Generalizations of 3-Sasakian manifolds and skew torsion

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Abstract

In the first part, we define and investigate new classes of almost 3-contact metric manifolds, with two guiding ideas in mind: first, what geometric objects are best suited for capturing the key properties of almost 3-contact metric manifolds, and second, the newly defined classes should admit 'good' metric connections with skew torsion. In particular, we introduce the *Reeb commutator function* and the *Reeb Killing function*, we define the new classes of *canonical almost* 3*-contact metric manifolds* and of 3-(α,δ)*-Sasaki manifolds* (including as special cases 3-Sasaki manifolds, quaternionic Heisenberg groups, and many others) and prove that the latter are hypernormal, thus generalizing a seminal result by Kashiwada. We study their behaviour under a new class of deformations, called $\mathcal{H}$-homothetic deformations, and prove that they admit an underlying quaternionic contact structure, from which we deduce the Ricci curvature. For example, a 3-(α,δ)-Sasaki manifold is Einstein either if $\alpha=\delta$ (the 3-α-Sasaki case) or if $\delta=(2n+3)\alpha$, where $\dim M=4n+3$.

The second part is actually devoted to finding these adapted connections. We start with a very general notion of φ*-compatible connections*, where φ denotes any element of the associated sphere of almost contact structures, and make them unique by a certain extra condition, thus yielding the notion of *canonical connection* (they exist exactly on canonical manifolds, hence the name). For 3-(α,δ)-Sasaki manifolds, we compute the torsion of this connection explicitly and we prove that it is parallel, we describe the holonomy, the ∇-Ricci curvature, and we show that the metric cone is a HKT-manifold. In dimension 7, we construct a cocalibrated G_2-structure inducing the canonical connection and we prove the existence of four generalized Killing spinors.

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Contents

1 Introduction and basic notions

1.1 Introduction and summary

Since their first definition by Kuo in 1970, almost 3-contact metric manifolds have been a steady, but difficult topic of research. They are a very natural objects to consider — they have three almost contact metric structures with orthonormal Reeb vector fields and compatibility relations modelled on the multiplication rules of the quaternions. Unfortunately, they turn out to be rather difficult to handle. Computations become quickly lengthy and complicated. Compared to other geometries (like almost hermitian manifolds or symplectic manifolds), their definition is not equivalent to the reduction of the frame bundle to a certain subgroup $G \subset \mathrm{O}(n)$, hence they do not admit a 'good' classification scheme into different classes. A deeper reason for most of the encountered problems seems to be that, together with almost contact metric structures, they do not possess an integrable Riemannian counterpart, in the sense that contact geometry does not appear in Berger's theorem on irreducible Riemannian holonomies. As a consequence, the Levi-Civita connection is not well-adapted to their geometric structure, and the quest for other connections (like hermitian connections for almost hermitian manifolds) turns out to be a challenging task, with many open questions.

The current paper has the goal to address two of the sketched problems. In the first part, we define and investigate new classes of almost 3-contact metric manifolds, with two guiding ideas in mind: first, what geometric objects are best suited for capturing the key properties of almost 3-contact metric manifolds, and second, the newly defined classes should admit good invariant connections. The second part is actually devoted to finding these adapted connections, with attention restricted to connections that are metric and with skew torsion. Such connections are by now a widely established tool for the successful investigation of most non-integrable geometries. As a side condition, the classes defined in the first part should include 3-Sasaki manifolds and quaternionic Heisenberg groups, and the results proved in the second part should reproduce some known partial results on these.

Part one (Sections 1–2)

Consider an almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$. If there exists a function $\delta \in C^\infty(M)$ such that $\eta_k([\xi_i, \xi_j]) = 2\delta\epsilon_{ijk}$ for any $i, j, k = 1, 2, 3$, we call it the *Reeb commutator function* of M. Of course, not any M will admit a Reeb commutator function; but if it does, this function δ encodes in a very succinct way the relative 'positions' of the Reeb vector fields ξ_i. If the Reeb vector fields are Killing (like for 3-α-Sasakian structures and many other almost 3-contact structures), we prove in Corollary 2.1.1 that the existence of the Reeb commutator function δ follows. This function will play a special role in our study. As a first class of manifolds admitting a Reeb commutator function, we introduce 3-*δ-cosymplectic manifolds*: they will be defined by the conditions $d\eta_i = -2\delta\eta_j \wedge \eta_k$ and $d\Phi_i = 0$, for every even permutation (i, j, k) of $(1, 2, 3)$, where δ is a real constant, and Φ_i denotes the fundamental 2-form given by $\Phi_i(X, Y) = g(X, \varphi_i Y)$. For $\delta = 0$ we get in fact a 3-cosymplectic manifold.

Any 3-Sasakian structure has constant Reeb commutator function $\delta = 1$, while for a quaternionic Heisenberg group the Reeb vector fields commute, hence $\delta = 0$. Actually, we place both

3-α-Sasakian manifolds and the quaternionic Heisenberg groups in the more general class of almost 3-contact metric manifolds $(M, \varphi_i, \xi_i, \eta_i, g)$ satisfying the following condition:

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k \tag{1.1}$$

for every even permutation (i, j, k) of $(1, 2, 3)$, where α and δ are real constants, and $\alpha \neq 0$. We call these manifolds 3-(α, δ)-*Sasaki manifolds*; they include 3-α-Sasaki manifolds as a special case ($\alpha = \delta$). As we shall see many geometric features are captured by equation (1.1). First we show that 3-(α, δ)-Sasaki manifolds are hypernormal, that is the Nijenhuis tensor fields $N_{\varphi_i} := [\varphi_i, \varphi_i] + d\eta_i \otimes \xi_i$ are all vanishing (Theorem 2.2.1): this is a generalization of a seminal result of Kashiwada stating that every 3-contact metric manifold (corresponding to $\alpha = \delta = 1$) is 3-Sasakian [Ka01]. Furthermore, every 3-(α, δ)-Sasaki manifold has Killing Reeb vector fields, with constant Reeb commutator function δ (Corollary 2.3.1). We also study the behavior of these structures under a new type of deformations, called $\mathcal{H}$-*homothetic deformations* (Section 2.3). We show that these deformations preserve the class of 3-(α, δ)-Sasaki structures with $\delta = 0$, called *degenerate*. In the non-degenerate case, the sign of the product $\alpha\delta$ is preserved. In particular all 3-(α, δ)-Sasaki structures with $\alpha\delta > 0$ can be deformed into a 3-Sasakian structure. Examples of 3-(α, δ)-Sasaki structures with $\alpha\delta < 0$ do exist as well: they can be defined on the canonical principal $SO(3)$-bundle of a quaternionic Kähler (not hyperKähler) manifold with negative scalar curvature [Ko75, Ta96]. It is also worth observing that 3-(α, δ)-Sasaki manifolds admit an underlying quaternionic contact structure which is quaternionic contact Einstein in the sense of the definition given in [IMV14]; this allows us to determine the Ricci tensor of the Riemannian metric g (Proposition 2.3.3). In particular, a 3-(α, δ)-Sasaki manifold is Einstein either if $\alpha = \delta$ (the well-known 3-α-Sasaki case) or if $\delta = (2n + 3)\alpha$, where $\dim M = 4n + 3$.

A second reason why we are interested in 3-(α, δ)-Sasaki manifolds is that they provide a large class of *canonical* almost 3-contact metric manifolds. The defining conditions of what we call a canonical structure will be justified by Theorem 4.1.1, where we prove that these are exactly the manifolds admitting a unique canonical connection. To define them, we need to introduce the auxiliary tensor fields A_{ij} $(i, j = 1, 2, 3)$

$$A_{ij}(X, Y) \; := \; g((\mathcal{L}_{\xi_j}\varphi_i)X, Y) + d\eta_j(X, \varphi_i Y) + d\eta_j(\varphi_i X, Y) \quad \forall X, Y \in \Gamma(\mathcal{H}), \; \mathcal{H} := \bigcap_{i=1}^{3} \ker \eta_i.$$

Here $\mathcal{L}_{\xi_j}$ denotes the Lie derivative with respect to ξ_j. We also put $A_i := A_{ii}$. We say that an almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ admits a *Reeb Killing function* if there exists a smooth function $\beta \in C^\infty(M)$ such that for every $X, Y \in \Gamma(\mathcal{H})$ and every even permutation (i, j, k) of $(1, 2, 3)$,

$$A_i(X, Y) = 0, \qquad A_{ij}(X, Y) = -A_{ji}(X, Y) = \beta\Phi_k(X, Y).$$

The intrinsic meaning of β is subtler than that of δ; one key property is that the Reeb Killing function controls the derivatives of the structure tensors φ_i, ξ_i, and η_i with respect to the canonical connection (see Remark 4.1.2). We call $(M, \varphi_i, \xi_i, \eta_i, g)$ a *canonical almost* 3*-contact metric manifold* if it admits a Reeb Killing function β, all ξ_i are Killing vector fields, the Nijenhuis tensors N_{φ_i} are skew-symmetric on $\mathcal{H}$, and $N_{\varphi_i} - \varphi_i \circ d\Phi_i = N_{\varphi_j} - \varphi_j \circ d\Phi_j$ on $\Gamma(\mathcal{H})$ for all $i, j = 1, 2, 3$. As a special case, a canonical almost 3-contact metric manifold will be called *parallel* if its Reeb Killing function β vanishes; why this case is of interest will again be explained in the second part. We prove that 3-(α, δ)-Sasaki manifolds are always canonical with Reeb Killing function $\beta = 2(\delta - 2\alpha)$ in Corollary 2.3.3, and that 3-δ-cosymplectic manifolds are always hypernormal, canonical and parallel (but not 3-(α, δ)-Sasakian by definition) in Corollary 2.1.2.

We end this summary with a figure reviewing the different almost 3-contact metric structures that we shall discuss, trying to catch their most relevant features (Figure 1). The very interesting isolated point S^7 shall be treated in Example 4.1.3.

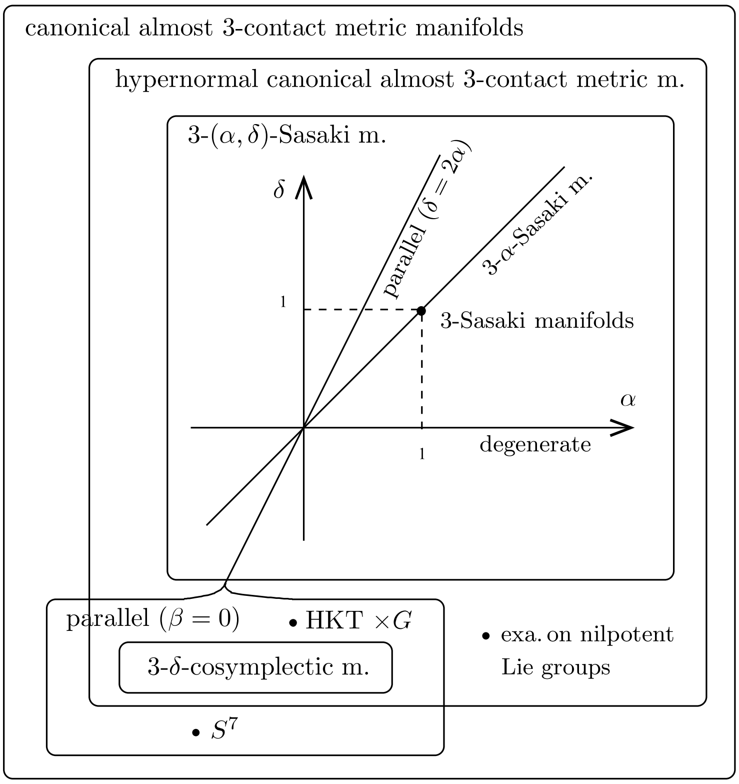

Figure 1: The different classes of almost 3-contact metric manifolds

Part two (Sections 3–5)

Given a G-structure on a Riemannian manifold (M, g), a *characteristic connection* denotes a metric connection with totally skew-symmetric torsion (briefly, with *skew torsion*) preserving the G-structure. For instance, an almost hermitian manifold (M, J, g) admits a (unique) hermitian connection with skew torsion if and only if the Nijenhuis tensor is totally skew-symmetric. Similar results hold for other geometries, for example hyperKähler manifolds with torsion, also known as HKT-manifolds [GP00].

In the context of almost contact geometry Friedrich and Ivanov [FI02] provided necessary and sufficient conditions for an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ to admit a characteristic connection (which then turns out to be unique): the Reeb vector field ξ has to be Killing and the tensor field N_φ has to be skew-symmetric (see Theorem 1.2.1 for details). For example the characteristic connection of a Sasakian manifold has torsion $T = \eta \wedge d\eta$.

It is well-known, however, that a unique characteristic connection cannot be defined in any naive way for almost 3-contact metric manifolds. This is easiest seen by looking at the 3-Sasaki situation: In this case each of the three Sasakian structures $(\varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$, admits a unique characteristic connection with torsion $T_i = \eta_i \wedge d\eta_i$. However, these three connections do not coincide, hence there does not exist a metric connection with skew torsion preserving all three Sasaki structures. To overcome this difficulty, a notion of canonical connection was proposed for a 7-dimensional 3-Sasakian manifold in [AF10a] by making a detour to the canonical (cocalibrated) G_2-structure associated to the 3-Sasakian structure. This canonical connection has torsion $T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i$ and preserves the *vertical* and *horizontal* distributions, denoted by $\mathcal{V}$ and $\mathcal{H}$ respectively, where $\mathcal{V}$ is the distribution spanned by the Reeb vector fields ξ_1, ξ_2, ξ_3. Furthermore, this connection has parallel torsion and admits a parallel spinor field that induces the three Riemannian Killing spinor fields of the 3-Sasaki manifold.

A second remarkable example of almost 3-contact metric manifolds admitting a canonical connection is given by quaternionic Heisenberg groups. In [AFS15] the authors study the geometry of these nilpotent Lie groups, describing natural left invariant almost 3-contact metric structures $(\varphi_i, \xi_i, \eta_i, g_\lambda)$, $\lambda > 0$, which can be defined in all dimensions $4n + 3$. It is shown that the metric connection with skew torsion $T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i - 4\lambda\eta_1 \wedge \eta_2 \wedge \eta_3$ preserves the horizontal and vertical distributions and equips the group with a naturally reductive homogeneous structure, thus highlighting again its importance. In the 7-dimensional case this connection can also be obtained by means of a cocalibrated G_2-structure.

The main objective of our study is to find good connections on almost 3-contact metric manifolds that generalize the canonical connections of the described examples. We begin with the observation that the choice of the Reeb vector fields ξ_i in the vertical distribution is somewhat arbitrary. Rather, any almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ carries a sphere Σ_M of almost contact metric structures $(\varphi_a, \xi_a, \eta_a, g)$, with $\varphi_a = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3$ for every $a = (a_1, a_2, a_3) \in S^2$. The horizontal and the vertical distributions are φ-invariant for every $\varphi \in \Sigma_M$. Now, if (φ, ξ, η, g) is a structure in Σ_M, a metric connection ∇ with skew torsion on M will be called a *φ-compatible connection* if it preserves the splitting $TM = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle and $(\nabla_X\varphi)Y = 0$ for all horizontal vector fields X, Y. In Theorem 3.1.1 we provide necessary and sufficient conditions for the existence of φ-compatible connections, one of which being the total skew-symmetry of the tensor field N_φ on $\mathcal{H}$; the other ones involve the Lie derivatives of the Riemannian metric g, and they are satisfied in the special case where the three Reeb vector fields are Killing. We also show that if M admits φ_i-compatible connections for every $i = 1, 2, 3$, then M admits φ-compatible connections for *every* structure $\varphi \in \Sigma_M$.

Despite the good behavior of φ-compatibility with respect to the associated sphere Σ_M, this notion is still too weak, since φ-compatible connections are not uniquely determined. They are parametrized by smooth functions $T(\xi_1, \xi_2, \xi_3) =: \gamma \in C^\infty(M)$, where T is the torsion of the connection. We call γ the *parameter function* of the connection.

A suggestion for requiring some further conditions on the connection comes from the case when the Reeb vector fields are Killing. In this case, given a φ-compatible connection ∇ with parameter function γ, the ∇-derivative of each ξ_i is completely determined by γ and the Reeb commutator function δ through

$$\nabla_X\xi_i = \tfrac{2\delta+\gamma}{2}(\eta_k(X)\xi_j - \eta_j(X)\xi_k)$$

for every vector field X, and (i, j, k) even permutation of $(1, 2, 3)$ (Proposition 3.2.2). This suggests the idea that one can require the covariant derivatives of the structure tensors φ_i to behave in a similar way. Therefore, we look for metric connections ∇ with skew torsion such that

$$\nabla_X\varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \tag{1.2}$$

for some smooth function β and for every even permutation (i, j, k) of $(1, 2, 3)$. In fact, in Theorem 4.1.1 we prove that an almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ admits a metric connection ∇ with skew torsion satisfying (1.2) if and only if it is canonical with Reeb Killing function β. If such a connection ∇ exists, it is unique and it is φ-compatible for every structure φ in the associated sphere Σ_M. The parameter function of ∇ is $\gamma = 2(\beta - \delta)$, δ being the Reeb commutator function. We call ∇ the *canonical connection* of M, and show that the covariant derivatives of the other structure tensors are given by

$$\nabla_X\xi_i = \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k), \qquad \nabla_X\eta_i = \beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k).$$

One can notice the analogy of (1.2) with the equation satisfied by the Levi-Civita connection of a quaternion-Kähler manifold (see Remark 4.1.1). There are various remarkable properties of canonical almost 3-contact metric manifolds and their canonical connection deserving special attention. First, for a canonical manifold $(M, \varphi_i, \xi_i, \eta, g)$, each structure (φ, ξ, η, g) in the sphere Σ_M admits a characteristic connection (Theorem 2.1.1). In particular, if ∇ is the canonical connection and ∇^i

the characteristic connection of the structure $(\varphi_i, \xi_i, \eta_i, g)$, their torsions T and T_i are related by (Theorem 4.2.1)

$$T - T_i = -\beta(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k)$$

where (i, j, k) is an even permutation of $(1, 2, 3)$. A surprising situation occurs for parallel canonical manifolds ($\beta = 0$): for them, the three characteristic connections ∇^i are identical, and they coincide with the canonical connection. Hence, all structure tensors φ_i, ξ_i, η_i are ∇-parallel. It is surprising that this fact was not discovered before.

Focusing on the canonical connection of a 3-(α, δ)-Sasaki manifold, for 3-Sasakian manifolds our canonical connection coincides, as desired, with the connection defined in the 7-dimensional case in [AF10a]. Similarly, on quaternionic Heisenberg groups our canonical connection coincides with the connection defined in [AFS15]. We also show that the canonical connection ∇ of a 3-(α, δ)-Sasaki manifold has parallel torsion (Theorem 4.4.1), we determine its Ricci tensor, and discuss the ∇-Einstein condition (Theorem 4.4.2).

A 'good' connection on M should induce a good connection on the cone. Recall that the metric cone of a 3-Sasaki manifold is hyper-Kähler. If the Reeb Killing function is constant and strictly negative, the canonical connection ∇ allows us to define a metric connection with skew torsion $\bar{\nabla}$ on the cone $(\bar{M}, \bar{g}) = (M \times \mathbb{R}^+, a^2 r^2 g + dr^2)$, $a = -\beta/2 > 0$, such that $\bar{\nabla} J_1 = \bar{\nabla} J_2 = \bar{\nabla} J_3 = 0$, where J_1, J_2, J_3 are almost hermitian structures naturally defined on $(\bar{M}, \bar{g})$, and such that $J_1 J_2 = J_3 = -J_2 J_1$: hence, we obtain a hyperhermitian structure. If furthermore, $(M, \varphi_i, \xi_i, \eta_i, g)$ is a 3-(α, δ)-Sasaki manifold, the cone is an HKT-manifold, a class of manifolds that is much larger than the class of hyper-Kähler manifolds (see Section 4.3).

Finally, we consider 7-dimensional 3-(α, δ)-Sasaki manifolds (Section 4.5) in order to investigate their relationship to G_2-geometry, which only exists in this dimension. We prove that the canonical connection coincides with the characteristic connection of a cocalibrated G_2-structure; as such, it admits a parallel spinor field ψ_0. We show that ψ_0 and the three Clifford products $\psi_i := \xi_i \cdot \psi_0$ are generalized Killing spinor fields, and we compute their generalized Killing numbers (for $\alpha = \delta$, they coincide with the three Riemannian Killing spinors of a 3-α-Sasaki manifold).

The appendix is devoted to the discussion of further examples. We describe left invariant almost 3-contact metric structures on various nilpotent Lie groups, providing examples of canonical structures which are not 3-(α, δ)-Sasaki, and non-canonical structures admitting φ_i-compatible connections.

Notation. Throughout this text, δ denotes the Reeb commutator function (Definition 2.1.1), which can be a real constant, and sometimes, it appears as factor in front of a differential form. It shouldn't be confused with a codifferential (we don't need any in this paper). For further ease of notation, we will often set $\eta_{ij} := \eta_i \wedge \eta_j$ etc., in particular when formulas tend to become heavy.

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1.2 Review of almost contact and 3-contact metric manifolds

We review some basic definitions and properties on almost contact metric manifolds. This serves mainly as a reference, but it is assorted by comments relevant to our work as we move on.

Definition 1.2.1. An *almost contact manifold* is a $(2n + 1)$-dimensional smooth manifold M endowed with a structure (φ, ξ, η), where φ is a $(1, 1)$-tensor field, ξ a vector field, and η a 1-form, such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

implying that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and φ has rank $2n$. The tangent bundle of M splits as $TM = \mathcal{H} \oplus \langle \xi \rangle$, where $\mathcal{H}$ is the $2n$-dimensional distribution defined by $\mathcal{H} = \mathrm{Im}(\varphi) = \ker \eta$.

The vector field ξ is called the *characteristic* or *Reeb vector field.* The almost contact structure is said to be *normal* if $N_\varphi := [\varphi,\varphi] + d\eta \otimes \xi$ vanishes, where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ [Bl10]. More precisely, for any vector fields X and Y, N_φ is given by

$$N_\varphi(X,Y) = [\varphi X,\varphi Y] + \varphi^2[X,Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] + d\eta(X,Y)\xi.$$

It is known that any almost contact manifold admits a compatible metric, that is a Riemannian metric g such that, for every $X,Y \in \mathfrak{X}(M)$, $g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y)$. Then $\eta = g(\cdot,\xi)$ and $\mathcal{H} = \langle\xi\rangle^\perp$. The manifold (M,φ,ξ,η,g) is called an *almost contact metric manifold.*

An α*-contact metric manifold* is defined as an almost contact metric manifold such that

$$d\eta \,=\, 2\alpha\Phi, \quad \alpha \in \mathbb{R}^*, \tag{1.3}$$

where Φ is the fundamental 2-form defined by $\Phi(X,Y) = g(X,\varphi Y)$; a 1-contact metric manifold is just called a *contact metric manifold* for short[1]; the 1-form η turns then out to be a *contact form,* in the sense that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. An α*-Sasakian manifold* is defined as a normal α-contact metric manifold, and again such a manifold with $\alpha = 1$ is called a *Sasakian manifold.* A more general class of α-Sasakian manifolds is given by *quasi-Sasakian manifolds*, defined as normal almost contact metric manifolds with closed 2-form Φ. We recall that the Reeb vector field of a (quasi)-Sasakian or α-Sasakian manifold is always Killing. As a comprehensive introduction to Sasakian geometry, we recommend the monography [BG08]. For some recent results, we refer to [CNY15].

We recall now some basic facts about connections with totally skew-symmetric torsion—we refer to [Ag06] for further details. If (M,g) is a Riemannian manifold, a metric connection ∇ with torsion T is said to have *totally skew-symmetric torsion*, or *skew torsion* for short, if the $(0,3)$-tensor field T defined by

$$T(X,Y,Z) = g(T(X,Y),Z)$$

is a 3-form. The relation between ∇ and the Levi-Civita connection ∇^g is then given by

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2}T(X,Y). \tag{1.4}$$

In [FI02] T. Friedrich ad S. Ivanov proved the following theorem concerning *characteristic connections* on almost contact metric manifolds, i. e. metric connections with skew torsion parallelizing all structure tensors.

Theorem 1.2.1. *Let (M,φ,ξ,η,g) be an almost contact metric manifold. It admits a metric connection ∇ with skew torsion and $\nabla\eta = \nabla\varphi = 0$ if and only if N_φ is totally skew-symmetric and if ξ is a Killing vector field. The connection ∇ is then uniquely determined and its torsion is given by*

$$T = \eta \wedge d\eta + N_\varphi + d^\varphi\Phi - \eta \wedge (\xi \lrcorner N_\varphi),$$

where $d^\varphi\Phi$ is defined as $d^\varphi\Phi(X,Y,Z) := -d\Phi(\varphi X,\varphi Y,\varphi Z)$.

For example, quasi-Sasakian manifolds admit a unique characteristic connection whose torsion is given by $T = \eta \wedge d\eta$. For later, let us observe that $\nabla\eta = \nabla\varphi = 0$ implies that the characteristic connection preserves the distributions $\mathcal{H}$ and $\mathcal{V}$ – a property we shall like to have later on for almost 3-contact manifolds as well. Further results on the characteristic connection of almost contact metric manifolds may be found in [Pu12, Pu13]; in particular, one finds there a detailed investigation for special classes of manifolds. The special situation of normal almost contact metric manifolds with Killing Reeb vector field was investigated in [CM14, HTY13], leading to a notion of Sasaki manifolds with torsion.

[1]The alternative name *almost α-Sasakian manifold* can be found in the literature for what we call an α-contact metric manifold, see for example [JV81]; however, we find this notion less suggestive.

Definition 1.2.2. An *almost 3-contact manifold* is a differentiable manifold M of dimension $4n+3$ endowed with three almost contact structures $(\varphi_i, \xi_i, \eta_i)$, $i = 1, 2, 3$, satisfying the following relations,

$$\begin{aligned} \varphi_k &= \varphi_i\varphi_j - \eta_j \otimes \xi_i = -\varphi_j\varphi_i + \eta_i \otimes \xi_j, \\ \xi_k = \varphi_i\xi_j &= -\varphi_j\xi_i, \quad \eta_k = \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i, \end{aligned} \tag{1.5}$$

for any even permutation (i, j, k) of $(1, 2, 3)$ [Bl10]. The tangent bundle of M splits as $TM = \mathcal{H} \oplus \mathcal{V}$, where

$$\mathcal{H} := \bigcap_{i=1}^{3} \ker \eta_i, \qquad \mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle.$$

In particular $\mathcal{H}$ has rank $4n$. We call any vector belonging to the distribution $\mathcal{H}$ *horizontal* and any vector belonging to the distribution $\mathcal{V}$ *vertical.* The manifold is said to be *hypernormal* if each almost contact structure (ϕ_i, ξ_i, η_i) is normal. In [YIK73] it was proved that if two of the almost contact structures are normal, then so is the third.

Any almost 3-contact manifold admits a Riemannian metric g which is compatible with each of the three structures. Then M is said to be an *almost 3-contact metric manifold* with structure $(\varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$. For ease of notation, we will just say that $(M, \varphi_i, \xi_i, \eta_i, g)$ is an almost 3-contact metric manifold, and it is self-understood that the index is running from 1 to 3. The subbundles $\mathcal{H}$ and $\mathcal{V}$ are orthogonal with respect to g and the three Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal—the structure group of the tangent bundle is in fact reducible to $\mathrm{Sp}(n) \times \{1\}$ [Ku70]; this implies in particular that each almost 3-contact manifold is spin. The following remarkable subclasses of almost 3-contact metric manifolds will be of particular importance to our work:

1) A 3-*(quasi)-Sasakian* resp. 3-*α-Sasakian manifold* is an almost 3-contact metric manifold for which each of the three structures is (quasi)-Sasakian resp. α-Sasakian [CND08, CND09]. A remarkable result due to T. Kashiwada states that if the three structures are contact metric structures, then the manifold is 3-Sasakian [Ka01]. Many results on the topology of 3-Sasakian manifolds are available, see for example [GS96].

2) A 3-*cosymplectic manifold* is an almost 3-contact metric manifold satisfying $d\eta_i = 0$, $d\Phi_i = 0$. It is known that such a structure is hypernormal (see [FIP04, Theorem 4.13]), so that each of the three structures is cosymplectic. It is also known that any 3-cosymplectic manifold is locally isometric to the Riemannian product of a hyper-Kähler manifold and a 3-dimensional flat abelian Lie group [CN07].

Later on, we shall see that the new classes of 3-(α, δ)-Sasaki manifolds (Definition 2.2.1) and 3-δ-cosymplectic manifolds (Definition 2.1.4) generalize 3-α-Sasaki and 3-cosymplectic manifolds, respectively, by using a vertical extra term.

1.3 The sphere of associated almost contact structures

Given an almost 3-contact metric manifold, one can define a sphere of almost contact structures containing $\pm\varphi_i$ $(i = 1, 2, 3)$ as antipodal points [CNY16]. This sphere is a canonical object to consider, since the choice of the elements φ_i is somewhat arbitrary.

Definition 1.3.1. For any almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ we define its associated sphere Σ_M of almost contact structures

$$\Sigma_M := \{\varphi_a := a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 \mid a = (a_1, a_2, a_3) \in S^2\}$$

as well as the associated bundle of endomorphisms

$$\Upsilon_M := \{\varphi_a := a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 \mid a = (a_1, a_2, a_3) \in \mathbb{R}^3\}.$$

For any $\varphi_a \in \Sigma_M$, its Reeb vector field and dual 1-form are defined respectively as

$$\xi_a := a_1\xi_1 + a_2\xi_2 + a_3\xi_3, \quad \eta_a := a_1\eta_1 + a_2\eta_2 + a_3\eta_3.$$

The Riemannian metric g is compatible with all the structures $(\varphi_a, \xi_a, \eta_a)$ (see [CNY16] for more details). When it has no importance, the index a will be omitted.

Observe that for any $\varphi \in \Sigma_M$, the distributions $\mathcal{H}$ and $\mathcal{V}$ are φ-invariant. Thus, φ encodes more geometric information than just the choice of an almost contact structure on M.

As seen in Theorem 1.2.1, a crucial property is whether the Nijenhuis tensor is skew-symmetric. Although it is not, for an almost contact metric structure φ in the associated sphere Σ_M, just the sum of the Nijenhuis tensors of the φ_i's, we shall prove next that it is skew-symmetric if this property holds for each N_{φ_i}. So, consider $\varphi \in \Sigma_M$, and define tensor fields $N_{i,j}$

$$N_{i,j} := [\varphi_i, \varphi_j] + d\eta_i \otimes \xi_j + d\eta_j \otimes \xi_i,$$

where

$$\begin{aligned}[\varphi_i, \varphi_j](X, Y) := [\varphi_i X, \varphi_j Y] &- \varphi_i[\varphi_j X, Y] - \varphi_j[X, \varphi_i Y] + [\varphi_j X, \varphi_i Y] \\ &- \varphi_j[\varphi_i X, Y] - \varphi_i[X, \varphi_j Y] + (\varphi_i\varphi_j + \varphi_j\varphi_i)[X, Y].\end{aligned}$$

In particular, $N_{i,i} = 2N_{\varphi_i}$. Notice that, using (1.5), one has $\forall X, Y \in \Gamma(\mathcal{H})$

$$\begin{aligned}N_{i,j}(X, Y) = [\varphi_i X, \varphi_j Y] &- \varphi_i[\varphi_j X, Y] - \varphi_j[X, \varphi_i Y] \\ &+ [\varphi_j X, \varphi_i Y] - \varphi_j[\varphi_i X, Y] - \varphi_i[X, \varphi_j Y].\end{aligned} \tag{1.6}$$

The following crucial equation was proved in [CNY16]:

$$N_\varphi = N_{\varphi_1} + N_{\varphi_2} + N_{\varphi_3} + a_1a_2N_{1,2} + a_1a_3N_{1,3} + a_2a_3N_{2,3}. \tag{1.7}$$

If the almost 3-contact metric structure is hypernormal, the tensor fields $N_{i,j}$ are all vanishing [CNY16], and thus $N_\varphi = 0$. We say that N_φ is *skew-symmetric on* $\mathcal{H}$ if the $(0,3)$-tensor field defined by

$$N_\varphi(X, Y, Z) = g(N_\varphi(X, Y), Z) \quad \forall X, Y, Z \in \Gamma(\mathcal{H})$$

is a 3-form on $\mathcal{H}$. We collect a few equivalent conditions for N_φ to be skew-symmetric on $\mathcal{H}$ which we will prove to be useful.

Lemma 1.3.1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold, $\varphi \in \Sigma_M$. The following conditions are equivalent:*

1) *N_φ is skew-symmetric on $\mathcal{H}$;*

2) *For any $X, Y \in \Gamma(\mathcal{H})$: $g((\nabla^g_X\varphi)X, Y) = g((\nabla^g_{\varphi X}\varphi)\varphi X, Y)$;*

3) *For any $X, Y, Z \in \Gamma(\mathcal{H})$:*

$$g((\nabla^g_X\varphi)Y + (\nabla^g_Y\varphi)X, Z) = g((\nabla^g_{\varphi X}\varphi)\varphi Y + (\nabla^g_{\varphi Y}\varphi)\varphi X, Z);$$

4) *For any $Y, Z \in \Gamma(\mathcal{H})$: $g((\nabla^g_{\varphi Z}\varphi)Z, Y) + g((\nabla^g_Z\varphi)\varphi Z, Y) = 0$.*

Proof. For the equivalence of 1), 2), and 3), see [DL14, Proposition 3.1]. If 2) holds, we get 4) by applying it to $X = Z + \varphi Z$. Conversely, applying 4) for $Z = X + \varphi X$, we obtain 2). □

We shall now prove that if each N_{φ_i} is skew-symmetric on $\mathcal{H}$, then the tensor fields $N_{i,j}$ are skew-symmetric on $\mathcal{H}$ as well. We proceed in two steps.

Lemma 1.3.2. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold. Let $i, j = 1, 2, 3$, with $i \neq j$. The tensor field $N_{i,j}$ is skew-symmetric on $\mathcal{H}$ if and only if*

$$g((\nabla^g_{\varphi_j Y}\varphi_i)Y + (\nabla^g_Y\varphi_i)\varphi_j Y, X) + g((\nabla^g_{\varphi_i Y}\varphi_j)Y + (\nabla^g_Y\varphi_j)\varphi_i Y, X) = 0$$

for every $X, Y \in \Gamma(\mathcal{H})$.

Proof. Since $N_{i,j} = N_{j,i}$ we can fix an even permutation (i, j, k) of $(1, 2, 3)$. By (1.6), one can check that

$$\begin{aligned} N_{i,j}(X,Y) \;\; = \;\; & (\nabla^g_{\varphi_i X}\varphi_j)Y - (\nabla^g_{\varphi_i Y}\varphi_j)X + (\nabla^g_{\varphi_j X}\varphi_i)Y - (\nabla^g_{\varphi_j Y}\varphi_i)X \\ & + \varphi_i(\nabla^g_Y(\varphi_j X)) - \varphi_i(\nabla^g_X(\varphi_j Y)) + \varphi_j(\nabla^g_Y(\varphi_i X)) - \varphi_j(\nabla^g_X(\varphi_i Y)) \end{aligned}$$

for every $X, Y \in \Gamma(\mathcal{H})$, and thus

$$\begin{aligned} g(N_{i,j}(X,Y),Y) \;\; = \;\; & - g((\nabla^g_{\varphi_i Y}\varphi_j)X, Y) - g((\nabla^g_{\varphi_j Y}\varphi_i)X, Y) + g(\varphi_i\nabla^g_Y(\varphi_j X), Y) \\ & + g(\nabla^g_X(\varphi_j Y), \varphi_i Y) + g(\varphi_j\nabla^g_Y(\varphi_i X), Y) + g(\nabla^g_X(\varphi_i Y), \varphi_j Y). \end{aligned}$$

Now, $g(\varphi_i Y, \varphi_j Y) = -g(Y, \varphi_i\varphi_j Y) = -g(Y, \varphi_k Y) = 0$. Then,

$$g(\nabla^g_X(\varphi_j Y), \varphi_i Y) + g(\nabla^g_X(\varphi_i Y), \varphi_j Y) = X(g(\varphi_i Y, \varphi_j Y)) = 0.$$

Furthermore,

$$\begin{aligned} & g(\varphi_i\nabla^g_Y(\varphi_j X), Y) + g(\varphi_j\nabla^g_Y(\varphi_i X), Y) \\ & = g(\varphi_i(\nabla^g_Y\varphi_j)X + \varphi_i\varphi_j(\nabla^g_Y X), Y) + g(\varphi_j(\nabla^g_Y\varphi_i)X + \varphi_j\varphi_i(\nabla^g_Y X), Y) \\ & = -g((\nabla^g_Y\varphi_j)X, \varphi_i Y) - g((\nabla^g_Y\varphi_i)X, \varphi_j Y), \end{aligned}$$

where we took into account that $(\varphi_i\varphi_j + \varphi_j\varphi_i)Y = 0$. We deduce that

$$\begin{aligned} & g(N_{i,j}(X,Y),Y) = \\ & \;\; = \;\; g((\nabla^g_{\varphi_i Y}\varphi_j)Y, X) + g((\nabla^g_{\varphi_j Y}\varphi_i)Y, X) + g((\nabla^g_Y\varphi_j)\varphi_i Y, X) + g((\nabla^g_Y\varphi_i)\varphi_j Y, X), \end{aligned}$$

which gives the result. □

Proposition 1.3.1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold such that N_{φ_1}, N_{φ_2}, and N_{φ_3} are skew-symmetric on $\mathcal{H}$. Then $N_{i,j}$ is skew-symmetric on $\mathcal{H}$ for every $i, j = 1, 2, 3$. In particular, N_φ is skew-symmetric on $\mathcal{H}$ for any φ in the associated sphere Σ_M.*

Proof. We first prove some auxiliary formulas. In the following we always consider vector fields $X, Y, Z \in \Gamma(\mathcal{H})$. Let (i, j, k) be an even permutation of $(1, 2, 3)$. Since $\varphi_i X = \varphi_j\varphi_k X$ and $\varphi_i^2 X = -X$, one easily checks that

$$g((\nabla^g_X\varphi_i)Y, Z) = g((\nabla^g_X\varphi_j)\varphi_k Y + \varphi_j(\nabla^g_X\varphi_k)Y, Z), \tag{1.8}$$

$$g((\nabla^g_X\varphi_i)\varphi_i Y + \varphi_i(\nabla^g_X\varphi_i)Y, Z) = 0. \tag{1.9}$$

N_{φ_i} being skew-symmetric on $\mathcal{H}$, we have by Lemma 1.3.1 4),

$$g((\nabla^g_{\varphi_i Y}\varphi_i)Y, X) + g((\nabla^g_Y\varphi_i)\varphi_i Y, X) = 0. \tag{1.10}$$

Applying the above formula for $Y = \varphi_j Z$ we obtain

$$g((\nabla^g_{\varphi_k Z}\varphi_i)\varphi_j Z, X) + g((\nabla^g_{\varphi_j Z}\varphi_i)\varphi_k Z, X) = 0. \tag{1.11}$$

From (1.10) and (1.8) it follows that

$$0 \; = \; g((\nabla^g_{\varphi_i Y}\varphi_j)\varphi_k Y + \varphi_j(\nabla^g_{\varphi_i Y}\varphi_k)Y, X) + g((\nabla^g_Y\varphi_j)\varphi_k\varphi_i Y + \varphi_j(\nabla^g_Y\varphi_k)\varphi_i Y, X),$$

and thus

$$0 = g((\nabla^g_{\varphi_i Y}\varphi_j)\varphi_k Y, X) + g((\nabla^g_Y \varphi_j)\varphi_j Y, X) - g((\nabla^g_{\varphi_i Y}\varphi_k)Y, \varphi_j X) - g((\nabla^g_Y \varphi_k)\varphi_i Y, \varphi_j X). \quad (1.12)$$

At this point, using formulas (1.8), (1.9), (1.10), (1.11), we have

$$\begin{aligned}
g((\nabla^g_{\varphi_i Y}\varphi_j)\varphi_k Y, X) &= -g((\nabla^g_{\varphi_k Y}\varphi_j)\varphi_i Y, X) = \\
&= -g((\nabla^g_{\varphi_k Y}\varphi_k)\varphi_i^2 Y, X) - g(\varphi_k(\nabla^g_{\varphi_k Y}\varphi_i)\varphi_i Y, X) \\
&= +g((\nabla^g_{\varphi_k Y}\varphi_k)Y, X) + g((\nabla^g_{\varphi_k Y}\varphi_i)\varphi_i Y, \varphi_k X) \\
&= -g((\nabla^g_Y \varphi_k)\varphi_k Y, X) - g(\varphi_i(\nabla^g_{\varphi_k Y}\varphi_i)Y, \varphi_k X) \\
&= -g(\varphi_j(\nabla^g_Y \varphi_k)\varphi_k Y, \varphi_j X) - g((\nabla^g_{\varphi_k Y}\varphi_i)Y, \varphi_j X) \\
&= -g((\nabla^g_Y \varphi_i)\varphi_k Y, \varphi_j X) + g((\nabla^g_Y \varphi_j)\varphi_k^2 Y, \varphi_j X) - g((\nabla^g_{\varphi_k Y}\varphi_i)Y, \varphi_j X) \\
&= -g((\nabla^g_Y \varphi_i)\varphi_k Y, \varphi_j X) - g((\nabla^g_{\varphi_k Y}\varphi_i)Y, \varphi_j X) + g(\varphi_j(\nabla^g_Y \varphi_j)Y, X).
\end{aligned}$$

Substituting the obtained expression for $g((\nabla^g_{\varphi_i Y}\varphi_j)\varphi_k Y, X)$ in (1.12), we have

$$0 = g((\nabla^g_Y \varphi_i)\varphi_k Y, \varphi_j X) + g((\nabla^g_{\varphi_k Y}\varphi_i)Y, \varphi_j X) + g((\nabla^g_{\varphi_i Y}\varphi_k)Y, \varphi_j X) + g((\nabla^g_Y \varphi_k)\varphi_i Y, \varphi_j X),$$

and thus $N_{k,i}$ is skew-symmetric on $\mathcal{H}$, owing to Lemma 1.3.2. The last claim about the skew-symmetry of N_φ now follows from identity (1.7), proved in [CNY16]. □

We end this section with a lemma that will subsequently be used several times. Although the formula does not look very neat, its qualitative claim is important: it states that the tensor fields N_{φ_i} of an almost 3-contact metric structure can be expressed in terms of the 1-forms η_i and the fundamental 2-forms Φ_i.

Lemma 1.3.3. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold. Then the following formula holds $\forall X, Y, Z \in \mathfrak{X}(M)$:*

$$\begin{aligned}
&N_{\varphi_i}(X, Y, Z) = \qquad (1.13)\\
&= -d\Phi_j(X, Y, \varphi_j Z) + d\Phi_j(\varphi_i X, \varphi_i Y, \varphi_j Z) + d\Phi_k(X, \varphi_i Y, \varphi_j Z) + d\Phi_k(\varphi_i X, Y, \varphi_j Z) \\
&\quad - \eta_i(X)[d\eta_j(\varphi_i Y, \varphi_j Z) + d\eta_k(Y, \varphi_j Z)] + \eta_i(Y)[d\eta_j(\varphi_i X, \varphi_j Z) + d\eta_k(X, \varphi_j Z)] \\
&\quad + \eta_j(Z)[d\eta_j(X, Y) - d\eta_j(\varphi_i X, \varphi_i Y)] - \eta_j(Z)[d\eta_k(X, \varphi_i Y) + d\eta_k(\varphi_i X, Y)]
\end{aligned}$$

where (i, j, k) is an even permutation of $(1, 2, 3)$.

Proof. As it is known, one can define three almost hermitian structures (J_i, G) on the product manifold $M \times \mathbb{R}$ as

$$J_i\Big(X, f\frac{d}{dt}\Big) = \Big(\varphi_i X - f\xi_i, \eta_i(X)\frac{d}{dt}\Big), \qquad G = g + dt^2,$$

where $X \in \mathfrak{X}(M)$ and f is a differentiable function on $M \times \mathbb{R}$. These almost hermitian structures satisfy $J_1 J_2 = J_3 = -J_2 J_1$. Denoting by Ω_i the associated Kähler forms, one has

$$\begin{aligned}
\Omega_i\Big(\Big(X, f\frac{d}{dt}\Big), \Big(Y, f'\frac{d}{dt}\Big)\Big) &= G\Big(\Big(X, f\frac{d}{dt}\Big), J_i\Big(Y, f'\frac{d}{dt}\Big)\Big) \\
&= g(X, \varphi_i Y - f'\xi_i) + f\eta_i(Y) \\
&= \Phi_i(X, Y) - \eta_i(X)f' + f\eta_i(Y).
\end{aligned}$$

Using the same notations Φ_i and η_i for differential forms on $M \times \mathbb{R}$ such that $\frac{d}{dt} \lrcorner \Phi_i = 0$ and $\eta_i(\frac{d}{dt}) = 0$, we have $\Omega_i = \Phi_i - \eta_i \wedge dt$, and thus

$$d\Omega_i = d\Phi_i - d\eta_i \wedge dt. \quad (1.14)$$

Now, the Nijenhuis tensors of the tensor fields J_i satisfy

$$\begin{aligned} G([J_i,J_i](X,Y),Z) = & -d\Omega_j(X,Y,J_jZ) + d\Omega_j(J_iX,J_iY,J_jZ) \\ & + d\Omega_k(X,J_iY,J_jZ) + d\Omega_k(J_iX,Y,J_jZ) \end{aligned} \tag{1.15}$$

for all vector fields X,Y,Z on $M\times\mathbb{R}$, where (i,j,k) is an even permutation of $(1,2,3)$ (see [Ka98, Lemma 3.2]). Taking $X,Y,Z\in\mathfrak{X}(M)$, the left-hand side in (1.15) coincides with $g(N_{\varphi_i}(X,Y),Z)$. Applying (1.14) and being $\frac{d}{dt}\lrcorner\, d\Phi_i = 0$ and $\frac{d}{dt}\lrcorner\, d\eta_i = 0$, we have

$$\begin{aligned} N_{\varphi_i}(X,Y,Z) = & -d\Phi_j(X,Y,\varphi_jZ) + \eta_j(Z)d\eta_j(X,Y) \\ & + d\Phi_j(\varphi_iX,\varphi_iY,\varphi_jZ) - \eta_i(X)d\eta_j(\varphi_iY,\varphi_jZ) \\ & - \eta_i(Y)d\eta_j(\varphi_jZ,\varphi_iX) - \eta_j(Z)d\eta_j(\varphi_iX,\varphi_iY) \\ & + d\Phi_k(X,\varphi_iY,\varphi_jZ) - \eta_i(Y)d\eta_k(\varphi_jZ,X) - \eta_j(Z)d\eta_k(X,\varphi_iY) \\ & + d\Phi_k(\varphi_iX,Y,\varphi_jZ) - \eta_i(X)d\eta_k(Y,\varphi_jZ) - \eta_j(Z)d\eta_k(\varphi_iX,Y) \end{aligned}$$

thus proving (1.13). □

Remark 1.3.1. As a consequence of the above lemma, we observe: if $(M,\varphi_i,\xi_i,\eta_i,g)$ is an almost 3-contact metric manifold such that $d\Phi_i(X,Y,Z)=0$ for all $i=1,2,3$ and for all horizontal vector fields X,Y,Z, then $N_{\varphi_i}(X,Y,Z)=0$ for all $X,Y,Z\in\Gamma(\mathcal{H})$. Hence, in this case conditions 1) and 3) in Definition 2.1.3 of canonical structures below are satisfied.

2 New classes of almost 3-contact metric manifolds

2.1 Remarkable functions and canonical almost 3-contact metric manifolds

Definition 2.1.1. We say that an almost 3-contact metric manifold $(M,\varphi_i,\xi_i,\eta_i,g)$ admits a *Reeb commutator function* if there exists a function $\delta\in C^\infty(M)$ satisfying

$$\eta_k([\xi_i,\xi_j]) \;=\; 2\delta\epsilon_{ijk} \text{ for every } \; i,j,k=1,2,3,$$

where ϵ_{ijk} is the totally skew-symmetric symbol. We shall call the function δ the *Reeb commutator function.*

Clearly, the existence of a constant Reeb commutator δ expresses that the three Reeb vector fields form a Lie algebra under the restriction of the commutator to $\mathcal{V}$, which is abelian in the case $\delta=0$, or isomorphic to $\mathfrak{so}(3)$ if $\delta\neq 0$.

Lemma 2.1.1 (Existence of a Reeb commutator function)**.** *Let $(M,\varphi_i,\xi_i,\eta_i,g)$ be an almost 3-contact metric manifold. Then the following conditions are equivalent:*

1) $(\mathcal{L}_{\xi_i}g)(\xi_j,\xi_k)=0$ *for every* $i,j,k=1,2,3$*;*

2) $\eta_k([\xi_i,\xi_j])=2\delta\epsilon_{ijk}$ *for some function* $\delta\in C^\infty(M)$ *and for every* $i,j,k=1,2,3$*;*

3) $\eta_k(\nabla^g_{\xi_i}\xi_j)=\delta\epsilon_{ijk}$ *for some function* $\delta\in C^\infty(M)$ *and for every* $i,j,k=1,2,3$.

Proof. The equivalence of 1) and 2) is consequence of the following equations, which hold for every $i,j,k=1,2,3$:

$$\begin{aligned} &(\mathcal{L}_{\xi_i}g)(\xi_j,\xi_k) = -\eta_k([\xi_i,\xi_j]) + \eta_j([\xi_k,\xi_i]), \\ &(\mathcal{L}_{\xi_i}g)(\xi_i,\xi_k) = \eta_i([\xi_k,\xi_i]), \quad (\mathcal{L}_{\xi_i}g)(\xi_k,\xi_k) = 2\,\eta_k([\xi_k,\xi_i]). \end{aligned}$$

Now, let us assume that 2) holds. Since $\eta_i([\xi_j,\xi_k])=\eta_j([\xi_k,\xi_i])=2\delta\epsilon_{ijk}$, we have

$$g(\nabla^g_{\xi_j}\xi_k,\xi_i) - g(\nabla^g_{\xi_k}\xi_j,\xi_i) = g(\nabla^g_{\xi_k}\xi_i,\xi_j) - g(\nabla^g_{\xi_i}\xi_k,\xi_j).$$

It follows that $g(\nabla^g_{\xi_j}\xi_i,\xi_k) = -g(\nabla^g_{\xi_i}\xi_j,\xi_k)$, and thus

$$2\delta\epsilon_{ijk} = \eta_k([\xi_i,\xi_j]) = 2\,\eta_k(\nabla^g_{\xi_i}\xi_j),$$

which implies 3). Conversely, 2) immediately follows from 3). □

Corollary 2.1.1. *Any almost 3-contact metric manifold $(M,\varphi_i,\xi_i,\eta_i,g)$ for which all ξ_i are Killing vector fields admits a Reeb commutator function δ.*

A second remarkable function catches subtle properties of the Lie derivatives. Given an almost 3-contact metric manifold $(M,\varphi_i,\xi_i,\eta_i,g)$, we introduce the tensor fields A_{ij}, $i,j=1,2,3$, defined on the subbundle $\mathcal{H}$ of TM by

$$A_{ij}(X,Y) \ := \ g((\mathcal{L}_{\xi_j}\varphi_i)X,Y) + d\eta_j(X,\varphi_i Y) + d\eta_j(\varphi_i X,Y). \tag{2.1}$$

We shall denote by A_i the tensor field A_{ii}. In Proposition 3.2.2, we will prove that an expression of this type appears as the covariant derivative of $\varphi\in\Sigma_M$ for any φ-compatible connection, thus partially explaining its relevance. In Section 2.3, we will encounter many manifolds for which $A_i=0$ and the tensor fields A_{ij} $(i\neq j)$ are skew-symmetric in i,j and proportional to Φ_k, where k is the only remaining index in $\{1,2,3\}$ different from i and j. The following definition captures these properties.

Definition 2.1.2. An almost 3-contact metric manifold $(M,\varphi_i,\xi_i,\eta_i,g)$ is said to admit a *Reeb Killing function* if there exists a smooth function $\beta\in C^\infty(M)$ such that for every $X,Y\in\Gamma(\mathcal{H})$ and every even permutation (i,j,k) of $(1,2,3)$,

$$A_i(X,Y)=0, \qquad A_{ij}(X,Y) = -A_{ji}(X,Y) = \beta\Phi_k(X,Y). \tag{2.2}$$

As a special case, M will be called a *parallel* almost 3-contact metric manifold if it has vanishing Reeb Killing function, $\beta=0$ or, equivalently, $A_{ij}=0$ $\forall i,j=1,2,3$.

Remark 2.1.1. The intrinsic meaning of the function β is not as obvious as for the Reeb commutator function, but it will become clearer as we proceed. Most importantly, we shall see later that it controls the derivatives of the structure tensors φ_i,ξ_i, and η_i with respect to the canonical connection (see Remark 4.1.2), and this is in fact the justification why manifolds with vanishing β are called parallel.

The following definition turns out to be tailor-made for our purposes:

Definition 2.1.3. Let $(M,\varphi_i,\xi_i,\eta_i,g)$ be an almost 3-contact metric manifold. We call it a *canonical almost 3-contact metric manifold* if the following conditions are satisfied:

1) each N_{φ_i} is skew-symmetric on $\mathcal{H}$,

2) each ξ_i is a Killing vector field,

3) for any $X,Y,Z\in\Gamma(\mathcal{H})$ and any $i,j=1,2,3$,

$$N_{\varphi_i}(X,Y,Z) - d\Phi_i(\varphi_i X,\varphi_i Y,\varphi_i Z) = N_{\varphi_j}(X,Y,Z) - d\Phi_j(\varphi_j X,\varphi_j Y,\varphi_j Z),$$

4) M admits a Reeb Killing function $\beta\in C^\infty(M)$.

As before, a *parallel* canonical almost 3-contact metric manifold is one with vanishing Reeb Killing function, $\beta=0$.

By Corollary 2.1.1, a canonical almost 3-contact metric manifold admits also a Reeb commutator function δ. We shall see in Theorem 4.1.1 that canonical almost 3-contact metric manifolds are exactly those admitting a canonical connection, thus explaining the name. In a first step, we prove that each of the three almost contact metric structures of a canonical almost 3-contact metric manifold admits a characteristic connection in the sense of Friedrich and Ivanov (Theorem 1.2.1):

Theorem 2.1.1 (Characteristic connections of canonical manifolds). *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a canonical almost 3-contact metric manifold. Then the following hold:*

1) *The three Nijenhuis tensors N_{φ_i} $(i = 1, 2, 3)$ are skew-symmetric on TM, and hence each almost contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$ admits a characteristic connection ∇^i.*

2) *Each almost contact metric structure (φ, ξ, η, g) in the associated sphere Σ_M admits a characteristic connection.*

Proof. 1) By assumption, the three Reeb vector fields ξ_i are Killing vector fields and the Nijenhuis tensors N_{φ_i} are skew-symmetric on $\mathcal{H}$. We prove that each N_{φ_i} is skew-symmetric on TM. First of all, the definition of the Nijenhuis tensor implies that for every $X, Y \in \Gamma(\mathcal{H})$ and for every even permutation (i, j, k) of $(1, 2, 3)$, the following equations hold:

$$\begin{aligned}
N_{\varphi_i}(X, Y) &= [\varphi_i X, \varphi_i Y] - [X, Y] - \varphi_i[\varphi_i X, Y] - \varphi_i[X, \varphi_i Y],\\
N_{\varphi_i}(X, \xi_i) &= -[X, \xi_i] - \varphi_i[\varphi_i X, \xi_i],\\
N_{\varphi_i}(X, \xi_j) &= [\varphi_i X, \xi_k] - [X, \xi_j] - \varphi_i[\varphi_i X, \xi_j] - \varphi_i[X, \xi_k],\\
N_{\varphi_i}(X, \xi_k) &= -[\varphi_i X, \xi_j] - [X, \xi_k] - \varphi_i[\varphi_i X, \xi_k] + \varphi_i[X, \xi_j],\\
N_{\varphi_i}(\xi_i, \xi_j) &= -[\xi_i, \xi_j] - \varphi_i[\xi_i, \xi_k],\\
N_{\varphi_i}(\xi_i, \xi_k) &= -[\xi_i, \xi_k] + \varphi_i[\xi_i, \xi_j],\\
N_{\varphi_i}(\xi_j, \xi_k) &= 0.
\end{aligned} \tag{2.3}$$

Let δ be the Reeb commutator function, i. e. $\eta_t([\xi_r, \xi_s]) = 2\delta\epsilon_{rst}$, with $r, s, t = 1, 2, 3$. Using the defining relations (1.5), one easily checks that

$$\begin{aligned}
&N_{\varphi_i}(\xi_i, \xi_j, \xi_i) = N_{\varphi_i}(\xi_i, \xi_j, \xi_j) = N_{\varphi_i}(\xi_i, \xi_k, \xi_i) = N_{\varphi_i}(\xi_i, \xi_k, \xi_k) = 0,\\
&N_{\varphi_i}(\xi_i, \xi_j, \xi_k) = N_{\varphi_i}(\xi_i, \xi_k, \xi_j) = 0.
\end{aligned}$$

For $X \in \Gamma(\mathcal{H})$ and $r, s = 1, 2, 3$, one checks that

$$(\mathcal{L}_X g)(\xi_r, \xi_s) = -(\mathcal{L}_{\xi_r} g)(X, \xi_s) - (\mathcal{L}_{\xi_s} g)(X, \xi_r).$$

Since the Reeb vector fields are Killing, this quantity vanishes, which is equivalent to $\eta_r([X, \xi_s]) + \eta_s([X, \xi_r]) = 0$. Therefore,

$$\begin{aligned}
&N_{\varphi_i}(X, \xi_i, \xi_i) = -\eta_i([X, \xi_i]) = 0,\\
&N_{\varphi_i}(X, \xi_j, \xi_j) = \eta_j([\varphi_i X, \xi_k]) - \eta_j([X, \xi_j]) + \eta_k([\varphi_i X, \xi_j]) + \eta_k([X, \xi_k]) = 0.
\end{aligned}$$

Similarly one shows the following identities:

$$\begin{aligned}
&N_{\varphi_i}(X, \xi_k, \xi_k) = 0, && N_{\varphi_i}(X, \xi_i, \xi_j) + N_{\varphi_i}(X, \xi_j, \xi_i) = 0,\\
&N_{\varphi_i}(X, \xi_j, \xi_k) = N_{\varphi_i}(X, \xi_k, \xi_j) = 0, && N_{\varphi_i}(X, \xi_i, \xi_k) + N_{\varphi_i}(X, \xi_k, \xi_i) = 0.
\end{aligned}$$

Since ξ_i is a Killing vector field, we have

$$\begin{aligned}
&N_{\varphi_i}(\xi_i, \xi_j, X) + N_{\varphi_i}(\xi_i, X, \xi_j)\\
&= -g([\xi_i, \xi_j], X) + g([\xi_i, \xi_k], \varphi_i X) - g([\xi_i, X], \xi_j) + g([\xi_i, \varphi_i X], \xi_k)\\
&= (\mathcal{L}_{\xi_i} g)(\xi_j, X) - (\mathcal{L}_{\xi_i} g)(\xi_k, \varphi_i X) = 0
\end{aligned}$$

and analogously, $N_{\varphi_i}(\xi_i, \xi_k, X) + N_{\varphi_i}(\xi_i, X, \xi_k) = 0$. Now the existence of a Reeb Killing function β (see eq. (2.2)) yields for $X, Y \in \Gamma(\mathcal{H})$

$$\begin{aligned}
&N_{\varphi_i}(X, Y, \xi_i) + N_{\varphi_i}(X, \xi_i, Y)\\
&= \eta_i([\varphi_i X, \varphi_i Y]) - \eta_i([X, Y]) + g([\xi_i, X], Y) - g([\xi_i, \varphi_i X], \varphi_i Y)\\
&= -d\eta_i(\varphi_i X, \varphi_i Y) + d\eta_i(X, Y) - g((\mathcal{L}_{\xi_i}\varphi_i)\varphi_i X, Y)\\
&= -A_i(\varphi_i X, Y) = 0.
\end{aligned}$$

Furthermore, one can compute

$$N_{\varphi_i}(X,Y,\xi_j) = -d\eta_j(\varphi_i X,\varphi_i Y) + d\eta_j(X,Y) - d\eta_k(\varphi_i X,Y) - d\eta_k(X,\varphi_i Y), \tag{2.4}$$

as well as

$$\begin{aligned} N_{\varphi_i}(X,\xi_j,Y) &= g([\varphi_i X,\xi_k],Y) - g([X,\xi_j],Y) + g([\varphi_i X,\xi_j],\varphi_i Y) + g([X,\xi_k],\varphi_i Y) \\ &= -g((\mathcal{L}_{\xi_j}\varphi_i)\varphi_i X,Y) - g((\mathcal{L}_{\xi_k}\varphi_i)X,Y). \end{aligned}$$

Therefore, using again the existence of a Reeb Killing function, we conclude

$$\begin{aligned} N_{\varphi_i}(X,Y,\xi_j) + N_{\varphi_i}(X,\xi_j,Y) &= -A_{ij}(\varphi_i X,Y) - A_{ik}(X,Y) \\ &= -\beta\Phi_k(\varphi_i X,Y) + \beta\Phi_j(X,Y) \\ &= -\beta g(\varphi_i X,\varphi_k Y) + \beta g(X,\varphi_j Y) = 0. \end{aligned}$$

Analogously one shows that $N_{\varphi_i}(X,Y,\xi_k) + N_{\varphi_i}(X,\xi_k,Y) = 0$. Finally,

$$N_{\varphi_i}(\xi_i,X,X) = -g([\xi_i,X],X) - g(\varphi_i[\xi_i,\varphi_i X],X) = g((\mathcal{L}_{\xi_i}\varphi_i)\varphi_i X,X) = A_i(\varphi_i X,X) = 0,$$

and furthermore

$$\begin{aligned} N_{\varphi_i}(\xi_j,X,X) &= g([\xi_k,\varphi_i X],X) - g([\xi_j,X],X) - g(\varphi_i[\xi_k,X],X) - g(\varphi_i[\xi_j,\varphi_i X],X) \\ &= g((\mathcal{L}_{\xi_k}\varphi_i)X,X) + g((\mathcal{L}_{\xi_j}\varphi_i)\varphi_i X,X) \ = \ A_{ik}(X,X) + A_{ij}(\varphi_i X,X) \\ &= -\beta\Phi_j(X,X) + \beta\Phi_k(\varphi_i X,X) = 0. \end{aligned}$$

Analogously, $N_{\varphi_i}(\xi_k,X,X) = 0$, completing the proof that each N_{φ_i} is skew-symmetric on TM. Since all ξ_i are Killing vector fields, the existence of a characteristic connection ∇^i for each φ_i now follows from Theorem 1.2.1.

2) Let (φ,ξ,η,g) be in the associated sphere Σ_M. Its Reeb vector field ξ is obviously Killing, and thus the main point is to prove that its Nijenhuis tensor N_φ is skew-symmetric on TM. By 1), each tensor N_{φ_i} is skew-symmetric on TM, and consequently Proposition 1.3.1 implies that each $N_{i,j}$, $i \neq j$, is skew-symmetric on $\mathcal{H}$. In view of (1.7), we only need to show that each $N_{i,j}$ is skew-symmetric on TM. In the following we fix an even permutation (i,j,k) of $(1,2,3)$ and denote by X,Y,Z horizontal vector fields. We proceed case by case as 1), hence we shall be brief. From the definition of $N_{i,j}$, taking into account that $(\varphi_i\varphi_j + \varphi_j\varphi_i)X = 0$, we obtain after a short calculation

$$\begin{aligned} N_{i,j}(X,Y) &= [\varphi_i X,\varphi_j Y] - [\varphi_j X,\varphi_i Y] - \varphi_i[\varphi_j X,Y] - \varphi_i[X,\varphi_j Y] \\ &\quad - \varphi_j[\varphi_i X,Y] - \varphi_j[X,\varphi_i Y], \\ N_{i,j}(X,\xi_i) &= -[\varphi_i X,\xi_k] - \varphi_i[\varphi_j X,\xi_i] + \varphi_i[X,\xi_k] - \varphi_j[\varphi_i X,\xi_i] \\ &= (\mathcal{L}_{\xi_k}\varphi_i)X - (\mathcal{L}_{\xi_i}\varphi_i)\varphi_j X - (\mathcal{L}_{\xi_i}\varphi_j)\varphi_i X, \\ N_{i,j}(X,\xi_j) &= [\varphi_j X,\xi_k] - \varphi_i[\varphi_j X,\xi_j] - \varphi_j[\varphi_i X,\xi_j] - \varphi_j[X,\xi_k] \\ &= -(\mathcal{L}_{\xi_k}\varphi_j)X - (\mathcal{L}_{\xi_j}\varphi_i)\varphi_j X - (\mathcal{L}_{\xi_j}\varphi_j)\varphi_i X, \\ N_{i,j}(X,\xi_k) &= [\varphi_i X,\xi_i] - [\varphi_j X,\xi_j] - \varphi_i[\varphi_j X,\xi_k] - \varphi_i[X,\xi_i] - \varphi_j[\varphi_i X,\xi_k] + \varphi_j[X,\xi_j] \\ &= -(\mathcal{L}_{\xi_i}\varphi_i)X + (\mathcal{L}_{\xi_j}\varphi_j)X - (\mathcal{L}_{\xi_k}\varphi_i)\varphi_j X - (\mathcal{L}_{\xi_k}\varphi_j)\varphi_i X, \\ N_{i,j}(\xi_i,\xi_j) &= \varphi_i[\xi_k,\xi_j] - \varphi_j[\xi_i,\xi_k], \\ N_{i,j}(\xi_i,\xi_k) &= [\xi_k,\xi_j] + \varphi_j[\xi_i,\xi_j], \\ N_{i,j}(\xi_j,\xi_k) &= [\xi_k,\xi_i] - \varphi_i[\xi_j,\xi_i]. \end{aligned} \tag{2.5}$$

Now, since M admits a Reeb commutator function, one easily checks that $N_{i,j}(\xi_r,\xi_s,\xi_t) = 0$ for every $r,s,t = 1,2,3$. Furthermore, since $\eta_r([X,\xi_s]) + \eta_s([X,\xi_r]) = 0$, we deduce $N_{i,j}(X,\xi_r,\xi_s) +$

$N_{i,j}(X,\xi_s,\xi_r)=0$. Next we compute

$$\begin{aligned}
&N_{i,j}(\xi_i,\xi_j,X)+N_{i,j}(\xi_i,X,\xi_j)\\
&=-g([\xi_k,\xi_j],\varphi_i X)+g([\xi_i,\xi_k],\varphi_j X)-g([\xi_k,\varphi_i X],\xi_j)+g([\xi_i,\varphi_j X],\xi_k)\\
&=(\mathcal{L}_{\xi_k}g)(\varphi_i X,\xi_j)-(\mathcal{L}_{\xi_i}g)(\xi_k,\varphi_j X)=0.
\end{aligned}$$

Analogously, using the equations in (2.5), one shows that

$$N_{i,j}(\xi_i,\xi_k,X)+N_{i,j}(\xi_i,X,\xi_k)=0,\qquad N_{i,j}(\xi_j,\xi_k,X)+N_{i,j}(\xi_j,X,\xi_k)=0.$$

Since M admits a Reeb Killing function β, we have

$$\begin{aligned}
&N_{i,j}(X,Y,\xi_i)+N_{i,j}(X,\xi_i,Y)\\
&=-d\eta_i(\varphi_i X,\varphi_j Y)-d\eta_i(\varphi_j X,\varphi_i Y)+d\eta_k(\varphi_i X,Y)+d\eta_k(X,\varphi_i Y)\\
&\quad+g((\mathcal{L}_{\xi_k}\varphi_i)X,Y)-g((\mathcal{L}_{\xi_i}\varphi_i)\varphi_j X,Y)-g((\mathcal{L}_{\xi_i}\varphi_j)\varphi_i X,Y)\\
&=A_{ik}(X,Y)-A_i(\varphi_j X,Y)+d\eta_i(\varphi_i\varphi_j X,Y)-A_{ji}(\varphi_i X,Y)+d\eta_i(\varphi_j\varphi_i X,Y)\\
&=-\beta\Phi_j(X,Y)+\beta\Phi_k(\varphi_i X,Y)=-\beta g(X,\varphi_j Y)+\beta g(\varphi_i X,\varphi_k Y)=0.
\end{aligned}$$

In the same way one shows that

$$N_{i,j}(X,Y,\xi_j)+N_{i,j}(X,\xi_j,Y)=0,\qquad N_{i,j}(X,Y,\xi_k)+N_{i,j}(X,\xi_k,Y)=0.$$

Finally,

$$\begin{aligned}
N_{i,j}(\xi_i,X,X)&=-g((\mathcal{L}_{\xi_k}\varphi_i)X,X)+g((\mathcal{L}_{\xi_i}\varphi_i)\varphi_j X,X)-g((\mathcal{L}_{\xi_i}\varphi_j)\varphi_i X,X)\\
&=-A_{ik}(X,X)+A_i(\varphi_j X,X)+d\eta_i(\varphi_j X,\varphi_i X)+d\eta_i(\varphi_i\varphi_j X,X)\\
&\quad-A_{ji}(\varphi_i X,X)+d\eta_i(\varphi_i X,\varphi_j X)+d\eta_i(\varphi_j\varphi_i X,X)\\
&=\beta\Phi_j(X,X)+\beta\Phi_k(\varphi_i X,X)=0,
\end{aligned}$$

and analogously $N_{i,j}(\xi_j,X,X)=N_{i,j}(\xi_k,X,X)=0$, thus completing the proof. □

We introduce now a slight generalization of 3-cosymplectic manifolds.

Definition 2.1.4. A 3-δ-*cosymplectic manifold* is an almost 3-contact metric manifold satisfying

$$d\eta_i=-2\delta\eta_j\wedge\eta_k,\qquad d\Phi_i=0, \tag{2.6}$$

for some $\delta\in\mathbb{R}$ and for every even permutation (i,j,k) of $(1,2,3)$.

When $\delta=0$, we get the notion of 3-cosymplectic manifolds. We shall describe the class of 3-δ-cosymplectic manifolds with $\delta\neq 0$, and show that all 3-δ-cosymplectic manifolds are parallel and canonical.

Proposition 2.1.1. *Let $(M,\varphi_i,\xi_i,\eta_i,g)$ be a 3-δ-cosymplectic manifold with $\delta\neq 0$. Then the structure is hypernormal, and the Levi-Civita connection satisfies*

$$(\nabla^g_X\varphi_i)Y=\delta\{\eta_j(X)\eta_j(Y)+\eta_k(X)\eta_k(Y)\}\xi_i-\delta\,\eta_i(Y)\{\eta_j(X)\xi_j+\eta_k(X)\xi_k\}, \tag{2.7}$$

$$\nabla^g_X\xi_i=\delta\{\eta_k(X)\xi_j-\eta_j(X)\xi_k\} \tag{2.8}$$

for every $X,Y\in\mathfrak{X}(M)$ and for every even permutation (i,j,k) of $(1,2,3)$. Furthermore, each ξ_i is a Killing vector field and M is locally isometric to the Riemannian product of a hyper-Kähler manifold and the 3-dimensional sphere of constant curvature δ^2.

Proof. The fact that the structure is hypernormal is a consequence of Lemma 1.3.3, which expressed N_{φ_i} in terms of η_i and Φ_i. More precisely, the defining relation (2.6) of 3-δ-cosymplectic manifolds, when plugged into the identity (1.13), yields after a short calculation that $N_{\varphi_i} = 0$.

By [Bl10, Lemma 6.1], the Levi-Civita connection of any hypernormal structure satisfies

$$\begin{aligned} 2g((\nabla^g_X \varphi_i)Y, Z) &= d\Phi_i(X, \varphi_i Y, \varphi_i Z) - d\Phi_i(X, Y, Z) \\ &\quad + d\eta_i(\varphi_i Y, X)\eta_i(Z) - d\eta_i(\varphi_i Z, X)\eta_i(Y) \end{aligned} \tag{2.9}$$

for every $X, Y, Z \in \mathfrak{X}(M)$. Together with equation (2.6), this yields

$$\begin{aligned} 2g((\nabla^g_X \varphi_i)Y, Z) &= -2\delta(\eta_j \wedge \eta_k)(\varphi_i Y, X)\eta_i(Z) + 2\delta(\eta_j \wedge \eta_k)(\varphi_i Z, X)\eta_i(Y) \\ &= 2\delta\{\eta_k(Y)\eta_k(X) + \eta_j(X)\eta_j(Y)\}\eta_i(Z) \\ &\quad + 2\delta\{-\eta_k(Z)\eta_k(X) - \eta_j(Z)\eta_j(X)\}\eta_i(Y), \end{aligned}$$

and hence we proved (2.7). Applying (2.7) for $Y = \xi_i$, we have

$$\nabla^g_X \xi_i = -\varphi_i^2(\nabla^g_X \xi_i) = \varphi_i((\nabla^g_X \varphi_i)\xi_i) = -\delta\{\eta_j(X)\xi_k - \eta_k(X)\xi_j\},$$

which proves (2.8). It follows that

$$g(\nabla^g_X \xi_i, Y) = \delta(\eta_k \wedge \eta_j)(X, Y)$$

for every vector fields X, Y, thus showing that ξ_i is Killing. We can also deduce that

$$\nabla^g_{\xi_i} \xi_i = 0, \quad \nabla^g_{\xi_i} \xi_j = -\nabla^g_{\xi_j} \xi_i = \delta \xi_k, \quad [\xi_i, \xi_j] = 2\delta \xi_k.$$

Then, $\mathcal{V}$ is an integrable distribution with totally geodesic leaves, globally spanned by Killing vector fields. Since $\mathcal{H}$ is integrable as well, the manifold is locally isometric to the Riemannian product of a manifold M' tangent to $\mathcal{H}$ and a 3-dimensional Lie group tangent to $\mathcal{V}$, which is isomorphic to $SO(3)$. Owing to (2.7), the almost 3-contact metric structure induces on M' a hyper-Kähler structure. Furthermore, the leaves of $\mathcal{V}$ have constant sectional curvature δ^2. □

Together with what was known before on 3-δ-cosymplectic manifolds with $\delta = 0$ (Section 1.2), we obtain:

Corollary 2.1.2. *Any 3-δ-cosymplectic manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ is a parallel canonical almost 3-contact metric manifold.*

Proof. Conditions 1)-3) of Definition 2.1.3 are satisfied since the structure is hypernormal, the fundamental 2-forms are closed, and the Reeb vector fields are Killing. Since M is locally isometric to the Riemannian product of a horizontal hyper-Kähler manifold and a vertical Lie group, we have $(\mathcal{L}_{\xi_i}\varphi_j)X = 0$ for every horizontal vector field X and for every $i, j = 1, 2, 3$. It follows that $A_{ij}(X, Y) = 0$ for every $X, Y \in \mathcal{H}$. In particular, this shows that the Reeb Killing functions β vanishes, i. e. it is a parallel canonical almost 3-contact metric manifold. □

For 3-δ-cosymplectic structures on Lie groups, see Example 5.1. Inspired by the previous result, we sketch a slightly more general construction of parallel canonical almost 3-contact metric manifolds:

Example 2.1.1 (Examples arising on HKT manifolds)**.** A hyper-Kähler with torsion manifold, briefly HKT-manifold, is defined as a hyperhermitian manifold (M, J_i, h) endowed with a metric connection ∇^c with skew-symmetric torsion such that $\nabla^c J_i = 0$ for all $i = 1, 2, 3$. This is equivalent to requiring that

$$J_1 d\Omega_1 = J_2 d\Omega_2 = J_3 d\Omega_3, \tag{2.10}$$

where Ω_i is the Kähler form of J_i. The unique metric connection with skew torsion parallelizing the complex structures has torsion $T_0 = -J_i d\Omega_i$. Let us consider a HKT-manifold (M, J_i, h)

and a 3-dimensional Lie group G with Lie algebra $\mathfrak{g}$ spanned by vector fields ξ_1, ξ_2, ξ_3 such that $[\xi_i, \xi_j] = 2\delta\xi_k$, for some $\delta \in \mathbb{R}$ and for every even permutation (i, j, k) of $(1, 2, 3)$. In particular, for $\delta = 0$ we have an abelian Lie group, while for $\delta \neq 0$, G is isomorphic to $SO(3)$. On the product manifold $M \times G$ one can define in a natural way an almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$, by

$$\varphi_i|_{TM} = J_i, \quad \varphi_i\xi_i = 0, \quad \varphi_i\xi_j = \xi_k, \quad \varphi_i\xi_k = -\xi_j,$$
$$\eta_i|_{TM} = 0, \quad \eta_i(\xi_i) = 1, \qquad \eta_i(\xi_j) = \eta_i(\xi_k) = 0,$$

and g the product metric of h and the left invariant Riemannian metric on G with respect to which ξ_1, ξ_2, ξ_3 are an orthonormal basis of $\mathfrak{g}$. We show that this structure is canonical with vanishing Reeb Killing function.

Since each structure J_i is integrable, one can easily verify that the almost 3-contact structure is hypernormal. Each fundamental 2-form Φ_i satisfies $\Phi_i(X, Y) = -\Omega_i(X, Y)$, so that (2.10) implies

$$d\Phi_i(\varphi_i X, \varphi_i Y, \varphi_i Z) = d\Phi_j(\varphi_j X, \varphi_j Y, \varphi_j Z)$$

for every $i, j = 1, 2, 3$ and $X, Y, Z \in \mathcal{H}$. Moreover, each ξ_i is a Killing vector field and the tensor fields A_{ij} are all vanishing.

By the previous examples, one could be tempted to believe that parallel canonical almost 3-contact metric manifolds are always locally isometric to products, and hence of limited interest. In Example 4.1.3, it is shown that S^7 carries in a natural way a parallel canonical almost 3-contact metric structure as well.

2.2 A generalization of Kashiwada's theorem and 3-(α, δ)-Sasaki manifolds

In [Ka01] T. Kashiwada proved that a 3-contact metric manifold is necessarily 3-Sasakian. We shall show that hypernormality is in fact a key property of a larger class of almost 3-contact metric manifolds, the so-called 3-(α, δ)-Sasaki manifolds.

Definition 2.2.1. An almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ will be called a 3-(α, δ)-*Sasaki manifold* if it satisfies

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k \tag{2.11}$$

for every even permutation (i, j, k) of $(1, 2, 3)$, where $\alpha \neq 0$ and δ are real constants. A 3-(α, δ)-Sasaki manifold will be called *degenerate* if $\delta = 0$ and *nondegenerate* otherwise—quaternionic Heisenberg groups are examples of degenerate 3-(α, δ)-Sasaki manifolds (Example 2.3.2). When $\alpha = \delta = 1$, we have a 3-contact metric manifold, and hence a 3-Sasaki manifold by Kashiwada's theorem [Ka01]. For $\alpha = \delta$, one easily verifies that the manifold is 3-α-Sasakian.

Remark 2.2.1. This definition captures two different aspects. First, the manifold is what one could call a 'horizontal 3-α-contact metric manifold' in the sense that it satisfies the α-contact condition (1.3) for horizontal vector fields,

$$d\eta_i(X, Y) = 2\alpha\Phi_i(X, Y) \quad \forall X, Y \in \Gamma(\mathcal{H}).$$

The second term proportional to $\eta_j \wedge \eta_k$ is reminiscent of the definition of 3-δ-cosymplectic manifolds, see equation (2.6); however, it is not a generalization of this notion, since α is not allowed to vanish and the fundamental 2-forms Φ_i need not be closed.

The following consequences are immediate. In particular, the second property interprets the constant δ as the Reeb commutator function, and thus yields a first hint why the distinction between degenerate and nondegenerate 3-(α, δ)-Sasaki manifolds is reasonable; the $\mathcal{H}$-homothetic deformations to be studied in the next section will give further justification for this distinction.

Lemma 2.2.1. *Any* 3-(α, δ)-*Sasaki manifold* $(M, \varphi_i, \xi_i, \eta_i, g)$ *satisfies:*

1) *Each ξ_i is an infinitesimal automorphism of the distribution $\mathcal{H}$, i. e.*

$$d\eta_r(X, \xi_s) = 0 \qquad X \in \Gamma(\mathcal{H}),\ r, s = 1, 2, 3;$$

2) *The constant δ is the Reeb commutator function,*

$$d\eta_r(\xi_s, \xi_t) = -2\delta\epsilon_{rst}, \qquad r, s, t = 1, 2, 3;$$

3) *The differentials $d\Phi_i$ are given by*

$$d\Phi_i = 2(\delta - \alpha)(\eta_k \wedge \Phi_j - \eta_j \wedge \Phi_k). \tag{2.12}$$

Proof. Only the last claim requires a proof. By differentiating (2.11), one obtains

$$2\alpha\, d\Phi_i + 2(\alpha - \delta)(d\eta_j \wedge \eta_k - \eta_j \wedge d\eta_k) = 0.$$

Applying again (2.11), the result follows since $\alpha \neq 0$. □

The proof that any 3-(α, δ)-Sasaki manifold has Killing Reeb vector fields and admits a constant Reeb Killing function requires more work, see Corollary 2.3.1 and Corollary 2.3.3. As a first crucial result, we prove the announced generalization of Kashiwada's theorem.

Theorem 2.2.1 (Generalized Kashiwada Theorem)**.** *Any* 3-(α, δ)*-Sasaki manifold is hypernormal.*

Proof. We shall compute the tensor fields N_{φ_i}, distinguishing case by case between horizontal and vertical vector fields. Again, the crucial ingredient is our Lemma 1.3.3, which expressed N_{φ_i} in terms of η_i and Φ_i. We always denote by X, Y, Z horizontal vector fields and by (i, j, k) an even permutation of $(1, 2, 3)$.

First, equation (2.12) implies $d\Phi_i(X, Y, Z) = 0$ and thus $N_{\varphi_i}(X, Y, Z) = 0\ \forall i = 1, 2, 3$ by Remark 1.3.1. Notice that, since

$$\xi_i \lrcorner\, \Phi_i = 0, \qquad \xi_j \lrcorner\, \Phi_i = -\eta_k, \qquad \xi_k \lrcorner\, \Phi_i = \eta_j,$$

equation (2.12) implies that

$$\xi_i \lrcorner\, d\Phi_i = 0, \quad \xi_j \lrcorner\, d\Phi_i = -2(\delta - \alpha)(\Phi_k + \eta_{ij}), \quad \xi_k \lrcorner\, d\Phi_i = 2(\delta - \alpha)(\Phi_j + \eta_{ki}). \tag{2.13}$$

Therefore, Lemma 1.3.3 implies after applying (2.11) and (2.13):

$$\begin{aligned}
N_{\varphi_i}(X, \xi_i, Z) &= -d\Phi_j(X, \xi_i, \varphi_j Z) + d\Phi_k(\varphi_i X, \xi_i, \varphi_j Z) + d\eta_j(\varphi_i X, \varphi_j Z) + d\eta_k(X, \varphi_j Z) \\
&= -2(\delta - \alpha)\Phi_k(\varphi_j Z, X) - 2(\delta - \alpha)\Phi_j(\varphi_j Z, \varphi_i X) \\
&\quad + 2\alpha\Phi_j(\varphi_i X, \varphi_j Z) + 2\alpha\Phi_k(X, \varphi_j Z) \\
&= 2\delta\Phi_j(\varphi_i X, \varphi_j Z) + 2\delta\Phi_k(X, \varphi_j Z) = -2\delta g(\varphi_i X, Z) - 2\delta g(X, \varphi_i Z) = 0, \\
N_{\varphi_i}(X, \xi_j, Z) &= d\Phi_j(\varphi_i X, \xi_k, \varphi_j Z) + d\Phi_k(\varphi_i X, \xi_j, \varphi_j Z) \\
&= -2(\delta - \alpha)\Phi_i(\varphi_j Z, \varphi_i X) + 2(\delta - \alpha)\Phi_i(\varphi_j Z, \varphi_i X) = 0, \\
N_{\varphi_i}(X, \xi_k, Z) &= -d\Phi_j(X, \xi_k, \varphi_j Z) - d\Phi_k(X, \xi_j, \varphi_j Z) \\
&= 2(\delta - \alpha)\Phi_i(\varphi_j Z, X) - 2(\delta - \alpha)\Phi_i(\varphi_j Z, X) = 0.
\end{aligned}$$

From the definition of N_{φ_i} (see equation (2.3)), we have

$$\begin{aligned}
N_{\varphi_i}(X, Y, \xi_i) &= -d\eta_i(\varphi_i X, \varphi_i Y) + d\eta_i(X, Y), \\
N_{\varphi_i}(X, Y, \xi_j) &= -d\eta_j(\varphi_i X, \varphi_i Y) + d\eta_j(X, Y) - d\eta_k(\varphi_i X, Y) - d\eta_k(X, \varphi_i Y), \\
N_{\varphi_i}(X, Y, \xi_k) &= -d\eta_k(\varphi_i X, \varphi_i Y) + d\eta_k(X, Y) + d\eta_j(\varphi_i X, Y) + d\eta_j(X, \varphi_i Y).
\end{aligned}$$

Using the fact that the structure is horizontal 3-α-contact, the above terms are all vanishing. On the other hand, one can easily verify that this is coherent with Lemma 1.3.3. Notice that (2.13) implies $d\Phi_r(X,\xi_s,\xi_t)=0$ for every $r,s,t=1,2,3$ and $X\in\Gamma(\mathcal{H})$. Taking also into account that $d\eta_r(X,\xi_s)=0$, we deduce from (1.13) that

$$N_{\varphi_r}(X,\xi_s,\xi_t)=N_{\varphi_r}(\xi_s,\xi_t,X)=0.$$

Finally, (1.13) implies together with $d\eta_r(\xi_s,\xi_t)=-2\delta\epsilon_{rst}$ that

$$N_{\varphi_i}(\xi_i,\xi_j,\xi_k)=N_{\varphi_i}(\xi_i,\xi_k,\xi_j)=N_{\varphi_i}(\xi_j,\xi_k,\xi_i)=0,$$

completing the proof that M is hypernormal. □

2.3 Properties and examples of 3-(α,δ)-Sasaki manifolds

We shall describe the behaviour of 3-(α,δ)-Sasaki structures under a special type of deformations, inspired by the classical $\mathcal{D}$-homothetic deformations of almost contact metric structures. Given an almost 3-contact metric structure $(\varphi_i,\xi_i,\eta_i,g)$, one can consider the deformed structure

$$\eta_i'=c\eta_i,\qquad \xi_i'=\frac{1}{c}\xi_i,\qquad \varphi_i'=\varphi_i,\qquad g'=ag+b\sum_{i=1}^{3}\eta_i\otimes\eta_i, \tag{2.14}$$

where a,b,c are real numbers such that $a>0$, $a+b>0$, $c\neq 0$. One can show that $(\varphi_i',\xi_i',\eta_i',g')$ is an almost 3-contact metric structure if and only if $c^2=a+b$. Indeed, each $(\varphi_i',\xi_i',\eta_i')$ is an almost contact structure and equations (1.5) are readily verified. As for the Riemannian metric g', if (i,j,k) is an even permutation of $(1,2,3)$, we have

$$\begin{aligned}
g'(\varphi_i'X,\varphi_i'Y)&=ag(\varphi_iX,\varphi_iY)+b\{\eta_j(\varphi_iX)\eta_j(\varphi_iY)+\eta_k(\varphi_iX)\eta_k(\varphi_iY)\}\\
&=ag(X,Y)-a\eta_i(X)\eta_i(Y)+b\{\eta_j(X)\eta_j(Y)+\eta_k(X)\eta_k(Y)\}\\
&=g'(X,Y)-(a+b)\eta_i(X)\eta_i(Y)\\
&=g'(X,Y)-\frac{a+b}{c^2}\eta_i'(X)\eta_i'(Y).
\end{aligned}$$

Therefore, g' is compatible with the structure $(\varphi_i',\xi_i',\eta_i')$ if and only if $c^2=a+b$. In particular, for $c=a$, we can choose $b=a(a-1)$.

Definition 2.3.1. The deformation (2.14) of an almost 3-contact metric manifold with real parameters a,b,c satisfying $a>0$, $a+b>0$, $c\neq 0$, $c^2=a+b$ will be called a *$\mathcal{H}$-homothetic deformation* of the original manifold, and an almost 3-contact metric manifold which can be obtained through a $\mathcal{H}$-homothetic deformation will be called *$\mathcal{H}$-homothetic* to the original manifold.

Proposition 2.3.1. *The $\mathcal{H}$-homothetic deformation of a 3-(α,δ)-Sasaki manifold is again a 3-(α',δ')-Sasaki manifold with*

$$\alpha'=\alpha\frac{c}{a},\qquad \delta'=\frac{\delta}{c}.$$

In particular, $\alpha'\delta'$ has the same sign as $\alpha\delta$, and the $\mathcal{H}$-homothetic deformation is degenerate if and only if the undeformed 3-(α,δ)-Sasaki manifold is degenerate.

Proof. The fundamental 2-form of the deformed structure is given by

$$\begin{aligned}
\Phi_i'(X,Y)=g'(X,\varphi_iY)&=ag(X,\varphi_iY)+b\{\eta_j(X)\eta_j(\varphi_iY)+\eta_k(X)\eta_k(\varphi_iY)\}\\
&=a\Phi_i(X,Y)+b\{-\eta_j(X)\eta_k(Y)+\eta_k(X)\eta_j(Y)\},
\end{aligned}$$

where (i,j,k) is an even permutation of $(1,2,3)$, and thus $\Phi_i' = a\Phi_i - b\,\eta_{jk}$. Then, if $(\varphi_i, \xi_i, \eta_i, g)$ is 3-(α,δ)-Sasaki, we have

$$\begin{aligned} d\eta_i' &= 2\alpha c\,\Phi_i + 2(\alpha-\delta)c\,\eta_{jk} = 2\alpha\frac{c}{a}(\Phi_i' + b\,\eta_{jk}) + 2(\alpha-\delta)c\,\eta_{jk} \\ &= 2\alpha\frac{c}{a}\Phi_i' + 2\Big(\alpha c\frac{b+a}{a} - \delta c\Big)\eta_{jk} = 2\alpha\frac{c}{a}\Phi_i' + 2\Big(\alpha\frac{c}{a} - \frac{\delta}{c}\Big)\eta_j' \wedge \eta_k', \end{aligned}$$

where we applied $c^2 = a + b$. □

Example 2.3.1 ($\mathcal{H}$-Deformations of 3-Sasaki manifolds I)**.** As a consequence of the above proposition, if $(\varphi_i, \xi_i, \eta_i, g)$ is a 3-Sasakian structure ($\alpha = \delta = 1$), the $\mathcal{H}$-deformed structure is a 3-(α',δ')-Sasaki structure with $\alpha' \neq \delta'$ whenever $b \neq 0$. Indeed

$$\alpha' - \delta' = \frac{c}{a} - \frac{1}{c} = \frac{c^2 - a}{ac} = \frac{b}{ac} \neq 0.$$

Another interesting special case of $\mathcal{H}$-Deformations of 3-Sasaki manifolds with interesting curvature properties will be discussed in Example 4.4.1.

Conversely, let us describe those 3-(α,δ)-Sasaki manifolds which admit a 3-$\tilde{\alpha}$-Sasakian $\mathcal{H}$-homothetic deformation (the tilde only serves to distinguish which parameters correspond to which manifold):

Lemma 2.3.1. *A 3-(α,δ)-Sasaki manifold is $\mathcal{H}$-homothetic to a 3-$\tilde{\alpha}$-Sasaki manifold if and only if it is nondegenerate with $\alpha\delta > 0$.*

Proof. Since 3-$\tilde{\alpha}$-Sasaki manifolds are nondegenerate, this condition is obvious by Proposition 2.3.1, so we can assume $\delta \neq 0$. In order to be 3-α-Sasakian, we must find admissible parameters a, b, c such that $\alpha' = \delta'$, which is equivalent to $\frac{\alpha}{\delta} = \frac{a}{c^2}$. Since $a > 0$, the necessary condition $\alpha\delta > 0$ follows. Furthermore, we can assume that $\alpha, \delta > 0$, since the $\mathcal{H}$-homothetic deformation with parameters $a = 1, c = -1, b = 0$ changes the signs of α and δ. For $\alpha, \delta > 0$, one checks that

$$a = 1,\ b = \frac{\delta}{\alpha} - 1,\ c = \sqrt{\frac{\delta}{\alpha}} \tag{2.15}$$

is the desired deformation. □

Remark 2.3.1. Observe that under $\mathcal{H}$-homothetic deformations given by (2.15), $\alpha\delta = \text{const.}$, so all 3-$(\alpha,\delta)$-Sasaki manifolds which are $\mathcal{H}$-homothetic under the given deformation to a fixed 3-α-Sasaki manifold lie on a 'hyperbola' in the (α,δ)-plane. For $\alpha = \delta$, the $\mathcal{H}$-homothetic deformation coincides with the identity.

Thus, the questions arises whether 3-(α,δ)-Sasaki manifolds with $\alpha\delta < 0$ exist at all. In order to answer this question positively, we need the notion of *negative* 3-*Sasakian manifolds.* They are defined as normal almost 3-contact manifolds $(M, \varphi_i, \xi_i, \eta_i)$ endowed with a compatible semi-Riemannian metric $\tilde{g}$, i. e. $\tilde{g}(\varphi_i X, \varphi_i Y) = \tilde{g}(X,Y) - \eta_i(X)\eta_i(Y)$ such that $\tilde{g}$ has signature $(3, 4n)$, where $4n+3$ is the dimension of M, and $d\eta_i(X,Y) = 2\tilde{g}(X, \varphi_i Y)$. It is known that quaternionic Kähler (not hyperKähler) manifolds with negative scalar curvature admit a canonically associated principal SO(3)-bundle $P(M)$ which is endowed with a negative 3-Sasakian structure [Ko75, Ta96].

If $(M, \varphi_i, \xi_i, \eta_i, \tilde{g})$ is a negative 3-Sasakian manifold, as in [Ta96] we consider the Riemannian metric

$$g = -\tilde{g} + 2\sum_{i=1}^{3} \eta_i \otimes \eta_i$$

which is compatible with the structure $(\varphi_i, \xi_i, \eta_i)$. A simple computation shows that

$$d\eta_i(X,Y) = -2g(X, \varphi_i Y) - 4(\eta_j \wedge \eta_k)(X,Y)$$

for every vector fields X, Y. Therefore $(M, \varphi_i, \xi_i, \eta_i, g)$ is a 3-(α, δ)-Sasaki manifold with $\alpha = -1$ and $\delta = 1$. Applying the $\mathcal{H}$-homothetic deformation (2.14), one obtains a 3-(α', δ')-Sasaki structure with $\alpha'\delta' < 0$, and $\alpha' \neq -\delta'$ whenever $b \neq 0$. Indeed

$$\alpha' + \delta' = -\frac{c}{a} + \frac{1}{c} = \frac{a - c^2}{ac} = -\frac{b}{ac} \neq 0.$$

Conversely, we have the following

Lemma 2.3.2. *A 3-(α, δ)-Sasaki manifold is $\mathcal{H}$-homothetic to a 3-$(\tilde{\alpha}, \tilde{\delta})$-Sasaki manifold with $\tilde{\alpha} = -\tilde{\delta} < 0$ if and only if it is nondegenerate with $\alpha\delta < 0$.*

Proof. By Proposition 2.3.1 an $\mathcal{H}$-homothetic deformation of a 3-$(\tilde{\alpha}, \tilde{\delta})$-Sasaki manifold with $\tilde{\alpha} = -\tilde{\delta}$, is 3-$(\alpha, \delta)$-Sasaki with $\alpha\delta < 0$. Conversely, given a 3-(α, δ)-Sasaki manifold with $\alpha\delta < 0$, first we can assume that $\alpha < 0$ and $\delta > 0$ since the $\mathcal{H}$-homothetic deformation with parameters $a = 1, c = -1, b = 0$ changes the signs of α and δ. Then, we need admissible parameters a, b, c such that $\alpha' = -\delta'$, which is equivalent to $\frac{\alpha}{\delta} = -\frac{a}{c^2}$. The desired deformation is

$$a = 1, \; b = -\frac{\delta}{\alpha} - 1, \; c = \sqrt{-\frac{\delta}{\alpha}}. \tag{2.16}$$

□

Remark 2.3.2. Under $\mathcal{H}$-homothetic deformations given by (2.16), $\alpha\delta = \text{const}$. Then, all 3-$(\alpha, \delta)$-Sasaki manifolds which are $\mathcal{H}$-homothetic under the given deformation to a fixed 3-$(\tilde{\alpha}, \tilde{\delta})$-Sasaki manifold, with $\tilde{\alpha} = -\tilde{\delta}$, lie on a 'hyperbola' in the (α, δ)-plane.

We will now further investigate the geometry of 3-(α, δ)-Sasaki manifolds. First we prove some preliminary formulas.

Lemma 2.3.3. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold. Then the associated fundamental 2-forms satisfy for all $X, Y \in \mathfrak{X}(M)$ and any cyclic permutation (i, j, k) of $(1, 2, 3)$ the following relations:*

$$\Phi_j(\varphi_i X, \varphi_i Y) = -\Phi_j(X, Y) + (\eta_i \wedge \eta_k)(X, Y), \tag{2.17}$$
$$\Phi_k(\varphi_i X, \varphi_i Y) = -\Phi_k(X, Y) - (\eta_i \wedge \eta_j)(X, Y), \tag{2.18}$$
$$\Phi_j(\varphi_i X, Y) = -\Phi_k(X, Y) - \eta_i(X)\eta_j(Y), \tag{2.19}$$
$$\Phi_k(\varphi_i X, Y) = \Phi_j(X, Y) - \eta_i(X)\eta_k(Y). \tag{2.20}$$

Proof. Applying (1.5), we have

$$\begin{aligned}
\Phi_j(\varphi_i X, \varphi_i Y) &= g(\varphi_i X, \varphi_j \varphi_i Y) = g(\varphi_i X, -\varphi_i \varphi_j Y + \eta_i(Y)\xi_j + \eta_j(Y)\xi_i) \\
&= -g(X, \varphi_j Y) + \eta_i(X)\eta_i(\varphi_j Y) - \eta_i(Y) g(X, \varphi_i \xi_j) \\
&= -\Phi_j(X, Y) + \eta_i(X)\eta_k(Y) - \eta_i(Y)\eta_k(X)
\end{aligned}$$

which proves (2.17). As regards (2.19), we have

$$\Phi_j(\varphi_i X, Y) = -g(X, \varphi_i \varphi_j Y) = -g(X, \varphi_k Y + \eta_j(Y)\xi_i) = -\Phi_k(X, Y) - \eta_i(X)\eta_j(Y).$$

Analogously, one shows (2.18) and (2.20). □

The following two propositions contain the necessary preparations for showing that 3-(α, δ)-Sasaki manifolds are canonical, as they yield remarkable identities for $\nabla^g \varphi_i$ and $\nabla^g \xi_i$.

Proposition 2.3.2. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α, δ)-Sasaki manifold. Then the Levi-Civita connection satisfies for all $X, Y \in \mathfrak{X}(M)$ and any cyclic permutation (i, j, k) of $(1, 2, 3)$:*

$$\begin{aligned}
(\nabla^g_X \varphi_i) Y = \alpha \, &[g(X, Y)\xi_i - \eta_i(Y) X] - 2(\alpha - \delta) \, [\eta_k(X)\varphi_j Y - \eta_j(X)\varphi_k Y] \\
&+ (\alpha - \delta) \, [\eta_j(X)\eta_j(Y) + \eta_k(X)\eta_k(Y)] \, \xi_i \\
&- (\alpha - \delta) \, \eta_i(Y) \, [\eta_j(X)\xi_j + \eta_k(X)\xi_k] \, .
\end{aligned} \tag{2.21}$$

Proof. Since the structure is hypernormal, the Levi-Civita connection satisfies by [Bl10, Lemma 6.1] the identity (2.9) used before. Let us evaluate its terms. To start with, the defining relation (2.12) of a 3-(α, δ)-Sasaki manifold and the preceding equations (2.17)-(2.20) yield

$$\begin{aligned}
&d\Phi_i(X, \varphi_i Y, \varphi_i Z) = \\
&= 2(\delta - \alpha)[\eta_k(X)\Phi_j(\varphi_i Y, \varphi_i Z) - \eta_j(X)\Phi_k(\varphi_i Y, \varphi_i Z) + \eta_k(\varphi_i Y)\Phi_j(\varphi_i Z, X) \\
&\quad - \eta_j(\varphi_i Y)\Phi_k(\varphi_i Z, X) + \eta_k(\varphi_i Z)\Phi_j(X, \varphi_i Y) - \eta_j(\varphi_i Z)\Phi_k(X, \varphi_i Y)] \\
&= 2(\delta - \alpha)[-\eta_k(X)\Phi_j(Y, Z) - \eta_k(X)(\eta_k \wedge \eta_i)(Y, Z) \\
&\quad + \eta_j(X)\Phi_k(Y, Z) + \eta_j(X)(\eta_i \wedge \eta_j)(Y, Z) \\
&\quad - \eta_j(Y)\Phi_k(Z, X) - \eta_j(Y)\eta_i(Z)\eta_j(X) + \eta_k(Y)\Phi_j(Z, X) - \eta_k(Y)\eta_i(Z)\eta_k(X) \\
&\quad - \eta_j(Z)\Phi_k(X, Y) + \eta_j(Z)\eta_i(Y)\eta_j(X) + \eta_k(Z)\Phi_j(X, Y) + \eta_k(Z)\eta_i(Y)\eta_k(X)] \\
&= d\Phi_i(X, Y, Z) + 4(\delta - \alpha)[-\eta_k(X)\Phi_j(Y, Z) + \eta_j(X)\Phi_k(Y, Z)] \\
&\quad + 4(\delta - \alpha)\eta_j(X)[\eta_i(Y)\eta_j(Z) - \eta_j(Y)\eta_i(Z)] \\
&\quad + 4(\delta - \alpha)\eta_k(X)[\eta_i(Y)\eta_k(Z) - \eta_k(Y)\eta_i(Z)].
\end{aligned}$$

On the other hand, again using the defining relation (2.11), we obtain

$$\begin{aligned}
&d\eta_i(\varphi_i Y, X)\eta_i(Z) - d\eta_i(\varphi_i Z, X)\eta_i(Y) = \\
&= \eta_i(Z)[2\alpha\Phi_i(\varphi_i Y, X) + 2(\alpha - \delta)(\eta_j \wedge \eta_k)(\varphi_i Y, X)] \\
&\quad - \eta_i(Y)[2\alpha\Phi_i(\varphi_i Z, X) + 2(\alpha - \delta)(\eta_j \wedge \eta_k)(\varphi_i Z, X)] \\
&= 2\alpha[g(X, Y)\eta_i(Z) - g(X, Z)\eta_i(Y)] \\
&\quad + 2(\alpha - \delta)\eta_i(Z)[-\eta_k(Y)\eta_k(X) - \eta_j(X)\eta_j(Y)] \\
&\quad - 2(\alpha - \delta)\eta_i(Y)[-\eta_k(Z)\eta_k(X) - \eta_j(X)\eta_j(Z)]
\end{aligned}$$

Inserting the above computations in (2.9), we conclude that

$$\begin{aligned}
g((\nabla^g_X \varphi_i)Y, Z) = {} & \alpha[g(X, Y)\eta_i(Z) - g(X, Z)\eta_i(Y)] \\
& + 2(\delta - \alpha)[\eta_k(X)g(\varphi_j Y, Z) - \eta_j(X)g(\varphi_k Y, Z)] \\
& - (\alpha - \delta)\eta_i(Z)[-\eta_k(Y)\eta_k(X) - \eta_j(X)\eta_j(Y)] \\
& + (\alpha - \delta)\eta_i(Y)[-\eta_k(Z)\eta_k(X) - \eta_j(X)\eta_j(Z)]
\end{aligned}$$

which implies (2.21). □

Corollary 2.3.1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α, δ)-Sasaki manifold. Then, for every $X \in \mathfrak{X}(M)$ and for every even permutation (i, j, k) of $(1, 2, 3)$,*

$$\nabla^g_X \xi_i = -\alpha\varphi_i X - (\alpha - \delta)\,[\eta_k(X)\xi_j - \eta_j(X)\xi_k], \tag{2.22}$$

$$\nabla^g_{\xi_i} \xi_i = 0, \quad \nabla^g_{\xi_i} \xi_j = -\nabla^g_{\xi_j} \xi_i = \delta\xi_k. \tag{2.23}$$

Consequently, each ξ_i is a Killing vector field. Furthermore, the distribution $\mathcal{V}$ is integrable with totally geodesic leaves.

Proof. Applying (2.21) for $Y = \xi_i$, we get

$$(\nabla^g_X \varphi_i)\xi_i = \alpha[\eta_i(X)\xi_i - X] + (\alpha - \delta)[\eta_j(X)\xi_j + \eta_k(X)\xi_k].$$

Applying φ_i on both hand-sides, since $(\nabla^g_X \varphi_i)\xi_i = -\varphi_i(\nabla^g_X \xi_i)$, we obtain (2.22). Equations (2.23) are immediate consequences of (2.22). It follows that the distribution $\mathcal{V}$ is integrable with totally geodesic leaves. In particular $[\xi_i, \xi_j] = 2\delta\xi_k$. Finally, from (2.22) we have

$$g(\nabla^g_X \xi_i, Y) = \alpha\Phi_i(X, Y) + (\alpha - \delta)(\eta_j \wedge \eta_k)(X, Y)$$

for every $X, Y \in \mathfrak{X}(M)$. Since $\nabla^g \xi_i$ is skew-symmetric, ξ_i is Killing. □

Corollary 2.3.2. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α,δ)-Sasaki manifold. Then for every even permutation (i,j,k) of $(1,2,3)$ we have*

$$\mathcal{L}_{\xi_i}\varphi_i = 0, \qquad \mathcal{L}_{\xi_i}\varphi_j = -\mathcal{L}_{\xi_j}\varphi_i = 2\delta\varphi_k. \tag{2.24}$$

Proof. For the first Lie derivative, notice that by (2.21) we have $\nabla^g_{\xi_i}\varphi_i = 0$. Then, applying also (2.22), for every vector field X we have

$$\begin{aligned}(\mathcal{L}_{\xi_i}\varphi_i)X &= [\xi_i, \varphi_i X] - \varphi_i[\xi_i, X] \\ &= \nabla^g_{\xi_i}(\varphi_i X) - \nabla^g_{\varphi_i X}\xi_i - \varphi_i(\nabla^g_{\xi_i}X) + \varphi_i(\nabla^g_X \xi_i) \\ &= (\nabla^g_{\xi_i}\varphi_i)X - \nabla^g_{\varphi_i X}\xi_i + \varphi_i(\nabla^g_X\xi_i) \\ &= \alpha\varphi_i^2 X + (\alpha-\delta)[\eta_k(\varphi_i X)\xi_j - \eta_j(\varphi_i X)\xi_k] \\ &\quad - \alpha\varphi_i^2 X - (\alpha-\delta)[\eta_k(X)\varphi_i\xi_j - \eta_j(X)\varphi_i\xi_k] = 0.\end{aligned}$$

Now, using (2.21) for the covariant derivative $\nabla^g\varphi_j$, for every vector field Y, we have

$$\begin{aligned}(\nabla^g_{\xi_i}\varphi_j)Y &= \alpha[\eta_i(Y)\xi_j - \eta_j(Y)\xi_i] - 2(\alpha-\delta)\varphi_k Y + (\alpha-\delta)[\eta_i(Y)\xi_j - \eta_j(Y)\xi_i] \\ &= -2(\alpha-\delta)\varphi_k Y + (2\alpha-\delta)[\eta_i(Y)\xi_j - \eta_j(Y)\xi_i].\end{aligned}$$

Therefore, applying also (2.22), we get

$$\begin{aligned}(\mathcal{L}_{\xi_i}\varphi_j)X &= (\nabla^g_{\xi_i}\varphi_j)X - \nabla^g_{\varphi_j X}\xi_i + \varphi_j(\nabla^g_X\xi_i) \\ &= -2(\alpha-\delta)\varphi_k X + (2\alpha-\delta)[\eta_i(X)\xi_j - \eta_j(X)\xi_i] + \alpha\varphi_i\varphi_j X \\ &\quad + (\alpha-\delta)\eta_k(\varphi_j X)\xi_j - \alpha\varphi_j\varphi_i X + (\alpha-\delta)\eta_j(X)\varphi_j\xi_k \\ &= 2\delta\varphi_k X + \alpha[\eta_j(X)\xi_i - \eta_i(X)\xi_j] + (2\alpha - \delta + \delta - \alpha)[\eta_i(X)\xi_j - \eta_j(X)\xi_i] \\ &= 2\delta\varphi_k X.\end{aligned}$$

Analogously, one shows that $\mathcal{L}_{\xi_j}\varphi_i = -2\delta\varphi_k$. ☐

Corollary 2.3.3. *Every 3-(α,δ)-Sasaki manifold is a canonical almost 3-contact metric manifold. In particular, it admits a constant Reeb Killing function $\beta = 2(\delta - 2\alpha)$.*

Proof. We proved that each ξ_i is a Killing vector field in Corollary 2.3.1 and that the structure is hypernormal in Theorem 2.2.1. Furthermore, by (2.12), $d\Phi_i(X,Y,Z) = 0$ for every $X,Y,Z \in \Gamma(\mathcal{H})$. Therefore, conditions 1)-3) of Definition 2.1.3 are satisfied. As regards condition 4), we show that M admits a (constant) Reeb Killing function $\beta = 2(\delta - 2\alpha)$. Indeed, for every $X, Y \in \Gamma(\mathcal{H})$

$$\begin{aligned}A_i(X,Y) &= g((\mathcal{L}_{\xi_i}\varphi_i)X, Y) + d\eta_i(X, \varphi_i Y) + d\eta_i(\varphi_i X, Y) \\ &= 2\alpha\Phi_i(X, \varphi_i Y) + 2\alpha\Phi_i(\varphi_i X, Y) = 0,\end{aligned}$$

where we have applying (2.24) and (2.11). Moreover, for every even permutation (i,j,k) of $(1,2,3)$ and for every $X, Y \in \Gamma(\mathcal{H})$ we have

$$\begin{aligned}A_{ij}(X,Y) &= g((\mathcal{L}_{\xi_j}\varphi_i)X, Y) + d\eta_j(X, \varphi_i Y) + d\eta_j(\varphi_i X, Y) \\ &= -2\delta g(\varphi_k X, Y) + 2\alpha\Phi_j(X, \varphi_i Y) + 2\alpha\Phi_j(\varphi_i X, Y) \\ &= 2\delta g(X, \varphi_k Y) + 2\alpha g(X, \varphi_j\varphi_i Y) - 2\alpha g(X, \varphi_i\varphi_j Y) \\ &= 2(\delta - 2\alpha)\Phi_k(X,Y).\end{aligned}$$

Analogously, one checks $A_{ji}(X,Y) = -2(\delta - 2\alpha)\Phi_k(X,Y)$. Hence, M is canonical. ☐

Definition 2.3.2. Accordingly, we will call a 3-(α,δ)-Sasaki manifold with $\delta = 2\alpha$ a *parallel* 3-(α,δ)-Sasaki manifold, compare Definition 2.1.2.

We close the section with some examples of degenerate 3-(α, δ)-Sasaki manifolds and the observation that any 3-(α, δ)-Sasaki manifold admits an underlying quaternionic contact Einstein structure; this will allow us to compute the Riemannian Ricci curvature of a 3-(α, δ)-Sasaki manifold.

Example 2.3.2 (Quaternionic Heisenberg groups)**.** The quaternionic Heisenberg group of dimension $4p+3$ is the connected, simply connected Lie group N_p with Lie algebra $\mathfrak{n}_p$ spanned by vector fields $\xi_1, \xi_2, \xi_3, \tau_r, \tau_{p+r}, \tau_{2p+r}, \tau_{3p+r}$, $r = 1, \ldots, p$, whose non-vanishing commutators are:

$$\begin{aligned} &[\tau_r, \tau_{p+r}] = \lambda\xi_1 && [\tau_r, \tau_{2p+r}] = \lambda\xi_2 && [\tau_r, \tau_{3p+r}] = \lambda\xi_3 \\ &[\tau_{2p+r}, \tau_{3p+r}] = \lambda\xi_1 && [\tau_{3p+r}, \tau_{p+r}] = \lambda\xi_2 && [\tau_{p+r}, \tau_{2p+r}] = \lambda\xi_3, \end{aligned}$$

where λ is a positive real number. As described in [AFS15], the Lie group N_p admits a left invariant almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g_\lambda)$, $i = 1, 2, 3$, where g_λ is the Riemannian metric with respect to which the above basis is orthonormal, η_i is the dual 1-form of ξ_i, and φ_i is given by

$$\varphi_i = \eta_j \otimes \xi_k - \eta_k \otimes \xi_j + \sum_{r=1}^{p} [\theta_r \otimes \tau_{ip+r} - \theta_{ip+r} \otimes \tau_r + \theta_{jp+r} \otimes \tau_{kp+r} - \theta_{kp+r} \otimes \tau_{jp+r}]$$

where θ_l, $l = 1, \ldots, 4p$, is the dual 1-form of τ_l, and (i, j, k) is an even permutation of $(1, 2, 3)$. The differential of each 1-form η_i is given by

$$d\eta_i = -\lambda \sum_{r=1}^{p} [\theta_r \wedge \theta_{ip+r} + \theta_{jp+r} \wedge \theta_{kp+r}],$$

and the fundamental 2-forms of the structure are

$$\Phi_i = -\eta_j \wedge \eta_k - \sum_{r=1}^{p} [\theta_r \wedge \theta_{ip+r} + \theta_{jp+r} \wedge \theta_{kp+r}].$$

Therefore,

$$d\eta_i = \lambda(\Phi_i + \eta_j \wedge \eta_k).$$

Then $(N_p, \varphi_i, \xi_i, \eta_i, g_\lambda)$ is a 3-(α, δ)-Sasaki manifold with $2\alpha = \lambda$ and $\delta = 0$.

The previous example suggests that there might be a deeper relation between 3-(α, δ)-Sasaki manifolds and quaternionic contact manifolds. We are now going to explain this relation.

A *quaternionic contact manifold* is a $(4n + 3)$-dimensional manifold with a distribution $\mathcal{H}$ of codimension 3 and a $\mathrm{Sp}(n)\mathrm{Sp}(1)$ structure locally defined by 1-forms $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3$. Then, $\mathcal{H} = \cap_{i=1}^{3} Ker(\tilde{\eta}_i)$ carries a positive definite symmetric tensor g, called the horizontal metric and a compatible rank-three bundle $\mathbb{Q}$ consisting of endomorphisms of $\mathcal{H}$ locally generated by three almost complex structures I_1, I_2, I_3, such that

i) $I_1 I_2 = I_3$,

ii) $g(I_i \cdot, I_i \cdot) = g(\cdot, \cdot)$,

iii) $2g(I_i X, Y) = d\tilde{\eta}_i(X, Y)$, $X, Y \in \Gamma(\mathcal{H})$.

In dimension at least eleven, such a manifold admits a unique distribution $\mathcal{V}$, called the vertical distribution, supplementary to $\mathcal{H}$, and a unique linear connection, called the Biquard connection, satisfying distinguished conditions [Bi00]. The vertical distribution is locally generated by the Reeb vector fields $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3$ such that

$$\tilde{\eta}_i(\tilde{\xi}_j) = \delta_{ij}, \qquad (\tilde{\xi}_i \lrcorner d\tilde{\eta}_j)|_{\mathcal{H}} = 0, \qquad (\tilde{\xi}_i \lrcorner d\tilde{\eta}_j)|_{\mathcal{H}} = -(\tilde{\xi}_j \lrcorner d\tilde{\eta}_i)|_{\mathcal{H}}.$$

In dimension 7 the existence of the Biquard connection still holds provided that one assumes the existence of the Reeb vector fields [Du06].

In [IMV14] the authors introduce *quaternionic contact Einstein manifolds*, briefly *qc-Einstein manifolds*, defined as quaternionic contact manifolds such that

$$\mathrm{Ric}(X,Y) = \frac{\mathrm{Scal}}{4n} g(X,Y), \qquad X,Y \in \Gamma(\mathcal{H}),$$

where Ric and Scal are respectively the qc Ricci tensor and the qc scalar curvature associated to the Biquard connection. In [IMV16] a qc-Einstein manifold is characterized as a quaternionic contact manifold whose structure satisfies

$$d\tilde{\eta}_i = 2\omega_i + S\tilde{\eta}_j \wedge \tilde{\eta}_k, \tag{2.25}$$

for every even permutation (i,j,k) of $(1,2,3)$, where S is a constant related to the qc scalar curvature by $8n(n+2)S = \mathrm{Scal}$, and each ω_i is a 2-form defined by

$$\xi \lrcorner \omega_i = 0, \qquad 2\omega_i(X,Y) = d\tilde{\eta}_i(X,Y) \qquad \xi \in \Gamma(\mathcal{V}),\ X,Y \in \Gamma(\mathcal{H}).$$

Now, the horizontal metric g can be extended to a metric h on M by requiring that $\mathcal{H}$ and $\mathcal{V}$ are orthogonal and $h(\tilde{\xi}_i, \tilde{\xi}_j) = \delta_{ij}$. Furthermore, one can consider a one-parameter family of (pseudo)Riemannian metrics h^λ, $\lambda \neq 0$, defined by

$$h^\lambda(X,Y) = h(X,Y),\ \ h^\lambda(X,\tilde{\xi}_i) = 0,\ \ h^\lambda(\tilde{\xi}_i,\tilde{\xi}_j) = \lambda h(\tilde{\xi}_i,\tilde{\xi}_j) = \lambda\delta_{ij},\ \ X,Y \in \Gamma(\mathcal{H}). \tag{2.26}$$

In [IMV16] it is proved that on a qc-Einstein manifold the (pseudo)Riemannian Ricci and scalar curvatures of h^λ are given by

$$\mathrm{Ric}^\lambda(A,B) = \Big(4n\lambda + \frac{S^2}{2\lambda}\Big) h^\lambda(A_{\mathcal{V}}, B_{\mathcal{V}}) + \big(2S(n+2) - 6\lambda\big) h^\lambda(A_{\mathcal{H}}, B_{\mathcal{H}}), \tag{2.27}$$

$$\mathrm{Scal}^\lambda = \frac{1}{\lambda}\Big(-12n\lambda^2 + 8n(n+2)S\lambda + \frac{3}{2}S^2\Big),$$

where, for a vector field A on M, $A_{\mathcal{V}}$ and $A_{\mathcal{H}}$ denote the orthogonal projections of A on $\mathcal{H}$ and $\mathcal{V}$, respectively.

Proposition 2.3.3. *Every 3-(α,δ)-Sasaki manifold $(M,\varphi_i,\xi_i,\eta_i,g)$ admits an underlying quaternionic contact structure which is qc-Einstein with $S = 2\alpha\delta$, and its Riemannian Ricci curvature is given by*

$$\mathrm{Ric}^g \ = \ 2\alpha\big(2\delta(n+2) - 3\alpha\big) g + 2(\alpha-\delta)\big((2n+3)\alpha - \delta\big)\sum_{i=1}^{3} \eta_i \otimes \eta_i.$$

In particular, a 3-(α,δ)-Sasaki manifold is Riemannian Einstein if and only if $\delta = \alpha$ or $\delta = (2n+3)\alpha$.

Proof. The underlying quaternionic contact structure is given by the horizontal distribution $\mathcal{H}$, the Riemannian metric g, the almost complex structures I_i and the 1-forms $\tilde{\eta}_i$ defined by

$$I_i := \varphi_i|_{\mathcal{H}}, \qquad \tilde{\eta}_i := -\frac{1}{\alpha}\eta_i.$$

Indeed, for all horizontal vector fields X,Y

$$d\tilde{\eta}_i(X,Y) = -\frac{1}{\alpha} d\eta_i(X,Y) = -2\Phi_i(X,Y) = 2g(I_iX,Y).$$

Let us show that this is in fact a qc-Einstein structure. For this, let ω_i be the 2-form defined by

$$\xi_r \lrcorner \omega_i = 0, \quad 2\omega_i(X,Y) = d\tilde{\eta}_i(X,Y), \qquad X,Y \in \Gamma(\mathcal{H}),$$

hence

$$\omega_i = -\Phi_i - \eta_j \wedge \eta_k.$$

Since $d\eta_i = 2\alpha\Phi_i + 2(\alpha-\delta)\eta_j \wedge \eta_k$, we have

$$\begin{aligned} d\tilde{\eta}_i &= -2\Phi_i - 2\frac{\alpha-\delta}{\alpha}\eta_j \wedge \eta_k \;=\; 2\omega_i + 2\eta_j \wedge \eta_k - 2\frac{\alpha-\delta}{\alpha}\eta_j \wedge \eta_k \\ &= 2\omega_i + 2\frac{\delta}{\alpha}\eta_j \wedge \eta_k \;=\; 2\omega_i + 2\alpha\delta\,\tilde{\eta}_j \wedge \tilde{\eta}_k. \end{aligned}$$

Therefore, (2.25) is satisfied with $S = 2\alpha\delta$. Now, the Reeb vector fields associated to the qc structure are $\tilde{\xi}_i := -\alpha\xi_i$, and the Riemannian metric g coincides with the Riemannian metric h^λ defined in (2.26) with $\lambda = \alpha^2$. Therefore, the Ricci tensor of g is

$$\mathrm{Ric}^g(A,B) = \left(4n\alpha^2 + 2\delta^2\right) g(A_{\mathcal{V}}, B_{\mathcal{V}}) + \left(4\alpha\delta(n+2) - 6\alpha^2\right) g(A_{\mathcal{H}}, B_{\mathcal{H}}).$$

Since $g(A_{\mathcal{V}}, B_{\mathcal{V}}) = \sum_{i=1}^3 \eta_i(A)\eta_i(B)$ and $g(A_{\mathcal{H}}, B_{\mathcal{H}}) = g(A,B) - \sum_{i=1}^3 \eta_i(A)\eta_i(B)$, applying (2.27) from [IMV16], we have

$$\begin{aligned} \mathrm{Ric}^g(A,B) &= \left(4\alpha\delta(n+2) - 6\alpha^2\right) g(A,B) + 2\left(\delta^2 - 2\alpha\delta(n+2) + (2n+3)\alpha^2\right) \sum_{i=1}^3 \eta_i(A)\eta_i(B) \\ &= \left(4\alpha\delta(n+2) - 6\alpha^2\right) g(A,B) + 2(\alpha-\delta)\big((2n+3)\alpha - \delta\big) \sum_{i=1}^3 \eta_i(A)\eta_i(B). \end{aligned}$$ □

In Remark 4.4.1, we will establish that the canonical connection of a 3-(α,δ)-Sasaki manifold is indeed a qc connection for the underlying quaternionic contact structure.

3 φ-compatible connections of almost 3-contact metric manifolds

3.1 General existence of φ-compatible connections

Definition 3.1.1. Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold, φ an almost contact metric structure in the associated sphere Σ_M. A linear connection ∇ on M will be called a *φ-compatible connection* if it is a metric connection with skew torsion preserving the splitting $TM = \mathcal{H} \oplus \mathcal{V}$, and such that

$$(\nabla_X \varphi)Y = 0 \qquad \forall X, Y \in \Gamma(\mathcal{H}). \tag{3.1}$$

Notice that, given a metric connection ∇ on M, the requirement that ∇ preserves the splitting of the tangent bundle is equivalent to any of the following conditions:

a) $\nabla_X Y \in \Gamma(\mathcal{H})$ for every $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(\mathcal{H})$;

b) $\nabla_X \xi_i \in \Gamma(\mathcal{V})$ for every $X \in \mathfrak{X}(M)$ and $i = 1, 2, 3$.

Therefore, if the splitting is preserved, $\mathcal{H}$ being φ-invariant, condition (3.1) is equivalent to

$$g((\nabla_X \varphi)Y, Z) = 0 \qquad \forall X, Y, Z \in \Gamma(\mathcal{H}). \tag{3.2}$$

Comparing to the characteristic connection of an almost contact metric manifold, we see that condition (3.1) is weaker than the requirement $\nabla\varphi = 0$, whereas the requirement to preserve the distributions $\mathcal{H}$ and $\mathcal{V}$ was, in the previous situation, an automatic consequence.

Remark 3.1.1. Notice that for any structure $\varphi \in \Sigma_M$, setting $J := \varphi|_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$, one has $J^2 = -I$. Therefore, $(\mathcal{H}, J)$ is an almost CR structure on M, compatible with the Riemannian metric g. In [DL14] the authors study metric connections with skew torsion on a Riemannian manifold (M, g) endowed with a compatible almost CR structure $(\mathcal{H}, J)$. The characteristic connections defined in that case are required to parallelize the structure $(\mathcal{H}, J)$. In our approach to almost 3-contact metric manifolds, the condition of φ-compatibility is weaker, since we do not require the parallelism of the tensor J along Reeb vector fields.

We shall determine necessary and sufficient conditions for an almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ to admit φ-compatible connections.

Theorem 3.1.1 (Existence of φ-compatible connections)**.** *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold, φ an almost contact metric structure in the associated sphere Σ_M. Then M admits a φ-compatible connection if and only if the following conditions are satisfied:*

i) *the tensor field N_φ is skew-symmetric on $\mathcal{H}$;*

ii) $(\mathcal{L}_{\xi_i} g)(X, Y) = 0$ *for every* $X, Y \in \Gamma(\mathcal{H})$ *and* $i = 1, 2, 3$*;*

iii) $(\mathcal{L}_X g)(\xi_i, \xi_j) = 0$ *for every* $X \in \Gamma(\mathcal{H})$ *and* $i, j = 1, 2, 3$.

The torsion of any φ-compatible connection satisfies ($X, Y, Z \in \Gamma(\mathcal{H})$, $i, j = 1, 2, 3$):

$$T(X, Y, Z) = N_\varphi(X, Y, Z) - d\Phi(\varphi X, \varphi Y, \varphi Z), \tag{3.3}$$

$$T(X, Y, \xi_i) = d\eta_i(X, Y), \tag{3.4}$$

$$T(X, \xi_i, \xi_j) = -g([\xi_i, \xi_j], X). \tag{3.5}$$

Remark 3.1.2. Thus, the φ-compatible connections are parametrized by smooth functions $T(\xi_1, \xi_2, \xi_3) := \gamma \in C^\infty(M)$. Once γ is fixed, the torsion of the corresponding φ-compatible connection is given by

$$\begin{aligned} T(X, Y, Z) &= N_\varphi(X_\mathcal{H}, Y_\mathcal{H}, Z_\mathcal{H}) - d\Phi(\varphi X_\mathcal{H}, \varphi Y_\mathcal{H}, \varphi Z_\mathcal{H}) \\ &+ \sum_i \{\eta_i(X) d\eta_i(Y_\mathcal{H}, Z_\mathcal{H}) + \eta_i(Y) d\eta_i(Z_\mathcal{H}, X_\mathcal{H}) + \eta_i(Z) d\eta_i(X_\mathcal{H}, Y_\mathcal{H})\} \\ &- \sum_{i,j} \{\eta_i(X)\eta_j(Y) g([\xi_i, \xi_j], Z_\mathcal{H}) + \eta_i(Y)\eta_j(Z) g([\xi_i, \xi_j], X_\mathcal{H}) \\ &\qquad + \eta_i(Z)\eta_j(X) g([\xi_i, \xi_j], Y_\mathcal{H})\} + \gamma \eta_{123}(X, Y, Z), \end{aligned}$$

where $X_\mathcal{H}$ denotes the horizontal part of a vector field X. Sometimes, we will call the chosen function γ the *parameter function* of the φ-compatible connection ∇.

Proof. First, one can easily check that for every $X, Y, Z \in \Gamma(\mathcal{H})$,

$$N_\varphi(X, Y, Z) = g((\nabla^g_{\varphi X}\varphi)Y - (\nabla^g_Y \varphi)\varphi X + (\nabla^g_X \varphi)\varphi Y - (\nabla^g_{\varphi Y}\varphi)X, Z). \tag{3.6}$$

Let us suppose that M admits a φ-compatible connection ∇, with torsion T. Then, from (1.4) and (3.2) we have

$$2g((\nabla^g_X \varphi)Y, Z) + T(X, \varphi Y, Z) + T(X, Y, \varphi Z) = 0 \tag{3.7}$$

for every $X, Y, Z \in \Gamma(\mathcal{H})$. Using (3.6) and (3.7) we get

$$N_\varphi(X, Y, Z) = T(X, Y, Z) - T(\varphi X, \varphi Y, Z) - T(\varphi X, Y, \varphi Z) - T(X, \varphi Y, \varphi Z), \tag{3.8}$$

which implies that N_φ is skew-symmetric on $\mathcal{H}$. Since $\nabla g = 0$ and the connection preserves the splitting of the tangent bundle, we have

$$\begin{aligned} &(\mathcal{L}_{\xi_i} g)(X, Y) = \\ &= \xi_i(g(X, Y)) - g(\nabla_{\xi_i} X - \nabla_X \xi_i - T(\xi_i, X), Y) - g(X, \nabla_{\xi_i} Y - \nabla_Y \xi_i - T(\xi_i, Y)) \\ &= T(\xi_i, X, Y) + T(\xi_i, Y, X) = 0, \end{aligned}$$

which proves ii). Analogously, we get iii). Before proving the converse, we verify equations (3.3), (3.4) and (3.5). Equations (3.4) and (3.5) are immediate consequence of the fact that ∇ preserves the splitting $TM = \mathcal{H} \oplus \mathcal{V}$. As regards (3.3), applying (3.7), for every $X, Y, Z \in \Gamma(\mathcal{H})$, we have

$$d\Phi(X, Y, Z) = -\overset{XYZ}{\mathfrak{S}}\, g((\nabla^g_X \varphi)Y, Z) = \overset{XYZ}{\mathfrak{S}}\, T(X, Y, \varphi Z), \tag{3.9}$$

where $\overset{XYZ}{\mathfrak{S}}$ denotes the cyclic sum over X, Y, Z. Applying (3.8) and (3.9), we obtain (3.3).

As for the converse, let us suppose that i)-iii) hold. Let T be the 3-form on M defined by (3.3)-(3.5) and

$$\begin{aligned} &T(\xi_i, X, Y) = -T(X, \xi_i, Y) = T(X, Y, \xi_i), \\ &T(\xi_i, \xi_j, X) = -T(\xi_i, X, \xi_j) = T(X, \xi_i, \xi_j), \\ &T(\xi_i, \xi_j, \xi_k) = \epsilon_{ijk}\gamma, \end{aligned}$$

for every $X, Y \in \Gamma(\mathcal{H})$ and $i, j, k = 1, 2, 3$, where γ is a smooth function on M. Let ∇ be the metric connection with totally skew-symmetric torsion T, which is given by (1.4). We prove that ∇ preserves the splitting $TM = \mathcal{H} \oplus \mathcal{V}$. Indeed, for every $X, Y \in \Gamma(\mathcal{H})$ and $i = 1, 2, 3$, we have

$$g(\nabla_X Y, \xi_i) \;=\; g(\nabla^g_X Y, \xi_i) - \frac{1}{2}\, g([X, Y], \xi_i) \;=\; \frac{1}{2}\, g(\nabla^g_X Y + \nabla^g_Y X, \xi_i) \;=\; -\frac{1}{2}\,(\mathcal{L}_{\xi_i} g)(X, Y)$$

which vanishes because of ii). Analogously, from iii), for every $X \in \Gamma(\mathcal{H})$ and $i, j = 1, 2, 3$,

$$g(\nabla_{\xi_j} X, \xi_i) = \frac{1}{2}\,(\mathcal{L}_X g)(\xi_i, \xi_j) = 0.$$

In order to prove (3.2), a simple computation using (3.3) and (3.6) gives

$$T(X, Y, Z) = g((\nabla^g_X \varphi)Y - (\nabla^g_Y \varphi)X, \varphi Z) - g((\nabla^g_{\varphi Z} \varphi)X, Y) \tag{3.10}$$

for every $X, Y, Z \in \Gamma(\mathcal{H})$. Applying the above formula and Lemma 1.3.1 3), we have

$$\begin{aligned} T(X, \varphi Y, Z) + T(X, Y, \varphi Z) \;&=\; g((\nabla^g_X \varphi)\varphi Y - (\nabla^g_{\varphi Y} \varphi)X, \varphi Z) - g((\nabla^g_{\varphi Z} \varphi)X, \varphi Y) \\ &\quad\; - g((\nabla^g_X \varphi)Y - (\nabla^g_Y \varphi)X, Z) + g((\nabla^g_Z \varphi)X, Y) \\ &=\; -\,2g((\nabla^g_X \varphi)Y, Z). \end{aligned}$$

Therefore,

$$g((\nabla_X \varphi)Y, Z) = g((\nabla^g_X \varphi)Y, Z) + \frac{1}{2}\,(T(X, \varphi Y, Z) + T(X, Y, \varphi Z)) = 0. \qquad \square$$

Remark 3.1.3. Condition iii) in Theorem 3.1.1 is equivalent to

$$d\eta_i(X, \xi_j) + d\eta_j(X, \xi_i) = 0,$$

or also

$$g(\nabla^g_{\xi_i} \xi_j, X) + g(\nabla^g_{\xi_j} \xi_i, X) = 0 \tag{3.11}$$

for every $X \in \Gamma(\mathcal{H})$ and $i, j = 1, 2, 3$.

A particularly simple situation occurs when N_φ vanishes on $\mathcal{H}$. We give some simple-to-check criteria when this happens.

Proposition 3.1.1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold, and $\varphi \in \Sigma_M$. Assume that M admits a φ-compatible connection ∇ with torsion T. Then, the following conditions are equivalent:*

a) $g((\nabla^g_X \varphi)Y, Z) = 0$ *for every* $X, Y, Z \in \Gamma(\mathcal{H})$,

b) $T(X, Y, Z) = 0$ *for every* $X, Y, Z \in \Gamma(\mathcal{H})$,

c) $d\Phi(X, Y, Z) = 0$ *for every* $X, Y, Z \in \Gamma(\mathcal{H})$.

If any of these conditions is satisfied, $N_\varphi(X, Y, Z) = 0$ for any $X, Y, Z \in \Gamma(\mathcal{H})$.

Proof. The equivalence of a) and b) follows from (3.10) and (3.7). Condition a) obviously implies c). Conversely, supposing that c) holds, from (3.3) it follows that T and N_φ coincide on $\mathcal{H}$. Hence, from (3.9), we have

$$N_\varphi(X, Y, \varphi Z) + N_\varphi(Y, Z, \varphi X) + N_\varphi(Z, X, \varphi Y) = 0$$

for every $X, Y, Z \in \Gamma(\mathcal{H})$. Since N_φ is skew-symmetric and $N_\varphi(\varphi X, Y) = -\varphi N_\varphi(X, Y)$, we deduce that $N_\varphi(X, Y, \varphi Z) = 0$. Therefore b), or equivalently a), holds. □

Assume that M admits φ_i-compatible connections, in the sense that conditions i)-iii) in Theorem 3.1.1 are satisfied for each structure $(\varphi_i, \xi_i, \eta, g)$. We would like to conclude that M admits φ-compatible connections for *any* $\varphi \in \Sigma_M$. Conditions ii) and iii) do not depend on the choice of φ, hence there is nothing to check. To verify condition i) for φ, we need to know that N_φ is skew-symmetric on $\mathcal{H}$ if this is true for each N_{φ_i}—but this is exactly the contents of our Proposition 1.3.1. Hence, we can state the following remarkable corollary from Proposition 1.3.1 and Theorem 3.1.1. It underlines once more that the associated sphere is a very canonical object.

Corollary 3.1.1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold. If M admits φ_i-compatible connections for every $i = 1, 2, 3$, then M admits φ-compatible connections for every almost contact metric structure φ in the associated sphere Σ_M.*

3.2 Effect of Killing Reeb vector fields

Looking back at the conditions for the existence of φ-compatible connections from Theorem 3.1.1, we see that conditions ii) and iii) on the Reeb vector fields ξ_i are rather weak—in particular, they don't need to be Killing. Nevertheless, it is a familiar fact from examples that this is the case in many interesting classes of almost 3-contact metric manifolds. Thus, we investigate the situation of Killing Reeb vector fields separately in this section.

Proposition 3.2.1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold, φ an almost contact metric structure in the associated sphere Σ_M. Given a φ-compatible connection ∇, the Reeb vector fields are ∇-parallel along the distribution $\mathcal{H}$ if and only if $(\mathcal{L}_{\xi_i} g)(X, \xi_j) = 0$ for any $X \in \Gamma(\mathcal{H})$ and $i, j = 1, 2, 3$.*

Proof. If ∇ is a φ-compatible connection, we deduce for every $X \in \Gamma(\mathcal{H})$ and $i, j = 1, 2, 3$ from (1.4), (3.5) and (3.11)

$$\begin{aligned} g(\nabla_X \xi_i, \xi_j) &= g(\nabla^g_X \xi_i, \xi_j) - \frac{1}{2} g([\xi_i, \xi_j], X) = g(\nabla^g_X \xi_i, \xi_j) - \frac{1}{2} g(\nabla^g_{\xi_i} \xi_j - \nabla^g_{\xi_j} \xi_i, X) \\ &= g(\nabla^g_X \xi_i, \xi_j) + g(\nabla^g_{\xi_j} \xi_i, X) = (\mathcal{L}_{\xi_i} g)(X, \xi_j). \end{aligned}$$

Since the distribution $\mathcal{V}$ is parallel with respect to the connection ∇, we have that $\nabla_X \xi_i = 0$ for every $X \in \Gamma(\mathcal{H})$ and $i = 1, 2, 3$, if and only if the Lie derivatives $(\mathcal{L}_{\xi_i} g)(X, \xi_j)$ are all vanishing. □

The following proposition shows that when the Reeb vector fields are Killing, the existence of φ-compatible connections and the existence of the Reeb commutator function (Definition 2.1.1), garanteed by Corollary 2.1.1, are intricately related. In return, the Reeb commutator function and the parameter function of a φ-compatible connection ∇ describe the ∇-derivative of one Reeb vector field through the other Reeb vector fields in a very symmetric expression.

Proposition 3.2.2. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold, φ an almost contact metric structure in the associated sphere. Assume that*

i) *the tensor field N_φ is skew-symmetric on $\mathcal{H}$;*

ii) *each ξ_i is a Killing vector field.*

Let δ be its Reeb commutator function. Then M admits φ-compatible connections. If ∇ is any φ-compatible connection with torsion T and parameter function γ, the following equations hold:

1) *For every $X \in \mathfrak{X}(M)$, and for every even permutation (i,j,k) of $(1,2,3)$,*

$$\nabla_X \xi_i = \tfrac{2\delta+\gamma}{2}(\eta_k(X)\xi_j - \eta_j(X)\xi_k). \tag{3.12}$$

2) *For every $i = 1,2,3$, and for every $X, Y \in \Gamma(\mathcal{H})$,*

$$g((\nabla_{\xi_i}\varphi)X, Y) = g((\mathcal{L}_{\xi_i}\varphi)X, Y) + d\eta_i(\varphi X, Y) + d\eta_i(X, \varphi Y). \tag{3.13}$$

Proof. First we prove that condition iii) in Theorem 3.1.1 is satisfied. Indeed, one can easily check that for every $X \in \Gamma(\mathcal{H})$ and $i,j = 1,2,3$,

$$(\mathcal{L}_X g)(\xi_i, \xi_j) = -(\mathcal{L}_{\xi_i} g)(X, \xi_j) - (\mathcal{L}_{\xi_j} g)(X, \xi_i) = 0.$$

Hence, by Theorem 3.1.1, M admits φ-compatible connections. Given a φ-compatible connection ∇ with torsion T, let $\gamma = T(\xi_1, \xi_2, \xi_3)$ be the parameter function. Recall that by Lemma 2.1.1, the Reeb commutator function δ satisfies

$$\eta_k([\xi_i, \xi_j]) = 2\,\eta_k(\nabla^g_{\xi_i}\xi_j) = 2\delta\epsilon_{ijk} \tag{3.14}$$

for every $i,j,k = 1,2,3$. In order to verify (3.12), by Proposition 3.2.1, we have $\nabla_X \xi_i = 0$ for every $X \in \Gamma(\mathcal{H})$. Recall that the distribution $\mathcal{V}$ is parallel with respect to the φ-compatible connection. Therefore, since $\nabla_{\xi_i}\xi_i = \nabla^g_{\xi_i}\xi_i \in \Gamma(\mathcal{V})$, formula (3.14) gives $\nabla_{\xi_i}\xi_i = 0$. Now, if (i,j,k) is an even permutation of $(1,2,3)$, for the covariant derivative $\nabla_{\xi_j}\xi_i$, we have

$$g(\nabla_{\xi_j}\xi_i, \xi_i) = 0, \quad g(\nabla_{\xi_j}\xi_i, \xi_j) = -g(\xi_i, \nabla_{\xi_j}\xi_j) = 0,$$

and applying again (3.14),

$$g(\nabla_{\xi_j}\xi_i, \xi_k) = g(\nabla^g_{\xi_j}\xi_i, \xi_k) + \frac{1}{2}T(\xi_j, \xi_i, \xi_k) = -\frac{2\delta+\gamma}{2}.$$

Therefore, $\nabla_{\xi_j}\xi_i = -\frac{2\delta+\gamma}{2}\xi_k$. Analogously, one checks that $\nabla_{\xi_k}\xi_i = \frac{2\delta+\gamma}{2}\xi_j$, completing the proof of (3.12).

We prove (3.13). Applying the Koszul formula for the Levi-Civita connection, for every $X, Y \in \Gamma(\mathcal{H})$ we have

$$\begin{aligned} 2g((\nabla^g_{\xi_i}\varphi)X, Y) &= 2g(\nabla^g_{\xi_i}(\varphi X), Y) + 2g(\nabla^g_{\xi_i}X, \varphi Y) \\ &= g([\xi_i, \varphi X], Y) - g([\varphi X, Y], \xi_i) + g([Y, \xi_i], \varphi X) \\ &\quad + g([\xi_i, X], \varphi Y) - g([X, \varphi Y], \xi_i) + g([\varphi Y, \xi_i], X). \end{aligned} \tag{3.15}$$

Since ξ_i is a Killing vector field, we get

$$\begin{aligned} 0 &= (\mathcal{L}_{\xi_i} g)(X, \varphi Y) + (\mathcal{L}_{\xi_i} g)(\varphi X, Y) \\ &= -g([\xi_i, X], \varphi Y) - g([\xi_i, \varphi Y], X) - g([\xi_i, \varphi X], Y) - g([\xi_i, Y], \varphi X). \end{aligned} \tag{3.16}$$

Therefore, from (3.15) and (3.16) it follows that

$$\begin{aligned} 2\,g((\nabla^g_{\xi_i}\varphi)X, Y) &= 2g([\xi_i, \varphi X], Y) + 2\,g([\xi_i, X], \varphi Y) - \eta_i([X, \varphi Y]) - \eta_i([\varphi X, Y]) \\ &= 2\,g((\mathcal{L}_{\xi_i}\varphi)X, Y)) + d\eta_i(X, \varphi Y) + d\eta_i(\varphi X, Y). \end{aligned}$$

Finally, applying the above formula and (3.4),

$$\begin{aligned} 2\,g((\nabla_{\xi_i}\varphi)X, Y) &= 2\,g((\nabla^g_{\xi_i}\varphi)X, Y) + T(\xi_i, \varphi X, Y) + T(\xi_i, \varphi X, Y) \\ &= 2\,g((\mathcal{L}_{\xi_i}\varphi)X, Y)) + 2\,d\eta_i(X, \varphi Y) + 2\,d\eta_i(\varphi X, Y), \end{aligned}$$

which proves (3.13). □

Remark 3.2.1. We recognize that the right hand side of eq. (3.13) in the previous Proposition is just, for $\varphi = \varphi_j$, the tensor field A_{ji} introduced in equation (2.1).

4 The canonical connection of an almost 3-contact metric manifold

4.1 General existence of the canonical connection

In the following we will provide a criterion allowing to define a unique metric connection with skew torsion on an almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$, called the *canonical connection*. Its crucial property is captured by equation (4.1), from which all others will follow. Hence, we start by explaining what singles out this particular condition.

Remark 4.1.1. Recall that a hyper-Kähler manifold can be defined as a Riemannian manifold of dimension $4n \geq 4$ admitting an anti-commuting pair I_1, I_2 of integrable complex structures, relative to both of which the metric is Kähler. This implies that I_1, I_2, and $I_3 := I_1 I_2$ are parallel for the Levi-Civita connection ∇^g. In contrast, on a general quaternion-Kähler manifold it is not possible to find individual structures I_1, I_2, I_3 that are parallel, but only a bundle of endomorphisms (namely, the one spanned by I_1, I_2, I_3) preserved as a whole; more precisely, there should exist 1-forms α_i such that

$$\nabla^g_X I_i = -\alpha_k(X) I_j + \alpha_j(X) I_k \quad \forall X \in \mathfrak{X}(M)$$

for every even permutation (i, j, k) of $(1, 2, 3)$. These equations were first considered by [Is74], see also [Sa99] for a very nice survey. Now, the analogy to equation (4.1) becomes obvious: A canonical connection is one mimicking the derivative equations of quaternion-Kähler geometry, with I_i replaced by φ_i and α_i replaced by $-\tilde{\beta}\eta_i$.

Theorem 4.1.1 (Existence of the canonical connection)**.** *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold. Then M admits a metric connection ∇ with skew torsion such that for some smooth function $\tilde{\beta}$,*

$$\nabla_X \varphi_i = \tilde{\beta}(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \forall X \in \mathfrak{X}(M) \tag{4.1}$$

for every even permutation (i, j, k) of $(1, 2, 3)$, if and only if it is a canonical almost 3-contact metric manifold.

If such a connection ∇ exists, it is unique and it is φ-compatible for every almost contact metric structure φ in the associated sphere Σ_M, and $\tilde{\beta}$ coincides with the Reeb Killing function β. The torsion of ∇ is given by (3.3)-(3.5) and the parameter function is

$$\gamma := T(\xi_1, \xi_2, \xi_3) = 2(\beta - \delta), \tag{4.2}$$

where δ is the Reeb commutator function.

Proof. Let us assume that M admits a metric connection ∇ with skew torsion satisfying (4.1). First we show that

$$\nabla_X \xi_i = \tilde{\beta}(\eta_k(X)\xi_j - \eta_j(X)\xi_k) \tag{4.3}$$

for every $X \in \mathfrak{X}(M)$ and for every even permutation (i, j, k) of $(1, 2, 3)$. Indeed, from (4.1) we have

$$(\nabla_X \varphi_i)\xi_i = -\tilde{\beta}(\eta_k(X)\xi_k + \eta_j(X)\xi_j). \tag{4.4}$$

Since $(\nabla_X \varphi_i)\xi_i = -\varphi_i(\nabla_X \xi_i)$ and $\eta_i(\nabla_X \xi_i) = 0$, applying φ_i on both sides of (4.4), we get (4.3). Therefore, being ∇ a metric connection preserving the distribution $\mathcal{V}$, it preserves the splitting $TM = \mathcal{H} \oplus \mathcal{V}$. On the other hand, since $\nabla_X \varphi_i = 0$ for every $X \in \Gamma(\mathcal{H})$, it turns out that ∇ is a φ_i-compatible connection for all $i = 1, 2, 3$. Then, conditions i)-iii) in Theorem 3.1.1 are satisfied. In particular, each N_{φ_i} is skew-symmetric on $\mathcal{H}$ and, by equation (3.3), the torsion T of ∇ satisfies

$$T(X, Y, Z) = N_{\varphi_i}(X, Y, Z) - d\Phi_i(\varphi_i X, \varphi_i Y, \varphi_i Z)$$

for every $X, Y, Z \in \Gamma(\mathcal{H})$ and $i = 1, 2, 3$, thus proving condition 3) of Definition 2.1.3 of a canonical almost 3-contact metric manifold.

In order to prove that each ξ_i is a Killing vector field, we already know, by the φ_i-compatibility, that $(\mathcal{L}_{\xi_i} g)(X,Y) = 0$ for every $X, Y \in \Gamma(\mathcal{H})$ and $i, j = 1, 2, 3$. Furthermore, since $\nabla_X \xi_i = 0$ for all $X \in \Gamma(\mathcal{H})$, Proposition 3.2.1 implies that $(\mathcal{L}_{\xi_i} g)(X, \xi_j) = 0$. Now, one can easily check that equation (4.3) implies

$$g(\nabla_{\xi_i}\xi_j, \xi_k) = \tilde{\beta}\epsilon_{ijk}$$

for all indices $i, j, k = 1, 2, 3$. Setting $\gamma := T(\xi_1, \xi_2, \xi_3)$, we have

$$g(\nabla^g_{\xi_i}\xi_j, \xi_k) = g(\nabla_{\xi_i}\xi_j, \xi_k) - \frac{1}{2}T(\xi_i, \xi_j, \xi_k) = \frac{2\tilde{\beta} - \gamma}{2}\epsilon_{ijk},$$

so that condition 3) in Lemma 2.1.1 is satisfied for $\delta = \frac{2\tilde{\beta}-\gamma}{2}$. Therefore, for all indices $i, j, k = 1, 2, 3$, we have $(\mathcal{L}_{\xi_i} g)(\xi_j, \xi_k) = 0$, or equivalently $\eta_k([\xi_i, \xi_j]) = 2\delta\epsilon_{ijk}$. This completes the proof that each ξ_i is Killing. Furthermore, notice that the linear connection ∇ is uniquely determined, the parameter function being given by $\gamma = 2(\tilde{\beta} - \delta)$.

Finally, we prove that $\tilde{\beta}$ is the Reeb Killing function. By Proposition 3.2.2, equation (3.13) holds. On the other hand, taking into account the definition of the tensor fields A_{ij} in (2.1), and applying (4.1), we have

$$\begin{aligned}
A_i(X,Y) &= g((\nabla_{\xi_i}\varphi_i)X, Y) = 0, \\
A_{ij}(X,Y) &= g((\nabla_{\xi_j}\varphi_i)X, Y) = -\tilde{\beta}g(\varphi_k X, Y) = \tilde{\beta}\Phi_k(X,Y), \\
A_{ji}(X,Y) &= g((\nabla_{\xi_i}\varphi_j)X, Y) = \tilde{\beta}g(\varphi_k X, Y) = -\tilde{\beta}\Phi_k(X,Y),
\end{aligned}$$

for every $X, Y \in \Gamma(\mathcal{H})$ and for every even permutation (i, j, k) of $(1, 2, 3)$. Hence, $\tilde{\beta}$ is the Reeb Killing function on M.

In order to prove the converse, let us assume that M is a canonical almost 3-contact metric manifold. First we notice that, since each ξ_i is a Killing vector field, Lemma 2.1.1 implies the existence of a Reeb commutator function δ, i. e. $\eta_k([\xi_i, \xi_j]) = 2\delta\epsilon_{ijk}$. By Proposition 3.2.2, M admits φ_i-compatible connections for all $i = 1, 2, 3$. We denote by ∇^i the φ_i-compatible connection with torsion T_i such that

$$T_i(\xi_1, \xi_2, \xi_3) = 2(\beta - \delta),$$

where β is the Reeb Killing function. Then, owing to condition 3) of Definition 2.1.3 and equations (3.3), (3.4), (3.5) for the torsion of a φ-compatible connection, we have $T_1 = T_2 = T_3$, hence the three connections coincide. We denote by ∇ this unique connection and we prove that it satisfies (4.1) with $\tilde{\beta} = \beta$. From Proposition 3.2.2, applying (3.12) with $\gamma = 2(\beta - \delta)$, we have

$$\nabla_X \xi_i = \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k)$$

for every $X \in \mathfrak{X}(M)$ and for every even permutation (i, j, k) of $(1, 2, 3)$. Using the above equation, one can check that

$$(\nabla_X \varphi_i)\xi_h = \beta(\eta_k(X)\varphi_j\xi_h - \eta_j(X)\varphi_k\xi_h) \tag{4.5}$$

for every $h = 1, 2, 3$. Indeed, for $h = i$, we obtain

$$(\nabla_X \varphi_i)\xi_i \; = \; -\varphi_i(\nabla_X \xi_i) \; = \; -\beta(\eta_k(X)\xi_k + \eta_j(X)\xi_j) \; = \; \beta(\eta_k(X)\varphi_j\xi_i - \eta_j(X)\varphi_k\xi_i).$$

Analogously, one verifies (4.5) for $h = j, k$. From equation (3.13) and the fact that M admits the Reeb Killing function β, for every $Y, Z \in \Gamma(\mathcal{H})$ we have

$$\begin{aligned}
g((\nabla_{\xi_i}\varphi_i)Y, Z) &= A_i(Y, Z) = 0, \\
g((\nabla_{\xi_j}\varphi_i)Y, Z) &= A_{ij}(Y, Z) = \beta\Phi_k(Y, Z) = -\beta g(\varphi_k Y, Z), \\
g((\nabla_{\xi_k}\varphi_i)Y, Z) &= A_{ik}(Y, Z) = -\beta\Phi_j(Y, Z) = \beta g(\varphi_j Y, Z),
\end{aligned}$$

where (i, j, k) is an even permutation of $(1, 2, 3)$. Then, since ∇ preserves the splitting of the tangent bundle, we have

$$(\nabla_{\xi_i}\varphi_i)Y = 0, \quad (\nabla_{\xi_j}\varphi_i)Y = -\beta\varphi_k Y, \quad (\nabla_{\xi_k}\varphi_i)Y = \beta\varphi_j Y. \tag{4.6}$$

On the other hand, since ∇ is φ_i-compatible, we have

$$(\nabla_X\varphi_i)Y = 0 \tag{4.7}$$

for all $X, Y \in \Gamma(\mathcal{H})$. Then, taking into account (4.5), (4.6), and (4.7), equation (4.1) is verified. □

Definition 4.1.1. If $(M, \varphi_i, \xi_i, \eta_i, g)$ is a canonical almost 3-contact metric manifold, the connection ∇ satisfying (4.1) of Theorem 4.1.1 will be called the *canonical connection of M*.

Remark 4.1.2. If $(M, \varphi_i, \xi_i, \eta_i, g)$ is a canonical almost 3-contact metric manifold with canonical connection ∇, the covariant derivatives of the structure tensors are completely determined by the Reeb Killing function β,

$$\begin{aligned}\nabla_X\varphi_i &= \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k),\\ \nabla_X\xi_i &= \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k),\\ \nabla_X\eta_i &= \beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k),\end{aligned}$$

where (i, j, k) is an even permutation of $(1, 2, 3)$. The first identity holds by definition, whereas the second and third identity follow from Proposition 3.2.2. In particular, we observe that each structure $(\varphi_i, \xi_i, \eta_i, g)$ is parallel along $\mathcal{H}$ and its Reeb vector field ξ_i. In analogy to quaternion-Kähler geometry, we may consider the *fundamental 4-form* Ψ

$$\Psi := \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2 + \Phi_3 \wedge \Phi_3.$$

It is independent of choice of basis and $\Psi^n \neq 0$. The last formulas imply immediately:

Corollary 4.1.1. *The canonical connection of a canonical almost 3-contact metric manifold $(M^{4n+3}, \varphi_i, \xi_i, \eta_i, g)$ leaves the associated bundle of endomorphisms Υ_M invariant and it satisfies*

$$\nabla\Psi = 0, \quad \nabla\eta_{123} = 0.$$

In particular, its holonomy algebra $\mathfrak{hol}(\nabla)$ is contained in $(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(4n) \oplus \mathfrak{so}(3)$.

By Remark 3.1.2, we know that η_{123} is one summand of the torsion of the canonical connection (and in fact of any φ-compatible connection). In general, however, the torsion will not be parallel. In Theorem 4.4.1, we shall prove that it is parallel for 3-(α, δ)-Sasaki manifolds.

Finally, let us look at the special case of vanishing Reeb Killing function β. The canonical connection satisfies then $\nabla\varphi_i = \nabla\xi_i = 0 \ \forall\ i$, i. e. all structure tensors are parallel; by uniqueness of the characteristic connection, we conclude:

Corollary 4.1.2. *The canonical connection ∇ of a parallel canonical almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ coincides with the characteristic connection ∇^i of each of its almost contact metric structures $(\varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$. Furthermore, its holonomy algebra satisfies $\mathfrak{hol}(\nabla) \subset \mathfrak{sp}(n)$.*

In general, it is not possible to derive a simple formula for the canonical torsion of a canonical almost 3-contact metric manifold. Here are some exceptional cases:

Example 4.1.1. We already know from Corollary 2.1.2 that any 3-δ-cosymplectic manifold is a parallel canonical almost 3-contact metric manifold. Therefore, the identities (3.3)-(3.5) and (4.2) imply directly that the torsion of the canonical connection is given by

$$T = -2\delta\, \eta_{123}.$$

Of course, this is not surprising if we recall that M is locally isometric to the product of a hyper-Kähler manifold with either a 3-dimensional flat group ($\delta = 0$, [CN07]) or a 3-dimensional sphere ($\delta \neq 0$, Proposition 2.1.1).

Example 4.1.2. Similarly, one shows that for the product of a HKT-manifold M with a certain Lie group G considered in Example 2.1.1, the torsion T of the canonical connection ∇ is given by

$$T = T_0 - 2\delta\, \eta_{123},$$

where we extend the 3-form T_0 on the product $M \times G$ in such a way that $\xi_i \lrcorner T_0 = 0$.

A well-celebrated theorem of Cartan and Schouten states that the only manifolds carrying a flat metric connection with torsion are compact Lie groups and the 7-sphere ([CS26]; see [D'AN68, Wo72a, Wo72b] for proofs and [AF10b] for a classification-free short proof). The following example shows that this flat connection is in fact the canonical connection of a natural almost 3-contact metric structure on S^7. We follow the notations used in [AF10b], hence we shall be brief.

Example 4.1.3 (S^7 as a non-hypernormal parallel canonical almost 3-contact metric manifold)**.** In dimension 7, the complex Spin(7)-representation $\Delta_7^{\mathbb{C}}$ is the complexification of a real 8-dimensional representation $\kappa : \mathrm{Spin}(7) \to \mathrm{End}(\Delta_7)$, since the real Clifford algebra $\mathcal{C}(7)$ is isomorphic to $\mathcal{M}(8) \oplus \mathcal{M}(8)$. Thus, we may identify $\mathbb{R}^8$ with the vector space Δ_7 and embed therein the sphere S^7 as the set of all (algebraic) spinors of length one, equipped with the induced metric g. Fix your favourite explicit realisation of the spin representation by skew matrices, $\kappa_i := \kappa(e_i) \in \mathfrak{so}(8) \subset \mathrm{End}(\mathbb{R}^8)$, $i = 1, \ldots, 7$. Define vector fields $e_1, \ldots, e_7$ on S^7 by

$$e_i(x) \;=\; \kappa_i \cdot x \text{ for } x \in S^7 \subset \Delta_7.$$

From the antisymmetry of $\kappa_1, \ldots, \kappa_7$, one deduces that they form an orthonormal global frame of TS^7. Hence, they constitute an explicit parallelisation of S^7 by Killing vector fields. We define an almost 3-contact metric structure by setting $\xi_i := e_i$ $(i = 1, 2, 3)$, $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$, $\mathcal{H} = \langle e_4, \ldots, e_7 \rangle$, and

$$\Phi_1 = -(e_{23} + e_{45} + e_{67}), \quad \Phi_2 = e_{13} - e_{46} + e_{57}, \quad \Phi_3 = -(e_{12} + e_{47} + e_{56}).$$

Furthermore, we define functions on S^7 by $\alpha_{ijk}(x) := -g(\kappa_i \kappa_j \kappa_k x, x)$: they are quadratic functions in the coordinates x_i and hence never constant (for i, j, k all different) and the properties of Clifford multiplication imply that they are totally skew-symmetric in all indices. The commutator of vector fields is inherited from the ambient space, hence $[e_i(x), e_j(x)] = 2\kappa_i\kappa_j x$ for $i \neq j$. This implies

$$[e_i(x), e_j(x)] = 2 \sum_{k=1}^{7} \alpha_{ijk}(x)\, e_k(x) \quad \forall i, j = 1, \ldots, 7,$$

and hence $\delta(x) := \alpha_{123}(x)$ is the Reeb commutator function of the almost 3-contact metric structure. One further checks that all N_{φ_i} are skew-symmetric but non-trivial. Hence, the structure is not hypernormal, but each (ξ_i, Φ_i, η_i) admits a characteristic connection. We now define a connection ∇ on TS^7 by $\nabla e_i(x) = 0\ \forall i$; observe that this implies that all tensor fields with constant coefficients like the Φ_i's are parallel as well. In particular, by its uniqueness, ∇ has to coincide with the characteristic connection of all three almost contact structures. This connection is trivially flat and metric and just the one claimed to exist by the Cartan-Schouten result. By Theorem 4.1.1, it coincides with the canonical connection of $(S^7, \xi_i, \Phi_i, \eta_i, g)$ and has vanishing Reeb Killing function β. Alltogether, we conclude that $(S^7, \xi_i, \Phi_i, \eta_i, g)$ is a non-hypernormal parallel canonical almost 3-contact metric manifold, as claimed. Just as an additional piece of information, let us observe that it is proved in [AF10b] that ∇ does not have parallel torsion (in fact, $T(e_i, e_j, e_k) = -g([e_i, e_j], e_k)$) and it is a characteristic G_2-connection of Fernandez-Gray class $\mathfrak{X}_1 \oplus \mathfrak{X}_3 \oplus \mathfrak{X}_4$.

For general β, the difference $\nabla - \nabla^i$ is computed in the next theorem.

4.2 Properties of the canonical connection

By Theorem 2.1.1, the three almost contact metric structures $(\varphi_i, \xi_i, \eta_i, g)$ of a canonical almost 3-contact metric manifold admit characteristic connections ∇^i, $i = 1, 2, 3$. As a first result, we compare them to the canonical connection.

Theorem 4.2.1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a canonical almost 3-contact metric manifold, ∇ its canonical connection, β its Reeb Killing function, and ∇^i the characteristic connections of the three almost contact metric structures $(\varphi_i, \xi_i, \eta_i, g)$. The connections ∇ and ∇^i are related by*

$$\nabla = \nabla^i - \frac{\beta}{2}(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k) \tag{4.8}$$

for every even permutation (i, j, k) of $(1, 2, 3)$.

Proof. By Theorem 1.2.1, the torsion T_i of ∇^i is given by

$$T_i = \eta_i \wedge d\eta_i + N_{\varphi_i} + d^{\varphi_i}\Phi_i - \eta_i \wedge (\xi_i \lrcorner N_{\varphi_i}). \tag{4.9}$$

We show that the torsions T and T_i of ∇ and ∇^i are related by

$$T - T_i = -\beta(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k), \tag{4.10}$$

where (i, j, k) is an even permutation of $(1, 2, 3)$. We will proceed case by case, computing the difference $T - T_i$ on horizontal and vertical vector fields. First of all, we deduce from (4.9) and the general expressions (3.3), (3.4) for the torsion of any φ-compatible connection

$$\begin{aligned} T(X, Y, Z) &= T_i(X, Y, Z) = N_{\varphi_i}(X, Y, Z) - d\Phi_i(\varphi_i X, \varphi_i Y, \varphi_i Z), \\ T(X, Y, \xi_i) &= T_i(X, Y, \xi_i) = d\eta_i(X, Y) \end{aligned}$$

for every $X, Y, Z \in \Gamma(\mathcal{H})$. Now, fixing an even permutation (i, j, k) of $(1, 2, 3)$ and using $\mathcal{L}_{\xi_k} g = 0$, for every $X, Y \in \Gamma(\mathcal{H})$ we have

$$\begin{aligned} d\Phi_i(X, Y, \xi_k) &= \xi_k(\Phi_i(X, Y)) - \Phi_i([X, Y], \xi_k) - \Phi_i([Y, \xi_k], X) - \Phi_i([\xi_k, X], Y) \\ &= -\xi_k(g(\varphi_i X, Y)) + g([X, Y], \xi_j) + g([\xi_k, Y], \varphi_i X) - g([\xi_k, X], \varphi_i Y) \\ &= -g([\xi_k, \varphi_i X], Y) - g([\xi_k, X], \varphi_i Y) + \eta_j([X, Y]) \\ &= -g((\mathcal{L}_{\xi_k}\varphi_i)X, Y) - d\eta_j(X, Y). \end{aligned}$$

One can verify that $g((\mathcal{L}_{\xi_k}\varphi_i)\varphi_i X, \varphi_i Y) = -g((\mathcal{L}_{\xi_k}\varphi_i)X, Y)$ so that

$$d\Phi_i(\varphi_i X, \varphi_i Y, \xi_k) = g((\mathcal{L}_{\xi_k}\varphi_i)X, Y) - d\eta_j(\varphi_i X, \varphi_i Y). \tag{4.11}$$

Using the expression (4.9) for the torsion T_i, and applying equations (2.4) and (4.11), we have

$$\begin{aligned} T_i(X, Y, \xi_j) &= N_{\varphi_i}(X, Y, \xi_j) - d\Phi_i(\varphi_i X, \varphi_i Y, \xi_k) \\ &= d\eta_j(X, Y) - d\eta_k(\varphi_i X, Y) - d\eta_k(X, \varphi_i Y) - g((\mathcal{L}_{\xi_k}\varphi_i)X, Y). \end{aligned}$$

On the other hand, the torsion T satisfies $T(X, Y, \xi_j) = d\eta_j(X, Y)$, and we have

$$T(X, Y, \xi_j) - T_i(X, Y, \xi_j) = A_{ik}(X, Y) = -\beta\Phi_j(X, Y),$$

where we used the fact that β is a Reeb Killing function. Analogously, one can check that $T(X, Y, \xi_k) - T_i(X, Y, \xi_k) = -\beta\Phi_k(X, Y)$, coherently with (4.10).

Now, using again (4.9), we have

$$T_i(X, \xi_i, \xi_j) = d\eta_i(\xi_j, X) = -g([\xi_j, X], \xi_i).$$

On the other hand, by (3.5), $T(X, \xi_i, \xi_j) = -g([\xi_i, \xi_j], X)$. Hence,

$$T(X, \xi_i, \xi_j) - T_i(X, \xi_i, \xi_j) = -(\mathcal{L}_{\xi_j} g)(\xi_i, X) = 0.$$

In the same way one shows that $T(X, \xi_i, \xi_k) - T_i(X, \xi_i, \xi_k) = 0$, and these relations are in accordance with (4.10), since for example

$$-\beta(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k)(X, \xi_i, \xi_j) = -\beta\Phi_j(X, \xi_i) = 0.$$

We shall compute now the difference $T - T_i$ on vector fields X, ξ_j, ξ_k, with $X \in \Gamma(\mathcal{H})$. First, from $\eta_r([X, \xi_r]) = 0$, we have

$$\begin{aligned} d\Phi_i(X, \xi_j, \xi_k) &= X(\Phi_i(\xi_j, \xi_k)) - \Phi_i([X, \xi_j], \xi_k) - \Phi_i([\xi_j, \xi_k], X) - \Phi_i([\xi_k, X], \xi_j) \\ &= -X(g(\xi_j, \xi_j)) + \eta_j([X, \xi_j]) - g([\xi_j, \xi_k], \varphi_i X) - \eta_k([\xi_k, X]) \\ &= -g([\xi_j, \xi_k], \varphi_i X). \end{aligned} \tag{4.12}$$

From (4.9) we have

$$T_i(X, \xi_j, \xi_k) = N_{\varphi_i}(X, \xi_j, \xi_k) + d\Phi_i(\varphi_i X, \xi_k, \xi_j) = -g([\xi_j, \xi_k], X),$$

where we used (4.12) and the fact that $N_{\varphi_i}(\xi_j, \xi_k) = 0$ (see (2.3)). Therefore, by (3.5), $T_i(X, \xi_j, \xi_k) = T(X, \xi_j, \xi_k)$, again coherently with (4.10). Finally, by (4.2) and (4.9), we have

$$T(\xi_i, \xi_j, \xi_k) - T_i(\xi_i, \xi_j, \xi_k) = 2\beta - 2\delta - d\eta_i(\xi_j, \xi_k) = 2\beta - 2\delta + 2\delta = 2\beta,$$

and one can easily check that

$$-\beta(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k)(\xi_i, \xi_j, \xi_k) = -\beta(\Phi_j(\xi_k, \xi_i) + \Phi_k(\xi_i, \xi_j)) = 2\beta,$$

which completes the proof of (4.10). Therefore, by (1.4) and (4.10), we get (4.8). □

Remark 4.2.1. Under the hypotheses Theorem 4.2.1, equation (4.10) implies that the torsion T of the canonical connection and the torsions T_1, T_2, T_3 of the three characteristic connections satisfy

$$3T \;=\; T_1 + T_2 + T_3 - 2\beta(\eta_1 \wedge \Phi_1 + \eta_2 \wedge \Phi_2 + \eta_3 \wedge \Phi_3).$$

We shall show now that 3-(α, δ)-Sasaki manifolds are the only canonical horizontal 3-α-contact metric manifolds with integrable distribution $\mathcal{V}$. By a result of B. Cappelletti-Montano, we have the following

Proposition 4.2.1 ([Ca09, Prop. 3.2]). *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold such that the distribution $\mathcal{V}$ is integrable and each ξ_i is a Killing vector field. Then, the following properties hold:*

i) $[\xi_i, \xi_j] = 2\delta\xi_k$ *for every even permutation (i, j, k) of $(1, 2, 3)$, and some constant δ;*

ii) *each ξ_i is an infinitesimal automorphism of the horizontal distribution $\mathcal{H}$, i.e. $[\xi_i, X] \in \Gamma(\mathcal{H})$ for every $X \in \Gamma(\mathcal{H})$;*

iii) *the distribution $\mathcal{V}$ has totally geodesic leaves.*

Observe that under the hypothesis that $\mathcal{V}$ is integrable, condition i) is equivalent to the existence of a *constant* Reeb commutator function, since the projection of the commutator to $\mathcal{H}$ vanishes.

Theorem 4.2.2. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a canonical almost 3-contact metric manifold with canonical connection ∇ and Reeb Killing function β. Assume that the following conditions hold:*

i) *the distribution $\mathcal{V}$ is integrable;*

ii) *$d\eta_i(X, Y) = 2\alpha\Phi_i(X, Y)$ for every $X, Y \in \Gamma(\mathcal{H})$ and $i = 1, 2, 3$, and for some real constant $\alpha \neq 0$.*

Then the structure admits a constant Reeb commutator δ and $(M, \varphi_i, \xi_i, \eta_i, g)$ is a 3-(α, δ)-Sasaki manifold.

Proof. Since the distribution $\mathcal{V}$ is integrable and the Reeb vector fields are Killing, from Proposition 4.2.1, each ξ_i is an infinitesimal automorphism of the horizontal distribution $\mathcal{H}$, and thus

$$d\eta_r(X, \xi_s) = 0, \qquad \forall X \in \Gamma(\mathcal{H}),\ r, s = 1, 2, 3.$$

Furthermore, $[\xi_i, \xi_j] = 2\delta\xi_k$ for every even permutation (i, j, k) of $(1, 2, 3)$ and some constant δ. Therefore,

$$d\eta_r(\xi_s, \xi_t) = -2\delta\epsilon_{rst}.$$

Taking into account condition ii) we deduce that the differential of each 1-form η_i is given by

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k$$

where (i, j, k) is an even permutation of $(1, 2, 3)$. □

4.3 The cone of a canonical almost 3-contact metric manifold

In [AH15] the authors studied cones of G manifolds endowed with a characteristic connection. Given a Riemannian manifold (M, g) equipped with a metric connection ∇ with skew-symmetric torsion T, the cone $(\bar{M}, \bar{g}) = (M \times \mathbb{R}^+, a^2r^2g + dr^2)$, $a > 0$, can be endowed with an appendant connection $\bar{\nabla} := \nabla^{\bar{g}} + \frac{1}{2}\bar{T}$, where $\bar{T}$ is the skew-symmetric torsion of $\bar{\nabla}$, defined by

$$\bar{T}(X, Y) = T(X, Y) \text{ for } X, Y \perp \partial_r, \quad \partial_r \lrcorner \bar{T} = 0.$$

The positive real number a will be called the *cone constant.*

Now, let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold. On the cone $(\bar{M}, \bar{g})$ one can consider three almost hermitian structures defined by

$$\begin{array}{lll} J_1(ar\partial_r) = \xi_1, & J_1(\xi_1) = -ar\partial_r, & J_1(V) = -\varphi_1(V) \text{ for } V \perp \xi_1, \partial_r, \\ J_2(ar\partial_r) = \xi_2, & J_2(\xi_2) = -ar\partial_r, & J_2(V) = -\varphi_2(V) \text{ for } V \perp \xi_2, \partial_r, \\ J_3(ar\partial_r) = -\xi_3, & J_3(\xi_3) = ar\partial_r, & J_3(V) = -\varphi_3(V) \text{ for } V \perp \xi_3, \partial_r, \end{array} \tag{4.13}$$

These structures satisfy $J_1J_2 = J_3 = -J_2J_1$, and hence $(\bar{M}, \bar{g}, J_1, J_2, J_3)$ is an almost hyperhermitian manifold. We will use the following result.

Theorem 4.3.1 ([AH15])**.** *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be an almost 3-contact metric manifold such that each structure $(\varphi_i, \xi_i, \eta_i, g)$ admits a characteristic connection ∇^i with skew torsion T_i. Let ∇ be a metric connection with totally skew-symmetric torsion on M. Then the appendant connection $\bar{\nabla}$ satisfies $\bar{\nabla}J_1 = \bar{\nabla}J_2 = \bar{\nabla}J_3 = 0$ if and only if there exists some positive constant a (the cone constant) such that the three tensors $S_i := T_i - 2a\eta_i \wedge \Phi_i$ coincide with the torsion T of ∇. Furthermore, if M is hypernormal, then $\bar{M}$ is an HKT manifold.*

Let us point out that in the preceding result, the non-existence of a characteristic connection for almost 3-contact metric manifolds was circumvented by requiring the property that the three difference tensors S_i should coincide—one then views them as the torsion of a connection and lifts it to the cone.

Corollary 4.3.1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a canonical almost 3-contact metric manifold, ∇ its canonical connection. Assume that its Reeb Killing function β is constant and negative. If ∇' is the metric connection on M with skew torsion T' given by*

$$T' := T + \beta(\eta_1 \wedge \Phi_1 + \eta_2 \wedge \Phi_2 + \eta_3 \wedge \Phi_3),$$

the appendant connection $\bar{\nabla}'$ on the cone $(\bar{M}, \bar{g})$, with cone constant $a = -\frac{\beta}{2}$ is a hermitian connection, i. e. it parallelizes the almost hermitian structures J_i, $i = 1, 2, 3$, defined by (4.13).

If, furthermore, $(M, \varphi_i, \xi_i, \eta_i, g)$, is a 3-$(\alpha, \delta)$-Sasaki manifold, then the cone $(\bar{M}, \bar{g})$ is an HKT manifold.

Proof. From Theorem 2.1.1, we know that each almost contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$ admits a characteristic connection ∇^i; Theorem 4.2.1 shows that the connections ∇ and ∇_i are related by (4.8). Now, taking $S_i := T_i + \beta\eta_i \wedge \Phi_i$, by (4.10) we get

$$S_i = T + \beta(\eta_1 \wedge \Phi_1 + \eta_2 \wedge \Phi_2 + \eta_3 \wedge \Phi_3),$$

so that the three tensors S_i, $i = 1, 2, 3$, coincide. We can thus apply Theorem 4.3.1. Consider the cone $(\bar{M}, \bar{g})$ corresponding to the cone constant $a := -\frac{\beta}{2}$. If ∇' is the metric connection on M with totally skew-symmetric torsion $T' := S_1 = S_2 = S_3$, the appendant connection $\bar{\nabla}'$ parallelizes the almost hermitian structures J_i, $i = 1, 2, 3$.

Theorem 2.2.1 states that any 3-(α, δ)-Sasaki manifold is hypernormal, hence the last claim follows from the corresponding statement in Theorem 4.3.1. □

Remark 4.3.1. Recall that for a 3-(α, δ)-Sasaki manifold, the Reeb Killing function β is automatically constant and may be computed from α and δ through $\beta = 2\delta - 4\alpha$. Thus, the condition $\beta < 0$ in Corollary 4.3.1 may be restated as $2\alpha > \delta$ in this situation.

Remark 4.3.2. For example, we know that any 3-Sasakian manifold is a 3-(α, δ)-Sasaki manifold with $\beta = -2 < 0$. Comparing the expression for T' above with the results of [AH15, Section 3.5], one sees that $\bar{\nabla}'$ will then coincide with the Levi-Civita connection of the natural hyper-Kähler structure on the cone $\bar{M}$. Similarly, the quaternionic Heisenberg group is a 3-(α, δ)-Sasaki manifold with $\beta = -2\lambda$, λ a positive non-zero parameter. In [AFS15, Thm 11] it was shown (by applying Theorem 4.3.1) that the cone of the 7-dimensional quaternionic Heisenberg group is a HKT manifold. Hence, Corollary 4.3.1 generalizes both results.

4.4 The canonical connection of a 3-(α, δ)-Sasaki manifold

We now look in detail at the canonical connection of a 3-(α, δ)-Sasaki manifold. Recall that such a manifold is always a canonical almost 3-contact metric manifold (Corollary 2.3.3), and hence the existence (and uniqueness) of the canonical connection is garanteed.

Remark 4.4.1 (∇ as a qc connection). By Proposition 2.3.3, we know that every 3-(α, δ)-Sasaki manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ admits an underlying quaternionic contact structure which is qc-Einstein with $S = 2\alpha\delta$, with almost complex structures $I_i := \varphi_i|_{\mathcal{H}}$ and 1-forms $\tilde{\eta}_i := -\frac{1}{\alpha}\eta_i$. In general, the condition for a metric connection ∇ to preserve the qc structure reduces to the requirement that ∇ preserves the splitting $TM = \mathcal{H} \oplus \mathcal{V}$ and has the additional properties

$$\nabla(I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3) = 0, \quad \nabla(\tilde{\xi_1} \otimes I_1 + \tilde{\xi_2} \otimes I_2 + \tilde{\xi_3} \otimes I_3) = 0.$$

The equations of Remark 4.1.2 imply that the canonical connection of a 3-(α, δ)-Sasaki manifold is indeed a qc connection (this was already observed for the quaternionic Heisenberg group in [AFS15]). The most commonly used such connection is the well-known Biquard connection.

Remark 4.4.2. The relations ((i, j, k) an even permutation of $(1, 2, 3)$)

$$\xi_i \lrcorner \Phi_i = 0, \qquad \xi_j \lrcorner \Phi_i = -\eta_k, \qquad \xi_k \lrcorner \Phi_i = \eta_j$$

holding for any 3-(α, δ)-Sasaki manifold imply that we can split the 2-forms Φ_i in their vertical and horizontal part,

$$\Phi_1 \ = \ -\eta_{23} + \Phi_1^{\mathcal{H}}, \quad \Phi_2 \ = \ \eta_{13} + \Phi_2^{\mathcal{H}}, \quad \Phi_3 \ = \ -\eta_{12} + \Phi_3^{\mathcal{H}}, \quad \Phi_i^{\mathcal{H}} \in \Lambda^2(\mathcal{H}) \text{ for } i = 1, 2, 3,$$

which we can alternatively summarize as $\Phi_i = -\eta_{jk} + \Phi_i^{\mathcal{H}}$ for even permutations. Furthermore, the defining condition of a 3-(α, δ)-Sasaki manifold may be reformulated as

$$d\eta_i \ = \ 2\alpha\, \Phi_i^{\mathcal{H}} - 2\delta\, \eta_{jk}$$

for even permutations. All in all, this distinction between horizontal and vertical contributions allows to identify the horizontal, vertical, and mixed parts of the torsion and its derivative more clearly.

Theorem 4.4.1. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α,δ)-Sasaki manifold. The torsion of its canonical connection ∇ is given by*

$$T \;=\; \sum_{i=1}^{3} \eta_i \wedge d\eta_i + 8(\delta-\alpha)\,\eta_{123} \;=\; 2\alpha \sum_{i=1}^{3} \eta_i \wedge \Phi_i^{\mathcal{H}} + 2(\delta - 4\alpha)\,\eta_{123}$$

and satisfies $\nabla T = 0$ as well as

$$\begin{aligned} dT &= 4\alpha^2 \sum_{i=1}^{3} \Phi_i \wedge \Phi_i + 8\alpha(\delta-\alpha) \overset{i,j,k}{\mathfrak{S}} \Phi_i \wedge \eta_{jk} \\ &= 4\alpha^2 \sum_{i=1}^{3} \Phi_i^{\mathcal{H}} \wedge \Phi_i^{\mathcal{H}} + 8\alpha(\delta-2\alpha) \overset{i,j,k}{\mathfrak{S}} \Phi_i^{\mathcal{H}} \wedge \eta_{jk}. \end{aligned}$$

Here, the symbol $\overset{i,j,k}{\mathfrak{S}}$ means the sum over all even permutations of $(1,2,3)$.

Proof. Each almost contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$ admits a characteristic connection ∇^i (Theorem 2.1.1) with torsion

$$T_i = \eta_i \wedge d\eta_i + d^{\varphi_i}\Phi_i,$$

since the structure is hypernormal. Applying (2.12), since $\eta_k \circ \varphi_i = \eta_j$ and $\eta_j \circ \varphi_i = -\eta_k$, and using also equations (2.17) and (2.18), we have

$$\begin{aligned} d^{\varphi_i}\Phi_i(X,Y,Z) &= 2(\alpha-\delta)\{(\eta_k \wedge \Phi_j)(\varphi_i X, \varphi_i Y, \varphi_i Z) - (\eta_j \wedge \Phi_k)(\varphi_i X, \varphi_i Y, \varphi_i Z)\} \\ &= 2(\alpha-\delta)\{-\eta_j(X)(\Phi_j + \eta_{ki})(Y,Z) \\ &\quad - \eta_j(Y)(\Phi_j + \eta_{ki})(Z,X) - \eta_j(Z)(\Phi_j + \eta_{ki})(X,Y)\} \\ &\quad - 2(\alpha-\delta)\{\eta_k(X)(\Phi_k + \eta_{ij})(Y,Z) \\ &\quad + \eta_k(Y)(\Phi_k + \eta_{ij})(Z,X) + \eta_k(Z)(\Phi_k + \eta_{ij})(X,Y)\} \\ &= 2(\alpha-\delta)(-\eta_j \wedge \Phi_j - \eta_{jki})(X,Y,Z) \\ &\quad - 2(\alpha-\delta)(\eta_k \wedge \Phi_k + \eta_{kij})(X,Y,Z) \\ &= 2(\delta-\alpha)(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k + 2\,\eta_{123})(X,Y,Z). \end{aligned}$$

Therefore, the torsion T_i is given by

$$T_i = \eta_i \wedge d\eta_i + 2(\delta-\alpha)(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k + 2\,\eta_{123}),$$

and by (4.10), the torsion of the canonical connection is

$$\begin{aligned} T &= T_i - 2(\delta-2\alpha)(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k) \\ &= \eta_i \wedge d\eta_i + 2\alpha(\eta_j \wedge \Phi_j + \eta_k \wedge \Phi_k) + 4(\delta-\alpha)\eta_{123} \\ &= \eta_i \wedge d\eta_i + \eta_j \wedge \{d\eta_j + 2(\delta-\alpha)\eta_{ki}\} + \eta_k \wedge \{d\eta_k + 2(\delta-\alpha)\eta_{ij}\} + 4(\delta-\alpha)\eta_{123} \\ &= \sum_{i=1}^{3} \eta_i \wedge d\eta_i + 8(\delta-\alpha)\eta_{123}\,. \end{aligned}$$

The alternative expression in terms of $\Phi_i^{\mathcal{H}}$ follows by substituting their definitions from Remark 4.4.2.

One can verify that both expressions for T are coherent with equations (3.3), (3.4), (3.5) and (4.2). In particular,

$$T(X,Y,Z) = T(X,\xi_i,\xi_j) = 0, \tag{4.14}$$

$$T(X,Y,\xi_i) = 2\alpha\Phi_i(X,Y), \quad T(\xi_i,\xi_j,\xi_k) = 2(\beta-\delta) = 2(\delta-4\alpha), \tag{4.15}$$

for every $X, Y, Z \in \Gamma(\mathcal{H})$, and where (i, j, k) is an even permutation of $(1, 2, 3)$. Now, since the canonical connection ∇ preserves the splitting $TM = \mathcal{H} \oplus \mathcal{V}$, from (4.14) we obtain

$$(\nabla_U T)(X, Y, Z) = (\nabla_U T)(X, \xi_i, \xi_j) = 0,$$

for every $U \in \mathfrak{X}(M)$ and $X, Y, Z \in \Gamma(\mathcal{H})$. By Remark 4.1.2, the covariant derivatives $\nabla\xi_i$ are given by

$$\nabla_U \xi_i = \beta(\eta_k(U)\xi_j - \eta_j(U)\xi_k) \tag{4.16}$$

for every $U \in \mathfrak{X}(M)$ and where $\beta = 2(\delta - 2\alpha)$. In particular, $\nabla_U \xi_i \in \langle \xi_j, \xi_k \rangle$ and thus, using also the second identity in (4.15), we get

$$(\nabla_U T)(\xi_i, \xi_j, \xi_k) = 0.$$

Furthermore, using the first identity in (4.15), and (4.16), we have

$$\begin{aligned}
&(\nabla_U T)(X, Y, \xi_i)\\
&= 2\alpha\nabla_U(\Phi_i(X, Y)) - 2\alpha\Phi_i(\nabla_U X, Y) - 2\alpha\Phi_i(X, \nabla_U Y) - T(X, Y, \nabla_U \xi_i)\\
&= 2\alpha(\nabla_U \Phi_i)(X, Y) + \beta\eta_k(U)T(X, Y, \xi_j) - \beta\eta_j(U)T(X, Y, \xi_k)\\
&= 2\alpha g(X, (\nabla_U \varphi_i)Y) + 2\beta\alpha\eta_k(U)\Phi_j(X, Y) - 2\beta\alpha\eta_j(U)\Phi_k(X, Y)\\
&= 2\alpha\{g(X, (\nabla_U \varphi_i)Y) + \beta g(X, \eta_k(U)\varphi_j Y - \eta_j(U)\varphi_k Y)\}
\end{aligned}$$

which vanishes because of (4.1). This completes the proof that $\nabla T = 0$. Finally, differentiating the expression for T, we obtain

$$\begin{aligned}
dT &= \sum_{i=1}^{3} d\eta_i \wedge d\eta_i + 8(\delta - \alpha) \overset{i,j,k}{\mathfrak{S}} d\eta_i \wedge \eta_{jk}\\
&= \overset{i,j,k}{\mathfrak{S}} \big(2\alpha\Phi_i + 2(\alpha - \delta)\eta_{jk}\big) \wedge \big(2\alpha\Phi_i + 2(\alpha - \delta)\eta_{jk}\big) + 16\alpha(\delta - \alpha) \overset{i,j,k}{\mathfrak{S}} \Phi_i \wedge \eta_{jk}\\
&= 4\alpha^2 \sum_{i=1}^{3} \Phi_i \wedge \Phi_i - 8\alpha(\alpha - \delta) \overset{i,j,k}{\mathfrak{S}} \Phi_i \wedge \eta_{jk}.
\end{aligned}$$

This completes the proof. □

Remark 4.4.3 (φ-compatible connections of 3-(α, δ)-Sasaki manifolds). An immediate computation based on Proposition 3.2.2 and the preceding Theorem 4.4.1 shows that for a φ-compatible connection ∇^γ with parameter function γ of a 3-(α, δ)-Sasaki manifold, the general expression for the torsion T_γ is

$$T_\gamma \;=\; \sum_{i=1}^{3} \eta_i \wedge d\eta_i + (8\delta - 4\alpha + \gamma)\,\eta_{123} \;=\; 2\alpha \sum_{i=1}^{3} \eta_i \wedge \Phi_i^{\mathcal{H}} + \gamma\,\eta_{123}.$$

The canonical connection corresponds to the choice $\gamma = 2(\delta - 4\alpha)$.

The following lemma is purely computational, hence we omit the proof. The formulas are, however, quite useful, for example in the next section.

Lemma 4.4.1.

1) $d\Phi_i^{\mathcal{H}} = 2\delta(\Phi_j^{\mathcal{H}} \wedge \eta_k - \Phi_k^{\mathcal{H}} \wedge \eta_j)$ *and* $\alpha\, d\Phi_i^{\mathcal{H}} = \delta\, d(\eta_{jk})$ *for even permutations,*

2) *The form* $\Psi^{\mathcal{H}} := \sum_{i=1}^{3} \Phi_i^{\mathcal{H}} \wedge \Phi_i^{\mathcal{H}}$ *satisfies* $d\Psi^{\mathcal{H}} = 0$,

3) $d\left[\sum_{i=1}^{3} \eta_i \wedge \Phi_i^{\mathcal{H}}\right] = 2\alpha\Psi^{\mathcal{H}} + 2\delta \overset{i,j,k}{\mathfrak{S}} \Phi_i^{\mathcal{H}} \wedge \eta_{jk}$,

4) $d\eta_{123} = 2\alpha \overset{i,j,k}{\mathfrak{S}} \Phi_i^{\mathcal{H}} \wedge \eta_{jk}$.

Since any 3-α-Sasakian manifold is 3-(α, δ)-Sasaki with $\delta = \alpha$, we have the following

Corollary 4.4.1. *Any 3-α-Sasakian manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ admits a canonical connection ∇ with torsion*

$$T = \eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3$$

which satisfies $\nabla T = 0$.

Remark 4.4.4 (Canonical connection of a 3-Sasakian manifold). The canonical connection defined in the above Corollary coincides with the linear connection considered in [AF10a] for 7-dimensional 3-Sasakian manifolds. In this case the canonical connection ∇ coincides with the characteristic connection of the canonical (cocalibrated) G_2-structure of the 3-Sasakian manifold. This fact will be generalized in Section 4.5.

Remark 4.4.5 (Connections on S^7). The 7-sphere carries a dazzling array of interesting metric connections with skew torsion—we saw one of them in Example 4.1.3, and of course the natural 3-Sasaki structure on S^7 is covered by the previous Corollary and Remark. The 7-sphere can be endowed with several natural metrics: the round metric, the family of Berger metrics, or the naturally reductive metric stemming from the realisation as the homogeneous space $\mathrm{Spin}(7)/G_2$. A thorough investigation of metric connections invariant under Lie groups was carried out in [DGP16] ($G = \mathrm{Spin}(6)$) and [Ch16] ($G = \mathrm{Spin}(7)$). A comparison of their results with ours shows that none of these connections is φ-compatible for the underlying 3-Sasaki structure, because they do not preserve the distributions $\mathcal{V}$ and $\mathcal{H}$ (for details, see [DGP16, 5.13–5.19] and [Ch16, Ex. 4.8]).

For parallel 3-(α, δ)-Sasaki manifolds, corresponding to $\delta = 2\alpha$, we can state the following

Corollary 4.4.2. *The canonical connection of a parallel 3-(α, δ)-Sasaki manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ has torsion*

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i + 8\alpha\, \eta_{123}$$

and satisfies $\nabla T = 0$.

Remark 4.4.6. Notice that any 3-α-Sasakian structure admits $\mathcal{H}$-homothetic deformations which are 3-(α', δ')-Sasaki with $\delta' = 2\alpha'$. Indeed, if $(M, \varphi_i, \xi_i, \eta_i, g)$ is a 3-α-Sasakian manifold, the $\mathcal{H}$-deformed structure (2.14) is 3-(α', δ')-Sasaki with $\alpha' = \alpha\frac{c}{a}$, $\delta' = \frac{\alpha}{c}$ by Proposition 2.3.1. Therefore, these coefficients satisfy $\delta' = 2\alpha'$ if and only if $a = 2c^2$. On the other hand, we know that $c^2 = a + b$. Hence, we can conclude that all the deformed structures

$$\eta_i' = c\eta_i, \qquad \xi_i' = \frac{1}{c}\xi_i, \qquad \varphi_i' = \varphi_i, \qquad g' = 2c^2 g - c^2 \sum_{i=1}^{3} \eta_i \otimes \eta_i$$

are 3-(α', δ')-Sasaki with $\alpha' = \frac{\alpha}{2c}$ and $\delta' = \frac{\alpha}{c} = 2\alpha'$, each one admitting a canonical connection which parallelizes the structure tensor fields.

We showed that the quaternionic Heisenberg group $(N_p, \varphi_i, \xi_i, \eta_i, g_\lambda)$ is a degenerate 3-(α, δ)-Sasaki manifold ($\delta = 0$) with $2\alpha = \lambda$. Therefore,

Corollary 4.4.3. *The quaternionic Heisenberg group $(N_p, \varphi_i, \xi_i, \eta_i, g_\lambda)$ admits a canonical connection ∇ with torsion T given by*

$$T = \eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3 - 4\lambda\eta_{123}$$

which satisfies $\nabla T = 0$.

Remark 4.4.7. The canonical connection of the quaternionic Heisenberg group determined in the above corollary coincides with the canonical connection defined in [AFS15]. In [AFS15] the authors prove that this connection ∇ parallelizes its torsion and curvature tensors, and the holonomy algebra of ∇ is isomorphic to $\mathfrak{su}(2)$. In the 7-dimensional case this connection ∇ is also the characteristic connection of a cocalibrated G_2-structure.

Remark 4.4.8. In [Ca09], Cappelletti-Montano investigated hypernormal almost 3-contact metric manifolds admitting metric connections with certain invariance properties, but only under the assumption of having skew-symmetric torsion on $\mathcal{H}$. He calls these manifolds *almost 3-contact metric manifolds with torsion*. One easily checks with the properties of 3-(α,δ)-Sasaki manifolds we compiled that these are always almost 3-contact metric manifolds with torsion by [Ca09, Thm 4.3].

Finally, we formulate the result for the Ricci curvature Ric of the canonical connection. The computation is rather lengthy, but standard, hence we omit it. Observe that the property of being symmetric follows for Ric from $\nabla T = 0$.

Theorem 4.4.2 (∇-Ricci curvature for 3-(α,δ)-Sasaki manifolds)**.** *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α,δ)-Sasaki manifold of dimension $4n+3$. The Ricci tensor of the canonical connection ∇ is for all $X, Y \in \mathfrak{X}(M)$ given by*

$$\begin{aligned}\mathrm{Ric} &= 4\alpha\{\delta(n+2)-3\alpha\}\, g + 4\alpha\{\delta(2-n)-5\alpha\}\sum_{i=1}^{3}\eta_i\otimes\eta_i \\ &= 4\alpha\{\delta(n+2)-3\alpha\}\,\mathrm{Id}_{\mathcal{H}} + 16\alpha(\delta-2\alpha)\,\mathrm{Id}_{\mathcal{V}}.\end{aligned}$$

In particular, the manifold is ∇-Einstein if and only if $\delta(2-n)=5\alpha$, and it is never ∇-Ricci flat.

For details on the notion of ∇-Einstein manifolds, we refer to [AF14]. Together with the expression for the Riemannian Ricci curvature stated in Proposition 2.3.3, we can conclude after a short calculation:

Corollary 4.4.4. *A 3-(α,δ)-Sasaki manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ is both Riemannian Einstein and ∇-Einstein if and only if $\dim M = 7$ and $\delta = 5\alpha$.*

Example 4.4.1 ($\mathcal{H}$-Deformations of 3-Sasaki manifolds II)**.** Consider now a 7-dimensional 3-Sasaki manifold $(M^7, \varphi_i, \xi_i, \eta_i, g)$, i.e. $\alpha_0 = \delta_0 = 1$, and the particular one-parameter family of $\mathcal{H}$-homothetic deformations given by $a > 0$ arbitrary, $b = 1-a$, $c = 1$, hence $\alpha = \frac{1}{a}$ and $\delta = 1$ for the resulting 3-(α,δ)-Sasaki manifold (Proposition 2.3.1). From Corollary 2.3.3, we conclude that its Reeb Killing function is $\beta = 2(1-2/a)$. This is a well-known family of deformations, see for example [FKMS97, Fri07]. We conclude at once that a assumes the following particular values:

a	properties
1	Einstein and 3-Sasakian
2	parallel 3-(α,δ)-Sasakian
5	Einstein and ∇-Einstein 3-(α,δ)-Sasakian

4.5 The canonical G_2-structure of a 7-dimensional 3-(α,δ)-Sasaki manifold

In this Section, we restrict our attention to the situation that $(M^7, \varphi_i, \xi_i, \eta_i, g)$ is a (simply connected) 7-dimensional 3-(α,δ)-Sasaki manifold. In the adapted frame $e_1 = \xi_1,\ e_2 = \xi_2,\ e_3 = \xi_3$, e_4 any vector field orthonormal to $\mathcal{V}$, $e_5 = \varphi_1 e_4$, $e_6 = \varphi_2 e_4$, $e_7 = \varphi_3 e_4$ with dual 1-forms $\eta_i,\ i = 1, \ldots, 7$, the horizontal fundamental forms are given by

$$\Phi_1^{\mathcal{H}} = -\eta_{45} - \eta_{67}, \quad \Phi_2^{\mathcal{H}} = -\eta_{46} + \eta_{57}, \quad \Phi_3^{\mathcal{H}} = -\eta_{47} - \eta_{56}.$$

We shall prove that as in the 7-dimensional 3-Sasaki case [AF10a], its canonical connection is, in fact, a G_2-connection:

Theorem 4.5.1. *Let $(M^7, \varphi_i, \xi_i, \eta_i, g)$ be a 7-dimensional 3-(α,δ)-Sasaki manifold. The 3-form*

$$\omega \ := \ \sum_{i=1}^{3} \eta_i \wedge \Phi_i^{\mathcal{H}} + \eta_{123} \ = \ -\eta_{145} - \eta_{167} - \eta_{246} + \eta_{257} - \eta_{347} - \eta_{356} + \eta_{123}$$

defines a cocalibrated G_2-structure, i. e. it is of Fernandez-Gray type $W_1 \oplus W_3$, that we shall call the canonical G_2-structure. *Its characteristic connection ∇ coincides with the canonical connection.*

Proof. As this is not a paper on G_2-manifolds, we shall be brief, the necessary details on G_2-manifolds and their characteristic connection may for example be found in [FI02, Ag06]. For commodity, let us split ω as

$$\omega \ = \ \omega_1 + \omega_2 \ \text{ with } \ \omega_1 \ = \ \sum_{i=1}^{3} \eta_i \wedge \Phi_i^{\mathcal{H}}, \quad \omega_2 \ = \ \eta_{123}.$$

One checks by an explicit calculation that their Hodge duals are given by

$$*\omega_1 \ = \ -\eta_{2367} - \eta_{2345} - \eta_{1357} + \eta_{1346} - \eta_{1256} - \eta_{1247} = \overset{i,j,k}{\mathfrak{S}} \ \Phi_i^{\mathcal{H}} \wedge \eta_{jk}, \quad *\omega_2 \ = \ \eta_{4567} \ = \ \frac{1}{6}\Psi^{\mathcal{H}}.$$

By Lemma 4.4.1, $*\omega_1$ and $*\omega_2$ are closed, hence $d * \omega \ = \ 0$, which is the condition for being a cocalibrated G_2-structure. It is proved in [FI02] that any cocalibrated G_2-manifold admits a unique characteristic connection ∇ with torsion

$$T \ = \ - * d\omega + \frac{1}{6}\langle d\omega, *\omega\rangle\omega.$$

Again, $d\omega_i$ $(i \ = \ 1,2)$ may be read off directly from Lemma 4.4.1, from which we conclude that $*d\omega_2 = 2\alpha\,\omega_1$ and $*d\omega_1 = 2\delta\,\omega_1 + 12\alpha\,\omega_2$. One checks that $\langle d\omega, *\omega\rangle = 24\,\alpha + 12\,\delta$, and thus

$$T = 2\alpha\omega_1 + 2(\delta - 4\alpha)\omega_2,$$

in full agreement with Theorem 4.4.1. □

This result allows us to use the full machinery of G_2-geometry for further investigations of the connection ∇. In particular, since G_2 is the stabilizer of a generic spinor inside Spin(7), there exists a ∇-parallel spinor field ψ_0. For a systematic investigation of G_2-manifolds via spinors, we refer to [ACFH15].

Definition 4.5.1 (Canonical spinor field)**.** Let Σ be the real spin bundle of M^7. The G_2-form ω acts via Clifford multiplication on Σ as a symmetric endomorphism field with eigenvalue -7 (multiplicity one) and eigenvalue $+1$ (multiplicity seven). Consequently, it defines (assuming M^7 simply connected) a unique *canonical spinor field* ψ_0 such that

$$\nabla\psi_0 \ = \ 0, \quad \omega \cdot \psi_0 = -7\psi_0, \quad |\psi_0| = 1.$$

Furthermore, a cocalibrated G_2-manifold with characteristic torsion T satisfies $T \cdot \psi_0 = -\frac{1}{6}\langle d\omega, *\omega\rangle\,\psi_0$, which in our situation means

$$T \cdot \psi_0 \ = \ -(4\alpha + 2\delta)\,\psi_0.$$

Definition 4.5.2. Recall that a spinor field ψ is called a *generalized Killing spinor* if there exists a symmetric endomorphism field S such that $\nabla^g_X\psi = S(X) \cdot \psi$; being symmetric, we may assume that S is given in diagonal form. We call the eigenvalues of S the *generalized Killing numbers* of ψ. If they coincide and are non-zero (i. e. S is a non-trivial multiple of the identity), we have a (classical) Riemannian Killing spinor. The following results on 7-dimensional compact simply connected spin manifolds are well-known (see [FK90, BFGK91, Ag06, CS06, ABK13]):

1) M^7 admits (at least) one generalized Killing spinor if it is a cocalibrated G_2-manifold;

2) M^7 admits exactly one resp. two resp. three Killing spinor(s) if it is a nearly parallel G_2-manifold resp. Einstein-α-Sasaki manifold resp. 3-α-Sasaki manifold.

Thus, our 7-dimensional 3-(α, δ)-Sasaki manifold should have at least one generalized Killing spinor. In fact, we will show that it has four of them: the canonical spinor ψ_0 and the three Clifford products $\xi_i \cdot \psi_0$, which are linearly independent due to general properties of the spin representation.

Theorem 4.5.2 (Existence of generalized Killing spinors)**.** *Let $(M^7, \varphi_i, \xi_i, \eta_i, g)$ be a simply connected 7-dimensional 3-(α, δ)-Sasaki manifold, ψ_0 its canonical spinor.*

1) *The canonical spinor field ψ_0 is a generalized Killing spinor:*

$$\nabla^g_X \psi_0 = -\frac{3\alpha}{2} X \cdot \psi_0 \quad \text{for } X \in \mathcal{H}, \quad \nabla^g_Y \psi_0 = \frac{2\alpha - \delta}{2} Y \cdot \psi_0 \quad \text{for } Y \in \mathcal{V}.$$

The two generalized Killing numbers coincide if and only if $\delta = 5\alpha$; in this case, the G_2-form ω defines a nearly parallel G_2-structure (Gray-Fernandez type W_1).

2) *The Clifford products $\psi_i := \xi_i \cdot \psi_0$, $i = 1, 2, 3$, are generalized Killing spinors:*

$$\nabla^g_{\xi_i} \psi_i = \frac{2\alpha - \delta}{2} \xi_i \cdot \psi_i, \quad \nabla^g_{\xi_j} \psi_i = \frac{3\delta - 2\alpha}{2} \xi_j \cdot \psi_i \ (i \neq j), \quad \nabla^g_X \psi_i = \frac{\alpha}{2} X \cdot \psi_i \quad \text{for } X \in \mathcal{H}.$$

Any two of the generalized Killing numbers coincide if and only if $\alpha = \delta$, i. e. if M^7 is 3-α-Sasakian.

Proof. By Theorem 4.4.1, we know that (M^7, ω) is not only cocalibrated, but that it also has parallel torsion. We are thus in the context of cocalibrated G_2-manifolds with parallel torsion, studied by Friedrich in [Fri07]. The shape of the torsion identifies the right subcase to consider (Section 10, $\mathfrak{hol}(\nabla) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2)$, the correct identification of parameters is, up to an irrelevant change of numeration, $a = 2\alpha$ and $b = 2(\delta - 5\alpha)$). Thus, the results of [Fri07, p.646] yield the stated equations for ψ_0. Let us write this equation in a more uniform way by introducing the two generalized Killing numbers $\mu_{\mathcal{H}}, \mu_{\mathcal{V}} \in \mathbb{R}$

$$\nabla^g_X \psi_0 = \mu_{\mathcal{H}} X \cdot \psi_0 \quad \text{for } X \in \mathcal{H}, \quad \nabla^g_Y \psi_0 = \mu_{\mathcal{V}} Y \cdot \psi_0 \quad \text{for } Y \in \mathcal{V}.$$

To compute the Levi-Civita derivatives of $\psi_i = \xi_i \cdot \psi_0$, we begin with

$$\nabla^g_X (\xi_i \cdot \psi_0) = \nabla^g_X (\xi_i) \cdot \psi_0 + \xi_i \cdot \nabla^g_X \psi_0 \quad \forall\, X \in \mathfrak{X}(M).$$

By Corollary 2.3.1, we know that $\nabla^g_X(\xi_i) = -\alpha \varphi_i^{\mathcal{H}}(X)$ for $X \in \mathcal{H}$, $\nabla^g_{\xi_i}(\xi_i) = 0$, and $\nabla^g_{\xi_j}(\xi_i) = -\delta\, \xi_k$ for even permutations. Hence, it makes sense to distinguish these three cases; in particular, the first part of the claim follows immediatly. For the other two cases, a computer-assisted computation in the spin representation is needed: one checks that

$$\varphi_i^{\mathcal{H}}(X) \cdot \psi_0 = X \cdot \xi_i \cdot \psi_0 \quad \text{for } X \in \mathcal{H}, \quad \xi_i \cdot \xi_j \cdot \psi_0 = \xi_k \cdot \psi_0 \quad \text{for even permutations.}$$

The remaining claims now follow from a short calculation[2]. ☐

5 Appendix: Examples on Lie groups

In this section we denote by G a $(4n + 3)$-dimensional Lie group with Lie algebra $\mathfrak{g}$ spanned by vector fields $\xi_1, \xi_2, \xi_3, \tau_r, \tau_{n+r}, \tau_{2n+r}, \tau_{3n+r}$, $r = 1, \ldots, n$. We also consider the left invariant almost

[2] Observe that the theorem corrects a small computation error for the generalized Killing number in case $X \in \mathcal{H}$ from [AFS15, Cor.9]; this was due to a wrong sign in the expression for $\varphi_i^{\mathcal{H}}(X) \cdot \psi_0$.

3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$, where g is the Riemannian metric with respect to which the basis is orthonormal, η_i is the dual 1-form of ξ_i, and φ_i is given by

$$\varphi_i = \eta_j \otimes \xi_k - \eta_k \otimes \xi_j + \sum_{r=1}^{n} [\theta_r \otimes \tau_{in+r} - \theta_{in+r} \otimes \tau_r + \theta_{jn+r} \otimes \tau_{kn+r} - \theta_{kn+r} \otimes \tau_{jn+r}] \tag{5.1}$$

where θ_l, $l = 1, \ldots, 4n$, is the dual 1-form of τ_l, and (i, j, k) is an even permutation of $(1, 2, 3)$. The fundamental 2-forms of the structure are given by

$$\Phi_i = -\eta_j \wedge \eta_k - \sum_{r=1}^{n} [\theta_r \wedge \theta_{in+r} + \theta_{jn+r} \wedge \theta_{kn+r}]. \tag{5.2}$$

All our examples will be on nilpotent Lie groups, except the following one:

Example 5.1. Let $\mathfrak{g}$ be the Lie algebra with nonvanishing commutators given by

$$[\xi_i, \xi_j] = 2\delta\xi_k$$

where $\delta \in \mathbb{R}$, $\delta \neq 0$, and (i, j, k) is any even permutation of $(1, 2, 3)$. It is isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^{4n}$. The differential of each 1-form η_i is given by

$$d\eta_i = -2\delta\eta_j \wedge \eta_k,$$

Since the 1-forms θ_l are closed, from (5.2) we have

$$d\Phi_i = -d\eta_j \wedge \eta_k + \eta_j \wedge d\eta_k = 2\delta\, \eta_k \wedge \eta_i \wedge \eta_k - 2\delta\, \eta_j \wedge \eta_i \wedge \eta_j = 0.$$

Then the corresponding Lie group $(G, \varphi_i, \xi_i, \eta_i, g)$ is a 3-δ-cosymplectic manifold.

Example 5.2. Here we will construct an example of a hypernormal canonical non-parallel almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ that is not 3-(α, δ)-Sasaki (see Figure 1). Let $\mathfrak{g}$ be the Lie algebra with nonvanishing commutators given by

$$[\tau_r, \tau_{n+r}] = \xi_1, \qquad [\tau_r, \tau_{2n+r}] = \xi_2, \qquad [\tau_r, \tau_{3n+r}] = \xi_3.$$

Therefore,

$$d\eta_i = -\sum_{r=1}^{n} \theta_r \wedge \theta_{in+r}, \qquad d\theta_l = 0,$$

for every $i = 1, 2, 3$ and $l = 1, \ldots, 4n$. First, let us check that the left invariant almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, g)$ defined on the Lie group G is not 3-(α, δ)-Sasaki, nor in fact 3-δ-cosymplectic. Indeed, we have

$$d\eta_i = \Phi_i + \eta_j \wedge \eta_k + \sum_{r=1}^{n} \theta_{jn+r} \wedge \theta_{kn+r}.$$

The differential of the fundamental 2-forms are given by

$$d\Phi_i = -d\eta_j \wedge \eta_k + \eta_j \wedge d\eta_k.$$

Therefore, for every $X, Y, Z \in \mathcal{H}$ we have $d\Phi_i(X, Y, Z) = 0$ and $N_{\varphi_i}(X, Y, Z) = 0$. Since each ξ_i is a Killing vector field, in order to prove that the structure is canonical, we show that it admits a Reeb Killing function. Notice that (5.1) implies

$$\theta_r \circ \varphi_i = -\theta_{in+r}, \qquad \theta_{in+r} \circ \varphi_i = \theta_r, \qquad \theta_{in+r} \circ \varphi_j = \theta_{kn+r} = -\theta_{jn+r} \circ \varphi_i,$$

for every even permutation (i,j,k) of $(1,2,3)$. Now, since $\mathcal{L}_{\xi_j}\varphi_i = 0$, for every $X, Y \in \mathcal{H}$, we have

$$\begin{aligned}
A_{ij}(X,Y) &= d\eta_j(X,\varphi_i Y) + d\eta_j(\varphi_i X, Y) \\
&= -\sum_{r=1}^{n}(\theta_r \wedge \theta_{jn+r})(X,\varphi_i Y) - \sum_{r=1}^{n}(\theta_r \wedge \theta_{jn+r})(\varphi_i X, Y) \\
&= \sum_{r=1}^{n}[\theta_r(X)\theta_{kn+r}(Y) - \theta_{jn+r}(X)\theta_{in+r}(Y) + \theta_{in+r}(X)\theta_{jn+r}(Y) - \theta_{kn+r}(X)\theta_r(Y)] \\
&= \sum_{r=1}^{n}(\theta_r \wedge \theta_{kn+r})(X,Y) + \sum_{r=1}^{n}(\theta_{in+r} \wedge \theta_{jn+r})(X,Y) \\
&= -\Phi_k(X,Y).
\end{aligned}$$

Analogously, one shows that $A_{ji}(X,Y) = \Phi_k(X,Y)$. Furthermore,

$$\begin{aligned}
A_i(X,Y) &= -\sum_{r=1}^{n}(\theta_r \wedge \theta_{in+r})(X,\varphi_i Y) - \sum_{r=1}^{n}(\theta_r \wedge \theta_{in+r})(\varphi_i X, Y) \\
&= -\sum_{r=1}^{n}[\theta_r(X)\theta_r(Y) + \theta_{in+r}(X)\theta_{in+r}(Y) - \theta_{in+r}(X)\theta_{in+r}(Y) - \theta_r(X)\theta_r(Y)] = 0.
\end{aligned}$$

Therefore, the structure admits constant Reeb Killing function $\beta = -1$, and it is canonical. Observe that this allows us to conclude that the structure is hypernormal: Indeed, we already showed that $N_{\varphi_i}(X,Y,Z) = 0$ for every $X,Y,Z \in \Gamma(\mathcal{H})$. Since the structure is canonical, by Theorem 2.1.1, each tensor field N_{φ_i} is skew-symmetric on TM. Furthermore, each ξ_i lies in the center of the Lie algebra. Then, taking into account equations (2.5) one easily verifies that $N_{\varphi_i} = 0$.

Using (3.3)-(3.5) and (4.2), one can check that the torsion T of the canonical connection is given by

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i - 2\,\eta_1 \wedge \eta_2 \wedge \eta_3.$$

By Theorem 4.2.1, the characteristic connection of the structure $(\varphi_i, \xi_i, \eta_i, g)$ has torsion

$$\begin{aligned}
T_i &= T - \eta_j \wedge \Phi_j - \eta_k \wedge \Phi_k \\
&= T - \eta_j \wedge (d\eta_j - \eta_k \wedge \eta_i - \sum_{r=1}^{n}\theta_{kn+r} \wedge \theta_{in+r}) - \eta_k \wedge (d\eta_k - \eta_i \wedge \eta_j - \sum_{r=1}^{n}\theta_{in+r} \wedge \theta_{jn+r}) \\
&= \eta_i \wedge d\eta_i + \sum_{r=1}^{n}(\eta_j \wedge \theta_{kn+r} - \eta_k \wedge \theta_{jn+r}) \wedge \theta_{in+r}.
\end{aligned}$$

We end with two examples of almost 3-contact metric manifolds $(M, \varphi_i, \xi_i, \eta_i, g)$ that admit φ_i-compatible connections despite not being canonical (and, furthermore, not hypernormal).

Example 5.3. Let $\mathfrak{g}$ be the Lie algebra with non-vanishing commutators

$$[\tau_r, \tau_{n+r}] = [\tau_{2n+r}, \tau_{3n+r}] = \xi_1.$$

Assuming the corresponding Lie group G to be connected and simply connected, it is the product $H^{2n}_{\mathbb{R}} \times \mathbb{R}^2$, where $H^{2n}_{\mathbb{R}}$ is the real Heisenberg group of dimension $4n+1$. The left invariant structure $(\varphi_i, \xi_i, \eta_i, g)$ satisfies

$$d\eta_1 = -\sum_{r=1}^{n}[\theta_r \wedge \theta_{n+r} + \theta_{2n+r} \wedge \theta_{3n+r}] = \Phi_1 + \eta_2 \wedge \eta_3, \qquad d\eta_2 = 0, \qquad d\eta_3 = 0.$$

Being also $d\theta_l = 0$, we have

$$d\Phi_1 = 0, \qquad d\Phi_2 = \eta_3 \wedge d\eta_1 = \eta_3 \wedge \Phi_1, \qquad d\Phi_3 = -d\eta_1 \wedge \eta_2 = -\Phi_1 \wedge \eta_2.$$

Then, for every $i = 1, 2, 3$, and for every $X, Y, Z \in \mathcal{H}$, we have $d\Phi_i(X, Y, Z) = 0$ and $N_{\varphi_i}(X, Y, Z) = 0$. Each ξ_i is a Killing vector field. Proposition 3.2.2 implies that the manifold admits φ_i-compatible connections for every $i = 1, 2, 3$. Nevertheless, the structure is not canonical. Indeed, one can easily verify that, for every $X, Y \in \mathcal{H}$,

$$A_{i2}(X, Y) = A_{i3}(X, Y) = 0, \quad i = 1, 2, 3,$$

$$A_1(X, Y) = 0, \quad A_{21}(X, Y) = 2\Phi_3(X, Y), \quad A_{31}(X, Y) = -2\Phi_2(X, Y).$$

One can also notice that this structure is not hypernormal. Indeed, using (2.3), we see that $N_{\varphi_1} = 0$, but

$$N_{\varphi_2}(X, Y, \xi_1) = N_{\varphi_3}(X, Y, \xi_1) = 2\,\Phi_1(X, Y) \quad \forall X, Y \in \mathcal{H}.$$

Example 5.4. Let $\mathfrak{g}$ be the Lie algebra with non-vanishing commutators

$$[\tau_r, \tau_{n+r}] = [\tau_{2n+r}, \tau_{3n+r}] = \xi_1, \qquad [\tau_r, \tau_{2n+r}] = [\tau_{3n+r}, \tau_{n+r}] = \xi_2.$$

The corresponding connected simply connected Lie group G is the product $H^{2n}_{\mathbb{C}} \times \mathbb{R}$, where $H^{2n}_{\mathbb{C}}$ is the complex Heisenberg group of real dimension $4n+2$. The left invariant structure $(\varphi_i, \xi_i, \eta_i, g)$ satisfies

$$d\eta_1 = -\sum_{r=1}^{p} [\theta_r \wedge \theta_{n+r} + \theta_{2n+r} \wedge \theta_{3n+r}] = \Phi_1 + \eta_2 \wedge \eta_3,$$
$$d\eta_2 = -\sum_{r=1}^{p} [\theta_r \wedge \theta_{2n+r} + \theta_{3n+r} \wedge \theta_{n+r}] = \Phi_2 + \eta_3 \wedge \eta_1,$$

and $d\eta_3 = 0$. Since $d\theta_l = 0$, we have

$$d\Phi_1 = -d\eta_2 \wedge \eta_3 = -\Phi_2 \wedge \eta_3, \qquad d\Phi_2 = \eta_3 \wedge d\eta_1 = \eta_3 \wedge \Phi_1,$$
$$d\Phi_3 = -d\eta_1 \wedge \eta_2 + \eta_1 \wedge d\eta_2 = -\Phi_1 \wedge \eta_2 + \eta_1 \wedge \Phi_2.$$

Again, for every $i = 1, 2, 3$, and for every $X, Y, Z \in \mathcal{H}$, we have $d\Phi_i(X, Y, Z) = 0$ and $N_{\varphi_i}(X, Y, Z) = 0$. Each ξ_i is a Killing vector field. From Proposition 3.2.2 the manifold admits φ_i-compatible connections for every $i = 1, 2, 3$. Nevertheless, the structure is not canonical. Indeed, for every $X, Y \in \mathcal{H}$,

$$A_{13}(X, Y) = A_{23}(X, Y) = 0, \quad A_{31}(X, Y) = -2\Phi_2(X, Y), \quad A_{32}(X, Y) = 2\Phi_1(X, Y).$$

The structure is not hypernormal. Indeed, using (2.3), we see that $N_{\varphi_3} = 0$, but

$$N_{\varphi_1}(X, Y, \xi_3) = N_{\varphi_2}(X, Y, \xi_3) = -2\,\Phi_3(X, Y) \quad \forall X, Y \in \mathcal{H}.$$

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