Orchestrating the discovery of the Greatest Common Divisor and the Least Common Multiple hidden in a digital spirograph

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This paper focuses on good practices in technology-rich mathematics education, and in particular, on the principles and heuristics that may guide the teacher's orchestration of fruitful activities. We present a teaching activity, framed by the Method of Varied Inquiry (MVI), which involved a small group of 10th grade students, with the aim to let them discover the Greatest Common Divisor (GCD) and the Least Common Multiple (LCM) as objects that mathematize a digital spirograph. Results are analysed and discussed to show how students' learning can be supported by the teacher's orchestration of the activity.

Keywords: Digital spirograph, Greatest Common Divisor and Least Common Multiple, Instrumental Orchestration, Method of Varied Inquiry.

Introduction

Addressing opportunities and constraints of learning mathematics in a technology-rich environment has happened to be even more relevant in times of an immense increase of distance learning and of unexpected changes in teaching practices. In this paper we focus on the principles and heuristics that may guide the design and implementation of a didactic activity developed during the pandemic. More precisely, we are interested in reflecting on the way students' learning can be supported through appropriate design of the activity and teacher's behaviour. For this reason, we refer to the notion of teacher's orchestration developed within the research field of mathematics education, particularly in the case of the use of technological resources in classroom (Trouche, 2004; Drijvers et al., 2010).

The idea of the teaching activity we present in this paper comes from a study in which Ferrara, Ferrari and Savioli (2020) focussed on the mathematics "in movement" that can be addressed using a spirograph (a drawing tool used to create curves for recreational purposes): the movement is the key aspect used to give meaning to the mathematical relationships embedded in the spirograph, such as the concept of the Greatest Common Divisor (GCD) and the Least Common Multiple (LCM). In their work, Ferrara and colleagues suggest some reflections on the mathematical modelling of the spirograph, showing and analysing a teaching experiment developed with 5th grade students. Our attention was attracted by the idea that the characteristics of the spirograph fit with the purpose to offer students a situation in which variation gives power to the development of mathematical thinking, especially in terms of conjectures and argumentations. With this respect, the potential of the spirograph can be exploited not only at the 5th grade, but also at high school level. For this reason, we chose to use the spirograph as a didactical instrument with 10th grade students. The free online availability of a digital spirograph (https://nathanfriend.io/inspiral-web/) contributed then to design a teaching activity, to be developed in the distance context due to the pandemic. We wanted to exploit the opportunity given by the spirograph, in its digital version, to foster students' investigation of hidden mathematics. The functioning of the spirograph, with its related schemes of use, and the characteristics of the obtained curves do not depend on the version of the instrument (digital or not).

The reason why we have used the digital spirograph was due to the constraints of the pandemic. A comparison between the two versions of the spirograph was not an aim of this study but could be investigated in the future. According to this, our assumption was that, throughout the use of the spirograph, students can behave like researchers collecting data, formulating hypotheses and conjectures, and feeling the need to argue their own ideas and to discuss them with their peers. This requires, however, effective teacher orchestration in terms of task design and classroom interventions. To design and implement the teaching activity discussed in this paper we refer to the Methods of Varied Inquiry (MVI) (Arzarello, 2016). Results of this activity are analysed to answer the following research question: how can the students' discovery of the GCD and LCM hidden in a digital spirograph be supported by the teacher's orchestration of the activity? To answer to the research question, we describe and analyse the teacher's orchestration of the teaching activity, referring to the theory of variation developed by Marton and colleagues (2004).

Theoretical framework

Through the metaphor of the Instrumental Orchestration, Trouche (2004) offered a theoretical perspective to describe how the teacher can compose coherent sets of instruments within the classroom in order to guide students' instrumental genesis (Artigue, 2002) and its possible benefits for learning. In the teacher's intentional and systematic organisation and use of the various artefacts available in a learning environment in a given mathematical task situation, Drijvers and colleagues (2010) then distinguished three main elements: a didactical configuration, an exploitation mode and a didactical performance. In terms of the metaphor of the musical orchestrations: setting up the didactical configuration can be compared with choosing musical instruments to be included in the band, and arranging them in space so that the different sounds result in polyphonic music; setting up the exploitation mode can be compared with determining the partition for each of the musical instruments involved, bearing in mind the anticipated harmonies that will emerge; the didactical performance can be compared to a musical performance, in which the actual interplay between conductor and musicians reveals the feasibility of the intentions and the success of their realisation. In the next sections we will present the teacher's orchestration of the teaching activity describing the didactical configuration, the exploitation mode and the didactical performance.

This paper focuses on a teaching activity that was framed by the MVI method. It was introduced by Arzarello (2016) as a method to help students to consider a topic from different points of view and to understand it in a deeper way, fostering the transition from forms of "natural" argumentation to forms of mathematical reasoning. Its approach is near to that of a controlled experiment in science, in which the scientist can vary one variable and observe how another variable changes accordingly. Assuming that teaching with variations, in a controlled and systematic way, helps students to construct mathematical concepts, the MVI is based on the theory of variation, according to which in varying the didactical situations four schemes should be taken into account: to experience something, we must experience something else to compare with it (contrast); to fully understand what "three" is, we must also experience varying appearance of three (generalization); to experience a certain aspect of something, and to separate this aspect from other aspects, it must vary while other aspects remain invariant (separation); if there are several aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously (fusion) (Marton et al., 2004, p.16).

Methods

As said before, to identify the principles and heuristics that may guide the teachers to orchestrate the students' discovery of the GCD and LCM hidden in a digital spirograph, we analysed the development of the teaching activity in terms of didactical configuration, exploitation mode and didactical performance. In our study, the didactical configuration, characterised by the teaching setting and the artefacts involved in it, is defined by the digital spirograph and by the MVI. The exploitation mode, determined by the role of the spirograph, is described by the design of the teaching activity with its stages and its tasks. Finally, the didactical performance, in which the actual interplay between the teacher, the students and the spirograph reveals the students' discovery of the GCD and LCM, is the basis for the results' discussion.

The teaching activity, carried out in a distance context, was entirely video-recorded and transcribed. The transcripts were analysed according to the Marton's theory of variation, with the attempt to identify the variation schemes as they emerge in the students' intervention.

In this paper we briefly show and discuss results coming from the final collective discussion conducted by the teacher. They are representative of how the teacher's orchestration can guide the students' development and evolution of the variation schemes, thus fostering the discovery of the GCD and LCM hidden in the spirograph.

Orchestration of the teaching activity

In this section we describe the teaching activity focusing on the first two elements of the teacher's orchestration: the didactical configuration and the exploitation mode. The last element, the didactical performance, will be described in the results section.

Didactical configuration

As already underlined, in our study the didactical configuration is defined by the digital spirograph and by the MVI. The artefact, as it is explained below, was chosen for its potential to mediate the mathematical meanings of GCD and LCM. The MVI is used to design the tasks, because it well exploits the characteristics of the spirograph as an artefact in which everything is played on varying wheels and rings.

The spirograph is an artefact that can be used to draw a particular kind of geometrical curves (roulette) called hypotrochoids and epitrochoids. It is composed of a set of gear rings and wheels of different sizes which, by their combination, allow to create aesthetically fascinating curves. On each wheel there are some small holes positioned on a spiral at a different distance from their centre. To use this artefact, you must first choose a ring and a wheel, fix the ring on a sheet and position the wheel internally or externally to it. In our case the wheel was always put inside the ring and so the obtained curves are hypotrochoids. By inserting a pen inside one of the small holes in the wheel, the wheel is rotated around the ring and the curve begins to be drawn. After a complete rotation of the wheel inside the ring, a portion of the curve is traced, which could have cusps, or small roundings similar to flower petals. After a certain number of revolutions, the wheel returns to the starting position and then the curve closes, regardless of the choice of the hole on the wheel and the combination of the gears chosen (Figure 1).

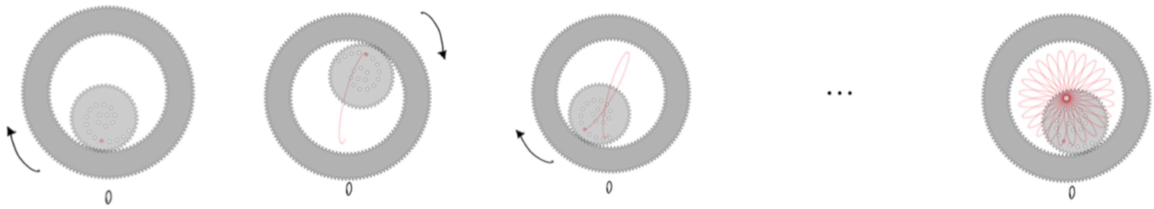


Figure 1: The drawing of the curve with the spirograph

Each curve that is drawn with the spirograph always has the following characteristics: a certain number of petals, a fixed distance between the petals (the number of ring teeth between two petals), a number of complete revolutions of the wheel inside the ring (up to the point where the wheel touches again the tooth of the ring from which it was started) necessary to complete the curve. Of these three characteristic elements, once the spirograph is removed, only the number of petals can be detected by the drawing.

What can be observed, using the spirograph, is that the number of petals, as well as the other characteristics, remains invariant if you change the hole, but instead varies if you change the choice of the ring-wheel combination. This is due to the mathematical relationships (involving the concepts of GCD and LCM) among the characteristics of the geometric curves and the numbers of teeth of the gears. Indeed, if R is the number of teeth of the ring, W is the number of teeth of the wheel, p is the number of petals, d is the distance between the petals and r is the number of revolutions of the wheel inside the ring, it can be seen that: $p \cdot d = R, r \cdot d = W, R \cdot r = W \cdot p$. Moreover, the number of petals corresponds to the number of times the wheel rotates around itself before returning to the starting point, while the curve closes when the repeated sum of the wheel teeth becomes a multiple of the number of the ring teeth. Hence, $R \cdot r = LCM(R, W)$. On the other hand, as $R \cdot r = W \cdot p$, it follows that $W \cdot p = LCM(R, W)$. So, $p = \frac{LCM(R,W)}{W}$ and $r = \frac{LCM(R,W)}{R}$. As a consequence, due to the relationship between GCD and LCM, it can be found that $d = GCD(R, W)$.

These considerations, which connect GCD and LCM to the hypotrochoids obtained with the spirograph, allow students to understand that the characteristics of the curves can be foreseen if the numbers of teeth of the chosen ring-wheel combination are known.

Exploitation mode

The goal of the activity is to use the digital spirograph as an instrument that allows students to untie the mathematical objects GCD and LCM from the idea of being useful objects only for the decomposition into prime numbers, but to see them as objects that mathematise the situation presented with the spirograph. The activity is divided into two phases, guided by appropriate tasks given to the students through an online word document to be filled and discussed: an initial phase of exploration of the instrument and a phase of discovery of the mathematics embedded in the spirograph. At the end of the two phases, to be done in small groups, a collective discussion is expected to be orchestrated by the teacher.

In the exploratory phase students are expected to answer to the questions "how is it done?" and "how does it work?" with the aim of making them familiar with the spirograph and discovering how it draws curves. The second phase of the activity can be divided into four sub-phases (described below), which takes into account the opportunity given by the spirograph to make variations and observe their consequences. To discover the mathematics of the spirograph, indeed, the tasks are designed to let students use different gears, observe and describe what was happening.

Variation of the small hole on the wheel - Students can be asked to choose a ring, a wheel and a small hole and to save the image obtained. Then keeping the same combination of the ring-wheel gears, students can be asked to choose another hole, to save the new image obtained and to compare the two images to describe what was happening. In this way, it is expected that the separation will emerge as a possible variation scheme of Marton's theory. Observing the two images, the students can also be pushed towards a first generalization: by changing the hole in the wheel, the characteristics of the curve obtained do not change, but that what changes is only its "size".

Variation of the wheel, keeping the ring fixed - Students can be asked to take the ring with 96 teeth, to choose a different wheel each, to draw the curve and to save the image. After completing the various drawings, students can fill a table in which, in addition to entering the choice of the wheel and the hole, they can also enter information relating to the characteristics of the curves obtained (the number of petals, the number of teeth between two petals and the number of complete revolutions of the wheel inside the ring) and to observe what happens. Varying the wheel while the ring is kept fixed can, again, boost the separation scheme to emerge. Moreover, as the characteristics of the curve vary when the wheel is changed, in this case, the students can experiment with the scheme of the contrast. The number of teeth of the wheel, in this way, can be seen as affecting the characteristics of the curve.

Variation of the ring, keeping the wheel fixed - Students can be asked to take the ring with 105 teeth and use the same wheels of the previous task. Again, students can fill a table entering the choice of the ring and the hole and the information relating to the characteristics of the curves. The comparison with the previous variation could allow the separation scheme to emerge again: the characteristics of the curve vary when the ring is changed while the wheel is keeping fixed, hence also the number of teeth of the ring affects the characteristic of the curve. This implies that the number of petals, the number of teeth between two petals and the number of complete revolutions of the wheel inside the ring, all depend on the number of the teeth of both the ring and the wheel.

Table 1: Summarising table of the characteristics of the curve according to the R-W choice

Ring (R)	Wheel (W)	Petals (p)	Distance (d)	Revolution (r)
96	36	8	12	3
105	36	35	3	12
96	52	24	4	13
105	52	105	1	52
96	63	32	3	21
105	63	5	21	3

Search for regularities - Students can be asked to take the rings with 96 and 105 teeth, use three wheels (such as those with 36, 52 and 63 teeth) and fill in a table to summarise the characteristics of the curves (see Table 1). With the questions, such as "Is it possible to find relationships between the numbers on each row? If so, which ones? Try to justify your answer", students can be helped to reflect and progress in the searching of regularities and in particular to find the dependence of the curve by both the ring and the wheel. The last sub-phase, thus, is meant to foster the fusion scheme to emerge, through the simultaneous experience with the critical and relevant aspects at stake.

The teacher, aware of the mathematics of the spirograph, can lead the collective discussion in order to guide the students in achieving their learning objective. Attention must be paid to the data shown in the table and the possibility of describing the regularities using mathematical formulas. For the collective discussion it is expected that the possible scheme of variation that can be reached is that of generalisation. From the discussion of experience, it will be possible to deduce: the independence of the characteristics of the curve from the choice of the hole; the dependence on both the number of teeth of the wheel and on the number of teeth of the ring; the mathematisation of the functioning of the spirograph, through the formulas that bring into play the GCD and LCM.

Results: the didactical performance

The teaching activity required three hours and involved 12 students. At the beginning of the lesson the teacher explained to the students that they would have been required to be divided into 3 sub-groups and accomplish two different tasks (related to the two phases described above) to be collectively discussed at end. The group work was conducted using the same online platform and sharing a unique document for each sub-group of students. Students were given 30 minutes to accomplish the first explorative phase and 60 minutes for the tasks of the second phase. We do not have enough space here to report the exact tasks given to the students, but the requests were made by the teacher according to what is described in the exploitation mode section. After a small break, the students were invited to leave the breakout rooms and to join the collective discussion guided by the teacher. In this section we focus on the final discussion concerning the second phase. In the first part of the discussion the teacher, showing and discussing a prepared picture, brought the students to share their considerations on what changes and what does not change if the combination of the ring-wheel gears remains fixed while the hole is changed. To guide the students to the shared conclusion that the petals size changes, but the number of the petals does not change, she did not re-voice those answers that are considered not pertinent to her aim. Afterwards, the discussion moved to the variation of the wheel and of the ring. The teacher boosted the students to experience the variation of one aspect at a time, keeping fixed the other, and the students arrived at the conclusion that the characteristics of the curves depend on both the ring and the wheel.

The excerpt below refers to the observations done when considering the obtained table (Table 1).

Teacher:	What observations did you make? Can you see any relationships between these numbers on each line?
Mattia:	8 times 12 gives 96, likewise 24 times 4 and 32 times 3 give 96
Teacher:	So, you are saying that: Petals ∗ Distance = Ring. Now we haven't embedded the wheel in any relation yet. Do you see any other relations in which the wheel appears?
Michele:	Distance * Revolutions = Wheel, in fact 12∗3,4∗13...
Giulia:	I did something a little different: Ring/Wheel ∗ Revolutions = Petals
Teacher:	We could also write it as: Ring ∗ Revolutions = Petals ∗ Wheel. Indeed, we see that 96 ∗ 3 = 288; and if I do Wheel ∗ Petals, so 36 ∗ 8...
Michele:	I always get 288
Teacher:	What does 288 have to do with the wheel and the ring?

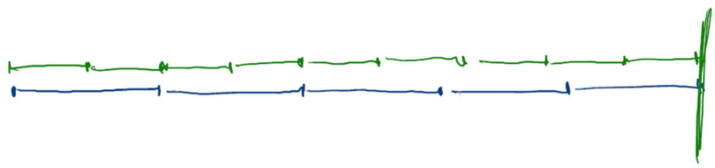
At this point, given the difficulty of the students, the teacher brought their attention to the fact that it is the revolution of the wheel around itself that draws the petals. In particular, in this case the curve is completed after 8 revolution of the wheel around itself and, at the end, the wheel has made 3 revolutions around the ring. The following excerpt starts with Michele's intervention after this last observation of the teacher.

Michele: The Least Common Multiple! 288 is the LCM between 96 and 36

Teacher: Why do you think it is the Least Common Multiple?

Michele: Because we have to fill the same number of wheels and rings, i.e. we have to occupy with a certain number of revolutions the wheel and the ring and then return to the same tooth […] we put in sequence all the teeth of the wheel and of the ring, and we see that… it is the same segment. We must then get to a certain point where the two segments formed by all the ring teeth and all the wheel teeth are equal

Teacher: That's like I have this kind of thing:



Michele: Yes, we have to fill the same quantity with 2 different segments. And so that's 288

Teacher: Now, the distance between one petal and another, if you were to read inside the ring, what will it represent?

Michele: Perhaps the Greatest Common Divisor. […] Basically, it is the same reasoning, but we have to divide, not to multiply. That is... in this case we have to find a segment that divides both the ring and the wheel equally. And that segment is the distance.

Discussion and conclusion

Results were analysed using the lens of the Marton's theory. At the beginning of the final discussion, the students focussed on the effect of the change of the hole on the obtained curves. Thanks to the teacher's decisions (to show and discuss a prepared picture and to ignore students' answers that are not pertinent to her aim) the students moved from the separation scheme, fostered by the task itself, to the generalisation given by the recognition that the choice of the hole does not affect the number of petals. In the following part of the discussion, the students' focus moved from the observation of particular drawings to the understanding that the ring-wheel choice affects the characteristics of the curve. Also in this case, the students' development of the separation scheme was fostered by the teacher's orchestration, and especially her decision to let students experience the variation of one aspect at a time, keeping fixed the others. Students' awareness of the meanings of the variations among the aspects in the table is evident in the first excerpt: the teacher's questions, aimed at fostering the reading of the table with a researcher lens, allowed students to develop the fusion scheme and to recognise the complex relationships among the various aspects. However, the intervention of the teacher was fundamental to give meaning to those relationships. In the last excerpt, indeed, it can be seen how Michele, thanks to the teacher's suggestion, succeeded in connecting the spirograph movement to the numbers in the table, thus endowing the LCM with both the products $R \cdot r$ and $W \cdot p$, and the GCD with the distance between two consecutive petals.

To conclude we can say that the analysis of the teacher's orchestration showed how students can be supported in the discovery of the mathematics hidden in the digital spirograph, thus answering our research question. Indeed, the students' learning was supported through an appropriate task design of the activity and an aware teacher's behaviour during the classroom intervention. In our case study, the teacher's orchestration was based on her willingness to exploit the opportunity given by the digital spirograph to offer students a situation in which variation gives power to the development of mathematical thinking. Results showed how students behaved like researchers, feeling the need to argue their own ideas and to discuss them with their peers. This was fostered by the teacher's decision to use the MVI as a method to help students to consider a topic from different points of view and to understand it in a deeper way.

Although this study was developed by taking into consideration only one case, its results can give a contribution to the discussion on good practices in technology-rich mathematics education. Moreover, they can be considered as a starting point to develop further investigations focusing on the principles and heuristics that may guide the teacher's orchestration of fruitful activities.

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