

BLOW-UP AND LIFESPAN ESTIMATES FOR A DAMPED WAVE EQUATION IN THE EINSTEIN-DE SITTER SPACETIME WITH NONLINEARITY OF DERIVATIVE TYPE

MAKRAM HAMOUDA¹, MOHAMED ALI HAMZA¹ AND ALESSANDRO PALMIERI^{2,3}

ABSTRACT. In this article, we investigate the blow-up for local solutions to a semilinear wave equation in the generalized Einstein - de Sitter spacetime with nonlinearity of derivative type. More precisely, we consider a semilinear damped wave equation with a time-dependent and not summable speed of propagation and with a time-dependent coefficient for the linear damping term with critical decay rate. We prove in this work that the results obtained in a previous work, where the damping coefficient takes two particular values 0 or 2, can be extended for any positive damping coefficient. We show the blow-up in finite time of local in time solutions and we establish upper bound estimates for the lifespan, provided that the exponent in the nonlinear term is below a suitable threshold and that the Cauchy data are nonnegative and compactly supported.

1. INTRODUCTION

We are interested in the semilinear damped wave equation when the speed of propagation is depending on time, namely the damped wave equations in Einstein - de Sitter spacetime, with time derivative nonlinearity which reads as follows:

$$(1.1) \quad \begin{cases} u_{tt} - t^{-2k} \Delta u + \frac{\mu}{t} u_t = |u_t|^p, & \text{in } \mathbb{R}^N \times [1, \infty), \\ u(x, 1) = \varepsilon f(x), \quad u_t(x, 1) = \varepsilon g(x), & x \in \mathbb{R}^N, \end{cases}$$

where $k \in [0, 1)$, $\mu \geq 0$, $p > 1$, $N \geq 1$ is the space dimension, $\varepsilon > 0$ is a parameter illustrating the size of the initial data, and f, g are supposed to be positive functions. Furthermore, we consider f and g with compact support on $B(0_{\mathbb{R}^N}, R)$, $R > 0$.

The problem (1.1) with time derivative nonlinearity being replaced by power nonlinearity is well understood in terms of blow-up phenomenon. Let us first recall the equation in this case. Under the usual Cauchy conditions, the semilinear wave equation with power nonlinearity is

$$(1.2) \quad u_{tt} - t^{-2k} \Delta u + \frac{\mu}{t} u_t = |u|^q, \quad \text{in } \mathbb{R}^N \times [1, \infty).$$

2010 *Mathematics Subject Classification.* 35L15, 35L71, 35B44.

Key words and phrases. Blow-up, Einstein-de Sitter spacetime, Glassey exponent, Lifespan, Critical curve, Nonlinear wave equations, Time-derivative nonlinearity.

The blow-up phenomenon for (1.2) is related to two particular exponents. The first exponent, $q_0(N, k)$, is the positive root of

$$((1 - k)N - 1)q^2 - ((1 - k)N + 1 + 2k)q - 2(1 - k) = 0,$$

and the second exponent is given by

$$q_1(N, k) = 1 + \frac{2}{N(1 - k)}.$$

Hence, the positive number $\max\left(q_0\left(N + \frac{\mu}{1 - k}, k\right), q_1(N, k)\right)$ seems to be a serious candidate for the critical power stating thus the threshold between the global existence and the blow-up regions, see e.g. [7, 22, 23, 27, 29].

Let us go back to (1.1) with $k = \mu = 0$. This case is in fact connected to the Glassey conjecture in which the critical exponent p_G is given by

$$(1.3) \quad p_G = p_G(N) := 1 + \frac{2}{N - 1}.$$

The above value p_G is creating a threshold (depending on p) between the region where we have the global existence of small data solutions (for $p > p_G$) and another where the blow-up of the solutions under suitable sign assumptions for the Cauchy data occurs (for $p \leq p_G$); see e.g. [14, 15, 17, 26, 32, 35].

Now, for $k < 0$ and $\mu = 0$, it is proven in [20] that the solution of (1.1), in the subcritical case ($1 < p \leq p_G(N(1 - k))$), blows up in finite time giving hence a lifespan estimate of the maximal existence time. This is equivalent to say that, for $1 < p \leq p_G(N(1 - k))$, we have the nonexistence of the solution of (1.1). However, the aforementioned result was recently improved in [18] thanks to the construction of adequate test functions. The new region obtained in [18] gives a plausible characterization of the critical exponent, namely

$$(1.4) \quad p \leq p_T(N, k) := 1 + \frac{2}{(1 - k)(N - 1) + k}.$$

Very recently, it is proved in [12] with different approaches, as an application of the case of mixed nonlinearities, that results similar to the above for the problem (1.1) with $k < 0$ and $\mu = 0$ hold.

We consider now the case $\mu > 0$ and $k = 0$ in (1.2). Hence, for a small μ , the solution of (1.2) behaves like a wave. In fact, the damping produces a shifting by $\mu > 0$ on the dimension N for the value of the critical power, see e.g. [16, 24, 30, 31], and [5, 6] for the case $\mu = 2$ and $N = 2, 3$. The global existence for $\mu = 2$ is proven in [5, 6, 21].

However, for μ large, the equation (1.2) is of a parabolic type and the behavior is like a heat-type equation; see e.g. [3, 4, 33].

On the other hand, for the solution of (1.1) with $\mu > 0$ and $k = 0$, in [19] a blow-up result is proved for $1 < p \leq p_G(N + 2\mu)$ and upper bound estimates for the lifespan are given as well. Later, this result was improved in [25], where $p_G(N + \mu)$ is found as upper bound for $\mu \geq 2$. Recently, an improvement is obtained in [10] stating that the critical value for p is given by $p_G(N + \mu)$ for all $\mu > 0$. This should be the optimal threshold value that needs to be rigorously proved by completing the present blow-up result with a global existence one when the exponent p is beyond the critical value.

We focus in this article on the blow-up of the solution of (1.1) for $k \in [0, 1)$. Our target is to give the upper bound, denoted here by $p_E = p_E(N, k, \mu)$, delimiting a new blow-up region for the Einstein - de Sitter spacetime equation (1.1).

First, as observed for the equation (1.2), where the damping produces a shift in q_0 in the dimensional parameter of magnitude $\frac{\mu}{1-k}$, we expect that the same phenomenon holds for (1.1). In other words, we predict that the upper bound $p_E = p_E(N, k, \mu)$ satisfies

$$(1.5) \quad p_E(N, k, \mu) = p_E\left(N + \frac{\mu}{1-k}, k, 0\right).$$

Using an explicit representation formula and Zhou's approach to proving the blow-up on a certain characteristic line, in [13], we proved that

$$(1.6) \quad p_E(N, k, 0) = p_T(N, k),$$

where p_T is defined by (1.4).

Now, in view of (1.5) and (1.6), we await, for the solution of (1.1) with $k \in [0, 1)$ and $\mu > 0$, that

$$(1.7) \quad p_E = p_E(N, k, \mu) := 1 + \frac{2}{(1-k)(N-1) + k + \mu}.$$

As we have mentioned, in [13] we proved that (1.7) holds true under some sign assumptions for the data for $\mu = 0$, but also for $\mu = 2$ (cf. Theorems 1.1 and 1.2). We aim in the present work to extend this result for all $\mu > 0$, and show that the upper bound value for p is in fact given by (1.7). We think that $p_E(N, k, \mu)$, for k small, characterizes the limiting value between the existence and nonexistence regions of the solution of (1.1). However, it is clear that this limiting exponent does not reach the optimal one in view of the very recent results in [28].

Finally, we recall here that the wave in (1.1) has a speed of propagation dependent of time. Therefore, this time-dependent speed of propagation term can be seen, after

rescaling (see (1.9) below), as a scale-invariant damping. Let $v(x, \tau) = u(x, t)$, where

$$(1.8) \quad \tau = \phi_k(t) := \frac{t^{1-k}}{1-k}.$$

Hence, we can easily see that $v(x, \tau)$ satisfies the following equation:

$$(1.9) \quad v_{\tau\tau} - \Delta v + \frac{\mu - k}{(1-k)\tau} \partial_\tau v = C_{k,p} \tau^{\mu_k(p-2)} |\partial_\tau v|^p, \quad \text{in } \mathbb{R}^N \times [1/(1-k), \infty),$$

where $\mu_k := \frac{-k}{1-k}$ and $C_{k,p} = (1-k)^{\mu_k(p-2)}$. Moreover, thanks to the above transformation, we can use the methods carried out in some earlier works [2, 9, 10, 11, 12] to build the proof of our main result.

The rest of the paper is arranged as follows. First, we state in Section 2 the weak formulation of (1.1) in the energy space, and then we give the main theorem. Section 3 is concerned with some technical lemmas that we will use to prove the main result. Finally, Section 4 is assigned to the proof of Theorem 2.2 which constitutes the main result of this article.

2. NONEXISTENCE RESULT

First, we define in the sequel the energy solution associated with (1.1).

Definition 2.1. *Let $f \in H^1(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$. The function u is said to be an energy solution of (1.1) on $[1, T)$ if*

$$\begin{cases} u \in \mathcal{C}([1, T), H^1(\mathbb{R}^N)) \cap \mathcal{C}^1([1, T), L^2(\mathbb{R}^N)), \\ u_t \in L^p_{loc}((1, T) \times \mathbb{R}^N), \end{cases}$$

satisfies, for all $\Phi \in C_0^\infty(\mathbb{R}^N \times [1, T))$ and all $t \in [1, T)$, the following equation:

$$(2.1) \quad \begin{aligned} & \int_{\mathbb{R}^N} u_t(x, t) \Phi(x, t) dx - \varepsilon \int_{\mathbb{R}^N} g(x) \Phi(x, 1) dx \\ & - \int_1^t \int_{\mathbb{R}^N} u_t(x, s) \Phi_t(x, s) dx ds + \int_1^t s^{-2k} \int_{\mathbb{R}^N} \nabla u(x, s) \cdot \nabla \Phi(x, s) dx ds \\ & + \int_1^t \int_{\mathbb{R}^N} \frac{\mu}{s} u_t(x, s) \Phi(x, s) dx ds = \int_1^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \Phi(x, s) dx ds, \end{aligned}$$

and the condition $u(x, 1) = \varepsilon f(x)$ is fulfilled in $H^1(\mathbb{R}^N)$.

A straightforward computation shows that (2.1) is equivalent to

$$(2.2) \quad \begin{aligned} & \int_{\mathbb{R}^N} \left[u_t(x, t) \Phi(x, t) - u(x, t) \Phi_t(x, t) + \frac{\mu}{t} u(x, t) \Phi(x, t) \right] dx \\ & \int_1^t \int_{\mathbb{R}^N} u(x, s) \left[\Phi_{tt}(x, s) - s^{-2k} \Delta \Phi(x, s) - \frac{\partial}{\partial s} \left(\frac{\mu}{s} \Phi(x, s) \right) \right] dx ds \\ & = \int_1^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx ds + \varepsilon \int_{\mathbb{R}^N} \left[-f(x) \Phi_t(x, 1) + (\mu f(x) + g(x)) \Phi(x, 1) \right] dx. \end{aligned}$$

Remark 2.1. Obviously, we can choose a test function Φ which is not compactly supported in view of the fact that the initial data f and g are supported on $B_{\mathbb{R}^N}(0, R)$. In fact, we have $\text{supp}(u) \subset \{(x, t) \in \mathbb{R}^N \times [1, \infty) : |x| \leq \phi_k(t) + R\}$.

The blow-up region and the lifespan estimate of the solutions of (1.1) constitute the objective of our main result which is the subject of the following theorem.

Theorem 2.2. *Let $\mu > 0$, $p \in (1, p_E(N, k, \mu)]$, $N \geq 1$ and $k \in [0, 1)$. Suppose that $f \in H^1(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$ are functions which are non-negative, with compact support on $B(0_{\mathbb{R}^N}, R)$, and non-vanishing everywhere. Then, there exists $\varepsilon_0 = \varepsilon_0(f, g, N, R, p, k, \mu) > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ the solution u to (1.1) which satisfies*

$$\text{supp}(u) \subset \{(x, t) \in \mathbb{R}^N \times [1, \infty) : |x| \leq \phi_k(t) + R\},$$

blows up in finite time T_ε , and

$$T_\varepsilon \leq \begin{cases} C \varepsilon^{-\frac{2(p-1)}{2 - ((1-k)(N-1) + k + \mu)(p-1)}} & \text{for } 1 < p < p_E(N, k, \mu), \\ \exp(C \varepsilon^{-(p-1)}) & \text{for } p = p_E(N, k, \mu), \end{cases}$$

where $p_E(N, k, \mu)$ is given by (1.7) and C is a positive constant independent of ε .

Remark 2.2. The results stated in Theorem 2.2 hold true for $k < 0$ and $\mu > 0$; see [1] where a more general model with mass term is studied.

Remark 2.3. After completing the first version of the present manuscript, we received a draft version of [28], where problem (1.1) is studied, among other things. In particular, for $\frac{n+1}{n+2} < k < 1$ and $\mu \in [0, (n+2)k - (n+1))$ the upper bound for p in the blow-up result is improved in [28] by proving the nonexistence of global solutions to (1.1) for $1 < p < 1 + \frac{1}{(1-k)n + \mu}$.

3. AUXILIARY RESULTS

It is worth mentioning that the choice of the test function, that we will use in the functionals that will be introduced later on, is crucial here. Naturally, in terms of dynamics of the solution of (1.1), the more accurate the choice of the test function is,

the better lifespan estimate we obtain. This is why we choose in the following to include all the linear terms inherited from (1.1). First, we introduce the function $\rho(t)$ [22] given by

$$(3.1) \quad \rho(t) := t^{\frac{1+\mu}{2}} K_{\frac{\mu-1}{2(1-k)}} \left(\frac{t^{1-k}}{1-k} \right), \quad \forall t \geq 1,$$

where $K_\nu(t)$ is the modified Bessel function of second kind defined as

$$(3.2) \quad K_\nu(t) = \int_0^\infty \exp(-t \cosh \zeta) \cosh(\nu \zeta) d\zeta, \quad \nu \in \mathbb{R}.$$

It is easy to see that $\rho(t)$ satisfies

$$(3.3) \quad \frac{d^2 \rho(t)}{dt^2} - t^{-2k} \rho(t) - \frac{d}{dt} \left(\frac{\mu}{t} \rho(t) \right) = 0, \quad \forall t \geq 1.$$

Second, we define the function $\varphi(x)$ by

$$(3.4) \quad \varphi(x) := \begin{cases} \int_{S^{N-1}} e^{x \cdot \omega} d\omega & \text{for } N \geq 2, \\ e^x + e^{-x} & \text{for } N = 1; \end{cases}$$

note that $\varphi(x)$ is introduced in [34] and satisfies $\Delta \varphi = \varphi$.

Hence, the function $\psi(x, t) := \varphi(x) \rho(t)$ verifies the following equation:

$$(3.5) \quad \partial_t^2 \psi(x, t) - t^{-2k} \Delta \psi(x, t) - \frac{\partial}{\partial t} \left(\frac{\mu}{t} \psi(x, t) \right) = 0.$$

In the following we enumerate some properties of the function $\rho(t)$ that we will use later on in the proof of our main result.

Lemma 3.1. *The next properties hold true for the function $\rho(t)$.*

- (i) *The function $\rho(t)$ is positive on $[1, \infty)$. Moreover, for all $t \geq 1$, there exists a constant C_1 such that $\rho(t)$ satisfies*

$$(3.6) \quad C_1^{-1} t^{\frac{k+\mu}{2}} \exp(-\phi_k(t)) \leq \rho(t) \leq C_1 t^{\frac{k+\mu}{2}} \exp(-\phi_k(t)),$$

where $\phi_k(t)$ is given by (1.8).

- (ii) *We have*

$$(3.7) \quad \lim_{t \rightarrow +\infty} \left(\frac{t^k \rho'(t)}{\rho(t)} \right) = -1.$$

Proof. First, we recall here the definition of $\rho(t)$, as in (3.1), and (1.8)

$$(3.8) \quad \rho(t) = t^{\frac{1+\mu}{2}} K_{\frac{\mu-1}{2(1-k)}} (\phi_k(t)), \quad \forall t \geq 1.$$

Hence, the positivity of $\rho(t)$ is straightforward thanks to (3.2). On the other hand, from [8], we have the following property for the function $K_\mu(t)$:

$$(3.9) \quad K_\mu(t) = \sqrt{\frac{\pi}{2t}} e^{-t} (1 + O(t^{-1})), \quad \text{as } t \rightarrow \infty.$$

Combining (3.8) and (3.9), and again remembering the definition of $\phi_k(t)$, given by (1.8), and the fact that $k < 1$, we conclude (3.6). The assertion (i) is thus proven.

Now, to prove (ii), using (3.8) we observe that

$$(3.10) \quad \frac{\rho'(t)}{\rho(t)} = \frac{\mu + 1}{2t} + t^{-k} \frac{K'_{\frac{\mu-1}{2(1-k)}}(\phi_k(t))}{K_{\frac{\mu-1}{2(1-k)}}(\phi_k(t))}.$$

Exploiting the well-known identity for the modified Bessel function,

$$(3.11) \quad \frac{d}{dz} K_\nu(z) = -K_{\nu+1}(z) + \frac{\nu}{z} K_\nu(z),$$

and combining (3.10) and (3.11) yields

$$(3.12) \quad \frac{\rho'(t)}{\rho(t)} = \frac{\mu}{t} - t^{-k} \frac{K_{1+\frac{\mu-1}{2(1-k)}}(\phi_k(t))}{K_{\frac{\mu-1}{2(1-k)}}(\phi_k(t))}.$$

From (3.9) and (3.12), and using the fact that $k \in [0, 1)$, we deduce (3.7).

This ends the proof of Lemma 3.1. \square

Throughout this article, the use of a generic parameter C is designed to denote a positive constant that might be dependent on p, q, k, N, R, f, g, μ but independent of ε . The value of the constant C may change from line to line. Nevertheless, when it is necessary, we will clearly mention the expression of C in terms of the parameters involved in our problem.

A classical estimate result for the function $\psi(x, t)$ is stated in the next lemma.

Lemma 3.2 ([34]). *Let $r > 1$. Then, there exists a constant $C = C(N, \mu, R, p, k, r) > 0$ such that*

$$(3.13) \quad \int_{|x| \leq \phi_k(t) + R} \left(\psi(x, t) \right)^r dx \leq C \rho^r(t) e^{r\phi_k(t)} (1 + \phi_k(t))^{\frac{(2-r)(N-1)}{2}}, \quad \forall t \geq 1.$$

Let u be a solution to (1.1) for which we introduce the following functionals:

$$(3.14) \quad \mathcal{U}(t) := \int_{\mathbb{R}^N} u(x, t) \psi(x, t) dx,$$

and

$$(3.15) \quad \mathcal{V}(t) := \int_{\mathbb{R}^N} u_t(x, t) \psi(x, t) dx.$$

The first lower bounds for $\mathcal{U}(t)$ and $\mathcal{V}(t)$ are respectively given by the following two lemmas where, for t large enough, we will prove that $\varepsilon^{-1}t^{-k}\mathcal{U}(t)$ and $\varepsilon^{-1}\mathcal{V}(t)$ are two bounded from below functions by positive constants.

Lemma 3.3. *Let u be a solution of (1.1). Assume in addition that the corresponding initial data satisfy the assumptions as in Theorem 2.2. Then, there exists $T_0 = T_0(k, \mu) > 2$ such that*

$$(3.16) \quad \mathcal{U}(t) \geq C_U \varepsilon t^k, \quad \text{for all } t \geq T_0,$$

where C_U is a positive constant that may depend on f, g, N, μ, R and k , but not on ε .

Proof. Let $t \in (1, T)$. Substituting in (2.2) $\Phi(x, t)$ by $\psi(x, t)$, we obtain

$$(3.17) \quad \begin{aligned} & \int_{\mathbb{R}^N} \left[u_t(x, t)\psi(x, t) - u(x, t)\psi_t(x, t) + \frac{\mu}{t}u(x, t)\psi(x, t) \right] dx \\ &= \int_1^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx ds + \varepsilon C(f, g), \end{aligned}$$

where

$$(3.18) \quad C(f, g) := \rho(1) \int_{\mathbb{R}^N} \left[\left(\mu - \frac{\rho'(1)}{\rho(1)} \right) f(x) + g(x) \right] \phi(x) dx.$$

Note that $C(f, g)$ is positive thanks to the fact that $\rho(1)$ and $\mu - \frac{\rho'(1)}{\rho(1)}$ are positive as well (in view of (3.12)) and the sign of the initial data. Hence, recall the definition of \mathcal{U} , as in (3.14), and (3.4), (3.17) gives

$$(3.19) \quad \mathcal{U}'(t) + \Gamma(t)\mathcal{U}(t) = \int_1^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx ds + \varepsilon C(f, g),$$

where

$$(3.20) \quad \Gamma(t) := \frac{\mu}{t} - 2 \frac{\rho'(t)}{\rho(t)}.$$

Neglecting the nonlinear term in (3.19), then multiplying the resulting equation from (3.19) by $\frac{t^\mu}{\rho^2(t)}$ and integrating on $(1, t)$, we get

$$(3.21) \quad \mathcal{U}(t) \geq \mathcal{U}(1) \frac{\rho^2(t)}{t^\mu \rho^2(1)} + \varepsilon C(f, g) \frac{\rho^2(t)}{t^\mu} \int_1^t \frac{s^\mu}{\rho^2(s)} ds.$$

From (3.1), the definition of $\phi_k(t)$, given by (1.8), and using the fact that $\mathcal{U}(1) > 0$, the estimate (3.21) implies that

$$(3.22) \quad \mathcal{U}(t) \geq \varepsilon C(f, g) t K^{\frac{\mu-1}{2(1-k)}}(\phi_k(t)) \int_{t/2}^t \frac{1}{s K^{\frac{\mu-1}{2(1-k)}}(\phi_k(s))} ds, \quad \forall t \geq 2.$$

In view of (3.9), we deduce the existence of $T_0 = T_0(k, \mu) > 2$ such that

(3.23)

$$\phi_k(t) K^{\frac{\mu-1}{2(1-k)}}(\phi_k(t)) > \frac{\pi}{4} e^{-2\phi_k(t)} \quad \text{and} \quad \phi_k(t)^{-1} K^{-\frac{\mu-1}{2(1-k)}}(\phi_k(t)) > \frac{1}{\pi} e^{2\phi_k(t)}, \quad \forall t \geq T_0/2.$$

Inserting (3.23) in (3.22) and using (1.8), we obtain that

$$(3.24) \quad \begin{aligned} \mathcal{U}(t) &\geq \varepsilon \frac{C(f, g)}{4} t^k e^{-2\phi_k(t)} \int_{t/2}^t \phi_k'(s) e^{2\phi_k(s)} ds \\ &\geq \varepsilon \frac{C(f, g)}{8} t^k [1 - e^{-2(\phi_k(t) - \phi_k(t/2))}], \quad \forall t \geq T_0. \end{aligned}$$

Thanks to (1.8) and the fact that $k < 1$, we observe that $t \mapsto 1 - e^{-2(\phi_k(t) - \phi_k(t/2))}$ is an increasing function on (T_0, ∞) , hence, its minimum is achieved at $t = T_0$. Therefore we deduce that

$$(3.25) \quad \mathcal{U}(t) \geq \varepsilon \kappa C(f, g) t^k, \quad \forall t \geq T_0,$$

where

$$\kappa := \frac{1}{8} \left(1 - \exp \left(-\frac{(2-2^k)T_0^{1-k}}{1-k} \right) \right).$$

Hence, Lemma 3.3 is now proved. \square

The next lemma gives the lower bound of the functional $\mathcal{V}(t)$.

Lemma 3.4. *Assume that the initial data are as in Theorem 2.2. For u an energy solution of (1.1), there exists $T_1 = T_1(k, \mu) > T_0$ such that*

$$(3.26) \quad \mathcal{V}(t) \geq C_{\mathcal{V}} \varepsilon, \quad \text{for all } t \geq T_1,$$

where $C_{\mathcal{V}}$ is a positive constant depending on f, g, N, μ, R and k , but not on ε .

Proof. Let $t \in [1, T)$. Recall the definitions of \mathcal{U} and \mathcal{V} , given respectively by (3.14) and (3.15), (3.4) and the identity

$$(3.27) \quad \mathcal{U}'(t) - \frac{\rho'(t)}{\rho(t)} \mathcal{U}(t) = \mathcal{V}(t).$$

Hence, the equation (3.19) yields

$$(3.28) \quad \mathcal{V}(t) + \left[\frac{\mu}{t} - \frac{\rho'(t)}{\rho(t)} \right] \mathcal{U}(t) = \int_1^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx ds + \varepsilon C(f, g).$$

A differentiation in time of the equation (3.28) gives

(3.29)

$$\mathcal{V}'(t) + \left[\frac{\mu}{t} - \frac{\rho'(t)}{\rho(t)} \right] \mathcal{U}'(t) - \left(\frac{\mu}{t^2} + \frac{\rho''(t)\rho(t) - (\rho'(t))^2}{\rho^2(t)} \right) \mathcal{U}(t) = \int_{\mathbb{R}^N} |u_t(x, t)|^p \psi(x, t) dx.$$

Now, thanks to (3.3) and (3.27), we deduce from (3.29) that

$$(3.30) \quad \mathcal{V}'(t) + \left[\frac{\mu}{t} - \frac{\rho'(t)}{\rho(t)} \right] \mathcal{V}(t) = t^{-2k} \mathcal{U}(t) + \int_{\mathbb{R}^N} |u_t(x, t)|^p \psi(x, t) dx,$$

that we rewrite as

$$(3.31) \quad \left(t^\mu \frac{\mathcal{V}(t)}{\rho(t)} \right)' = \frac{t^\mu}{\rho(t)} \left(t^{-2k} \mathcal{U}(t) + \int_{\mathbb{R}^N} |u_t(x, t)|^p \psi(x, t) dx \right), \quad \forall t \geq 1.$$

An integration of (3.31) over $(1, t)$ implies that

$$(3.32) \quad t^\mu \frac{\mathcal{V}(t)}{\rho(t)} = \frac{\mathcal{V}(1)}{\rho(1)} + \int_1^t \frac{s^\mu}{\rho(s)} \left(s^{-2k} \mathcal{U}(s) + \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx \right) ds, \quad \forall t \geq 1.$$

Thanks to the fact that $\mathcal{V}(1) \geq 0$, $\rho(1) > 0$ and using the lower bound of \mathcal{U} as in (3.16), we infer that

$$(3.33) \quad \mathcal{V}(t) \geq \frac{\rho(t)}{t^\mu} \int_1^t \frac{s^\mu}{\rho(s)} \left(C_U \varepsilon s^{-k} + \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx \right) ds, \quad \forall t \geq T_0.$$

Therefore the estimate (3.33) gives

$$(3.34) \quad \mathcal{V}(t) \geq C_U \varepsilon \frac{\rho(t)}{t^\mu} \int_{t/2}^t \frac{s^{-k+\mu}}{\rho(s)} ds, \quad \forall t \geq 2T_0.$$

For convenience, we rewrite (3.23) as follows:

$$(3.35) \quad \sqrt{\phi_k(t)} K_{\frac{\mu-1}{2(1-k)}}(\phi_k(t)) > \frac{\sqrt{\pi}}{2} e^{-\phi_k(t)} \quad \text{and} \quad \frac{1}{\sqrt{\phi_k(t)}} K_{\frac{\mu-1}{2(1-k)}}^{-1}(\phi_k(t)) > \frac{1}{\sqrt{\pi}} e^{\phi_k(t)}, \quad \forall t \geq T_0/2.$$

Using the expressions of $\rho(t)$ and $\phi_k(t)$, given respectively by (3.1) and (1.8), we deduce that

$$(3.36) \quad \begin{aligned} \mathcal{V}(t) &\geq \varepsilon C_U \left(\frac{1}{2} \right)^{\frac{\mu}{2}+1} e^{-\phi_k(t)} \int_{t/2}^t \phi_k'(s) e^{\phi_k(s)} ds \\ &\geq \varepsilon C_U \left(\frac{1}{2} \right)^{\frac{\mu}{2}+1} [1 - e^{-(\phi_k(t) - \phi_k(t/2))}], \quad \forall t \geq 2T_0. \end{aligned}$$

Analogously as in Lemma 3.3, we have

$$(3.37) \quad \mathcal{V}(t) \geq C_V \varepsilon, \quad \forall t \geq T_1 := 2T_0,$$

where

$$C_V := C_U \left(\frac{1}{2} \right)^{\frac{\mu}{2}+1} \left(1 - \exp \left(-\frac{(1 - 2^{k-1})(2T_0)^{1-k}}{1 - k} \right) \right).$$

This completes the proof of Lemma 3.4. \square

4. PROOF OF THEOREM 2.2.

This section is dedicated to proving the main result in Theorem 2.2 which exposes the blow-up dynamics of the solution of (1.1). Hence, to prove the blow-up result for (1.1) we will use (3.28) and (3.30). For this purpose, we multiply (3.28) by $\alpha \frac{\rho'(t)}{\rho(t)}$, and subtract the resulting equation from (3.30). Therefore we obtain for a certain $\alpha \geq 0$, whose range will be fixed afterward,

$$(4.1) \quad \begin{aligned} \mathcal{V}'(t) + \left[\frac{\mu}{t} - (1 + \alpha) \frac{\rho'(t)}{\rho(t)} \right] \mathcal{V}(t) &= -\varepsilon \alpha \frac{\rho'(t)}{\rho(t)} C(f, g) + \left[t^{-2k} + \alpha \frac{\rho'(t)}{\rho(t)} \left(\frac{\mu}{t} - \frac{\rho'(t)}{\rho(t)} \right) \right] \mathcal{U}(t) \\ &+ \int_{\mathbb{R}^N} |u_t(x, t)|^p \psi(x, t) dx - \alpha \frac{\rho'(t)}{\rho(t)} \int_1^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx ds, \quad \forall t \geq 1. \end{aligned}$$

Using (3.7), we can choose $\tilde{T}_2 \geq T_1$ (T_1 is given in Lemma 3.4) such that

$$(4.2) \quad \begin{aligned} \mathcal{V}'(t) + \left[\frac{\mu}{t} - (1 + \alpha) \frac{\rho'(t)}{\rho(t)} \right] \mathcal{V}(t) &\geq \frac{\varepsilon \alpha t^{-k}}{2} C(f, g) + (1 - 4\alpha) t^{-2k} \mathcal{U}(t) \\ &+ \int_{\mathbb{R}^N} |u_t(x, t)|^p \psi(x, t) dx + \frac{\alpha t^{-k}}{2} \int_1^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx ds, \quad \forall t \geq \tilde{T}_2. \end{aligned}$$

From now on the parameter α is chosen in $(1/7, 1/4)$. Thanks to (3.16), the estimate (4.2) leads to the following lower bound:

$$(4.3) \quad \begin{aligned} \mathcal{V}'(t) + \left[\frac{\mu}{t} - (1 + \alpha) \frac{\rho'(t)}{\rho(t)} \right] \mathcal{V}(t) &\geq \frac{\varepsilon \alpha t^{-k}}{2} C(f, g) + \int_{\mathbb{R}^N} |u_t(x, t)|^p \psi(x, t) dx \\ &+ \frac{\alpha t^{-k}}{2} \int_1^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx ds, \quad \forall t \geq \tilde{T}_2. \end{aligned}$$

Now, we introduce the following functional:

$$H(t) := C_2 \varepsilon + \frac{1}{16} \int_{\tilde{T}_3}^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx ds,$$

where $C_2 := \min(\alpha C(f, g)/4(1 + \alpha), C_{\mathcal{V}})$ ($C_{\mathcal{V}}$ is given by Lemma 3.4) and we choose $\tilde{T}_3 > \tilde{T}_2$ such that

$$(4.4) \quad \frac{\alpha}{2} C(f, g) - C_2 t^k \left(\frac{\mu}{t} - (1 + \alpha) \frac{\rho'(t)}{\rho(t)} \right) \geq 0,$$

and

$$(4.5) \quad \frac{\alpha}{2} - \frac{1}{16} t^k \left(\frac{\mu}{t} - (1 + \alpha) \frac{\rho'(t)}{\rho(t)} \right) \geq 0,$$

for all $t \geq \tilde{T}_3$ (this is possible thanks to (3.7), the definition of C_2 and the fact that $\alpha \in (1/7, 1/4)$).

Let

$$\mathcal{F}(t) := \mathcal{V}(t) - H(t),$$

which satisfies

$$\begin{aligned}
(4.6) \quad & \mathcal{F}'(t) + \left[\frac{\mu}{t} - (1 + \alpha) \frac{\rho'(t)}{\rho(t)} \right] \mathcal{F}(t) \geq \frac{15}{16} \int_{\mathbb{R}^N} |u_t(x, t)|^p \psi(x, t) dx \\
& + \left[\frac{\alpha}{2} - \frac{1}{16} \left(\frac{\mu}{t^{1-k}} - (1 + \alpha) \frac{t^k \rho'(t)}{\rho(t)} \right) \right] t^{-k} \int_{\tilde{T}_3}^t \int_{\mathbb{R}^N} |u_t(x, s)|^p \psi(x, s) dx ds \\
& + \left[\frac{\alpha}{2} C(f, g) - C_2 \left(\frac{\mu}{t^{1-k}} - (1 + \alpha) \frac{t^k \rho'(t)}{\rho(t)} \right) \right] \varepsilon t^{-k}, \quad \forall t \geq \tilde{T}_3.
\end{aligned}$$

Thanks to (4.4) and (4.5), we easily conclude that

$$(4.7) \quad \mathcal{F}'(t) + \left[\frac{\mu}{t} - (1 + \alpha) \frac{\rho'(t)}{\rho(t)} \right] \mathcal{F}(t) \geq 0, \quad \forall t \geq \tilde{T}_3.$$

Multiplying (4.7) by $\frac{t^\mu}{\rho^{1+\alpha}(t)}$ and integrating over (\tilde{T}_3, t) , we get

$$(4.8) \quad \mathcal{F}(t) \geq \mathcal{F}(\tilde{T}_3) \frac{\tilde{T}_3^\mu \rho^{1+\alpha}(t)}{t^\mu \rho^{1+\alpha}(\tilde{T}_3)}, \quad \forall t \geq \tilde{T}_3.$$

Hence, we see that $\mathcal{F}(\tilde{T}_3) = \mathcal{V}(\tilde{T}_3) - C_2 \varepsilon \geq \mathcal{V}(\tilde{T}_3) - C_V \varepsilon \geq 0$ in view of Lemma 3.4 and the definition of C_2 implying that $C_2 \leq C_V$.

Therefore we deduce that

$$(4.9) \quad \mathcal{V}(t) \geq H(t), \quad \forall t \geq \tilde{T}_3.$$

Now, employing the Hölder inequality and the estimates (3.13) and (3.15), we obtain

$$\begin{aligned}
(4.10) \quad H'(t) & \geq \frac{1}{16} \mathcal{V}^p(t) \left(\int_{|x| \leq \phi_k(t)+R} \psi(x, t) dx \right)^{-(p-1)} \\
& \geq C \mathcal{V}^p(t) \rho^{-(p-1)}(t) e^{-(p-1)\phi_k(t)} (\phi_k(t))^{-\frac{(N-1)(p-1)}{2}}.
\end{aligned}$$

In view of (3.6), we see that

$$(4.11) \quad H'(t) \geq C \mathcal{V}^p(t) t^{-\frac{[(N-1)(1-k)+k+\mu](p-1)}{2}}, \quad \forall t \geq \tilde{T}_3.$$

From the above estimate and (4.9), we have

$$(4.12) \quad H'(t) \geq C H^p(t) t^{-\frac{[(N-1)(1-k)+k+\mu](p-1)}{2}}, \quad \forall t \geq \tilde{T}_3.$$

Since $H(\tilde{T}_3) = C_2 \varepsilon > 0$, we easily obtain the blow-up in finite time for the functional $H(t)$, and consequently the one for $\mathcal{V}(t)$ due to (4.9).

The proof of Theorem 2.2 is now achieved.

ACKNOWLEDGMENTS

The authors are deeply thankful to the anonymous reviewer for the valuable remarks that improved the paper. A. Palmieri is supported by the Japan Society for the

Promotion of Science (JSPS) – JSPS Postdoctoral Fellowship for Research in Japan (Short-term) (PE20003).

REFERENCES

- [1] M.F. Ben Hassen, M. Hamouda, M. A. Hamza and H.K. Teka *Nonexistence result for the generalized Tricomi equation with the scale-invariant damping, mass term and time derivative nonlinearity*. To appear in *Asymptotic Analysis*, (2021), pp. 1–21. [5](#)
- [2] W. Chen, S. Lucente and A. Palmieri, *Nonexistence of global solutions for generalized Tricomi equations with combined nonlinearity*. *Nonlinear Analysis: Real World Applications*, **61** (2021), 103354. [4](#)
- [3] M. D’Abbicco, *The threshold of effective damping for semilinear wave equations*. *Math. Methods Appl. Sci.* **38** (6) (2015), 1032–1045. [3](#)
- [4] M. D’Abbicco and S. Lucente, *A modified test function method for damped wave equations*. *Adv. Nonlinear Stud.* **13** (4) (2013), 867–892. [3](#)
- [5] M. D’Abbicco and S. Lucente, *NLWE with a special scale invariant damping in odd space dimension*, *Discrete Contin. Dyn. Syst.* 2015, *Dynamical systems, differential equations and applications*. 10th AIMS Conference. Suppl., 312–319. [2](#)
- [6] M. D’Abbicco, S. Lucente and M. Reissig, *A shift in the Strauss exponent for semilinear wave equations with a not effective damping*. *J. Differential Equations*, **259** (10) (2015), 5040–5073. [2](#)
- [7] A. Galstian and K. Yagdjian, *Finite lifespan of solutions of the semilinear wave equation in the Einstein-de Sitter spacetime*. *Rev. Math. Phys.* **32** (7) (2020), 2050018. doi: 10.1142/S0129055X2050018X [2](#)
- [8] R.E. Gaunt, *Inequalities for modified Bessel functions and their integrals*. *J. Math. Anal. Appl.*, **420** (2014), 373–386. [7](#)
- [9] M. Hamouda and M.A. Hamza, *Blow-up for wave equation with the scale-invariant damping and combined nonlinearities*. *Math Meth. Appl. Sci.* **44** (1) (2021), 1127–1136. [4](#)
- [10] M. Hamouda and M.A. Hamza, *Improvement on the blow-up of the wave equation with the scale-invariant damping and combined nonlinearities*. *Nonlinear Anal. Real World Appl.*, **59** (2021), 103275, <https://doi.org/10.1016/j.nonrwa.2020.103275>. [3](#), [4](#)
- [11] M. Hamouda and M.A. Hamza, *A blow-up result for the wave equation with localized initial data: the scale-invariant damping and mass term with combined nonlinearities*. (2020) arXiv:2010.05455. [4](#)
- [12] M. Hamouda and M.A. Hamza, *Blow-up and lifespan estimate for the generalized Tricomi equation with mixed nonlinearities*. *Advances in Pure and Applied Mathematics*, 2021, vol. **12**, special issue, pp. 54–70. DOI: 10.21494/ISTE.OP.2021.0698. [2](#), [4](#)
- [13] M. Hamouda, M.A. Hamza and A. Palmieri, *A note on the nonexistence of global solutions to the semilinear wave equation with nonlinearity of derivative-type in the generalized Einstein – de Sitter spacetime*. *Communications on Pure & Applied Analysis*, **20** (11) (2021), 3703–3721. [3](#)
- [14] K. Hidano and K. Tsutaya, *Global existence and asymptotic behavior of solutions for nonlinear wave equations*, *Indiana Univ. Math. J.*, **44** (1995), 1273–1305. [2](#)
- [15] K. Hidano, C. Wang and K. Yokoyama, *The Glassey conjecture with radially symmetric data*, *J. Math. Pures Appl.*, **98** (5) (2012), 518–541. [2](#)
- [16] M. Ikeda and M. Sobajima, *Life-span of solutions to semilinear wave equation with time-dependent damping for specially localized initial data*. *Math. Ann.* **372** (3-4) (2018), 1017–1040. [2](#)

- [17] F. John, *Blow-up for quasilinear wave equations in three space dimensions*, Comm. Pure Appl. Math., **34** (1981), 29–51. [2](#)
- [18] N.-A. Lai and N.M. Schiavone, *Blow-up and lifespan estimate for generalized Tricomi equations related to Glassey conjecture*. arXiv: 2007.16003v2 (2020). [2](#)
- [19] N.-A. Lai and H. Takamura, *Nonexistence of global solutions of nonlinear wave equations with weak time-dependent damping related to Glassey’s conjecture*. Differential Integral Equations, **32** (1-2) (2019), 37–48. [3](#)
- [20] S. Lucente and A. Palmieri, *A blow-up result for a generalized Tricomi equation with nonlinearity of derivative type*. Milan J. Math., **89** (2021), 45–57. [2](#)
- [21] A. Palmieri, *A global existence result for a semilinear wave equation with scale-invariant damping and mass in even space dimension*. Math. Methods Appl. Sci. **42** (8) (2019), 2680–2706. [2](#)
- [22] A. Palmieri, *Blow – up results for semilinear damped wave equations in Einstein - de Sitter spacetime*. Z. Angew. Math. Phys., **72** (2021), 64. [2](#), [6](#)
- [23] A. Palmieri, *Lifespan estimates for local solutions to the semilinear wave equation in Einstein-de Sitter spacetime*. Preprint, arXiv: 2009.04388 (2020). [2](#)
- [24] A. Palmieri and M. Reissig, *A competition between Fujita and Strauss type exponents for blow-up of semi-linear wave equations with scale-invariant damping and mass*. J. Differential Equations, **266** (2-3) (2019), 1176–1220. [2](#)
- [25] A. Palmieri and Z. Tu, *A blow-up result for a semilinear wave equation with scale-invariant damping and mass and nonlinearity of derivative type*. Calc. Var. Partial Differential Equations, **60** (2021), 72. [3](#)
- [26] T. C. Sideris, *Global behavior of solutions to nonlinear wave equations in three space dimensions*, Comm. Partial Differential Equations, **8** (12) (1983), 1291–1323. [2](#)
- [27] K. Tsutaya and Y. Wakasugi, *Blow up of solutions of semilinear wave equations in Friedmann-Lemaître-Robertson-Walker spacetime*. J. Math. Phys., **61** (2020), 091503. [2](#)
- [28] K. Tsutaya and Y. Wakasugi, *On Glassey’s conjecture for semilinear wave equations in Friedmann-Lemaître-Robertson-Walker spacetime*. Boundary Value Problems, **2021** (2021), 94. [3](#), [5](#)
- [29] K. Tsutaya and Y. Wakasugi, *On heatlike lifespan of solutions of semilinear wave equations in Friedmann-Lemaître-Robertson-Walker spacetime*. J. Math. Anal. Appl., **500** (2021), 125133. [2](#)
- [30] Z. Tu, and J. Lin, *A note on the blowup of scale invariant damping wave equation with sub-Strauss exponent, preprint, arXiv:1709.00866v2, 2017*. [2](#)
- [31] Z. Tu, and J. Lin, *Life-span of semilinear wave equations with scale-invariant damping: critical Strauss exponent case*. Differential Integral Equations, **32** (5-6) (2019), 249–264. [2](#)
- [32] N. Tzvetkov, *Existence of global solutions to nonlinear massless Dirac system and wave equation with small data*, Tsukuba J. Math., **22** (1998), 193–211. [2](#)
- [33] K. Wakasugi, *Critical exponent for the semilinear wave equation with scale invariant damping*. In: M. Ruzhansky , V. Turunen (Eds.) Fourier Analysis. Trends in Mathematics. Birkhäuser, Cham (2014). https://doi.org/10.1007/978-3-319-02550-6_19. [3](#)
- [34] B. Yordanov and Q. S. Zhang, *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal., **231** (2006), 361–374. [6](#), [7](#)
- [35] Y. Zhou, *Blow-up of solutions to the Cauchy problem for nonlinear wave equations*, Chin. Ann. Math., **22B** (3) (2001), 275–280. [2](#)

¹ DEPARTMENT OF BASIC SCIENCES, DEANSHIP OF PREPARATORY YEAR AND SUPPORTING STUDIES, IMAM ABDULRAHMAN BIN FAISAL UNIVERSITY, P.O. BOX 1982, 34212 DAMMAM, SAUDI ARABIA.

² DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PISA, LARGO B. PONTECORVO 5, 56127 PISA, ITALY.

³ *Current address:* MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, Aoba, SENDAI 980-8578, JAPAN.

Email address: mmhamouda@iau.edu.sa (M. Hamouda)

Email address: mahamza@iau.edu.sa (M.A. Hamza)

Email address: alessandro.palmieri.math@gmail.com (A. Palmieri)