheaf locally free and $c_1(T_{\mathcal{F}}) = 0$, then $T_{\mathcal{F}}$ is either split or stable.

In [5], O. Calvo-Andrade, M. Corrêa and M. Jardim showed that if \mathcal{F} is a codimension one distribution on \mathbb{P}^3 with $T_{\mathcal{F}}$ locally free and $c_1(T_{\mathcal{F}}) = 0$, then $T_{\mathcal{F}}$ is either split or stable. We prove that this in fact holds for any Fano threefold X with Picard number $\rho(X) = 1$.

Theorem 1.4. *Let \mathcal{F} be a codimension one distribution on a Fano threefold X with Picard number $\rho(X) = 1$. If $T_{\mathcal{F}}$ is locally free and $c_1(T_{\mathcal{F}}) = 0$, then $T_{\mathcal{F}}$ is either split or stable.*

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Notation and conventions We always work over the field \mathbb{C} of complex numbers. Given a complex variety X , we denote by TX the tangent bundle $(\Omega_X^1)^*$ and to simplify the notation, given a distribution \mathcal{F} let us write $Z := \text{Sing}(\mathcal{F})$.

Assume that the Picard Group of X is \mathbb{Z} . We will denote $E(t) = E \otimes_{\mathcal{O}_X} \mathcal{O}_X(t)$ for $t \in \mathbb{Z}$ when E is a vector bundle on X , and $\mathcal{O}_X(1)$ denotes its ample generator.

If F is a sheaf on X , we denote by $h^i(X, F)$ the dimension of the complex vector space $H^i(X, F)$.

Remark 1.5. [18, Proposition 1.10] For any holomorphic vector bundle E of rank r ,

$$\bigwedge^k E \simeq \bigwedge^{r-k} E^* \otimes \det E.$$

In particular, if E is a rank 2 reflexive sheaf, then $E^* = E \otimes (\det E)^*$.

2. Preliminaries

In this section we recall some basic facts of the theory. Throughout this section, unless otherwise noted, X denotes a smooth weighted projective complete intersection Fano threefold with Picard number one.

2.1. Fano manifolds

A compact complex manifold X is *Fano* if its anticanonical line bundle $\mathcal{O}_X(-K_X) \simeq \bigwedge^n TX$ is ample.

Definition 2.2. We say that $X \subset \mathbb{P}(a_0, \dots, a_N)$ is a smooth n -dimensional *weighted complete intersection* in a weighted projective space, when X is the scheme-theoretic zero locus of $c = N - n$ weighted homogeneous polynomials f_1, \dots, f_c of degrees d_1, \dots, d_c .

Smooth weighted projective complete intersection Fano threefold, with Picard rank one, have been classified by Iskovskikh [19,20] and Mukai [24]. They are:

- (a) the projective space \mathbb{P}^3 ;
- (b) a quadric hypersurface $Q^3 = X_2 \subset \mathbb{P}^4$;
- (c) a cubic hypersurface $X_3 \subset \mathbb{P}^4$;
- (d) an intersection $X_{2,2}$ of two quadric hypersurfaces in \mathbb{P}^5 ;
- (e) a hypersurface of degree 4 in the weighted projective space $X_4 \subset \mathbb{P}(1, 1, 1, 1, 2)$;
- (f) a hypersurface of degree 6 in the weighted projective space $X_6 \subset \mathbb{P}(1, 1, 1, 2, 3)$;
- (g) an intersection $X_{2,3}$ of a quadric and a cubic in \mathbb{P}^5 ;
- (h) an intersection $X_{2,2,2}$ of three quadrics in \mathbb{P}^6 ;
- (i) an intersection of a quadratic cone and a hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 1, 1, 2)$.

A basic invariant of a Fano manifold is its index.

Definition 2.3. The *index* of X is the maximal integer $\iota_X > 0$ dividing $-K_X$ in $\text{Pic}(X)$, i.e. $-K_X = \iota_X \cdot H$, with H ample.

Theorem 2.4. [21] *Let X be Fano, $\dim(X) = n$. Then the index ι_X is at most $n + 1$; moreover, if $\iota_X = n + 1$, then $X \simeq \mathbb{P}^n$, and if $\iota_X = n$, then X is a quadric hypersurface $Q^n \subset \mathbb{P}^{n+1}$.*

By Theorem 2.4, a Fano threefold X can have $\iota_X \in \{1, 2, 3, 4\}$. Then, $\iota_X = 4$ implies $X = \mathbb{P}^3$, while $\iota_X = 3$, implies that X is a smooth quadric hypersurface Q^3 in \mathbb{P}^4 . In case $\iota_X = 2$, the variety X is called a del Pezzo threefold, while $\iota_X = 1$, the variety X is called a prime Fano threefold.

By [12, Theorem 3.3.4],

$$K_X \cong \mathcal{O}_X \left(\sum_{j=1}^c d_j - \sum_{i=0}^N a_i \right) \tag{2.5}$$

In particular, when X is Fano, its index is

$$\iota_X := \sum_{i=0}^N a_i - \sum_{j=1}^c d_j. \tag{2.6}$$

Let S_t be the t -th graded part of $S/(f_1, \dots, f_c)$. By [11, Lemma 7.1],

$$H^i(X, \mathcal{O}_X(t)) \cong \begin{cases} S_t & \text{if } i = 0; \\ 0 & \text{if } 1 \leq i \leq n - 1; \\ S_{-t+\iota_X} & \text{if } i = n. \end{cases} \tag{2.7}$$

Let Y be an n -dimensional Fano manifold with $\rho(Y) = 1$, and denote by $\mathcal{O}_Y(1)$ the ample generator of $\text{Pic}(Y)$. Let $X \in |\mathcal{O}_Y(d)|$ be a smooth divisor. We have the following exact sequences:

$$0 \rightarrow \Omega_Y^q(t-d) \rightarrow \Omega_Y^q(t) \rightarrow \Omega_Y^q(t)|_X \rightarrow 0, \tag{2.8}$$

and

$$0 \rightarrow \Omega_X^{q-1}(t-d) \rightarrow \Omega_Y^q(t)|_X \rightarrow \Omega_X^q(t) \rightarrow 0, \tag{2.9}$$

where this last one is a twisted exterior power of cotangent-conormal sequence on X .

2.10. Cohomology of cotangent sheaf

In this subsection we address important facts about the cohomology of X with coefficients in an analytic coherent sheaf.

Let p, q and t be integers, with p and q non-negative. Then,

$$h^p(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(t)) = \begin{cases} \binom{t+n-q}{t} \binom{t-1}{q} & \text{for } p = 0, 0 \leq q \leq n \text{ and } t > q, \\ 1 & \text{for } t = 0 \text{ and } 0 \leq p = q \leq n, \\ \binom{-t+q}{-t} \binom{-t-1}{n-q} & \text{for } p = n, 0 \leq q \leq n \text{ and } t < q - n, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout the work, we refer to this formula as classical Bott’s formula.

In [28], Snow showed a vanishing theorem for the cohomology of $\Omega^q(t)$ for quadric hypersurfaces $X = Q^n$ in \mathbb{P}^{n+1} and for a Grassmann manifold $X = Gr(s, m)$ of s -dimensional subspaces of \mathbb{C}^m .

Theorem 2.11. [28] [Bott’s formula for Quadric] *Let X be a nonsingular quadric hypersurface of dimension n .*

- (1) *If $-n + q \leq k \leq q$ and $k \neq 0$ and $k \neq -n + 2q$, then $H^p(X, \Omega^q(k)) = 0$ for all p ;*
- (2) *$H^p(X, \Omega^q) \neq 0$ if and only if $p = q$;*
- (3) *$H^p(X, \Omega^q(-n + 2q)) \neq 0$ if and only if $p = n - q$;*
- (4) *If $k > q$, then $H^p(X, \Omega^q(k)) \neq 0$ if and only if $p = 0$;*
- (5) *If $k < -n + q$, then $H^p(X, \Omega^q(k)) \neq 0$ if and only if $p = n$.*

In [12], Dolgachev generalized the classical Bott’s formula for the cohomology of twisted sheaves of differentials to the case of weighted projective spaces $(\overline{\Omega}_{\mathbb{P}}^q(t))$ as follows:

Let $\mathbb{P} = \mathbb{P}(a_0, \dots, a_N) = \text{Proj}(S(a_0, \dots, a_N))$ be a weighted projective space.

Consider the sheaves of $\mathcal{O}_{\mathbb{P}}$ -modules $\overline{\Omega}_{\mathbb{P}}^q(t)$ defined in [12, Section 2.1.5] for $q, t \in \mathbb{Z}, q \geq 0$. If $U \subset \mathbb{P}$ denotes the smooth locus of \mathbb{P} , and $\mathcal{O}_U(t)$ is the line bundle obtained by restricting $\mathcal{O}_{\mathbb{P}}(t)$ to U , then $\overline{\Omega}_{\mathbb{P}}^q(t)|_U = \Omega_U^q \otimes \mathcal{O}_U(t)$. The cohomology groups $H^p(\mathbb{P}, \overline{\Omega}_{\mathbb{P}}^q(t))$ are described by:

Theorem 2.12. [12, Section 2.3.2] *Let $h^p(\mathbb{P}, \overline{\Omega}_{\mathbb{P}}^q(t)) = \dim H^p(\mathbb{P}, \overline{\Omega}_{\mathbb{P}}^q(t))$. Then:*

- $h^0(\mathbb{P}, \overline{\Omega}_{\mathbb{P}}^q(t)) = \sum_{i=0}^q \left((-1)^{i+q} \sum_{\#J=i} \dim_{\mathbb{C}}(S_{t-a_J}) \right)$, where $J \subset \{0, \dots, N\}$ and $a_J := \sum_{i \in J} a_i$;
- $h^0(\mathbb{P}, \overline{\Omega}_{\mathbb{P}}^q(t)) = 0$ if $t < \min\{\sum_{j \in J} a_{i_j} \mid \#J = q\}$;
- $h^p(\mathbb{P}, \overline{\Omega}_{\mathbb{P}}^q(t)) = 0$ if $p \notin \{0, q, N\}$;
- $h^p(\mathbb{P}, \overline{\Omega}_{\mathbb{P}}^p(t)) = 0$ if $t \neq 0$ and $p \notin \{0, N\}$.

In particular, if $q \geq 1$, then

$$h^0(\mathbb{P}, \Omega_{\mathbb{P}}^q(t)) = 0 \text{ for any } t \leq q. \tag{2.13}$$

Notice that $\mathbb{P}(a_0, \dots, a_N) = \mathbb{P}^N$ is a projective space we obtain the classical Bott's formulas.

Now assume that \mathbb{P} has only isolated singularities, let $d > 0$ be such that $\mathcal{O}_{\mathbb{P}}(d)$ is a line bundle generated by global sections, and $X \in |\mathcal{O}_{\mathbb{P}}(d)|$ a smooth hypersurface. We will use the cohomology groups $H^p(\mathbb{P}, \overline{\Omega}_{\mathbb{P}}^q(t))$ to compute some cohomology groups $H^p(X, \Omega_X^q(t))$. Note that X is contained in the smooth locus of \mathbb{P} , so we have an exact sequence as in (2.9):

$$0 \rightarrow \Omega_X^{q-1}(t-d) \rightarrow \overline{\Omega}_{\mathbb{P}}^q(t)|_X \rightarrow \Omega_X^q(t) \rightarrow 0. \tag{2.14}$$

Tensoring the sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-d) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_X \rightarrow 0,$$

with the sheaf $\overline{\Omega}_{\mathbb{P}}^q(t)$, and noting that $\overline{\Omega}_{\mathbb{P}}^q(t) \otimes \mathcal{O}_{\mathbb{P}}(-d) \cong \overline{\Omega}_{\mathbb{P}}^q(t-d)$, we get an exact sequence as in (2.8):

$$0 \rightarrow \overline{\Omega}_{\mathbb{P}}^q(t-d) \rightarrow \overline{\Omega}_{\mathbb{P}}^q(t) \rightarrow \overline{\Omega}_{\mathbb{P}}^q(t)|_X \rightarrow 0. \tag{2.15}$$

Now, we will conclude this section by recalling the cohomological computations on weighted complete intersections X . The next theorem in terms of cohomology of X , is due to Flenner:

Theorem 2.16. [14, Satz 8.11] *Let X be a weighted complete intersection. Then,*

- $h^q(X, \Omega_X^q) = 1$ for $0 \leq q \leq n$, $q \neq \frac{n}{2}$.
- $h^p(X, \Omega_X^q(t)) = 0$ in the following cases
 - $0 < p < n$, $p + q \neq n$ and either $p \neq q$ or $t \neq 0$;
 - $p + q > n$ and $t > q - p$;
 - $p + q < n$ and $t < q - p$.

2.17. aCM and aB schemes

A closed subscheme $Y \subset \mathbb{P}^n$ is *arithmetically Cohen-Macaulay (aCM)* if its homogeneous coordinate ring $S(Y) = k[x_0, \dots, x_n]/I(Y)$ is a Cohen-Macaulay ring.

Equivalently, Y is aCM if $H_*^p(\mathcal{O}_Y) = 0$ for $1 \leq p \leq \dim(Y) - 1$ and $H_*^1(\mathcal{I}_Y) = 0$ (cf. [6]). From the long exact sequence of cohomology associated to the short exact sequence

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$$

one also deduces that Y is aCM if and only if $H_*^p(\mathcal{I}_Y) = 0$ for $1 \leq p \leq \dim(Y)$.

Similarly, a closed subscheme is *arithmetically Buchsbaum (aB)* if its homogeneous coordinate ring is a Buchsbaum ring (see [29]). Clearly, every aCM scheme is arithmetically Buchsbaum, but the converse is not true: the union of two disjoint lines is arithmetically Buchsbaum, but not aCM.

2.18. Rank 2 vector bundles on quadrics

Let E be rank 2 vector bundle on a smooth quadric $Q^n \subset \mathbb{P}^{n+1}$, with $n \geq 3$. We set

$$R = \bigoplus_{t \geq 0} H^0(Q^n, \mathcal{O}_{Q^n}(t)) \quad \text{and} \quad \mathfrak{m} = \bigoplus_{t > 0} H^0(Q^n, \mathcal{O}_{Q^n}(t)),$$

and also

$$H_*^i(Q^n, E) = \bigoplus_{t \in \mathbb{Z}} H^i(Q^n, E(t)) \quad \text{for } i = 0, \dots, n,$$

which are modules of finite length over the ring R .

Definition 2.19. We say that E is k -Buchsbaum, with $k \geq 0$, if for all integers p, q such that $1 \leq p \leq q - 1$ and $3 \leq q \leq n$ it holds

$$\mathfrak{m}^k \cdot H_*^p(Q', E|_{Q'}) = 0,$$

where Q' is a general q -dimensional linear section of Q^n , i.e. Q' is a quadric hypersurface cut out on Q^n by a general linear space $L \subset \mathbb{P}^{n+1}$ of dimension $q + 1$, that is $Q' = Q_n \cap L$.

Remark 2.20. Note that E is 0-Buchsbaum if and only if E has no intermediate cohomology, i.e. $H^i(Q^n, E(t)) = 0$ for every $t \in \mathbb{Z}$ and $1 \leq i \leq n - 1$. Such a bundle is also called *arithmetically Cohen-Macaulay*.

Observe also that E is 1-Buchsbaum if and only if E has every intermediate cohomology module with trivial structure over R . Such a bundle is also called *arithmetically Buchsbaum*.

Now, we recall the definition and some properties of spinor bundles on Q^n . For more details see [25,26].

Let S_k be spinor variety which parametrizes the family of $(k - 1)$ -planes in Q^{2k-1} or one of the two disjoint families of k -planes in Q^{2k} .

We have $\dim(S_k) = (k(k + 1))/2$, $\text{Pic}(S_k) = \mathbb{Z}$ and $h^0(S_k, \mathcal{O}(1)) = 2^k$. Spinor varieties are rational homogeneous manifolds of rank 1. When $n = 2k - 1$ is odd, consider for all $x \in Q^{2k-1}$ the variety

$$\{\mathbb{P}^{k-1} \in Gr(k - 1, 2k)/x \in \mathbb{P}^{k-1} \subset Q^{2k-1}\}.$$

This variety is isomorphic to S_{k-1} and we denote it by $(S_{k-1})_x$. Then we have a natural embedding

$$(S_{k-1})_x \rightarrow S_k.$$

Considering the linear spaces spanned by these varieties, we have for all $x \in Q^{2k-1}$ a natural inclusion $H^0((S_{k-1})_x, \mathcal{O}(1))^* \rightarrow H^0(S_k, \mathcal{O}(1))^*$ and then an embedding

$$s : Q^{2k-1} \rightarrow Gr(2^{k-1} - 1, 2^k - 1),$$

in the Grassmannian of $(2^{k-1} - 1)$ -subspaces of \mathbb{P}^{2^k-1} .

It is well known that $S_1 \simeq \mathbb{P}^1$, $S_2 \simeq \mathbb{P}^3$. The embedding $s : Q^3 \rightarrow Gr(1, 3)$ corresponds to a hyperplane section.

Definition 2.21. Let U be the universal bundle of the Grassmannian. We call $s^*U = S$ the *spinor bundle* on Q^{2k-1} , whose rank is 2^{k-1} .

The spinor bundle S on Q^3 is just the restriction of the universal sub-bundle on the 4-dimensional quadric.

Remark 2.22. The spinor bundle on Q^3 is a globally generated vector bundle of rank 2, by construction.

The next theorem shows the existence of bundles that are indecomposable on quadric.

Theorem 2.23. [25, Theorem 2.1] *The spinor bundle on Q^3 is stable.*

We have that S is the unique indecomposable bundle of rank 2 on Q^3 with $c_1(S) = -1$ and $c_2(S) = 1$, (see [4]).

2.24. Holomorphic distributions

Let us now recall facts about holomorphic distributions on complex projective varieties. For more details about distributions and foliations see [1,5,8–10,13].

Definition 2.25. Let X be a smooth complex manifold. A *codimension k distribution* \mathcal{F} on X is given by an exact sequence

$$\mathcal{F} : 0 \longrightarrow T_{\mathcal{F}} \xrightarrow{\phi} TX \xrightarrow{\pi} N_{\mathcal{F}} \longrightarrow 0, \quad (2.25)$$

where $T_{\mathcal{F}}$ is a coherent sheaf of rank $r_{\mathcal{F}} := \dim(X) - k$, and $N_{\mathcal{F}} := TX/\phi(T_{\mathcal{F}})$ is a torsion free sheaf.

The sheaves $T_{\mathcal{F}}$ and $N_{\mathcal{F}}$ are called the *tangent* and the *normal* sheaves of \mathcal{F} , respectively. The *singular set* of the distribution \mathcal{F} is defined by

$$\text{Sing}(\mathcal{F}) = \{x \in X \mid (N_{\mathcal{F}})_x \text{ is not a free } \mathcal{O}_{X,x} \text{- module}\}.$$

The *conormal* sheaf of \mathcal{F} is $N_{\mathcal{F}}^*$.

A distribution \mathcal{F} is said to be *locally free* if $T_{\mathcal{F}}$ is a locally free sheaf.

By definition, $\text{Sing}(\mathcal{F})$ is the singular set of the sheaf $N_{\mathcal{F}}$. It is a closed analytic subvariety of X of codimension at least two, since by definition $N_{\mathcal{F}}$ is torsion free.

Definition 2.26. A *foliation* is an integrable distribution, that is a distribution

$$\mathcal{F} : 0 \longrightarrow T_{\mathcal{F}} \xrightarrow{\phi} TX \xrightarrow{\pi} N_{\mathcal{F}} \longrightarrow 0$$

whose tangent sheaf is closed under the Lie bracket of vector fields, i.e. $[\phi(T_{\mathcal{F}}), \phi(T_{\mathcal{F}})] \subset \phi(T_{\mathcal{F}})$.

Clearly, every distribution of codimension $\dim(X) - 1$ is integrable.

From now on, we will consider codimension one distributions on Fano threefold X with Picard number one. Thus, sequence (2.25) simplifies to

$$\mathcal{F} : 0 \longrightarrow T_{\mathcal{F}} \xrightarrow{\phi} TX \xrightarrow{\pi} I_{Z/X}(r) \longrightarrow 0, \quad (2.26)$$

where $T_{\mathcal{F}}$ is a rank 2 reflexive sheaf and r is an integer such that $r = c_1(TX) - c_1(T_{\mathcal{F}})$. Observe that $N_{\mathcal{F}} = I_{Z/X}(r)$ where Z is the singular scheme of \mathcal{F} ; indeed, $N_{\mathcal{F}}$ is by definition a torsion free sheaf, which in this case will have rank one, hence an ideal sheaf, which defines the singular locus of the distribution.

A codimension one distribution on X can also be represented by a section $\omega \in H^0(\Omega_X^1(r))$, given by the dual of the morphism $\pi : TX \rightarrow N_{\mathcal{F}}$. On the other hand, such section induces a sheaf map $\omega : \mathcal{O}_X \rightarrow \Omega_X^1(r)$. Taking duals, we get a cosection

$$\omega^* : (\Omega_X^1(r))^* = TX(-r) \rightarrow \mathcal{O}_X$$

whose image is the ideal sheaf $I_{Z/X}$ of the singular scheme. The kernel of ω^* is the tangent sheaf \mathcal{F} of the distribution twisted by $\mathcal{O}(-r)$.

Remark 2.27. From this point of view, the integrability condition is equivalent to $\omega \wedge d\omega = 0$.

3. Tangent sheaf vs. singular scheme

In this section, all the distributions that we will consider will be defined for Fano manifolds as described in Section 2.1, of codimension one, and with singular set of codimension ≥ 2 .

For codimension one foliations in \mathbb{P}^3 , L. Giraldo and A. J. Pan Collantes showed in [16] that the tangent sheaf is locally free if and only if the singular scheme is a curve, i.e. a Cohen-Macaulay scheme of pure dimension one, cf. [[16], Theorem 3.2]. More recently, O. Calvo-Andrade, M. Corrêa, and M. Jardim obtained in [5] the following generalization of this theorem.

Lemma 3.1. [5, Lemma 2.1] *The tangent sheaf of a codimension one distribution is locally free if and only if its singular locus has pure codimension 2.*

We characterize when the tangent sheaf splits (i.e. direct sum of line bundles) in terms of the geometry of the singular scheme of the distribution.

The projective space case has been already considered. Indeed, when $\iota_X = 4$, L. Giraldo and A. J. Pan-Collantes showed in [16], that the tangent sheaf of a foliation of dimension 2 on \mathbb{P}^3 , splits if and only if its singular scheme Z is aCM [16, Theorem 3.3].

Let us now consider the quadric hypersurface Q^3 in \mathbb{P}^4 , i.e. $\iota_X = 3$. Let E a bundle on Q^3 . It is well known that if E splits on Q^3 , then:

$$H^i(Q^3, E(t)) = 0 \text{ for } 0 < i < 3, \text{ for all } t \in \mathbb{Z}.$$

The following result characterizes when the tangent sheaf of a distribution of dimension 2 on Q^3 , is split or spinor.

Theorem 3.2. *Let \mathcal{F} be a distribution on Q^3 of codimension one, such that the tangent sheaf $T_{\mathcal{F}}$ is locally free. If $T_{\mathcal{F}}$ either splits as a sum of line bundles or is a spinor bundle, then Z is arithmetically Buchsbaum, with $h^1(Q^3, \mathcal{I}_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(\mathcal{I}_Z)$. Conversely, if Z is arithmetically Buchsbaum with $h^1(Q^3, \mathcal{I}_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(\mathcal{I}_Z)$ and $h^2(T_{\mathcal{F}}(-2)) = h^2(T_{\mathcal{F}}(-1 - c_1(T_{\mathcal{F}}))) = 0$, then $T_{\mathcal{F}}$ either split or is a spinor bundle.*

Proof. Suppose that $T_{\mathcal{F}}$ either split or is a spinor bundle. By considering the short exact sequence (2.26), after tensoring with $\mathcal{O}_{Q^3}(t)$, and taking the long exact sequence of cohomology we get:

$$\begin{aligned} \dots \rightarrow H^1(Q^3, T_{\mathcal{F}}(t)) \rightarrow H^1(Q^3, TQ^3(t)) \rightarrow H^1(Q^3, \mathcal{I}_Z(r+t)) \rightarrow \\ \rightarrow H^2(Q^3, T_{\mathcal{F}}(t)) \rightarrow H^2(Q^3, TQ^3(t)) \rightarrow H^2(Q^3, \mathcal{I}_Z(r+t)) \rightarrow \dots \end{aligned} \tag{3.2}$$

Since $T_{\mathcal{F}}$ has no intermediate cohomology, we have that

$$H^1(Q^3, T_{\mathcal{F}}(t)) = H^2(Q^3, T_{\mathcal{F}}(t)) = 0 \text{ for all } t \in \mathbb{Z},$$

thus,

$$H^1(Q^3, TQ^3(t)) \simeq H^1(Q^3, \mathcal{I}_Z(r+t)).$$

By Bott's formula for quadrics (Theorem 2.11), we have that $H^1(Q^3, TQ^3(t)) = 0$ for all $t \neq -2$, i.e. $H^1(Q^3, TQ^3(-2)) \neq 0$. Hence, $H^1(Q^3, \mathcal{I}_Z(r-2)) \neq 0$ and Z is arithmetically Buchsbaum.

To prove the converse, suppose that Z satisfies the properties in the statement and that $h^2(T_{\mathcal{F}}(-2)) = h^2(T_{\mathcal{F}}(-1 - c_1(T_{\mathcal{F}}))) = 0$.

Consider the long exact cohomology sequence (3.2) for all $t \neq -2$. By Theorem 2.11, we obtain that $H^1(Q^3, TQ^3(t)) = 0$, for all $t \neq -2$. Applying Serre duality and Bott's formula, respectively, we get $H^0(Q^3, \mathcal{I}_Z(r+t)) = H^3(\mathcal{O}_{Q^3}(-t-r-3)) = 0$, for all $r \neq 2$. Thus, we have $H^1(Q^3, T_{\mathcal{F}}(t)) = 0$.

By Serre duality, we conclude that

$$0 = H^1(Q^3, T_{\mathcal{F}}(t)) = H^2(Q^3, T_{\mathcal{F}}(s)), \text{ for all } s \neq -1 - c_1(T_{\mathcal{F}}),$$

where $s = -t-3-c_1(T_{\mathcal{F}})$. As by hypothesis $h^2(T_{\mathcal{F}}(-1-c_1(T_{\mathcal{F}}))) = 0$, we conclude that $H^2(Q^3, T_{\mathcal{F}}(s)) = 0$ for all s . Then, for all $t \neq -2$, $T_{\mathcal{F}}$ either split or is a spinor bundle.

Now, consider the long exact cohomology sequence (3.2), with $t = -2$. By Bott's formula, we have that $H^0(Q^3, TQ^3(-2)) = 0$ and $H^2(Q^3, TQ^3(-2)) = 0$. By hypothesis, we get $h^2(T_{\mathcal{F}}(-2)) = 0$ and $h^1(Q^3, \mathcal{I}_Z(r-2)) = 1$. Moreover, applying Serre duality and Bott's formula, respectively, we get $H^0(Q^3, \mathcal{I}_Z(r-2)) = H^3(\mathcal{O}_{Q^3}(-1-r)) = 0$, for all $r \neq 2$. So, from the exact sequence

$$0 \rightarrow H^1(Q^3, T_{\mathcal{F}}(-2)) \rightarrow H^1(Q^3, TQ^3(-2)) \simeq \mathbb{C} \xrightarrow{\beta} H^1(Q^3, \mathcal{I}_Z(r-2)) \simeq \mathbb{C} \rightarrow 0$$

we conclude that $H^1(Q^3, T_{\mathcal{F}}(-2)) = 0$, since β is injective and $\ker(\beta) = H^1(Q^3, T_{\mathcal{F}}(-2))$. Then, for $t = -2$, $T_{\mathcal{F}}$ either split or is a spinor bundle. Therefore, $T_{\mathcal{F}}$ either split or is a spinor bundle, for all $t \in \mathbb{Z}$. \square

Based on the result in [5, Theorem 11.8], we can construct examples of codimension one distribution on Q^3 .

Example 3.3. The bundle Spinor on Q^3 twisted by $\mathcal{O}_{Q^3}(1-t)$, is the tangent sheaf of a codimension one distribution \mathcal{F} , for all $t \geq 0$. Indeed, since the spinor bundle S is globally generated, $S(t)$ is globally generated, for all $t \geq 0$. Moreover, its rank 2 reflexive sheaf on Q^3 , and $S(t) \otimes TQ^3$ is also globally generated, since TQ^3 is globally generated. Now, apply [5, Theorem 11.8] with $L = \mathcal{O}_{Q^3}$ to obtain the desired codimension one distribution

$$0 \rightarrow S^*(-t) \simeq S(1-t) \rightarrow TQ^3 \rightarrow \mathcal{I}_Z(r) \rightarrow 0.$$

If $\iota_X = 2$, we characterize when the tangent sheaf of a distribution of dimension 2 on a smooth weighted projective complete intersection del Pezzo Fano threefold X , has no intermediate cohomology. More precisely, we prove the following results:

Lemma 3.4. *Let X be a smooth weighted projective complete intersection del Pezzo Fano threefold. Then, $H^2(X, \Omega_X^1(t)) = 0$ for $t > 4$, and $H^1(X, \Omega_X^2(t)) = 0$ for $t > 10$.*

We omit the proof of the previous lemma, being a computation of the cohomology groups $H^p(X, \Omega_X^q(t))$ with $p, q \in \{1, 2\}$ and $p \neq q$, where X is each one of the varieties with index two described in Section 2.1. For this calculation, we use the sequences and cohomology formulas of the Section 2.10. We summarize the computations in Table 1.

Theorem 3.5. *Let \mathcal{F} be a distribution of codimension one on a smooth weighted projective complete intersection del Pezzo Fano threefold X , such that the tangent sheaf $T_{\mathcal{F}}$ is locally free. If $T_{\mathcal{F}}$ has no intermediate cohomology, then $H^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -6$ and $t > 8$. Conversely, if $H^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -6$ and $t > 8$, and $H^2(X, T_{\mathcal{F}}(t)) = 0$ for $t \leq 8$ and $H^1(X, T_{\mathcal{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathcal{F}})$, then $T_{\mathcal{F}}$ has no intermediate cohomology.*

Table 1
Values of t for which $H^1(X, \Omega_X^2(t)) = H^2(X, \Omega_X^1(t)) = 0$.

X	ι_X	$H^2(X, \Omega_X^1(t))$	t	$H^1(X, \Omega_X^2(t))$	t
X_3	2	0	$t > 1$	0	$t > 4$
$X_{2,2}$	2	0	$t > 0$	0	$t > 2$
X_4	2	0	$t > 2$	0	$t > 6$
X_6	2	0	$t > 4$	0	$t > 10$

Proof. Suppose that $T_{\mathcal{F}}$ has no intermediate cohomology. By considering the short exact sequence (2.26), after tensoring with $\mathcal{O}_X(t)$, and taking the long exact sequence of cohomology we get:

$$\begin{aligned} \cdots \rightarrow H^1(X, T_{\mathcal{F}}(t)) \rightarrow H^1(X, TX(t)) \rightarrow H^1(X, \mathcal{I}_Z(r+t)) \rightarrow \\ \rightarrow H^2(X, T_{\mathcal{F}}(t)) \rightarrow H^2(X, TX(t)) \rightarrow H^2(X, \mathcal{I}_Z(r+t)) \rightarrow \cdots \end{aligned} \tag{3.5}$$

Since $T_{\mathcal{F}}$ has no intermediate cohomology, we have that

$$H^1(X, T_{\mathcal{F}}(t)) = H^2(X, T_{\mathcal{F}}(t)) = 0 \text{ for all } t \in \mathbb{Z}.$$

Thus, we get $H^1(X, TX(t)) \simeq H^1(X, \mathcal{I}_Z(r+t))$.

By Remark 1.5, we have that $H^1(X, TX(t)) \simeq H^1(X, \Omega_X^2(t+2))$ and by Lemma 3.4, we get $H^1(X, TX(t)) = 0$, for $t > 8$. Moreover, by using Serre duality, we obtain $H^1(X, TX(t)) \simeq H^2(X, \Omega_X^1(-t-2))$ and thus, by Lemma 3.4, we get $H^1(X, TX(t)) = 0$, for $t < -6$. Therefore, $H^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -6$ and $t > 8$.

Conversely, suppose that $h^2(T_{\mathcal{F}}(t)) = 0$ for $t \leq 8$. Consider the long exact cohomology sequence (3.5). Applying Serre duality and Theorem 2.16, respectively, we get $H^2(X, TX(t)) = 0$ for $t > -2$. Since by hypothesis, $h^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -6$ and $t > 8$, we get $H^2(X, T_{\mathcal{F}}(t)) = 0$ for $t > 8$. Thus, $H^2(X, T_{\mathcal{F}}(t)) = 0$ for all $t \in \mathbb{Z}$.

By Serre Duality and by Remark 1.5, we conclude that

$$0 = H^2(X, T_{\mathcal{F}}(t)) = H^1(X, T_{\mathcal{F}}(s)),$$

where $s = -t - \iota_X - c_1(T_{\mathcal{F}})$. As by hypothesis $h^1(T_{\mathcal{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathcal{F}})$, we conclude that $H^1(X, T_{\mathcal{F}}(t)) = 0$ for all $t \in \mathbb{Z}$. Therefore, $T_{\mathcal{F}}$ has no intermediate cohomology. \square

In [3], Arrondo and Costa classify all aCM bundles on the Fano threefolds $X_d \subset P^{d+1}$ $d = 3, 4, 5$ of index 2. Such bundles are isomorphic to a twist of either S_L , or S_C , or S_E , where $L \subset X_d$ is a line, $C \subset X_d$ is a conic and $E \subset X_d$ is an elliptic curve of degree $d + 2$, respectively. Moreover, the authors showed that $S_L(1), S_C$ and $S_E(1)$ are globally generated vector bundles.

By using the result [5, Theorem 11.8], we can construct examples of codimension one distribution on X_d . Consider $\mathcal{E} \simeq S_L(1); S_C; S_E(1)$. We have that \mathcal{E} is generated by global sections.

Example 3.6. Let \mathcal{E} be a globally generated rank 2 reflexive sheaf on X_d . Then $\mathcal{E}^*(-t)$ is the tangent sheaf of a codimension one distribution \mathcal{F} , for all $t \geq 0$. Indeed, since \mathcal{E} is globally generated, $\mathcal{E}(t)$ is globally generated, for all $t \geq 0$. Then $\mathcal{E}(t) \otimes TX_d$ is also globally generated, since TX_d is globally generated. Now, apply the Theorem [5, Theorem 11.8] with $L = \mathcal{O}_{X_d}$ to obtain the desired codimension one distribution

$$0 \rightarrow \mathcal{E}^*(-t) \rightarrow TX_d \rightarrow \mathcal{I}_Z(r) \rightarrow 0.$$

Finally, if $\iota_X = 1$, we characterize when the tangent sheaf of a distribution of dimension 2 on a smooth weighted projective complete intersection prime Fano threefold X , has no intermediate cohomology. More precisely, we prove the following results.

Table 2
 Values of t for which $H^1(X, \Omega_X^2(t)) = H^2(X, \Omega_X^1(t)) = 0$.

X	ι_X	$H^2(X, \Omega_X^1(t))$	t	$H^1(X, \Omega_X^2(t))$	t
$X_{2,3}$	1	0	$t > 2$ if $X_{2,3} \subset X_2$ and $t > 3$ if $X_{2,3} \subset X_3$	0	$t > 5$ if $X_{2,3} \subset X_2$ and $t > 5$ if $X_{2,3} \subset X_3$
$X_{2,2,2}$	1	0	$t > 3$	0	$t > 5$
Y	1	0	$t > 5$ if $Y \subset C_2$ and $t > 5$ if $Y \subset X_4$	0	$t > 9$ if $Y \subset C_2$ and $t > 3$ if $Y \subset X_4$

Lemma 3.7. *Let X be a smooth weighted projective complete intersection prime Fano threefold. Then, $H^1(X, \Omega_X^2(t)) = 0$ for $t > 5$, and $H^2(X, \Omega_X^1(t)) = 0$ for $t > 3$.*

We will omit the proof of the previous lemma, being a very similar computation of the del Pezzo Fano’s case. We summarize the computations in Table 2.

Theorem 3.8. *Let \mathcal{F} be a distribution of codimension one on a smooth weighted projective complete intersection prime Fano threefold X , such that the tangent sheaf $T_{\mathcal{F}}$ is locally free. If $T_{\mathcal{F}}$ has no intermediate cohomology, then $H^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -4$ and $t > 4$. Conversely, if $H^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -4$ and $t > 4$, and $H^2(X, T_{\mathcal{F}}(t)) = 0$ for $t \leq 4$ and $H^1(X, T_{\mathcal{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathcal{F}})$, then $T_{\mathcal{F}}$ has no intermediate cohomology.*

Proof. Suppose that $T_{\mathcal{F}}$ has no intermediate cohomology. Consider, for each $t \in \mathbb{Z}$, the exact sequence

$$0 \rightarrow T_{\mathcal{F}}(t) \rightarrow TX(t) \rightarrow \mathcal{I}_Z(r+t) \rightarrow 0$$

and the long exact sequence of cohomology (3.5).

Since $T_{\mathcal{F}}$ has no intermediate cohomology, we get

$$H^1(X, T_{\mathcal{F}}(t)) = H^2(X, T_{\mathcal{F}}(t)) = 0 \text{ for all } t \in \mathbb{Z}.$$

Thus, we have $H^1(X, TX(t)) \simeq H^1(X, \mathcal{I}_Z(r+t))$.

By Remark 1.5, we get that $H^1(X, TX(t)) \simeq H^1(X, \Omega_X^2(t+1))$ and by Lemma 3.7, we get $H^1(X, TX(t)) = 0$, for $t > 4$. Moreover, using Serre duality, we obtain that $H^1(X, TX(t)) \simeq H^2(X, \Omega_X^1(-t-1))$ and thus, by Lemma 3.7, we get $H^1(X, TX(t)) = 0$, for $t < -4$. Therefore, $H^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -4$ and $t > 4$.

Conversely, suppose that $h^2(T_{\mathcal{F}}(t)) = 0$ for $t \leq 4$. Consider the long exact cohomology sequence (3.5). Applying Serre duality and Theorem 2.16, respectively, we get $H^2(X, TX(t)) = 0$ for $t > -1$. Since by hypothesis, $h^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -4$ and $t > 4$, we get $H^2(X, T_{\mathcal{F}}(t)) = 0$ for $t > 4$. Thus, we have $H^2(X, T_{\mathcal{F}}(t)) = 0$ for all $t \in \mathbb{Z}$.

By Serre Duality and by Remark 1.5, we obtain that

$$0 = H^2(X, T_{\mathcal{F}}(t)) = H^1(X, T_{\mathcal{F}}(s)),$$

where $s = -t - \iota_X - c_1(T_{\mathcal{F}})$. As by hypothesis $h^1(T_{\mathcal{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathcal{F}})$, we conclude that $H^1(X, T_{\mathcal{F}}(t)) = 0$ for all $t \in \mathbb{Z}$. Therefore, $T_{\mathcal{F}}$ has no intermediate cohomology. \square

In [23], Madonna classifies aCM bundles on a prime Fano threefold $X_{2g-2} := X$ of genus g . Let l be a line in X and consider that \mathcal{E} is associated to a line l in X with $c_1 = -1, c_2 = 1$. It holds that \mathcal{E} is generated by its global sections. Again, by using the result [5, Theorem 11.8], we can construct examples of codimension one distribution on X .

Example 3.9. Let \mathcal{E} be a globally generated rank 2 reflexive sheaf on a prime Fano threefold X . Then $\mathcal{E}^*(-t)$ is the tangent sheaf of a codimension one distribution \mathcal{F} , for all $t \geq 0$.

4. Conormal sheaf vs. singular scheme

In this section, all the distributions that we will consider will be defined for Fano threefolds, of codimension two, and with singular set of codimension ≥ 2 . Moreover, we will assume that the Picard number is one.

Looking at Definition 2.25 we can alternatively define a foliation through a coherent subsheaf $N_{\mathcal{F}}^*$ of Ω_X^1 such that

- (1) $N_{\mathcal{F}}^*$ is integrable ($dN_{\mathcal{F}}^* \subset N_{\mathcal{F}}^* \wedge \Omega_X^1$) and
- (2) the quotient $\Omega_X^1/N_{\mathcal{F}}^*$ is torsion free.

The codimension of \mathcal{F} is the generic rank of $N_{\mathcal{F}}^*$ and let us denote $r = c_1(\Omega_X^1) - c_1(N_{\mathcal{F}}^*)$.

We characterize when the conormal sheaf is split, in terms of the geometry of the singular scheme of the distribution. M. Corrêa, M. Jardim and R. Vidal Martins, showed in [8] that the conormal sheaf of a foliation of dimension one on \mathbb{P}^n splits if and only if its singular scheme is arithmetically Buchsbaum with $h^1(\mathcal{I}_Z(d-1)) = 1$ being the only nonzero intermediate cohomology [8, Theorem 5.2].

Based in this result, we prove the following:

Theorem 4.1. *Let \mathcal{F} be an one-dimensional distribution on a smooth weighted projective complete intersection Fano threefold X . If $N_{\mathcal{F}}^*$ is aCM, then Z is arithmetically Buchsbaum, with $h^1(X, \mathcal{I}_Z(r)) = 1$ being the only nonzero intermediate cohomology for $H^i(\mathcal{I}_Z)$.*

Proof. Suppose that $N_{\mathcal{F}}^*$ is aCM. For the case $\iota_X = 4$, i.e. $X \simeq \mathbb{P}^3$, the result follows from Theorem 5.2 in [8].

Consider, for each $t \in \mathbb{Z}$, the exact sequence

$$0 \rightarrow N_{\mathcal{F}}^*(t) \rightarrow \Omega_X^1(t) \rightarrow \mathcal{I}_Z(r+t) \rightarrow 0, \tag{4.2}$$

where r is a integer such that $r = c_1(\Omega_X^1) - c_1(N_{\mathcal{F}}^*)$. Taking the long exact sequence of cohomology we get:

$$\begin{aligned} \dots \rightarrow H^1(X, N_{\mathcal{F}}^*(t)) \rightarrow H^1(X, \Omega_X^1(t)) \rightarrow H^1(X, \mathcal{I}_Z(r+t)) \rightarrow \\ \rightarrow H^2(X, N_{\mathcal{F}}^*(t)) \rightarrow H^2(X, \Omega_X^1(t)) \rightarrow H^2(X, \mathcal{I}_Z(r+t)) \rightarrow \dots \end{aligned} \tag{4.3}$$

Since $N_{\mathcal{F}}^*$ has no intermediate cohomology, we have that

$$H^1(X, N_{\mathcal{F}}^*(t)) = H^2(X, N_{\mathcal{F}}^*(t)) = 0, \text{ for all } t \in \mathbb{Z}.$$

Thus, we get $H^1(X, \Omega_X^1(t)) \simeq H^1(X, \mathcal{I}_Z(r+t))$. By Theorem 2.16, $H^1(X, \Omega_X^1(t)) = 0$ for all $t \neq 0$, and $h^1(X, \Omega_X^1) = 1$. Then, we have $H^1(X, \mathcal{I}_Z(r)) \neq 0$ and Z is arithmetically Buchsbaum. \square

If X is a Fano threefold, meaning that $K_X^{-1} = \bigwedge^3 TX$ is ample, then the Kodaira vanishing Theorem shows that $H^q(X, \mathcal{O}_X) = 0$ and $H^q(X, \mathcal{O}(K_X^{-1})) = 0$ for $q > 0$, see [27].

A similar result for converse of Theorem 5.2 in [8] is the following:

Theorem 4.4. *Let \mathcal{F} be an one-dimensional distribution on a smooth weighted projective complete intersection Fano threefold X , with index $\iota_X \in \{1, 2, 3\}$. If Z is arithmetically Buchsbaum with $h^1(X, \mathcal{I}_Z(r)) = 1$*

being the only nonzero intermediate cohomology for $H^i(I_Z)$, and $h^2(N_{\mathcal{F}}^*) = h^2(N_{\mathcal{F}}^*(-c_1(N_{\mathcal{F}}^*) - \iota_X)) = 0$, then $N_{\mathcal{F}}^*$ is aCM.

Proof. Suppose that $h^2(N_{\mathcal{F}}^*) = h^2(N_{\mathcal{F}}^*(-c_1(N_{\mathcal{F}}^*) - \iota_X)) = 0$ and that Z is arithmetically Buchsbaum with $h^1(Q^3, I_Z(r)) = 1$ being the only nonzero intermediate cohomology.

Consider the long exact cohomology sequence (4.3), for all $t \neq 0$. By Theorem 2.16, we have that $H^1(X, \Omega_X^1(t)) = 0$, for all $t \neq 0$. Applying Serre duality and the Kodaira vanishing Theorem, respectively, we get

$$H^0(X, \mathcal{I}_Z(r+t)) = H^3(\mathcal{O}_X(-r-t-\iota_X)) = 0 \text{ for } r \neq -\iota_X.$$

Thus, we get $H^1(X, N_{\mathcal{F}}^*(t)) = 0$ for $t \neq 0$.

By Serre duality and by Remark 1.5, we conclude that

$$0 = H^1(X, N_{\mathcal{F}}^*(t)) = H^2(X, (N_{\mathcal{F}}^*)^*(-t - \iota_X)) = H^2(X, N_{\mathcal{F}}^*(-t - \iota_X - c_1(N_{\mathcal{F}}^*))).$$

Let $s = -c_1(N_{\mathcal{F}}^*) - t - \iota_X$ and $t \neq 0$. Thus, we get

$$H^2(X, N_{\mathcal{F}}^*(s)) = 0 \text{ for } s \neq -\iota_X - c_1(N_{\mathcal{F}}^*).$$

Since by hypothesis $h^2(N_{\mathcal{F}}^*(s)) = 0$ for $s = -\iota_X - c_1(N_{\mathcal{F}}^*)$, we conclude that $H^2(X, N_{\mathcal{F}}^*(t)) = 0$. Then, for all $t \neq 0$, $N_{\mathcal{F}}^*$ is aCM.

Now, consider the following piece of the long exact cohomology sequence (4.3), for $t = 0$:

$$\dots \rightarrow H^1(X, N_{\mathcal{F}}^*) \rightarrow H^1(X, \Omega_X^1) \simeq \mathbb{C} \xrightarrow{\beta} H^1(X, \mathcal{I}_Z(r)) \simeq \mathbb{C} \rightarrow 0.$$

The map β is surjective and injective. Thus, $\ker(\beta) = H^1(X, N_{\mathcal{F}}^*)$ is trivial, i.e. $H^1(X, N_{\mathcal{F}}^*) = 0$. As by hypothesis, $h^2(N_{\mathcal{F}}^*) = 0$, we conclude that for $t = 0$, $N_{\mathcal{F}}^*$ is aCM. Therefore, $N_{\mathcal{F}}^*$ is aCM for all $t \in \mathbb{Z}$. \square

5. Properties of the singular locus of distributions

In this section we analyze the properties of the singular schemes of codimension one holomorphic distributions on a threefold X .

5.1. Numerical invariants

Let \mathcal{F} be a codimension one distribution on threefold X given as in the exact sequence (2.26), with tangent sheaf $T_{\mathcal{F}}$ and singular scheme Z .

Following [5, section 2.1], let \mathcal{Q} be the maximal subsheaf of $\mathcal{O}_{Z/X}$ of codimension > 2 , so that one has an exact sequence of the form

$$0 \rightarrow \mathcal{Q} \rightarrow \mathcal{O}_{Z/X} \rightarrow \mathcal{O}_{C/X} \rightarrow 0 \tag{5.2}$$

where $C \subset X$ is a (possibly empty) subscheme of pure codimension 2.

The quotient sheaf is the structure sheaf of a subscheme $C \subset Z \subset X$ of pure dimension 1.

Definition 5.3. If Z is a 1-dimensional subscheme, then Z has a maximal pure dimension 1 subscheme C defining a sequence

$$0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{I}_C \rightarrow \mathcal{Q} \rightarrow 0, \tag{5.4}$$

where \mathcal{Q} is the maximal 0-dimensional subsheaf of \mathcal{O}_Z .

If X is a Kähler manifold of dimension n , and $Z \subset X$ is an analytic subset of codimension k , then

$$c_k(\mathcal{I}_Z) = (-1)^k (k-1)! [Z], \tag{5.5}$$

for more details, see [15].

In the next theorem we obtain the following expressions for the Chern class of the tangent sheaf in terms of the cycle of C . We will denote $H = c_1(\mathcal{O}_X(1))$.

Theorem 5.6. *Let \mathcal{F} be a codimension one distribution on a threefold X , with $\rho(X) = 1$, given as in the exact sequence (2.26), with tangent sheaf $T_{\mathcal{F}}$ and singular scheme Z . Then,*

$$c_2(T_{\mathcal{F}}) = c_2(TX) - rH \cdot K_X^{-1} + r^2 H^2 - [C],$$

and

$$c_3(T_{\mathcal{F}}) = c_3(TX) - c_3(I_{Z/X}) + rH \cdot [C] - K_X^{-1} \cdot [C] - rH \cdot c_2(TX) + r^2 H^2 \cdot K_X^{-1} - H^3 r^3.$$

Proof. Considering the exact sequence (2.26), we use that $c(TX) = c(T_{\mathcal{F}}) \cdot c(I_{Z/X}(r))$ to obtain

$$\begin{aligned} c_1(TX) &= c_1(T_{\mathcal{F}}) + c_1(I_{Z/X}(r)), \\ c_2(TX) &= c_1(T_{\mathcal{F}}) \cdot c_1(I_{Z/X}(r)) + c_2(T_{\mathcal{F}}) + c_2(I_{Z/X}(r)), \\ c_3(TX) &= c_3(T_{\mathcal{F}}) + c_3(I_{Z/X}(r)) + c_1(T_{\mathcal{F}}) \cdot c_2(I_{Z/X}(r)) + c_2(T_{\mathcal{F}}) \cdot c_1(I_{Z/X}(r)). \end{aligned} \tag{5.7}$$

The first equation gives $c_1(T_{\mathcal{F}}) = c_1(TX) - rH$. From the exact sequence (5.4), it follows that $c_2(I_{Z/X}(r)) = c_2(I_{C/X}(r)) = [C]$, thus substitution into the second equation yields

$$c_2(T_{\mathcal{F}}) = c_2(TX) - rH \cdot K_X^{-1} + H^2 r^2 - [C].$$

Moreover, the substituting the expressions for the first and second Chern classes into the third equation we obtain

$$c_3(TX) = c_3(T_{\mathcal{F}}) + c_3(I_{Z/X}(r)) + K_X^{-1} \cdot [C] - 2rH \cdot [C] + r \cdot c_2(TX) - r^2 H^2 \cdot K_X^{-1} + H^3 r^3. \tag{5.8}$$

Note that

$$c_3(I_{Z/X}(r)) = c_3(I_{Z/X}) + rH \cdot c_2(I_{Z/X}) + H^2 r^3, \tag{5.9}$$

while

$$c_2(I_{Z/X}) = [C] - H^2 r^2. \tag{5.10}$$

Substituting (5.10) into the equation (5.9), we obtain

$$c_3(I_{Z/X}(r)) = c_3(I_{Z/X}) + rH \cdot [C], \tag{5.11}$$

and thus

$$c_3(T_{\mathcal{F}}) = c_3(TX) - c_3(I_{Z/X}) + rH \cdot [C] - K_X^{-1} \cdot [C] - rH \cdot c_2(TX) + r^2 H^2 \cdot K_X^{-1} - H^3 r^3. \quad \square \tag{5.12}$$

5.13. Connectedness

Now, we will see when the singular locus of a codimension one distribution is connected. In [5], O. Calvo-Andrade, M. Corrêa and M. Jardim present a homological criterion for connectedness of the singular scheme of codimension 1 distributions on \mathbb{P}^3 .

We will extend this criterion for other smooth weighted projective complete intersection Fano threefold with Picard number one. Before, we need to prove the vanishing of $H^1(TX(-r))$ and $H^2(TX(-r))$.

Lemma 5.14. *Let X be a smooth weighted projective complete intersection Fano threefold with Picard number $\rho(X) = 1$. Then, $H^1(TX(-r)) = 0$ for $r > 6$ and $H^2(TX(-r)) = 0$ for $r \neq \iota_X$.*

Proof. Suppose X a smooth weighted projective complete intersection Fano threefold. If $\iota_X = 4$, i.e. $X \simeq \mathbb{P}^3$, by classical Bott’s formula we have that $H^1(TX(-r)) = 0$ for all r and $H^2(TX(-r)) = 0$ for $r \neq 2$. If $\iota_X = 3$, i.e. $X \simeq Q^3$, by Bott’s formula for quadric, we have that $H^1(TX(-r)) = 0$ for $r \neq 2$ and $H^2(TX(-r)) = 0$ for $r \neq 3$. If $\iota_X = 2$, by using Serre duality we get that $H^1(TX(-r)) = H^2(\Omega_X^1(r-2))$. By Table 1, comparing the values of t for which $H^2(\Omega_X^1(t)) = 0$ with $\iota_X = 2$, we can see that the common vanishing of cohomology group for these varieties, occurs when $t > 4$. Then, $H^2(\Omega_X^1(r-2)) = 0$ for $r > 6$ and $H^2(TX(-r)) = 0$ for $r \neq 2$ by Theorem 2.16. If $\iota_X = 1$, by using Serre duality we get $H^1(TX(-r)) = H^2(\Omega_X^1(r-1))$. By Table 2, comparing the values of t for which $H^2(\Omega_X^1(t)) = 0$ with $\iota_X = 1$, we can see that the common vanishing of cohomology group for these varieties, occurs when $t > 3$. Then, $H^2(\Omega_X^1(r-1)) = 0$ for $r > 4$ and $H^2(TX(-r)) = 0$ for $r \neq 1$ by Theorem 2.16. Now, comparing the values of r for which $H^1(TX(-r)) = 0$, we can see that the common vanishing of cohomology group considering all indices of X , occurs when $r > 6$ and $H^2(TX(-r)) = 0$ for $r \neq \iota_X$. \square

Theorem 5.15. *Let \mathcal{F} be a codimension one distribution with singular scheme Z and let X be a smooth weighted projective complete intersection Fano threefold with Picard number $\rho(X) = 1$. If $h^2(T_{\mathcal{F}}(-r)) = 0$ and $C \subset X$, $C \neq \emptyset$, then Z is connected and of pure dimension 1, so that $T_{\mathcal{F}}$ is locally free. Conversely, for $r \neq \iota_X$, if $Z = C$ is connected, then $T_{\mathcal{F}}$ is locally free and $h^2(T_{\mathcal{F}}(-r)) = 0$.*

Proof. Twisting the exact sequence (2.26) by $\mathcal{O}_X(-r)$ and passing to cohomology we obtain,

$$H^1(TX(-r)) \rightarrow H^1(I_{Z/X}) \rightarrow H^2(T_{\mathcal{F}}(-r)) \rightarrow H^2(TX(-r)).$$

By Lemma above, we get that $H^1(TX(-r)) = 0$ for $r > 6$.

If $h^2(T_{\mathcal{F}}(-r)) = 0$, then $h^1(I_{Z/X}) = 0$, for $r > 6$. It follows from the sequence

$$0 \rightarrow I_{Z/X} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{Z/X} \rightarrow 0$$

that

$$H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{O}_{Z/X}) \rightarrow 0,$$

hence $h^0(\mathcal{O}_{Z/X}) = 1$. From the sequence (5.2), we get

$$0 \rightarrow H^0(\mathcal{Q}) \rightarrow H^0(\mathcal{O}_{Z/X}) \rightarrow H^0(\mathcal{O}_{C/X}) \rightarrow 0$$

Thus either $h^0(\mathcal{O}_{C/X}) = 1$, and $\mathcal{Q} = 0$ and C is connected, or $\text{length}(\mathcal{Q}) = 1$ and C is empty. This second possibility is not valid because by hypothesis $C \neq \emptyset$. It follows that $Z = C$ must be connected and of pure dimension 1, and thus, by Lemma 3.1, $T_{\mathcal{F}}$ is locally free.

Conversely, assume that $Z = C$ is connected. Thus Z must be of pure dimension 1, and Lemma 3.1 implies that $T_{\mathcal{F}}$ is locally free. It also follows that $h^1(I_{Z/X}) = 0$, using Serre duality and the formula (2.7). Since $h^2(TX(-r)) = 0$ for $r \neq \iota_X$, we conclude that $h^2(T_{\mathcal{F}}(-r)) = 0$, as desired. \square

Corollary 5.16. *If \mathcal{F} is a codimension one distribution on X whose tangent sheaf splits as a sum of line bundles, then its singular scheme is connected.*

Proof. Assuming that $T_{\mathcal{F}} = \mathcal{O}_X(r_1) \oplus \mathcal{O}_X(r_2)$, then clearly $h^2(T_{\mathcal{F}}(-r)) = 0$, where $r = r_1 + r_2$. The result follows from Theorem 5.15. \square

Corollary 5.17. *Let \mathcal{F} be a codimension one distribution on X with locally free tangent sheaf. If $T_{\mathcal{F}}^*$ is ample, then its singular scheme is connected.*

Proof. We have, by Serre duality,

$$H^2(T_{\mathcal{F}}(-r)) \simeq H^1(T_{\mathcal{F}}^*(r) \otimes K_X) = H^1(T_{\mathcal{F}}^*(r - c_1(TX)) \otimes \mathcal{O}_X(c_1(TX)) \otimes K_X).$$

Observe that $T_{\mathcal{F}}^*(r - c_1(TX)) \otimes \mathcal{O}_X(c_1(TX)) \otimes K_X = T_{\mathcal{F}}^* \otimes \det(T_{\mathcal{F}}^*) \otimes \mathcal{O}_X(c_1(TX)) \otimes K_X$; since $T_{\mathcal{F}}^*$ and $\mathcal{O}_X(c_1(TX))$ are ample, then, by Griffiths Vanishing Theorem [17], we get

$$h^2(T_{\mathcal{F}}(-r)) = h^1(T_{\mathcal{F}}^* \otimes \det(T_{\mathcal{F}}^*) \otimes \mathcal{O}_X(c_1(TX)) \otimes K_X) = 0.$$

The result follows from Theorem 5.15. \square

6. Stability

In this section we will prove that If $T_{\mathcal{F}}$ is locally free with $c_1(T_{\mathcal{F}}) = 0$, then $T_{\mathcal{F}}$ is either split or stable. Firstly, we will prove this for quadrics.

Theorem 6.1. *Let \mathcal{F} be a codimension one distribution on Q^3 . If $c_1(T_{\mathcal{F}}) = 0$, then $T_{\mathcal{F}}$ is either split or stable.*

Proof. Since $c_1(T_{\mathcal{F}}) = 0$, we have the sequence

$$\mathcal{F} : 0 \rightarrow T_{\mathcal{F}} \rightarrow TQ^3 \rightarrow \mathcal{I}_{Z/Q^3}(3) \rightarrow 0.$$

Suppose that \mathcal{F} is non stable and non split. Let $\alpha \in H^0(T_{\mathcal{F}})$ be a nonzero section. This section induces a section σ on TQ^3 . Letting $\mathcal{S}_{\Gamma} := \text{coker } \alpha$, we have the following diagram after applying the Snake Lemma:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_{Q^3} & \xlongequal{\quad} & \mathcal{O}_{Q^3} & & \\
 & & \downarrow \alpha & & \downarrow \sigma & & \\
 0 & \longrightarrow & T_{\mathcal{F}} & \longrightarrow & TQ^3 & \longrightarrow & \mathcal{I}_Z(3) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathcal{I}_\Gamma & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{I}_Z(3) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Dualizing the last line of diagram, we obtain

$$0 \rightarrow \mathcal{O}_{Q^3}(-3) \rightarrow \mathcal{G}^* \rightarrow \mathcal{O}_{Q^3} \rightarrow \mathcal{E}xt^1(\mathcal{I}_Z(3), \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^1(\mathcal{I}_\Gamma, \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^2(\mathcal{I}_Z(3), \mathcal{O}_{Q^3}) \rightarrow 0.$$

By the sequence $0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_{Q^3} \rightarrow \mathcal{O}_Z \rightarrow 0$ we have the long exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{O}_Z, \mathcal{O}_{Q^3}) \rightarrow \text{Hom}(\mathcal{O}_{Q^3}, \mathcal{O}_{Q^3}) \rightarrow \text{Hom}(\mathcal{I}_Z, \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^1(\mathcal{O}_Z, \mathcal{O}_{Q^3}) \rightarrow \dots$$

Because Z is a curve of codimension 2 we have $\mathcal{E}xt^i(\mathcal{O}_Z, \mathcal{O}_{Q^3}) = 0$ for $i < 2$, and thus

$$\mathcal{O}_{Q^3} = \text{Hom}(\mathcal{O}_{Q^3}, \mathcal{O}_{Q^3}) \simeq \text{Hom}(\mathcal{I}_Z, \mathcal{O}_{Q^3}).$$

Dualizing the first line of diagram, we obtain

$$\begin{aligned}
 & 0 \rightarrow \mathcal{O}_{Q^3}(-3) \rightarrow \Omega_{Q^3}^1 \rightarrow \mathcal{F}^* \rightarrow \\
 & \rightarrow \mathcal{E}xt^1(\mathcal{I}_Z(3), \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^1(TQ^3, \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{Q^3}) \rightarrow \\
 & \rightarrow \mathcal{E}xt^2(\mathcal{I}_Z(3), \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^2(TQ^3, \mathcal{O}_{Q^3}) \rightarrow \dots
 \end{aligned}$$

A sheaf E is locally free if and only if $\mathcal{E}xt^i(E, \mathcal{O}_X) = 0$ for each $i \geq 1$. Thus,

$$\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{Q^3}) = \mathcal{E}xt^2(TQ^3, \mathcal{O}_{Q^3}) = 0$$

and by the sequence above we get $\mathcal{E}xt^2(\mathcal{I}_Z(3), \mathcal{O}_{Q^3}) = 0$.

Twisting the sequence 6.1 by $\mathcal{O}_{Q^3}(3)$ and dualizing, we obtain

$$\begin{aligned}
 \dots & \rightarrow \mathcal{E}xt^1(\mathcal{O}_Z(3), \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^1(\mathcal{O}_{Q^3}(3), \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^1(\mathcal{I}_Z(3), \mathcal{O}_{Q^3}) \rightarrow \\
 & \rightarrow \mathcal{E}xt^2(\mathcal{O}_Z(3), \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^2(\mathcal{O}_{Q^3}(3), \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^1(\mathcal{I}_Z(3), \mathcal{O}_{Q^3}) \rightarrow \\
 & \rightarrow \mathcal{E}xt^3(\mathcal{O}_Z(3), \mathcal{O}_{Q^3}) \rightarrow \mathcal{E}xt^3(\mathcal{O}_{Q^3}(3), \mathcal{O}_{Q^3}) \rightarrow \dots
 \end{aligned}$$

We have $\mathcal{E}xt^i(\mathcal{O}_{Q^3}(3), \mathcal{O}_{Q^3}) = 0$ for each $i \geq 1$ and $\mathcal{E}xt^2(\mathcal{I}_Z(3), \mathcal{O}_{Q^3}) = 0$ by previous computations. Therefore, $\mathcal{E}xt^3(\mathcal{O}_Z(3), \mathcal{O}_{Q^3}) = 0$ and

$$\mathcal{E}xt^1(\mathcal{I}_Z(3), \mathcal{O}_{Q^3}) \simeq \mathcal{E}xt^2(\mathcal{O}_Z(3), \mathcal{O}_{Q^3}) = \omega_Z(-3),$$

where ω_Z is the dualizing sheaf of the curve Z .

Similarly, from the sequence $0 \rightarrow \mathcal{I}_\Gamma \rightarrow \mathcal{O}_{Q^3} \rightarrow \mathcal{O}_\Gamma \rightarrow 0$, we get

$$\mathcal{E}xt^1(\mathcal{I}_\Gamma, \mathcal{O}_{Q^3}) \simeq \mathcal{E}xt^2(\mathcal{O}_\Gamma, \mathcal{O}_{Q^3}).$$

Since Γ is a codimension two, we have the local fundamental isomorphism

$$\mathcal{E}xt^2(\mathcal{O}_\Gamma, \mathcal{O}_{Q^3}) \simeq \text{Hom}(\det(\mathcal{I}_\Gamma/\mathcal{I}_\Gamma^2), \mathcal{O}_{Q^3}) = \mathcal{O}_{Q^3}^* \otimes \mathcal{O}_{Q^3} = \mathcal{O}_{Q^3}.$$

Therefore, $\mathcal{E}xt^1(\mathcal{I}_\Gamma, \mathcal{O}_{Q^3}) = \mathcal{O}_{Q^3}$. Thus, we obtain

$$\text{Supp}(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{Q^3})) \subset \text{Sing}(\mathcal{I}_\Gamma) = (\alpha = 0) \subset \text{Supp}(\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_{Q^3})) = \text{Sing}(\mathcal{G}).$$

Since Γ is smooth, the Chern classes of $T_{\mathcal{F}}$ are given by $c_1(T_{\mathcal{F}}) = 0$ and $c_2(T_{\mathcal{F}}) = \text{deg}(\Gamma) = d$.

Since $\omega_{Q^3} \simeq \mathcal{O}_{Q^3}(-3)$, we have $\omega_\Gamma \simeq \mathcal{O}_\Gamma(c_1 - 3)$. If Γ is an irreducible nonsingular curve of genus g , then ω_Γ is the canonical sheaf, which has degree $2g - 2$. Therefore,

$$2g - 2 = d(c_1 - 3) = c_2(c_1 - 3) = -3c_2.$$

By using the Theorem 5.6, we have $c_2(T_{\mathcal{F}}) = 4H^2 - H \cdot [Z]$. Since Γ is a curve, it has at least $\text{deg}(\Gamma) = 1$, thus $H \cdot [Z]$ is at most three. Then, if $H \cdot [Z] = 0$ ($\text{deg}(\Gamma) = 4$), we have $2g - 2 = -12 \Rightarrow g = -5$ and if $H \cdot [Z] = 2$ ($\text{deg}(\Gamma) = 2$), then $2g - 2 = -6 \Rightarrow g = -2$.

On the other hand, since $T_{\mathcal{F}}$ is locally free, it has $c_3(T_{\mathcal{F}}) = 0$. By Riemann-Roch theorem $c_3(I_Z) = (2g - 2)H^3 + c_1(Q^3)c_2(I_Z)$. Since Z is a curve, we have $c_2(I_Z) = [Z]$ and we know that $c_1(Q^3) = 3H$. Then, $c_3(I_Z) = (2g - 2)H^3 + 3H \cdot [Z]$. Replacing these values in the Theorem 5.6, we have

$$(2g - 2)H^3 = -10H^3 - 3H \cdot [Z]. \tag{6.1}$$

Now, by using the formula (6.1), if $H \cdot [Z] = 0$, we have a curve of genus $g = -4$ and if $H \cdot [Z] = 2$, we have a curve of genus $g = -7$. It is a contradiction. Therefore, $T_{\mathcal{F}}$ is either stable or split. \square

Remark 6.2. Note that if the subfoliation has only isolated singularities then $T_{\mathcal{F}}$ has to be split. Indeed, $T_{\mathcal{F}}$ would have a section that does not vanish on a curve, so it would have to be split.

Theorem 6.3. *Let X be a Fano threefold with Picard number one and \mathcal{F} be a distribution of codimension 1. If $T_{\mathcal{F}}$ is locally free with $c_1(T_{\mathcal{F}}) = 0$, then $T_{\mathcal{F}}$ is either split or stable.*

Proof. If $\iota_X = 4$, then the result follows from [5, Theorem 9.5]. If $\iota_X = 3$, it follows from Theorem 6.1. If $\iota_X = 2$, suppose that $T_{\mathcal{F}}$ has a global section, so such section induce a global section of TX . It follows from [22, Corollary 7.4], that the only Fano of index two with global field is X_5 . And by [22, Lemma 7.2], every global field of X_5 has only isolated singularities. Thus, $T_{\mathcal{F}}$ is split by Remark 6.2. It is a contradiction. If $\iota_X = 1$, follows from the proof of Theorem 8.1 in [22] that X is birational to X_5 . As in X_5 , every global field of X has only isolated singularities. Thus, $T_{\mathcal{F}}$ is split by Remark 6.2. Once again this is a contradiction. \square

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