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MULTIPLE SOLUTIONS FOR PERTURBED QUASILINEAR ELLIPTIC PROBLEMS

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Dedicated to the memory of Edward Fadell and Sufian Husseini

Abstract. We investigate the existence of multiple solutions for the (p,q)–quasilinear elliptic problem

$$\begin{cases} -\Delta_p u - \Delta_q u \;=\; g(x,u) + \varepsilon\, h(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < q < +\infty$, Ω is an open bounded domain of $\mathbb{R}^N$, the nonlinearity $g(x,u)$ behaves at infinity as $|u|^{q-1}$, $\varepsilon \in \mathbb{R}$ and $h \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

In spite of the possible lack of a variational structure of this problem, from suitable assumptions on $g(x,u)$ and appropriate procedures and estimates, the existence of multiple solutions can be proved for small perturbations.

1. Introduction

In the last years an increasing interest has been devoted to the study of the (p,q)–quasilinear elliptic problem

$$(P_\varepsilon) \qquad \begin{cases} -\Delta_p u - \Delta_q u \;=\; g(x,u) + \varepsilon\, h(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, $1 < p < q < +\infty$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ if $r \in \{p,q\}$, $\varepsilon \in \mathbb{R}$, $g(x,u)$ and $h(x,u)$ are given continuous functions on $\overline{\Omega} \times \mathbb{R}$.

Usually, multiple solutions of variational problems are obtained by means of topological tools such as, for example, the Krasnoselskii genus (see [11]), the Ljusternik–Schnirelman category (see [35]) or the Fadell–Husseini relative cohomological index (see [17]), if a symmetry group acts on the related functional. Unluckily, for our setting such approaches are not allowed for two main reasons: solutions of (P_ε) may not be critical points of a functional and if $q \neq 2$ the spectral properties of the q–Laplacian operator $-\Delta_q$ in the Sobolev space $W_0^{1,q}(\Omega)$ are still mostly unknown.

More precisely, if on the growth of the perturbation term $h(x,u)$ no hypothesis is assumed, problem (P_ε) may lose its variational structure on the Sobolev space $W_0^{1,q}(\Omega)$; so, it is useful the notion of essential values as introduced in [13, 14] for

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perturbations of nonsmooth functionals. On the other hand, a complete description of the spectrum $\sigma(\Delta_q)$ of $-\Delta_q$ in $W_0^{1,q}(\Omega)$ does not exist even if in [32] a sequence of eigenvalues is introduced by means of the pseudo–index theory related to the cohomological index of Fadell and Rabinowitz (see [18]) or other definitions of eigenvalues are given (see [10, 19, 27, 31]). In particular, if $N \geq 2$ we are not aware of whether the unbounded and increasing sequences of the known eigenvalues cover the whole spectrum $\sigma(-\Delta_q)$ or not; moreover, the eigenvalues do not provide a decomposition of the Banach space $W_0^{1,q}(\Omega)$. In order to overcome such difficulties, sequences of so–called quasi–eigenvalues $(\eta_m^0)_m$, $(\nu_m^0)_m$ and $(\nu_m)_m$ can be introduced so that they satisfy "good" properties and, when $q = 2$, they agree with the sequence of standard eigenvalues of $-\Delta$ in $W_0^{1,2}(\Omega)$ (see Subsection 2.1 for more details).

Anyway, results about problem (P_0) have been obtained in [9, 29, 30]. Furthermore, some multiplicity results have been proved at least for some particular models of the perturbed elliptic problem (P_ε) if $\varepsilon \neq 0$. More precisely, if $p = q = 2$ multiple solutions of (P_ε) exist when $g(x,\cdot)$ is odd and superlinear at infinity but subcritical (see [23]) and, if both $g(x,\cdot)$ and $h(x,\cdot)$ are odd, also when $g(x,\cdot)$ is asymptotically linear at infinity (see [5, 24]). On the other hand, if $p = q \neq 2$, problem (P_ε) has been studied if both $g(x,\cdot)$ and $h(x,\cdot)$ are odd when $g(x,\cdot)$ is "superlinear" but subcritical (see [24]).

In the general case $p \neq q$, for the perturbed equation (P_ε) only recently an existence theorem has been proved if $g(x,u)$ behaves as $|u|^{q-1}$ at infinity (see [6]) and, to our knowledge, no multiplicity result has been stated so far.

In this paper, we assume that

$$g(x,t) \ = \ \lambda_\infty |t|^{q-2}t + f(x,t) \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}$$

for some $\lambda_\infty \in \mathbb{R}$ and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$.

Thus, problem (P_ε) can be written as

$$(P_{\varepsilon,\infty}) \qquad \begin{cases} -\Delta_p u - \Delta_q u = \lambda_\infty |u|^{q-2}u + f(x,u) + \varepsilon\ h(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and for the nonlinear term $f(x,t)$ we consider the following hypotheses:

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$;

(f_2) there exists

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{q-2}t} \ = \ 0 \quad \text{uniformly in } \overline{\Omega};$$

(f_3) there exists

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{q-2}t} \ = \lambda_0 \in \mathbb{R} \setminus \{0\} \quad \text{uniformly in } \overline{\Omega};$$

(f_4) $f(x,\cdot)$ is odd for all $x \in \overline{\Omega}$.

Then, our main multiplicity result can be stated as follows.

Theorem 1.1. *Assume that $f(x,t)$ satisfies hypotheses (f_1)–(f_4) and the perturbation term $h(x,t)$ is such that*

(h_1) $h \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$*;*

(h_2) $h(x,\cdot)$ *is odd for all* $x \in \overline{\Omega}$.

Moreover, suppose that two integers n, $k \in \mathbb{N}$ exist such that $k \geq n$ and one of the following assumptions hold:

(i) $\quad \lambda_0 + \lambda_\infty < \eta_n^0$, $\quad \nu_k^0 < \lambda_\infty$,

(ii) $\quad \lambda_\infty < \eta_n^0$, $\quad \frac{q}{p}\,\nu_k < \lambda_0 + \lambda_\infty$,

where $(\eta_m^0)_m$, $(\nu_m^0)_m$, $(\nu_m)_m$ are quasi–eigenvalues (see Subsection 2.1). Then, if $\lambda_\infty \notin \sigma(-\Delta_q)$, a constant $\bar{\varepsilon} > 0$ exists such that for all $|\varepsilon| \leq \bar{\varepsilon}$ problem $(P_{\varepsilon,\infty})$ has at least $k - n + 1$ distinct pairs of nontrivial solutions.

It is worth to point out that $q > p$ is not an assumption, indeed the roles of p and q are interchangeable. Moreover, it is understood that by a solution we mean a weak solution, i.e., a function $u \in W_0^{1,q}(\Omega)$ solving problem $(P_{\varepsilon,\infty})$ in the sense of distributions. We note also that, under our assumptions, such weak solutions belong to $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in]0,1]$ (see, e.g., [21, Remark 1.3]).

Direct consequence of Theorem 1.1 is the following new result about the single q–Laplacian problem.

Corollary 1.2. *Assume that (f_1)–(f_4) and (h_1)–(h_2) hold. If $\lambda_\infty \notin \sigma(-\Delta_q)$ and $n, k \in \mathbb{N}$ exist such that $k \geq n$ and*

$$\min\{\lambda_0 + \lambda_\infty, \lambda_\infty\} < \eta_n^0 \leq \nu_k^0 < \max\{\lambda_0 + \lambda_\infty, \lambda_\infty\},$$

then $\bar{\varepsilon} > 0$ exists such that for all $|\varepsilon| \leq \bar{\varepsilon}$ problem

$$\begin{cases} -\Delta_q u = \lambda_\infty |u|^{q-2} u + f(x,u) + \varepsilon\, h(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $k - n + 1$ distinct pairs of nontrivial solutions.

Remark 1.3. From the monotonicity of both the sequences $(\nu_m^0)_m$, $(\nu_m)_m$ and the estimates in [4, Proposition 2.9], [6, Proposition 2.1] (see also (2.6) and (2.12) in the next section), it follows that

$$\eta_n^0 \leq \nu_k^0, \quad \eta_n^0 \leq \nu_k, \quad \eta_n \leq \nu_k \qquad \text{for all } k \geq n.$$

Therefore, since $p < q$, the two conditions (i) and (ii) stated in Theorem 1.1 can be written, respectively, as the chains of inequalities

$$\lambda_0 + \lambda_\infty < \eta_n^0 \leq \nu_k^0 < \lambda_\infty$$

and

$$\lambda_\infty < \eta_n^0 < \frac{q}{p}\,\nu_k < \lambda_0 + \lambda_\infty,$$

while the inequality $\eta_n^0 \leq \nu_k^0$ in Corollary 1.2 is not an assumption.

Remark 1.4. In the assumptions of Theorem 1.1 the unperturbed problem $(P_{0,\infty})$ has been studied in [3] if $p = 2$ and in [12] if $p \neq 2$. In these settings assumption (f_1) can be replaced by the weaker hypothesis

$(f_1)'$ $f(x,t)$ is a Carathéodory function (i.e., $f(\cdot,t)$ is measurable in Ω for all $t \in \mathbb{R}$ and $f(x,\cdot)$ is continuous in $\mathbb{R}$ for a.e. $x \in \Omega$) such that

$$\sup_{|t| \leq r} |f(\cdot,t)| \in L^\infty(\Omega) \qquad \text{for all } r > 0.$$

At last, we note that the theses in Theorem 1.1 and Corollary 1.2 hold also when the limit in (f_3) is infinite. In fact, the following results can be stated.

Proposition 1.5. *Assume that $f(x,t)$ satisfies hypotheses (f_1)–(f_2), (f_4) and*

$(f_3)'$ *there exists*

$$\lim_{t\to 0}\frac{f(x,t)}{|t|^{q-2}t} = -\infty \quad \textit{uniformly in } \overline{\Omega}.$$

Moreover, assume that $\lambda_\infty \notin \sigma(-\Delta_q)$*, the perturbation term* $h(x,t)$ *is such that* (h_1)–(h_2) *are verified and* $k \in \mathbb{N}$ *exists such that*

$(i)'$ $\nu_k^0 < \lambda_\infty$.

Then, $\bar\varepsilon > 0$ *exists such that for all* $|\varepsilon| \le \bar\varepsilon$ *problem* $(P_{\varepsilon,\infty})$ *has at least* k *distinct pairs of nontrivial solutions.*

Proposition 1.6. *Assume that* $f(x,t)$ *satisfies hypotheses* (f_1)–(f_2)*,* (f_4) *and*

$(f_3)''$ *there exists*

$$\lim_{t\to 0}\frac{f(x,t)}{|t|^{q-2}t} = +\infty \quad \textit{uniformly in } \overline{\Omega}.$$

If $\lambda_\infty \notin \sigma(-\Delta_q)$ *and the perturbation term* $h(x,t)$ *verifies conditions* (h_1)–(h_2)*, then* $\bar\varepsilon > 0$ *exists such that for all* $|\varepsilon| \le \bar\varepsilon$ *problem* $(P_{\varepsilon,\infty})$ *has infinitely many pairs of nontrivial solutions.*

Here, we deal with problem $(P_{\varepsilon,\infty})$ if $\varepsilon \neq 0$ and go further the results in [3, 5, 12] in different ways. Indeed, now the problem is non–homogeneonus, perturbed and can be set on a Banach (not Hilbert) space, requiring so greater effort both in the use of critical point theorems and in the procedures for the $W_0^{1,q}(\Omega)$–norm estimates on the solutions, since orthogonal decompositions are not available anymore. We point out that our approach requires continuous odd perturbations: actually, our technique works also when the symmetry is broken, but leads to more restrictive results (cf. [5]). More precisely, our strategy is the following: we start from a multiplicity result concerning nontrivial solutions of the symmetric unperturbed problem $(P_{0,\infty})$ by using the pseudo–index theory (see Theorem 3.2); then, by means of cut functions, for ε small we introduce perturbations of the functional associated to $(P_{0,\infty})$ which have essential values near to the critical levels of the solutions. Such essential values turn out to be critical ones and suitable procedures - deeply different from those ones in [5] where the decomposition of $W_0^{1,2}(\Omega)$ by means of eigenvalues is exploited - allow us to prove that the L^∞–norm of the family of critical points of the cut–perturbed functionals is bounded thanks to the results in Section 4. Hence, finally, solutions of $(P_{\varepsilon,\infty})$ can be found.

This paper is organized as follows: in Section 2 we present the variational and topological tools we are going to use, in Section 3 we state a multiplicity result for the unperturbed problem $(P_{0,\infty})$, in Section 4 we present a regularity result for solutions of problems which involve the (p,q)–Laplacian operator and, finally, Theorem 1.1 and its variants are proved in Section 5.

2. Preliminary material

Throughout this paper, we will use the following notations:

- $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$;
- $(X, \|\cdot\|_X)$ Banach space, $(X', \|\cdot\|_{X'})$ its dual, I C^1–functional on X;
- $I^b = \{u \in X : I(u) \le b\}$ sublevel of I corresponding to $b \in \mathbb{R}$;
- $I_b = \{u \in X : I(u) \ge b\}$ superlevel of I corresponding to $b \in \mathbb{R}$;

- $K_b = \{u \in X : I(u) = b, \ \mathrm{d}I(u) = 0\}$ set of the critical points of I in X at the critical level $b \in \mathbb{R}$;
- $|\cdot|_s$ norm in the Lebesgue space $L^s(\Omega)$, $1 \le s \le +\infty$;
- $\|\cdot\|_q$ norm in $W_0^{1,q}(\Omega)$, i.e., $\|u\|_q = |\nabla u|_q$ for all $u \in W_0^{1,q}(\Omega)$;
- $q^* = \frac{qN}{N-q}$ if $q < N$, $q^* = +\infty$ otherwise;
- $B_R = \{u \in W_0^{1,q}(\Omega) : \|u\|_q < R\}$, $\overline{B}_R = \{u \in W_0^{1,q}(\Omega) : \|u\|_q \le R\}$ and $S_R = \{u \in W_0^{1,q}(\Omega) : \|u\|_q = R\}$ for any $R > 0$;
- C is any positive real number, possibly different from line to line.

Moreover, for simplicity, by $(\beta_m)_m$ we denote any infinitesimal sequence which depends only on a given sequence of functions and by $(\beta_m(\varphi))_m$ any infinitesimal sequence which depends also on a fixed function φ.

2.1. **Sequences of quasi–eigenvalues.** Firstly, we want to introduce sequences of quasi–eigenvalues which have the "right" properties for investigating the existence of solutions of problem $(P_{\varepsilon,\infty})$.

To this aim, starting point is $\lambda_1^{(q)}$, first eigenvalue of operator $-\Delta_q$ in $W_0^{1,q}(\Omega)$, which is characterized by

$$\lambda_1^{(q)} \ = \inf_{u \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{|\nabla u|_q^q}{|u|_q^q}.$$

It is well known that it is positive, simple, isolated and has a unique positive eigenfunction $\varphi_1^{(q)}$ having unitary L^q–norm (see [1, 27]). Then, taking $\eta_1^0 = \lambda_1^{(q)}$ and $\psi_1^0 = \varphi_1^{(q)}$, an increasing diverging sequence $(\eta_m^0)_m$ of positive real numbers can be defined as in [8, Section 5] so that for each $m \in \mathbb{N}$ a corresponding function $\psi_m^0 \in W_0^{1,q}(\Omega)$ exists which has the following properties:

$$|\psi_m^0|_q = 1, \qquad |\nabla \psi_m^0|_q^q = \eta_m^0 \tag{2.1}$$

and $\psi_m^0 \neq \psi_n^0$ if $m \neq n$. Moreover, $(\psi_m^0)_m$ generates the whole space $W_0^{1,q}(\Omega)$ and is such that

$$W_0^{1,q}(\Omega) = Y_m^0 \oplus Z_m^0 \quad \text{ for all } m \in \mathbb{N}, \tag{2.2}$$

where $Y_m^0 = \mathrm{span}\{\psi_1^0, \dots, \psi_m^0\}$ and its closed infinite dimensional topological complement Z_m^0 can be explicitely described. Notice that, for all $m \in \mathbb{N}$, the following inequality holds:

$$\eta_m^0 |u|_q^q \le |\nabla u|_q^q \quad \text{ for all } u \in Z_{m-1}^0 \tag{2.3}$$

(cf. [8, Lemma 5.4]).

On the other hand, in order to deal with finite dimensional subspaces, another sequence of quasi–eigenvalues $(\nu_m^0)_m$ can be introduced as in [25]. More precisely, taking any $m \in \mathbb{N}$ and setting

$$\mathbb{W}_m^0 \ = \ \{Y \subset W_0^{1,q}(\Omega) : \ Y \text{ subspace}, \ \varphi_1^{(q)} \in Y \ \text{ and } \ \dim Y \ge m\}, \tag{2.4}$$

we define

$$\nu_m^0 = \inf_{Y \in \mathbb{W}_m^0} \sup_{u \in Y\setminus\{0\}} \frac{|\nabla u|_q^q}{|u|_q^q}. \tag{2.5}$$

By definition, it is $\nu_1^0 = \lambda_1^{(q)}$, moreover it can be proved that $(\nu_m^0)_m$ is an increasing diverging sequence such that

$$\eta_m^0 \le \nu_m^0 \quad \text{for all } m \in \mathbb{N} \tag{2.6}$$

(see [4, Proposition 2.9]).

Following similar ideas, the previous constructions can be extended to the (p,q)–Laplacian operator $-\Delta_p - \Delta_q$ in $W_0^{1,q}(\Omega)$ (see [12, Subsection 2.3]). In fact, overcoming the lack of homogeneity of such an operator, if

$$\eta_1 \ := \inf_{\substack{u\in W_0^{1,q}(\Omega)\\ |u|_q=1}} \left(|\nabla u|_p^p + |\nabla u|_q^q\right) \ \geq \ \lambda_1^{(q)},$$

a first function $\psi_1 \in W_0^{1,q}(\Omega)$ can be found so that

$$|\psi_1|_q = 1, \qquad |\nabla\psi_1|_p^p + |\nabla\psi_1|_q^q = \eta_1.$$

Then, we can define an increasing diverging sequence $(\eta_m)_m$ of positive real numbers and a corresponding sequence of functions $(\psi_m)_m$ such that $\psi_m \neq \psi_n$ if $m \neq n$ and

$$|\psi_m|_q = 1, \qquad |\nabla\psi_m|_p^p + |\nabla\psi_m|_q^q = \eta_m \qquad \text{for all } m \in \mathbb{N}. \tag{2.7}$$

As proved in [12, Lemma 2.6], these functions generate $W_0^{1,q}(\Omega)$ and are such that

$$W_0^{1,q}(\Omega) = Y_m \oplus Z_m \quad \text{ for all } m \in \mathbb{N}, \tag{2.8}$$

with $Y_m = \operatorname{span}\{\psi_1,\dots,\psi_m\}$ and Z_m its topological complement. In particular, the following inequalities hold:

$$\eta_m|u|_q^q \leq |\nabla u|_p^p + |\nabla u|_q^q \quad \text{ for all } u \in Z_{m-1} \cap \{u \in W_0^{1,q}(\Omega) : |u|_q \leq 1\} \tag{2.9}$$

and

$$\eta_m|u|_q^p \leq |\nabla u|_p^p + |\nabla u|_q^q \quad \text{ for all } u \in Z_{m-1} \setminus \{u \in W_0^{1,q}(\Omega) : |u|_q \leq 1\}.$$

Furthermore, taking any $m \in \mathbb{N}$ and setting

$$\mathbb{W}_m \ = \ \{Y \subset W_0^{1,q}(\Omega) : \ Y \text{ subspace}, \ \psi_1 \in Y \ \text{ and } \ \dim Y \geq m\}, \tag{2.10}$$

we define

$$\nu_m \ = \ \inf_{Y\in\mathbb{W}_m} \sup_{u\in Y\setminus\{0\}} \frac{|\nabla u|_p^p + |\nabla u|_q^q}{|u|_q^q}. \tag{2.11}$$

Again, $(\nu_m)_m$ is an increasing diverging sequence such that

$$\eta_m^0 \leq \nu_m, \quad \eta_m \leq \nu_m \qquad \text{for all } m \in \mathbb{N} \tag{2.12}$$

(see [6, Proposition 2.1]).

2.2. **Variational tools.** As pointed out in Section 1, our first step is proving the existence of multiple solutions for the unperturbed problem $(P_{0,\infty})$. To this aim, since such a problem has a variational structure, we need the "right" variational framework that we present here even if it consists essentially of widely known definitions and results.

Firstly, we recall that a C^1 functional $I : X \to \mathbb{R}$ satisfies the *Palais–Smale condition at level* c, $c \in \mathbb{R}$, briefly $(PS)_c$, if any sequence $(u_m)_m \subseteq X$ such that

$$\lim_{m\to+\infty} I(u_m) = c \quad \text{ and } \quad \lim_{m\to+\infty} \|\mathrm{d}I(u_m)\|_{X'} = 0$$

converges in X, up to subsequences.

Taking $-\infty \leq a < b \leq +\infty$, we say that I satisfies (PS) in $]a,b[$ if so is at each level $c \in]a,b[$.

When dealing with symmetric functionals, multiple solutions can be found not only by means of the genus theory (see, e.g., [34, 36] and references therein) but also by using the pseudo–index related to the genus (see [2, Theorem 2.9] which has been applied on Banach spaces in [4, Theorem 2.6]). Thus, we recall some notions of the index theory on a Banach space X for an even functional, i.e., when the symmetry group $\mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\}$ acts (see, e.g., [33]).

Let us define

$$\Sigma \;=\; \{A \subseteq X : A \text{ closed and symmetric w.r.t. the origin, i.e., } -u \in A \text{ if } u \in A\}$$

and

$$\mathcal{H} = \{h \in C(X,X) :\, h \text{ odd}\}.$$

Taking $A \in \Sigma$, $A \neq \emptyset$, the *genus* of A is defined as

$$\gamma(A) = \inf\{m \in \mathbb{N} :\; \exists \psi \in C(A, \mathbb{R}^m \setminus \{0\}) \text{ s.t. } \psi(-u) = -\psi(u) \text{ for all } u \in A\},$$

if such an infimum exists, otherwise $\gamma(A) = +\infty$. Assume $\gamma(\emptyset) = 0$.

The index theory $(\Sigma, \mathcal{H}, \gamma)$ related to $\mathbb{Z}_2$ is also called *genus* on X and satisfies the following properties (for more details, we refer to [33, Section 1] and [36, Section II.5]).

Proposition 2.1. *Taking A, $B \in \Sigma$, we have that:*

- (i_1) $\gamma(A) = 0 \Leftrightarrow A = \emptyset$*;*
- (i_2) $A \subseteq B \Rightarrow \gamma(A) \le \gamma(B)$ *(monotonicity property);*
- (i_3) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$ *(subadditivity property);*
- (i_4) $\gamma(A) \le \gamma(\overline{h(A)})$ *for all* $h \in \mathcal{H}$ *(supervariancy property);*
- (i_5) *if A is compact, $\delta > 0$ exists such that $\gamma(N_\delta(A)) = \gamma(A)$, where*
$$N_\delta(A) \;=\; \{x \in X :\, d(x, A) \le \delta\}$$
is the closed δ–neighbourhood of A (continuity property);
- (i_6) *if $h : X \to X$ is an odd homeomorphism, $B \subseteq X$ is an open bounded symmetric neighbourhood of* 0 *and V is a finite dimensional subspace of X, then*
$$\gamma(V \cap h(\partial B)) = \dim V;$$
- (i_7) *if W is a closed subspace of X with* $\operatorname{codim} W < +\infty$ *and $A \in \Sigma$ is such that* $\gamma(A) > \operatorname{codim} W$*, then $A \cap W \neq \emptyset$.*

Let $(\Sigma, \mathcal{H}, \gamma)$ be the genus on X and consider an even functional $I : X \to \mathbb{R}$. Then, taking any $m \in \mathbb{N}$ and setting

$$b_m \;=\; \inf_{A \in \Sigma_m} \sup_{u \in A} I(u),$$

with

$$\Sigma_m = \{A \in \Sigma :\, \gamma(A) \ge m\}, \tag{2.13}$$

the following characterization holds (for more details, see [34, Remarks 8.7(i)] and [4, Lemma 2.5]).

Lemma 2.2. *If $m \in \mathbb{N}$ is such that $\Sigma_m \neq \emptyset$ and $b_m \in \mathbb{R}$, then*

$$b_m \;=\; \inf\left\{b \in \mathbb{R} :\, \gamma\left(I^b\right) \ge m\right\}.$$

Now, if we consider $(\Sigma, \mathcal{H}, \gamma)$ the genus on $X = W_0^{1,q}(\Omega)$ and for any $m \in \mathbb{N}$ the corresponding set Σ_m as in (2.13), then we can define the sequence of mini–max levels $(\mu_m)_m$ such as

$$\mu_m \;=\; \inf_{A \in \Sigma_m} \sup_{u \in A \setminus \{0\}} \frac{|\nabla u|_p^p + |\nabla u|_q^q}{|u|_q^q}, \tag{2.14}$$

which generalizes the sequence of eigenvalues of $-\Delta_q$ in $W_0^{1,q}(\Omega)$ introduced in [28]. Then, the following estimates hold.

Proposition 2.3. *We have that*

$$\eta_m \ \leq \ \mu_m \ \leq \ \nu_m \qquad \text{for all } m \in \mathbb{N}, \tag{2.15}$$

with sequences $(\eta_m)_m$ and $(\nu_m)_m$ as in Subsection 2.1.

Proof. In order to prove the first inequality in (2.15), as in the proof of [4, Proposition 2.9] we argue by contradiction and assume that an integer $m \in \mathbb{N}$ exists such that $\mu_m < \eta_m$. Hence, we can take $\sigma > 0$ so that

$$\mu_m + \sigma < \eta_m. \tag{2.16}$$

Then, we consider the even C^1 functional

$$\Psi : u \in W_0^{1,q}(\Omega) \mapsto \Psi(u) = |\nabla u|_p^p + |\nabla u|_q^q \in \mathbb{R}$$

and the manifold $\mathcal{S}_q = \{u \in W_0^{1,q}(\Omega) : \ |u|_q = 1\}$. Since from (2.7) and (2.8) with $m-1$ it follows that

$$\eta_m = \min_{\substack{u \in Z_{m-1} \\ |u|_q = 1}} \Psi(u),$$

with Z_{m-1} such that (2.9) holds, we have that

$$Z_{m-1} \cap \mathcal{S}_q \subseteq \left(\Psi\big|_{\mathcal{S}_q}\right)_{\eta_m}$$

which implies

$$\{u \in \mathcal{S}_q : \ \Psi(u) < \eta_m\} \subseteq \mathcal{S}_q \setminus Z_{m-1}. \tag{2.17}$$

On the other hand, taking $\mathcal{I}(u) = \frac{\Psi(u)}{|u|_q^q}$ in $W_0^{1,q}(\Omega) \setminus \{0\}$, we have that

$$\mathcal{I}\big|_{\mathcal{S}_q} = \Psi\big|_{\mathcal{S}_q} \quad \text{and} \quad \mu_m \ = \ \inf\left\{b \in \mathbb{R} : \ \gamma\left(\mathcal{I}^b\right) \geq m\right\}.$$

Then, $\bar{b} \in \mathbb{R}$ exists such that

$$\mu_m \leq \ \bar{b} \ < \mu_m + \sigma \qquad \text{and} \qquad \gamma\left(\mathcal{I}^{\bar{b}}\right) \geq m. \tag{2.18}$$

Thus, (2.16) and (2.18) imply that

$$\mathcal{I}^{\bar{b}} \subseteq \mathcal{I}^{\mu_m + \sigma} \subseteq \mathcal{I}^{\eta_m};$$

where from (2.17) it results that $\mathcal{I}^{\bar{b}} \cap Z_{m-1} = \emptyset$. Hence, from property (i_7) and (2.8) with $m-1$, it has to be

$$\gamma\left(\left(\Psi\big|_{\mathcal{S}_q}\right)^{\bar{b}}\right) \ \leq \ m-1,$$

in contradiction with (2.18).
Moreover, since Proposition 2.1 implies that $\mathbb{W}_m \subset \Sigma_m$, the second inequality in (2.15) follows from definitions (2.11) and (2.14). □ □

Now, back to the abstract setting and following [7], we say that the pseudo–index related to the genus $(\Sigma, \mathcal{H}, \gamma)$ on a Banach space X, to an even functional $I : X \to \mathbb{R}$ and to a set $S \in \Sigma$ is the triplet $(S, \mathcal{H}^*, \gamma^*)$ such that $\mathcal{H}^* \subset \mathcal{H}$ is a group of odd homeomorphisms on X and $\gamma^* : \Sigma \longrightarrow \mathbb{N} \cup \{+\infty\}$ is the map defined by

$$\gamma^*(A) = \min_{h \in \mathcal{H}^*} \gamma(h(A) \cap S) \quad \text{for all } A \in \Sigma.$$

The following mini–max theorem holds (for the proof, see [2, Theorem 2.9]).

Theorem 2.4. *Let $a, b, c_0, c_\infty \in \overline{\mathbb{R}}$ be such that*

$$-\infty \leq a < c_0 < c_\infty < b \leq +\infty$$

and consider a C^1 even functional I, the genus $(\Sigma, \mathcal{H}, \gamma)$ on X, a set $S \in \Sigma$ and take $(S, \mathcal{H}^*, \gamma^*)$ pseudo–index theory related to the genus, I and S such that

$$\mathcal{H}^* = \{h \in \mathcal{H} : \ h \text{ bounded homeomorphism s.t. } h(u) = u \text{ if } u \notin I^{-1}(]a, b[)\}.$$

Assume that:

(i) *functional I satisfies (PS) in $]a, b[$;*
(ii) *$S \subseteq I^{-1}([c_0, +\infty[)$;*
(iii) *an integer $\tilde{k} \in \mathbb{N}$ and $\tilde{A} \in \Sigma$ exist such that $\tilde{A} \subseteq I^{c_\infty}$ and $\gamma^*(\tilde{A}) \geq \tilde{k}$.*

Then, for every $i \in \{1, \ldots, \tilde{k}\}$ the mini–max level

$$c_i = \inf_{A \in \Sigma_i^*} \sup_{u \in A} I(u), \qquad \text{with } \Sigma_i^* = \{A \in \Sigma : \gamma^*(A) \geq i\}, \tag{2.19}$$

is a critical value for I. Moreover, we have that

$$c_0 \leq c_1 \leq \ldots \leq c_{\tilde{k}} \leq c_\infty$$

and, if $c = c_i = \ldots = c_{i+r}$ with $1 \leq i \leq i + r \leq \tilde{k}$, then $\gamma(K_c) \geq r + 1$.

Remark 2.5. In Theorem 2.4(iii) a lower bound for the pseudo–index of a suitable set $\tilde{A}$ is required. In order to have such a bound at least in our setting, we consider $(\Sigma, \mathcal{H}, \gamma)$ genus on X and two closed subspaces V and W of X so that

$$\dim V < +\infty \quad \text{ and } \quad \text{codim } W < +\infty.$$

Then, for every open bounded symmetric neighbourhood B of 0 in X, taking every odd bounded homeomorphism $h : X \to X$ it results

$$\gamma(V \cap h(\partial B \cap W)) \geq \dim V - \text{codim } W$$

(for more details, see [2, Theorem A.2] and [4, Theorem 2.7]).

2.3. **Essential values.** As pointed out in Section 1, problem $(P_{\varepsilon,\infty})$ may not have a variational structure on $W_0^{1,q}(\Omega)$. Hence, as in [23], we use the auxiliary notion of essential value as introduced in the study of perturbations of nonsmooth functionals (see [13, 14]) and, in particular, the definition of odd–essential value introduced in [5, Section 2]. Even if the notion of essential value is topological, an essential value is candidate to be a critical level and is stable under small perturbations. Furthermore, critical levels arising from standard mini–max procedures are essential.

Definition 2.6. Let $I : X \to \mathbb{R}$ be a continuous even functional and consider $a, b \in \bar{\mathbb{R}}$ such that $a \leq b$. The pair (I^b, I^a) is *odd–trivial* if for each neighbourhood $[\alpha', \alpha'']$ of a and $[\beta', \beta'']$ of b in $\bar{\mathbb{R}}$, an odd continuous map $\varphi : I^{\beta'} \times [0, 1] \to I^{\beta''}$ exists such that

(i) $\varphi(x, 0) = x$ for all $x \in I^{\beta'}$;
(ii) $\varphi(I^{\beta'} \times \{1\}) \subseteq I^{\alpha''}$;
(iii) $\varphi(I^{\alpha'} \times [0, 1]) \subseteq I^{\alpha''}$.

Definition 2.7. Let $I : X \to \mathbb{R}$ be even and continuous. A number $c \in \mathbb{R}$ is an *odd–essential value* of I if for each $\varepsilon > 0$ some values $a, b \in]c - \varepsilon, c + \varepsilon[$, $a < b$, exist such that the pair (I^b, I^a) is not odd–trivial.

The following theorem states that small perturbations of a continuous even functional preserve the odd–essential values (for the proof, see [14, Theorem 3.1] or also [13, Theorem 2.6]).

Theorem 2.8. *Let $c \in \mathbb{R}$ be an odd–essential value of a continuous even functional $I : X \to \mathbb{R}$. Then, for every $\eta > 0$ a radius $\delta > 0$ exists such that every even functional $G \in C(X, \mathbb{R})$ with*

$$\sup\{|I(u) - G(u)| : u \in X\} \ < \ \delta$$

admits an odd–essential value in $]c - \eta, c + \eta[$.

Now, we focus on smooth functionals and recall some results which link critical and essential values, stating in particular that the critical values arising from mini–max procedures are essential if all the involved deformations are of the "same kind" (cf. [14, Theorems 3.7 and 3.9]).

Theorem 2.9. *Let $I : X \to \mathbb{R}$ be a C^1 even functional. If $c \in \mathbb{R}$ is an odd–essential value of I such that $(PS)_c$ holds, then c is a critical value of I in X.*

Remark 2.10. In general, the reverse implication does not hold. In fact, a critical value c is not necessarily an essential one even when $(PS)_c$ is satisfied (cf., e.g., [14, Example 3.12]).

Theorem 2.11. *Consider Γ non empty family of non empty symmetric subsets of X, $I \in C^1(X, \mathbb{R})$ even functional and $d \in \mathbb{R} \cup \{-\infty\}$. Assume that*

$$\overline{\varphi(C \times \{1\})} \in \Gamma$$

for every $C \in \Gamma$ and for every odd deformation $\varphi : X \times [0, 1] \longrightarrow X$ with $\varphi(u, t) = u$ on $I^d \times [0, 1]$.
Then, setting

$$c \ = \ \inf_{C \in \Gamma} \sup_{u \in C} I(u),$$

if $d < c < +\infty$ the level c is an odd–essential value of I in X.

Remark 2.12. Theorem 2.11 implies that the critical levels found in Theorem 2.4 are also odd–essential values of I.
Indeed, taking $\Gamma = \Sigma_i$, with $i \in \{1, \dots, \tilde{k}\}$, as defined in Theorem 2.4, and assuming $d = 0$, for any odd homeomorphism $\varphi : X \times [0, 1] \to X$ such that $\varphi(u, t) = u$ if $I(u) \leq 0$, we have that for each $C \in \Sigma_i$ the set $\overline{\varphi(C \times \{1\})}$ is close and symmetric. Moreover, from the supervariancy property of γ^*, we have that

$$\gamma^* \left(\overline{\varphi(C \times \{1\})} \right) = \gamma^* \left(\varphi \left(\overline{C \times \{1\}} \right) \right) = \gamma^* \left(\overline{C \times \{1\}} \right) = \gamma^*(C) \geq i;$$

hence, $\overline{\varphi(C \times \{1\})} \in \Sigma_i$ and Theorem 2.11 applies.

3. The unperturbed problem

In order to investigate the existence of solutions of the perturbed problem $(P_{\varepsilon,\infty})$, the starting point is studying the unperturbed problem $(P_{0,\infty})$.

To this aim, firstly we recognize the variational structure of the equation in $(P_{0,\infty})$. In fact, if assumption (f_1) is satisfied, we can define the C^1 real function

$$F(x, t) \ = \ \int_0^t f(x, s)\, \mathrm{d}s \quad \text{for all } (x, t) \in \overline{\Omega} \times \mathbb{R}$$

which is so that $F(x,0) = 0$ for all $x \in \overline{\Omega}$. Then, (f_1) and (f_2) imply that taking any $\sigma > 0$ a constant $C_\sigma > 0$ exists such that

$$|f(x,t)| \le \sigma |t|^{q-1} + C_\sigma \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R} \tag{3.1}$$

and so

$$|F(x,t)| \le \frac{\sigma}{q}|t|^q + C_\sigma |t| \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.2}$$

Hence, from a classical variational principle we have that the weak solutions of problem $(P_{0,\infty})$ are the critical points of the C^1–functional $J : W_0^{1,q}(\Omega) \to \mathbb{R}$ defined as

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{q}\int_\Omega |\nabla u|^q \, dx - \frac{\lambda_\infty}{q}\int_\Omega |u|^q \, dx - \int_\Omega F(x,u) \, dx \tag{3.3}$$

with

$$\begin{aligned} \langle dJ(u), v\rangle = & \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla v \, dx \\ & - \lambda_\infty \int_\Omega |u|^{q-2} u \, v \, dx - \int_\Omega f(x,u) v \, dx \end{aligned}$$

for all u, $v \in W_0^{1,q}(\Omega)$ (see, e.g., [16, Theorem 9 and p. 355]).

Now, we are ready to state that the functional J satisfies the (PS) condition (for the proof, see [6, Proposition 3.1]).

Proposition 3.1. *Assume that* (f_1)–(f_2) *hold and* $\lambda_\infty \notin \sigma(-\Delta_q)$. *Then, the functional* J *in* (3.3) *satisfies* (PS) *in* $\mathbb{R}$.

We note that such a statement does not need the behaviour of $f(x,t)$ near the origin, so it holds independently of the chosen hypothesis (f_3) or $(f_3)'$ or $(f_3)''$. On the other hand, such an information will be crucial in order to obtain the geometric assumptions required in Theorem 2.4 and then in the following multiplicity result which is stated in [12, Theorem 1.1].

Theorem 3.2. *Assume that* $f(x,t)$ *satisfies hypotheses* (f_1)–(f_4) *and that two integers* n, $k \in \mathbb{N}$ *exist such that* $k \ge n$ *and either* (i) *or* (ii) *in Theorem 1.1 holds. Then, if* $\lambda_\infty \notin \sigma(-\Delta_q)$, *problem* $(P_{0,\infty})$ *admits at least* $k - n + 1$ *distinct pairs of nontrivial solutions.*

Proof. The proof is essentially in [12] but, neverthless, we give it here with slight different details since we need some of its steps later on.
From assumption (f_4) and Proposition 3.1 it follows that the functional J in (3.3) is even and satisfies (PS) in $\mathbb{R}$. Moreover, we have that $J(0) = 0$, so critical levels different from zero give nontrivial solutions.
First of all, we note that taking every $\sigma > 0$ and $s > 0$ hypotheses (f_1)–(f_3) and [6, Lemma 3.2] imply the existence of $k_\sigma > 0$, $k_\sigma = k_\sigma(s)$, such that

$$-k_\sigma |t|^{s+q} + \frac{\lambda_0 - \sigma}{q}|t|^q \le F(x,t) \le \frac{\lambda_0 + \sigma}{q}|t|^q + k_\sigma |t|^{s+q} \tag{3.4}$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$.
Now, we have to prove our statement in two different ways according to assuming hypothesis (i) or (ii).
Firstly, let condition (i) be satisfied and fix $\sigma_0 > 0$ such that

$$\lambda_0 + \lambda_\infty + \sigma_0 < \eta_n^0 \le \nu_k^0 < \nu_k^0 + 2\sigma_0 < \lambda_\infty \tag{3.5}$$

(see Remark 1.3).

Step 1 Let η_n^0 be as in (i) and consider Z_{n-1}^0 so that both (2.2) is true with $m = n-1$ and (2.3) holds with $m = n$. We claim that two constants $\rho = \rho(\sigma_0) > 0$ and $c_0 = c_0(\sigma_0) > 0$ can be found so that

$$J(u) \geq c_0 \quad \text{for all } u \in S_\rho \cap Z_{n-1}^0. \tag{3.6}$$

Indeed, taking any $s \in [0, q^* - q[$ from (3.4) with $\sigma = \sigma_0$ it follows that

$$\int_\Omega F(x,u)\,\mathrm{d}x \leq \frac{\lambda_0 + \sigma_0}{q}|u|_q^q + k_{\sigma_0}|u|_{s+q}^{s+q} \quad \text{for all } u \in W_0^{1,q}(\Omega)$$

and then, from the Sobolev inequalities, we obtain that

$$J(u) \geq \frac{1}{q}\|u\|_q^q - \frac{\lambda_0 + \lambda_\infty + \sigma_0}{q}|u|_q^q - C\|u\|_q^{s+q} \quad \text{for all } u \in W_0^{1,q}(\Omega). \tag{3.7}$$

Thus, (2.3) with $m = n$ and (3.7) imply that

$$J(u) \geq \frac{1}{q}\left(1 - \frac{\lambda_0 + \lambda_\infty + \sigma_0}{\eta_n^0}\right)\|u\|_q^q - C\|u\|_q^{s+q} \quad \text{for all } u \in Z_{n-1}^0.$$

Hence, from (3.5) a suitable couple of real numbers $\rho(\sigma_0) > 0$ and $c_0(\sigma_0) > 0$ exist such that (3.6) holds.

Step 2 Let ν_k^0 be as in (i), σ_0 as in (3.5) and consider $\mathbb{W}_k^0$ as in (2.4) with $m = k$. We claim that a k–dimensional space $Y_k^{\sigma_0} \in \mathbb{W}_k^0$ and a constant $c_\infty > c_0$ exist such that

$$J(u) \leq c_\infty \quad \text{for all } u \in Y_k^{\sigma_0}. \tag{3.8}$$

Indeed, (3.2) and direct computations imply that a constant $C > 0$ exists such that

$$J(u) \leq \frac{1}{p}|\nabla u|_p^p + \frac{1}{q}|\nabla u|_q^q - \frac{\lambda_\infty}{q}|u|_q^q + \frac{\sigma_0}{q}|u|_q^q + C|u|_q \qquad \text{for all } u \in W_0^{1,q}(\Omega).$$

Thus, from definition (2.5) with $m = k$ a subspace $Y_k^{\sigma_0} \in \mathbb{W}_k^0$ exists, which can be chosen with $\dim Y_k^{\sigma_0} = k$, and is such that

$$\|u\|_q^q \;\leq\; (\nu_k^0 + \sigma_0)|u|_q^q \qquad \text{for all } u \in Y_k^{\sigma_0}. \tag{3.9}$$

Notice that, as $p < q$ and Ω bounded imply that $|\nabla u|_p^p \leq C|\nabla u|_q^p$, then from (3.9) and direct computations we obtain that

$$J(u) \leq \frac{1}{q}\left(\nu_k^0 + 2\sigma_0 - \lambda_\infty\right)|u|_q^q + C_1|u|_q^p + C_2|u|_q \qquad \text{for all } u \in Y_k^{\sigma_0}$$

for suitable C_1, $C_2 > 0$. Thus, from $q > p$ and (3.5) it follows that $J(u) \to -\infty$ if $\|u\|_q \to +\infty$ with $u \in Y_k^\sigma$. Hence, $c_\infty = c_\infty(\sigma)$ exists, with $c_\infty > c_0$ and c_0 as in *Step 1*, such that (3.8) holds.

Step 3 Taking Z_{n-1}^0, ρ, c_0 as in (3.6) and $Y_k^{\sigma_0}$, c_∞ as in (3.8), we can consider the pseudo–index theory $(S_\rho \cap Z_{n-1}^0, \mathcal{H}^*, \gamma^*)$ related to the genus, $S_\rho \cap Z_{n-1}^0$ and J in $W_0^{1,q}(\Omega)$. Then, from Remark 2.5 applied to $V = Y_k^{\sigma_0}$, $W = Z_{n-1}^0$ and $\partial B = S_\rho$, we get

$$\gamma\left(Y_k^{\sigma_0} \cap h\left(S_\rho \cap Z_{n-1}^0\right)\right) \geq \dim Y_k^{\sigma_0} - \text{ codim } Z_{n-1}^0 \qquad \text{for all } h \in \mathcal{H}^*,$$

which implies

$$\gamma^*(Y_k^{\sigma_0}) \geq k - n + 1.$$

Hence, if we apply Theorem 2.4 to J in $W_0^{1,q}(\Omega)$ with $S = S_\rho \cap Z_{n-1}^0$ and $\tilde{A} = Y_k^{\sigma_0}$, it follows that J has at least $k-n+1$ distinct pairs of critical points corresponding to at most $k-n+1$ distinct critical values c_i, where c_i is as in (2.19). We note

that $c_i \geq c_0 > 0$ for all $i \in \{1, \dots, k-n+1\}$; hence all the corresponding critical points are nontrivial.
Now, assume that (ii) holds and fix $\sigma_1 > 0$ such that

$$\lambda_\infty + \sigma_1 < \eta_n^0 < \frac{q}{p}(\nu_k + 2\sigma_1) < \lambda_\infty + \lambda_0 \tag{3.10}$$

(see Remark 1.3).
Step 1 Fixing any $s \in [0, q^* - q[$ from (3.4) with $\sigma = \frac{q}{p}\sigma_1$ and the Sobolev Embedding Theorem it follows that

$$J(u) \leq \frac{1}{p}(|\nabla u|_p^p + |\nabla u|_q^q) - \frac{\lambda_0 + \lambda_\infty - \frac{q}{p}\sigma_1}{q}|u|_q^q + C\|u\|_q^{s+q} \tag{3.11}$$

for all $u \in W_0^{1,q}(\Omega)$. Then, taking ν_k as in (ii) so that (2.11) applies with $m = k$, from definition (2.10) with $m = k$ a subspace $Y_k^{\sigma_1} \in \mathbb{W}_k$ exists with $\dim Y_k^{\sigma_1} = k$ and such that

$$|\nabla u|_p^p + |\nabla u|_q^q \leq (\nu_k + \sigma_1)\,|u|_q^q \qquad \text{for all } u \in Y_k^{\sigma_1}. \tag{3.12}$$

Then, from (3.10) – (3.12) and direct computations, taking any $u \in Y_k^{\sigma_1}$ we have that

$$\begin{aligned} -J(u) &\geq \frac{1}{q(\nu_k + \sigma_1)}\left(\lambda_0 + \lambda_\infty - \frac{q}{p}(\nu_k + 2\sigma_1)\right)(|\nabla u|_p^p + |\nabla u|_q^q) - C\|u\|_q^{s+q} \\ &\geq \frac{1}{q(\nu_k + \sigma_1)}\left(\lambda_0 + \lambda_\infty - \frac{q}{p}(\nu_k + 2\sigma_1)\right)\|u\|_q^q - C\|u\|_q^{s+q}. \end{aligned}$$

Hence, again from (3.10), two constants $\rho = \rho(\sigma_1) > 0$ and $c_0 = c_0(\sigma_1) > 0$ exist such that

$$-J(u) \geq c_0 \quad \text{ for all } u \in S_\rho \cap Y_k^{\sigma_1}. \tag{3.13}$$

Step 2 Now, taking η_n^0 as in (ii) so that (2.3) holds with $m = n$, from (3.2) with $\sigma = \sigma_1$, Sobolev Embedding Theorem and direct computations we prove that

$$-J(u) \leq -\frac{1}{q}\left(1 - \frac{\lambda_\infty + \sigma_1}{\eta_n^0}\right)\|u\|_q^q + C\|u\|_q \quad \text{for all } u \in Z_{n-1}^0.$$

Hence, from (3.10) we get that

$$-J(u) \to -\infty \text{ as } \|u\|_q \to +\infty \quad \text{if } u \in Z_{n-1}^0,$$

which implies that a constant $c_\infty > c_0$ exists, with c_0 as in the previous step, such that

$$-J(u) \leq c_\infty \quad \text{ for all } u \in Z_{n-1}^0. \tag{3.14}$$

Step 3 From estimates (3.13) and (3.14) we are able to apply Theorem 2.4 to functional $-J$ with $\tilde{A} = Z_{n-1}^0$, $S = S_\rho \cap Y_k^{\sigma_1}$ and essentially making use of the same remarks in *Step 3* of case (i). □ □

As pointed out in Section 1, we can also consider the case in which the limit λ_0 in (f_3) is infinite, thus the following results can be stated for the unperturbed problem $(P_{0,\infty})$.

Proposition 3.3. *Assume that $f(x,t)$ satisfies hypotheses (f_1)–(f_2), $(f_3)'$ and (f_4). Moreover, $\lambda_\infty \notin \sigma(-\Delta_q)$ and $k \in \mathbb{N}$ exists such that $(i)'$ holds. Then, problem $(P_{0,\infty})$ has at least k distinct pairs of nontrivial solutions.*

Proof. Thanks to (2.6), we have that $\eta_1^0 \leq \nu_k^0$ and, taking a "good" $\lambda_0 \in \mathbb{R}$ so that $\lambda_0 + \lambda_\infty < \eta_1^0$, and $\sigma_0 > 0$ so that (3.5) holds with $n = 1$, according to assumption $(f_3)'$ we can obtain the right estimate of (3.4) so that, by reasoning as in the proof of Theorem 3.2, case (i), Theorem 2.4 applies with $\tilde{A} = Y_k^{\sigma_0}$, $S = S_\rho$ for a suitable $\rho > 0$. Hence, J has at least k distinct pairs of nontrivial critical points. □ □

Proposition 3.4. *Assume that $f(x,t)$ satisfies hypotheses (f_1)–(f_2), $(f_3)''$ and (f_4). If $\lambda_\infty \notin \sigma(-\Delta_q)$ then problem $(P_{0,\infty})$ has infinitely many pairs of nontrivial solutions.*

Proof. Since $\eta_m^0 \nearrow +\infty$ an integer n exists so that $\eta_n^0 > \lambda_\infty$. Then, taking any $k \geq n$ and a large enough $\lambda_0 > 0$ so that $\frac{q}{p}\nu_k < \lambda_0 + \lambda_\infty$, for a constant $\sigma_1 > 0$ which satisfies (3.10) and according to assumption $(f_3)''$, we can obtain the left estimate of (3.4) so that the arguments in the proof of Theorem 1.1, case (ii) can be repeated. Then, Theorem 2.4 applies to $-J$ with $\tilde{A} = Z_{n-1}^0$, $S = S_\rho \cap Y_k^{\sigma_1}$ for a suitable $\rho > 0$ and $k - n + 1$ distinct pairs of nontrivial critical points exist. The end of the proof follows from the arbitrariness of $k \geq n$. □ □

4. Regularity of solutions of (p,q)–Laplacian equations

In this section we prove some regularity results for (p,q)–Laplacian problems. Classical results concerning the regularity of solutions of quasilinear elliptic equations can be found in [15, 21, 26, 37], while a result for (p,q)–Laplacian problems but on $\mathbb{R}^N$ is in [22]. Our proofs follow essentially those ones stated in [24, Section 4] for the q–Laplacian operator, which generalize [20, Theorem 8.15] concerning the Laplace operator.

Lemma 4.1. *Let $l \in C(\overline{\Omega} \times \mathbb{R})$ be such that*

$$|l(x,t)| \leq C(1 + |t|^r) \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R} \tag{4.1}$$

with $1 < r < q^* - 1$. Then, if $u \in W_0^{1,q}(\Omega)$ is a weak solution of

$$(P) \qquad \begin{cases} -\Delta_p u - \Delta_q u \;=\; l(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

it results that $u \in L^\gamma(\Omega)$ for all $\gamma \geq 1$.

Proof. The proof follows from Sobolev Embedding Theorem if $q \geq N$. Hence, we have to prove the statement if $1 < q < N$.
Let $u \in W_0^{1,q}(\Omega)$ be a solution of (P) and we note that, taking $s > 0$ and $\alpha > 1$, it results

$$\nabla(|u|^s u) = (s+1)|u|^s \nabla u, \qquad |u|^s |\nabla u|^\alpha = \frac{\alpha^\alpha}{(\alpha+s)^\alpha} |\nabla(|u|^{\frac{s}{\alpha}} u)|^\alpha.$$

So, first we multiply the equation in (P) by $|u|^s u$, then we integrate it on Ω and obtain that

$$\begin{aligned} &(s+1)\frac{p^p}{(p+s)^p}|\nabla(|u|^{\frac{s}{p}}u)|_p^p + (s+1)\frac{q^q}{(q+s)^q}|\nabla(|u|^{\frac{s}{q}}u)|_q^q \\ &\quad = \int_\Omega l(x,u)|u|^s u \,\mathrm{d}x \end{aligned}$$

which, together with (4.1), implies that

$$|\nabla(|u|^{\frac{s}{q}}u)|_q^q \leq \frac{(s+q)^q}{(s+1)q^q} \int_\Omega l(x,u)|u|^s u \,\mathrm{d}x \leq C\left(|u|_{r+s+1}^{r+s+1} + 1\right). \tag{4.2}$$

Thus, from Sobolev inequality it follows that

$$\left| |u|^{\frac{s}{q}} u \right|_{q^*}^{q} \leq C \left(|u|_{r+s+1}^{r+s+1} + 1 \right)$$

where it results $(\frac{s}{q}+1)q^* = \frac{N(s+q)}{N-q}$. So, we have that

$$|u|_{\frac{N(s+q)}{N-q}}^{s+q} \leq C \left(|u|_{r+s+1}^{r+s+1} + 1 \right)$$

which implies the following statement:

$$u \in L^{r+s+1}(\Omega) \quad \Rightarrow \quad u \in L^{\frac{N(s+q)}{N-q}}(\Omega). \tag{4.3}$$

So, since $u \in W_0^{1,q}(\Omega)$ implies $u \in L^{q^*}(\Omega)$, if we set $s_0 = q^* - (r+1)$ then from (4.3) we get $u \in L^{q_1}(\Omega)$, with $q_1 = \frac{N(s_0+q)}{N-q}$. From now on, as in [24, Lemma 4.5] by iteration we are able to complete the proof. □ □

Lemma 4.2. *Let $\gamma > N$ be such that $l \in L^{\frac{\gamma}{q}}(\Omega)$. Then, if $u \in W_0^{1,q}(\Omega)$ is a weak solution of*

$$(P_1) \qquad \begin{cases} -\Delta_p u - \Delta_q u \ = \ l(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

it results that $u \in L^\infty(\Omega)$ and a constant $C = C(\gamma, q, N, \Omega) > 0$ exists such that $|u|_\infty \leq C |l|_{\frac{\gamma}{q}}^{\frac{1}{q-1}}$.

Proof. The proof is trivial for $q > N$; hence, let us consider $1 < q < N$ (the case $q = N$ is similar).

For simplicity, let us set $k = |l|_{\frac{\gamma}{q}}^{\frac{1}{q-1}}$ and, for $\beta \geq 1$, $M > k$ we define

$$H(z) = \begin{cases} z^\beta - k^\beta & \text{if } k \leq z \leq M, \\ \beta M^{\beta-1}(z-M) + M^\beta - k^\beta & \text{if } z > M. \end{cases} \tag{4.4}$$

Moreover, setting $w = u^+ + k$, with $u^+ = \max\{u, 0\}$, we have that

$$\nabla w = \begin{cases} \nabla u & \text{if } u \geq 0, \\ 0 & \text{if } u < 0, \end{cases}$$

and, by using

$$v = G(w) = \int_k^w |H'(z)|^q \, dz$$

as test function in the equation in (P_1), we obtain that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla G(w) \, dx + \int_\Omega |\nabla u|^{q-2} \nabla u \cdot \nabla G(w) \, dx = \int_\Omega l(x) G(w) \, dx.$$

Since

$$\nabla G(w) = G'(w) \nabla w = \begin{cases} G'(w) \nabla u & \text{if } u \geq 0, \\ 0 & \text{if } u < 0, \end{cases}$$

then, taking $\Omega^+ = \{x \in \Omega : u(x) \geq 0\}$, we get

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla G(w) \, dx = \int_{\Omega^+} |\nabla u|^p |H'(w)|^q \, dx \geq 0$$

and

$$\int_\Omega |\nabla u|^{q-2} \nabla u \cdot \nabla G(w) \, dx = \int_{\Omega^+} |\nabla u|^q |H'(w)|^q \, dx = \int_\Omega |\nabla H(w)|^q \, dx$$

which imply

$$\int_\Omega |\nabla H(w)|^q \,\mathrm{d}x \le \int_\Omega l(x)G(w)\,\mathrm{d}x.$$

So, since from definition (4.4) we get $|H'(z)| \le |H'(w)|$ for all $k \le z \le w$ and then $0 \le G(w) \le wG'(w)$, we obtain

$$\int_\Omega |\nabla H(w)|^q \,\mathrm{d}x \le \int_\Omega |l(x)|wG'(w)\,\mathrm{d}x.$$

From now on, the proof follows word by word from that in [24, Lemma 4.6]. □ □

5. The perturbed problem

Now, we are ready to prove Theorem 1.1, that is the critical points of the unperturbed problem $(P_{0,\infty})$ allow us to find solutions of the perturbed problem $(P_{\varepsilon,\infty})$ if ε is small enough.

of Theorem 1.1. For simplicity, our proof is divided in some steps; more precisely, we will provide:

1. the construction of truncated perturbed functionals $J_{j,\varepsilon}$ and the existence of some critical points $u_i^{j,\varepsilon} \in W_0^{1,q}(\Omega)$;
2. the boundedness of the $W_0^{1,q}$-norm of $u_i^{j,\varepsilon}$;
3. the boundedness of the L^∞-norm of $u_i^{j,\varepsilon}$;
4. the existence of weak solutions of the perturbed problem $(P_{\varepsilon,\infty})$.

Step 1. As in [23], for any $j \in \mathbb{N}$ we consider a continuous cut function $\gamma_j : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma_j(t) = \begin{cases} 0 & \text{if } |t| \ge j+1 \\ 1 & \text{if } |t| \le j \end{cases}$$

with $0 < \gamma_j(t) < 1$ if $j < |t| < j+1$. Then, we set

$$h_j(x,t) = \gamma_j(t)h(x,t) \qquad \text{and} \qquad H_j(x,t) = \int_0^t h_j(x,s)\,\mathrm{d}s,$$

and note that a constant $\varepsilon_1(j) > 0$ exists such that it results

$$\varepsilon_1(j)|h_j(x,t)| < 1, \qquad \varepsilon_1(j)|H_j(x,t)| < 1 \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{5.1}$$

Now, taking any $j \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$, we can define the (truncated) perturbed functional

$$J_{j,\varepsilon} : u \in W_0^{1,q}(\Omega) \;\mapsto\; J_{j,\varepsilon}(u) = J(u) - \varepsilon \int_\Omega H_j(x,u)\,\mathrm{d}x \in \mathbb{R}.$$

Firstly, we assume that condition (i) of Theorem 1.1 holds. Then, from Theorem 3.2 at least $\bar{m}$ distinct critical levels c_i of the even functional J exist with

$$1 \le \bar{m} \le k - n + 1$$

and

$$0 < c_0 \le c_{i_1} < \ldots < c_{i_{\bar{m}}} \le c_\infty < +\infty,$$

where c_0 and c_∞ are as in the proof of Theorem 3.2, case (i).
From Remark 2.12 such critical levels are also odd–essential ones for J in $W_0^{1,q}(\Omega)$ and from Theorem 2.8 there exists $\varepsilon_2(j) \in]0, \varepsilon_1(j)[$ such that, if $|\varepsilon| \le \varepsilon_2(j)$, then $J_{j,\varepsilon}$ has at least $\bar{m}$ odd essential values $d_i^{j,\varepsilon}$, $i \in \{1, \ldots, \bar{m}\}$, such that

$$\frac{c_0}{2} < d_1^{j,\varepsilon} < \ldots < d_{\bar{m}}^{j,\varepsilon} < c_\infty + 1.$$

Moreover, since for any $j \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ the nonlinear term $f(x,t) + \varepsilon h_j(x,t)$ satisfies assumptions (f_1) and (f_2), as in Proposition 3.1 we obtain that $J_{j,\varepsilon}$ satisfies the (PS) condition in $\mathbb{R}$.
Hence, taking $j \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ such that $|\varepsilon| \le \varepsilon_2(j)$, from Theorem 2.9 it follows that for all $i \in \{1,\dots,\bar{m}\}$ the corresponding level $d_i^{j,\varepsilon}$ is also critical for $J_{j,\varepsilon}$, so $u_i^{j,\varepsilon} \in W_0^{1,q}(\Omega)$ exists such that

$$\begin{aligned}\int_\Omega |\nabla u_i^{j,\varepsilon}|^{p-2}\nabla u_i^{j,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega |\nabla u_i^{j,\varepsilon}|^{q-2}\nabla u_i^{j,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x \\ = \lambda_\infty \int_\Omega |u_i^{j,\varepsilon}|^{q-2}u_i^{j,\varepsilon}\,\varphi\,\mathrm{d}x + \int_\Omega f(x,u_i^{j,\varepsilon})\varphi\,\mathrm{d}x + \varepsilon\int_\Omega h_j(x,u_i^{j,\varepsilon})\varphi\,\mathrm{d}x\end{aligned} \tag{5.2}$$

for all $\varphi \in W_0^{1,q}(\Omega)$, i.e., $u_i^{j,\varepsilon} \in W_0^{1,q}(\Omega)$ is so that

$$-\Delta_p u_i^{j,\varepsilon} - \Delta_q u_i^{j,\varepsilon} \;=\; l_{j,\varepsilon}(x,u_i^{j,\varepsilon}) \quad \text{in } \Omega \tag{5.3}$$

as weak solution, with

$$l_{j,\varepsilon}(x,t) = \lambda_\infty|t|^{q-2}t + f(x,t) + \varepsilon h_j(x,t) \quad \text{for all } (x,t)\in\overline{\Omega}\times\mathbb{R}. \tag{5.4}$$

Step 2. We claim that a constant $C_q > 0$ exists such that

$$\|u_i^{j,\varepsilon}\|_q \le C_q \quad \text{for all } j\in\mathbb{N}, |\varepsilon|\le\varepsilon_2(j), i\in\{1,\dots,\bar{m}\}. \tag{5.5}$$

Indeed, arguing by contradiction, let us assume that for some $i \in \{1,\dots,\bar{m}\}$ the set

$$A_i = \{\|u_i^{j,\varepsilon}\|_q : j\in\mathbb{N}, |\varepsilon|\le\varepsilon_2(j)\}$$

is unbounded, that is, a sequence $(u^{j_m,\varepsilon_m})_m \subset W^{1,q}(\Omega)$ exists, with $u^{j_m,\varepsilon_m} := u_i^{j_m,\varepsilon_m}$ and $|\varepsilon_m| \le \varepsilon_2(j_m)$ for all $m\in\mathbb{N}$, such that

$$\|u^{j_m,\varepsilon_m}\|_q \to +\infty \quad \text{as } m\to+\infty. \tag{5.6}$$

Setting

$$w_{j_m,\varepsilon_m} \;=\; \frac{u^{j_m,\varepsilon_m}}{\|u^{j_m,\varepsilon_m}\|_q}, \tag{5.7}$$

we have that $(w_{j_m,\varepsilon_m})_m$ is bounded in $W_0^{1,q}(\Omega)$, so $w\in W_0^{1,q}(\Omega)$ exists such that, up to subsequences, it results

$$w_{j_m,\varepsilon_m} \rightharpoonup w \quad \text{weakly in } W_0^{1,q}(\Omega) \tag{5.8}$$

and

$$w_{j_m,\varepsilon_m} \to w \quad \text{strongly in } L^q(\Omega). \tag{5.9}$$

Now, taking

$$\varphi_{j_m,\varepsilon_m} \;=\; \frac{w_{j_m,\varepsilon_m} - w}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}$$

in (5.2) with $j = j_m$ and $\varepsilon = \varepsilon_m$, we obtain that

$$\begin{aligned}
&\int_\Omega \frac{|\nabla w_{j_m,\varepsilon_m}|^{p-2}}{\|u^{j_m,\varepsilon_m}\|_q^{q-p}} \nabla w_{j_m,\varepsilon_m} \cdot \nabla(w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x \\
&\quad + \int_\Omega |\nabla w_{j_m,\varepsilon_m}|^{q-2} \nabla w_{j_m,\varepsilon_m} \cdot \nabla(w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x \\
&= \lambda_\infty \int_\Omega |w_{j_m,\varepsilon_m}|^{q-2} w_{j_m,\varepsilon_m}\ (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x \\
&\quad + \int_\Omega \frac{f(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon}\|_q^{q-1}} (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x \\
&\quad + \varepsilon_m \int_\Omega \frac{h_j(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}} (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x.
\end{aligned}$$

Then, from (5.1), (5.6) and (5.9) we have that

$$\varepsilon_m \int_\Omega \frac{h_j(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}} (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m,$$

and, by means also of Hölder inequality and (3.1), direct computations give

$$\begin{aligned}
&\int_\Omega \frac{|\nabla w_{j_m,\varepsilon_m}|^{p-2}}{\|u^{j_m,\varepsilon_m}\|_q^{q-p}} \nabla w_{j_m,\varepsilon_m} \cdot \nabla(w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m, \\
&\int_\Omega |w_{j_m,\varepsilon_m}|^{q-2} w_{j_m,\varepsilon_m}\ (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m, \\
&\int_\Omega \frac{f(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon}\|_q^{q-1}} (w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m,
\end{aligned}$$

which imply that

$$\int_\Omega |\nabla w_{j_m,\varepsilon_m}|^{q-2} \nabla w_{j_m,\varepsilon_m} \cdot \nabla(w_{j_m,\varepsilon_m} - w)\, \mathrm{d}x = \beta_m.$$

Hence, from this last limit and (5.8) it follows that

$$w_{j_m,\varepsilon_m} \ \to \ w \quad \text{strongly in } W_0^{1,q}(\Omega), \tag{5.10}$$

which gives also $w \neq 0$ for definition (5.7).
Finally, taking any $\varphi \in W_0^{1,q}(\Omega)$ and applying again (5.2), with $j = j_m$ and $\varepsilon = \varepsilon_m$, on the test function $\frac{\varphi}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}$, we obtain that

$$\begin{aligned}
&\int_\Omega \frac{|\nabla w_{j_m,\varepsilon_m}|^{p-2}}{\|u^{j_m,\varepsilon_m}\|_q^{q-p}} \nabla w_{j_m,\varepsilon_m} \cdot \nabla\varphi\, \mathrm{d}x + \int_\Omega |\nabla w_{j_m,\varepsilon_m}|^{q-2} \nabla w_{j_m,\varepsilon_m} \cdot \nabla\varphi\, \mathrm{d}x \\
&= \lambda_\infty \int_\Omega |w_{j_m,\varepsilon_m}|^{q-2} w_{j_m,\varepsilon_m}\ \varphi\, \mathrm{d}x \ + \ \int_\Omega \frac{f(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}\ \varphi\, \mathrm{d}x \\
&\quad + \varepsilon_m \int_\Omega \frac{h_j(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}\ \varphi\, \mathrm{d}x.
\end{aligned} \tag{5.11}$$

Thus, from (5.1) and (5.6) we have that

$$\left| \varepsilon_m \int_\Omega \frac{h_j(x, u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}} \varphi\, \mathrm{d}x \right| \ \le \ \beta_m \|\varphi\|_q,$$

and from Hölder inequality, (5.6) and (5.7) we obtain that

$$\left|\int_\Omega \frac{|\nabla w_{j_m,\varepsilon_m}|^{p-2}}{\|u^{j_m,\varepsilon_m}\|_q^{q-p}}\nabla w_{j_m,\varepsilon_m}\cdot\nabla\varphi\,\mathrm{d}x\right| \;\le\; \beta_m\|\varphi\|_q.$$

On the other hand, reasoning as in the proof of [6, Proposition 3.1], by means of (5.6) we are able to prove also that

$$\lim_{m\to+\infty}\int_\Omega \frac{f(x,u^{j_m,\varepsilon_m})}{\|u^{j_m,\varepsilon_m}\|_q^{q-1}}\;\varphi\,\mathrm{d}x = 0;$$

hence, from (5.9), (5.10) and passing to the limit in (5.11), for the arbitrariness of φ we get that $\lambda_\infty \in \sigma(-\Delta_q)$, against our assumption. Thus, (5.5) has to hold.

Step 3. We claim that $C_\infty > 0$ exists such that

$$|u_i^{j,\varepsilon}|_\infty \le C_\infty \quad \text{for all } j\in\mathbb{N}, |\varepsilon|\le\varepsilon_2(j), i\in\{1,\dots,\bar m\}. \tag{5.12}$$

To this aim, first of all we note that from (5.5) such a claim is obvious if $q > N$. Thus, we have to prove (5.12) if $1 < q < N$ (analogous arguments still hold if $q = N$) and our idea is applying Lemma 4.2 to equation (5.3), taking $l(x) = l_{j,\varepsilon}(x, u_i^{j,\varepsilon}(x))$ so that for a "good" $\gamma > N$ it has to be $l \in L^{\frac{\gamma}{q}}(\Omega)$ and an estimate of such a function in $L^{\frac{\gamma}{q}}(\Omega)$ can be given uniformly with respect to $j\in\mathbb{N}$, ε so that $|\varepsilon|\le\varepsilon_2(j)$, and $i\in\{1,\dots,\bar m\}$.

Indeed, taking any $j\in\mathbb{N}$, ε so that $|\varepsilon|\le\varepsilon_2(j)$, $i\in\{1,\dots,\bar m\}$, from (3.1) with, e.g., $\sigma = 1$, and the estimates in (5.1), we have that definition (5.4) implies

$$|l_{j,\varepsilon}(x,t)| \le C(1+|t|^{q-1}) \quad \text{for all } (x,t)\in\overline{\Omega}\times\mathbb{R} \tag{5.13}$$

with $C = C(\lambda_\infty)$ but independent of j and ε as $\varepsilon_2(j)\le\varepsilon_1(j)$. Then, Lemma 4.1 applies to (5.3) and

$$u_i^{j,\varepsilon}\in L^\gamma(\Omega) \quad \text{for all } \gamma\ge 1, \tag{5.14}$$

or better, by reasoning as in the proof of Lemma 4.1, in our setting with $r = q-1$ and $s = N$, estimate (4.2) becomes

$$|\nabla(|u_i^{j,\varepsilon}|^{\frac{N}{q}}u_i^{j,\varepsilon})|_q^q \le C\left(|u_i^{j,\varepsilon}|_{q+N}^{q+N}+1\right) \tag{5.15}$$

with C independent of j,ε,i. Then, since (5.14) gives $u_i^{j,\varepsilon}\in L^{q+N}(\Omega)$, from (5.15) it follows that $|u_i^{j,\varepsilon}|^{\frac{N}{q}}u_i^{j,\varepsilon}\in W_0^{1,q}(\Omega)$ and then, from (5.15) and the Sobolev inequality, we have that

$$|u_i^{j,\varepsilon}|_{(q+N)\frac{q^*}{q}}^{q+N} = \left||u_i^{j,\varepsilon}|^{\frac{N}{q}}u_i^{j,\varepsilon}\right|_{q^*}^q \le C\left(|u_i^{j,\varepsilon}|_{q+N}^{q+N}+1\right) \tag{5.16}$$

with this new C still independent of j,ε,i.

We claim that

$$|u_i^{j,\varepsilon}|_{(q+N)\frac{q^*}{q}} \le C^* \tag{5.17}$$

with $C^*>0$ independent of j,ε,i. Indeed, if $|u_i^{j,\varepsilon}|_{q+N}\le 1$ then (5.17) is a direct consequence of (5.16). If, on the contrary, we have that $|u_i^{j,\varepsilon}|_{q+N}\ge 1$ then (5.16) implies that

$$|u_i^{j,\varepsilon}|_{(q+N)\frac{q^*}{q}} \le C|u_i^{j,\varepsilon}|_{q+N}. \tag{5.18}$$

So, taking $\alpha\in]0,1[$ such that

$$\frac{\alpha}{q}+\frac{(1-\alpha)q}{(q+N)q^*} = \frac{1}{q+N},$$

by means of the interpolation inequality

$$|u|_{q+N} \leq |u|_q^\alpha \, |u|^{1-\alpha}_{(q+N)\frac{q^*}{q}}$$

and (5.18) we infer that

$$|u_i^{j,\varepsilon}|_{(q+N)\frac{q^*}{q}} \leq C|u_i^{j,\varepsilon}|_q^\alpha \, |u_i^{j,\varepsilon}|^{1-\alpha}_{(q+N)\frac{q^*}{q}}$$

which implies

$$|u_i^{j,\varepsilon}|_{(q+N)\frac{q^*}{q}} \leq C|u_i^{j,\varepsilon}|_q.$$

Hence, from this last estimate together with the Poincaré inequality and (5.5) it follows (5.17).

At last, since $L^{(q+N)\frac{q^*}{q}}(\Omega) \subset L^{(q+N)\frac{q-1}{q}}(\Omega)$, from (5.17) and (5.13) we obtain the existence of a constant $C^{**} > 0$ such that

$$\int_\Omega |l_{j,\varepsilon}(x, u_i^{j,\varepsilon})|^{\frac{q+N}{q}} \leq C^{**}$$

with C^{**} independent of j, ε, i. Thus, Lemma 4.2 applies and (5.12) holds.

Step 4. Since (5.12) implies that for any $j > C_\infty$ it is

$$h_j(x, u_i^{j,\varepsilon}(x)) = h(x, u_i^{j,\varepsilon}(x)) \quad \text{for all } x \in \Omega,$$

problem $(P_{\varepsilon,\infty})$ has at least $k - n + 1$ pairs of nontrivial weak solutions.

On the other hand, if we assume that condition (ii) of Theorem 1.1 holds, the same arguments introduced in the previous case apply but considering as starting point the critical levels of the even functional J which exist thanks to Theorem 3.2, case (ii). □ □

of Propositions 1.5 and 1.6. By means of the same arguments introduced in the proof of Theorem 1.1 we have that Proposition 1.5 follows from Proposition 3.3 while Proposition 1.6 is a consequence of Proposition 3.4. □ □

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