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Blow-up results for semilinear damped wave equations in Einstein-de Sitter spacetime

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Abstract. We prove by using an iteration argument some blow-up results for a semilinear damped wave equation in generalized Einstein-de Sitter spacetime with a time-dependent coefficient for the damping term and power nonlinearity. Then, we conjecture an expression for the critical exponent due to the main blow-up results, which is consistent with many special cases of the considered model and provides a natural generalization of Strauss exponent. In the critical case, we consider a non-autonomous and parameter dependent Cauchy problem for a linear ODE of second order, whose explicit solutions are determined by means of special functions' theory.

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1. Introduction

In recent years, the wave equation in Einstein-de Sitter spacetime has been considered in [9, 10] in the linear case and in [11, 12, 29] in the semilinear case. Let us consider the semilinear wave equation with power nonlinearity in a *generalized Einstein-de Sitter spacetime*, that is, the equation with singular coefficients

$$\varphi_{tt} - t^{-2k}\Delta\varphi + 2t^{-1}\varphi_t = |\varphi|^p, \tag{1.1}$$

where $k \in [0, 1)$ and $p > 1$. This model is the semilinear wave equation in Einstein-de Sitter spacetime with power nonlinearity for $k = 2/3$ and $n = 3$. It has been proved in [12, 29] that for

$$1 < p \leqslant \max\left\{p_0\big(k, n + \tfrac{2}{1-k}\big), p_1(k, n)\right\}$$

a local in time solution to the corresponding nonsingular Cauchy problem (with initial data prescribed at the initial time $t = 1$) blows up in finite time, provided that the initial data fulfill certain integral sign conditions. More specifically, in [12] the subcritical case for (1.1) is investigated, while in [29] the critical case and the upper bound estimates for the lifespan are studied. Here and throughout the paper $p_0(k, n)$ is the positive root of the quadratic equation

$$\left(\tfrac{n-1}{2} - \tfrac{k}{2(1-k)}\right)p^2 - \left(\tfrac{n+1}{2} + \tfrac{3k}{2(1-k)}\right)p - 1 = 0, \tag{1.2}$$

when the coefficient for p^2 is not positive, we set formally $p_0(k, n) \doteq \infty$, while

$$p_1(k, n) \doteq 1 + \frac{2}{(1-k)n}. \tag{1.3}$$

Note that $p_1(k, n)$ is related to the Fujita exponent $p_{\mathrm{Fuj}}(n) \doteq 1 + \frac{2}{n}$. Indeed, according to this notation, it holds $p_1(k, n) = p_{\mathrm{Fuj}}\big((1-k)n\big)$ and $p_1(0, n) = p_{\mathrm{Fuj}}(n)$. On the other hand, $p_0(k, n)$ is a generalization of the Strauss exponent for the classical semilinear wave equation, since $p_0(0,n) = p_{\mathrm{Str}}(n)$, where $p_{\mathrm{Str}}(n)$ is the positive root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$.

In this paper, we generalize the model (1.1) by taking a nonnegative multiplicative constant μ for the damping term. More specifically, we investigate the blow-up dynamic for the nonsingular Cauchy problem

$$\begin{cases} u_{tt} - t^{-2k}\Delta u + \mu\, t^{-1} u_t = |u|^p & x \in \mathbb{R}^n,\ t \in (1,T), \\ u(1,x) = \varepsilon u_0(x) & x \in \mathbb{R}^n, \\ u_t(1,x) = \varepsilon u_1(x) & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

where $k \in [0,1)$, $p > 1$, μ is the nonnegative multiplicative constant in the time-dependent coefficient for the damping term and $\varepsilon > 0$ describes the size of the initial data. Let us point out that the not damped case $\mu = 0$ can be treated as well via our approach.

More precisely, we will focus on proving blow-up results whenever the exponent p belongs to the range

$$1 < p \leqslant \max\left\{p_0\big(k, n + \tfrac{\mu}{1-k}\big), p_1(k,n)\right\},$$

under suitable sign assumptions for u_0, u_1. According to (1.2), the shift $p_0\big(k, n + \frac{\mu}{1-k}\big)$ of $p_0(k,n)$ is nothing but the positive root to the quadratic equation

$$\left(\tfrac{n-1}{2} + \tfrac{\mu-k}{2(1-k)}\right) p^2 - \left(\tfrac{n+1}{2} + \tfrac{\mu+3k}{2(1-k)}\right) p - 1 = 0. \tag{1.5}$$

Therefore, the critical exponent $p_0\big(k, n + \frac{\mu}{1-k}\big)$ for (1.4) is obtained by the corresponding exponent in the not damped case via a formal shift in the dimension of magnitude $\frac{\mu}{1-k}$.

Let us provide an overview on the methods that we are going to use to prove the main results in this paper. In the subcritical case $1 < p < \max\left\{p_0\big(k, n + \frac{\mu}{1-k}\big), p_1(k,n)\right\}$, we employ a standard iteration argument based on a multiplier argument (see also [19, 20, 21] for further details on the multiplier argument). This approach is based on the employment of two time-dependent functionals related to a local solution u to (1.4) and generalizes the method from [36] for the semilinear wave equation with scale-invariant damping. The first functional is the space average of u and its dynamic will be considered for the iterative argument. On the other hand, we will work with a positive solution of the adjoint linear equation in order to prove the positivity of the second auxiliary functional. Hence, this second functional will also provide a first lower bound estimate for the first functional, allowing us to begin with the iteration procedure. In the critical case we should sharpen our iteration frame by considering a different time-dependent functional, so that a slicing procedure may be applied. In comparison to what happens in the subcritical case, a more precise analysis of the adjoint linear equation is necessary in the critical case $p = p_0\big(k, n + \frac{\mu}{1-k}\big)$. This approach follows the one developed in [29] which is in turn a generalization of the ideas introduced by Wakasa and Yordanov in [38, 39] and developed in different frameworks in [31, 32, 23, 3, 4]. Whereas in the other critical case $p = p_1(k,n)$, we can still work with the space average of a local in time solution as functional, although a slicing procedure has to be applied in order to deal with logarithmic factors in the lower bound estimates.

1.1. Notations

Throughout this paper we use the following notations: $\phi_k(t) \doteq \frac{t^{1-k}}{1-k}$ denotes the primitive of the speed of propagation $a_k(t) = t^{-k}$ that vanishes at $t = 0$, while the amplitude of the light cone is given by the function

$$A_k(t) \doteq \int_1^t \tau^{-k} \mathrm{d}\tau = \phi_k(t) - \phi_k(1); \tag{1.6}$$

B_R denotes the ball in $\mathbb{R}^n$ with radius R around the origin; $f \lesssim g$ means that there exists a positive constant C such that $f \leqslant Cg$ and, similarly, for $f \gtrsim g$; I_ν and K_ν denote the modified Bessel function of first and second kind of order ν, respectively; finally, as in the introduction, $p_0(k,n)$ is the positive solution to (1.2) and $p_1(k,n)$ is defined by (1.3).

1.2. Main results

Before stating the main theorems, let us introduce a suitable notion of energy solution to the semilinear Cauchy problem (1.4).

Definition 1.1. *Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We say that*

$$u \in \mathscr{C}\big([1,T), H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([1,T), L^2(\mathbb{R}^n)\big) \cap L^p_{\mathrm{loc}}\big([1,T)\times\mathbb{R}^n\big)$$

is an energy solution to (1.4) *on* $[1,T)$ *if* u *fulfills* $u(1,\cdot) = \varepsilon u_0$ *in* $H^1(\mathbb{R}^n)$ *and the integral relation*

$$\begin{aligned}
&\int_{\mathbb{R}^n} \partial_t u(t,x)\psi(t,x)\,\mathrm{d}x - \varepsilon\int_{\mathbb{R}^n} u_1(x)\psi(1,x)\,\mathrm{d}x - \int_1^t\int_{\mathbb{R}^n}\partial_t u(s,x)\psi_s(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_1^t\int_{\mathbb{R}^n} s^{-2k}\nabla u(s,x)\cdot\nabla\psi(s,x)\,\mathrm{d}x\,\mathrm{d}s + \int_1^t\int_{\mathbb{R}^n}\mu s^{-1}\partial_t u(s,x)\psi(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_1^t\int_{\mathbb{R}^n} |u(s,x)|^p\psi(s,x)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{1.7}$$

for any test function $\psi \in \mathscr{C}_0^\infty([1,T)\times\mathbb{R}^n)$ and any $t \in (1,T)$.

We point out that performing a further step of integration by parts in (1.7), we find the integral relation

$$\begin{aligned}
&\int_{\mathbb{R}^n}\partial_t u(t,x)\psi(t,x)\,\mathrm{d}x - \int_{\mathbb{R}^n} u(t,x)\psi_s(t,x)\,\mathrm{d}x + \int_{\mathbb{R}^n}\mu\, t^{-1}u(t,x)\psi(t,x)\,\mathrm{d}x \\
&\quad - \varepsilon\int_{\mathbb{R}^n} u_1(x)\psi(1,x)\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^n} u_0(x)\psi_s(1,x)\,\mathrm{d}x - \varepsilon\int_{\mathbb{R}^n}\mu\, u_0(x)\psi(1,x)\,\mathrm{d}x \\
&\quad + \int_1^t\int_{\mathbb{R}^n} u(s,x)\left(\psi_{ss}(s,x) - s^{-2k}\Delta\psi(s,x) - \mu s^{-1}\psi_s(s,x) + \mu s^{-2}\psi(s,x)\right)\mathrm{d}x\,\mathrm{d}s \\
&= \int_1^t\int_{\mathbb{R}^n}|u(s,x)|^p\psi(s,x)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{1.8}$$

for any $\psi \in \mathscr{C}_0^\infty([1,T)\times\mathbb{R}^n)$ and any $t \in (1,T)$.

Remark 1.2. *Let us point out that if the Cauchy data have compact support, say $\operatorname{supp} u_j \subset B_R$ for $j = 0,1$ and for some $R > 0$, then, for any $t \in (1,T)$ and any local solution u to* (1.4) *the support condition*

$$\operatorname{supp} u(t,\cdot) \subset B_{R+A_k(t)}$$

is satisfied, where A_k is defined by (1.6)*. Consequently, in Definition 1.1 it is possible to consider test functions which are not compactly supported, i.e., $\psi \in \mathscr{C}^\infty([1,T)\times\mathbb{R}^n)$.*

Theorem 1.3 (Subcritical case). *Let $\mu \geqslant 0$ and let the exponent of the nonlinear term p satisfy*

$$1 < p < \max\left\{p_0\big(k, n + \tfrac{\mu}{1-k}\big), p_1(k,n)\right\}.$$

Let us assume that $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ are nonnegative and nontrivial functions with supports contained in B_R for some $R > 0$. Let

$$u \in \mathscr{C}\big([1,T), H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([1,T), L^2(\mathbb{R}^n)\big) \cap L^p_{\mathrm{loc}}\big([1,T)\times\mathbb{R}^n\big)$$

be an energy solution to (1.4) *according to Definition 1.1 with lifespan $T = T(\varepsilon)$ and satisfying the support condition $\operatorname{supp} u(t,\cdot) \subset B_{A_k(t)+R}$ for any $t \in (1,T)$.*

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0,u_1,n,p,k,\mu,R)$ such that for any $\varepsilon \in (0,\varepsilon_0]$ the energy solution u blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon) \leqslant \begin{cases} C\varepsilon^{-\frac{p(p-1)}{\theta(n,k,\mu,p)}} & \text{if } p < p_0\big(k, n+\frac{\mu}{1-k}\big), \\ C\varepsilon^{-\left(\frac{2}{p-1}-(1-k)n\right)^{-1}} & \text{if } p < p_1(k,n), \end{cases} \tag{1.9}$$

holds, where the positive constant C is independent of ε and

$$\theta(n,k,\mu,p) \doteq 1-k+\Big((1-k)\tfrac{n+1}{2}+\tfrac{\mu+3k}{2}\Big)p-\Big((1-k)\tfrac{n-1}{2}+\tfrac{\mu-k}{2}\Big)p^2.$$

In order to properly state the results in the critical case, let us explicitly provide the threshold for μ which yields the transition from a dominant $p_0\big(k,n+\frac{\mu}{1-k}\big)$ to the case in which $p_1(k,n)$ is the highest exponent. Due to the fact that $p_0\big(k,n+\frac{\mu}{1-k}\big)$ is the biggest solution of (1.5), we have that $p_1(k,n)>p_0\big(k,n+\frac{\mu}{1-k}\big)$ if and only if

$$\Big(\tfrac{n-1}{2}+\tfrac{\mu-k}{2(1-k)}\Big)p_1(k,n)^2-\Big(\tfrac{n+1}{2}+\tfrac{\mu+3k}{2(1-k)}\Big)p_1(k,n)-1>0.$$

By straightforward computations, it follows that $p_1(k,n)>p_0\big(k,n+\frac{\mu}{1-k}\big)$ for $\mu>\mu_0(k,n)$, where

$$\mu_0(k,n) \doteq \frac{(1-k)^2n^2+(1-k)(1+2k)n+2}{n(1-k)+2}. \tag{1.10}$$

Note that for $k=0$ the splitting value $\mu_0(k,n)$ does coincide with the one for the semilinear wave equation with scale-invariant damping in the flat case from the work [18].

Theorem 1.4 (Critical case: part I). *Let $0\leqslant\mu\leqslant\mu_0(k,n)$ such that $\mu\leqslant k$ or $\mu\geqslant 2-k$. We consider $p=p_0\big(k,n+\frac{\mu}{1-k}\big)$. Let us assume that $u_0\in H^1(\mathbb{R}^n)$ and $u_1\in L^2(\mathbb{R}^n)$ are nonnegative and nontrivial functions with supports contained in B_R for some $R>0$. Let*

$$u\in\mathscr{C}\big([1,T),H^1(\mathbb{R}^n)\big)\cap\mathscr{C}^1\big([1,T),L^2(\mathbb{R}^n)\big)\cap L^p_{\text{loc}}\big([1,T)\times\mathbb{R}^n\big)$$

be an energy solution to (1.4) *according to Definition 1.1 with lifespan $T=T(\varepsilon)$ and satisfying the support condition* $\operatorname{supp} u(t,\cdot)\subset B_{A_k(t)+R}$ *for any $t\in(1,T)$.*

Then, there exists a positive constant $\varepsilon_0=\varepsilon_0(u_0,u_1,n,p,k,\mu,R)$ such that for any $\varepsilon\in(0,\varepsilon_0]$ the energy solution u blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon)\leqslant\exp\Big(C\varepsilon^{-p(p-1)}\Big)$$

holds, where the positive constant C is independent of ε.

Theorem 1.5 (Critical case: part II). *Let $\mu\geqslant\mu_0(k,n)$ and $p=p_1(k,n)$. Let us assume that $u_0\in H^1(\mathbb{R}^n)$ and $u_1\in L^2(\mathbb{R}^n)$ are nonnegative and nontrivial functions with supports contained in B_R for some $R>0$. Let*

$$u\in\mathscr{C}\big([1,T),H^1(\mathbb{R}^n)\big)\cap\mathscr{C}^1\big([1,T),L^2(\mathbb{R}^n)\big)\cap L^p_{\text{loc}}\big([1,T)\times\mathbb{R}^n\big)$$

be an energy solution to (1.4) *according to Definition 1.1 with lifespan $T=T(\varepsilon)$ and satisfying the support condition* $\operatorname{supp} u(t,\cdot)\subset B_{A_k(t)+R}$ *for any $t\in(1,T)$.*

Then, there exists a positive constant $\varepsilon_0=\varepsilon_0(u_0,u_1,n,p,k,\mu,R)$ such that for any $\varepsilon\in(0,\varepsilon_0]$ the energy solution u blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon)\leqslant\exp\Big(C\varepsilon^{-(p-1)}\Big)$$

holds, where the positive constant C is independent of ε.

The remaining part of the paper is organized as follows: the proof of the result in the subcritical case (cf. Theorem 1.3) is carried out in Section 2; in Section 3 we prove Theorem 1.4 by generalizing the approach introduced in [38]; finally, we show the proof of Theorem 1.5 in Section 4 via a standard slicing procedure.

2. Subcritical case

In this section we are going to prove Theorem 1.3. Let u be a local in time solution to (1.4) and let us assume that the assumptions from the statement of Theorem 1.3 on p and on the data are fulfilled. We will follow the multiplier approach introduced by [22] and then improved by [36], to derive a suitable iteration frame for the time-dependent functional

$$U_0(t) \doteq \int_{\mathbb{R}^n} u(t,x)\,\mathrm{d}x. \tag{2.11}$$

In order to obtain a first lower bound estimate for U_0 we will introduce a second time-dependent functional, following the main ideas of the pioneering paper [40] and adapting them to the case with time-depend coefficients as in [15, 12, 36, 33].

The section is organized as follows: in Section 2.1 we determine a suitable positive solution to the adjoint homogeneous linear equation with separate variables, then, we use this function to derive a lower bound estimate for U_0 in Section 2.3; in Sections 2.2 and 2.4 the derivation of the iteration frame and its application in an iterative argument are dealt with, respectively.

2.1. Solution of the adjoint homogeneous linear equation

In this section, we shall determine a particular positive solution to the adjoint homogeneous linear equation

$$\Psi_{ss} - s^{-2k}\Delta\Psi - \mu\, s^{-1}\Psi_s + \mu\, s^{-2}\Psi = 0. \tag{2.12}$$

First of all, we recall the remarkable function

$$\varphi(x) \doteq \begin{cases} \int_{\mathbb{S}^{n-1}} \mathrm{e}^{x\cdot\omega}\mathrm{d}\sigma_\omega & \text{if } n \geqslant 2, \\ \cosh x & \text{if } n = 1, \end{cases} \tag{2.13}$$

introduced in [40] for the study of the critical semilinear wave equation. The main properties of this function that will used throughout this paper are the following: φ is a positive and smooth function that satisfies $\Delta\varphi = \varphi$ and asymptotically behaves like $c_n|x|^{-\frac{n-1}{2}}\mathrm{e}^{|x|}$ as $|x| \to \infty$, where c_n is a positive constant depending on n.

If we look for a solution to (2.12) with separate variables, that is, we consider the ansatz $\Psi(s,x) = \varrho(s)\varphi(x)$, then, it suffices to find a positive solution to the ODE

$$\varrho'' - s^{-2k}\varrho - \mu s^{-1}\varrho' + \mu s^{-2}\varrho = 0. \tag{2.14}$$

We perform the change of variable $\tau = \phi_k(s)$. By using

$$\varrho' = s^{-k}\frac{\mathrm{d}\varrho}{\mathrm{d}\tau}, \qquad \varrho'' = s^{-2k}\frac{\mathrm{d}^2\varrho}{\mathrm{d}\tau^2} - ks^{-1-k}\frac{\mathrm{d}\varrho}{\mathrm{d}\tau},$$

it follows with straightforward computations that ϱ solves (2.14) if and only if

$$\frac{\mathrm{d}^2\varrho}{\mathrm{d}\tau^2} - \frac{k+\mu}{1-k}\,\frac{1}{\tau}\,\frac{\mathrm{d}\varrho}{\mathrm{d}\tau} + \left(\frac{\mu}{(1-k)^2}\,\frac{1}{\tau^2} - 1\right)\varrho = 0. \tag{2.15}$$

To further simplify the previous equation, we carry out the transformation $\varrho(\tau) = \tau^\sigma\zeta(\tau)$, where $\sigma \doteq \frac{1+\mu}{2(1-k)}$. Hence, using

$$\frac{\mathrm{d}\varrho}{\mathrm{d}\tau}(\tau) = \sigma\tau^{\sigma-1}\zeta(\tau) + \tau^\sigma\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}(\tau), \qquad \frac{\mathrm{d}^2\varrho}{\mathrm{d}\tau^2} = \sigma(\sigma-1)\tau^{\sigma-2}\zeta(\tau) + 2\sigma\tau^{\sigma-1}\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}(\tau) + \tau^\sigma\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2}(\tau),$$

we get that ϱ is a solution to (2.15) if and only if ζ solves

$$\tau^2\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2} + \left(2\sigma - \frac{k+\mu}{1-k}\right)\tau\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} + \left[\sigma\left(\sigma - 1 - \frac{k+\mu}{1-k}\right) + \frac{\mu}{(1-k)^2} - \tau^2\right]\zeta = 0. \tag{2.16}$$

Due to the choice of the parameter σ, equation (2.16) is nothing but a modified Bessel equation of order $\gamma \doteq \frac{\mu-1}{2(1-k)}$, that is, (2.16) can be rewritten as

$$\tau^2 \frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2} + \tau \frac{\mathrm{d}\zeta}{\mathrm{d}\tau} - (\gamma^2 + \tau^2)\zeta = 0.$$

If we pick the modified Bessel function of the second kind K_γ as solution to the previous equation, then, up to a negligible multiplicative constant, we find

$$\rho(s) \doteq s^{\frac{1+\mu}{2}} \mathrm{K}_\gamma\big(\phi_k(s)\big) \tag{2.17}$$

as a positive solution to (2.14) and, in turn,

$$\Psi(s,x) \doteq \rho(s)\varphi(x) = s^{\frac{1+\mu}{2}} \mathrm{K}_\gamma\big(\phi_k(s)\big)\varphi(x) \tag{2.18}$$

as a positive solution of the adjoint equation (2.12).

In the next sections, we will need to employ the asymptotic behavior of the function $\varrho = \varrho(t)$ for $t \to \infty$. Since $\mathrm{K}_\gamma(z) = \sqrt{\pi/(2z)}\mathrm{e}^{-z}\big(1 + O(z^{-1})\big)$ as $z \to \infty$ for $z > 0$ (cf. [25, Equation (10.25.3)]), then, the following asymptotic estimate holds

$$\varrho(t) = \sqrt{\frac{\pi}{2}}\, t^{\frac{k+\mu}{2}}\, \mathrm{e}^{-\phi_k(t)}\big(1 + O(t^{-1+k})\big) \qquad \text{for } t \to \infty. \tag{2.19}$$

The solution Ψ of the adjoint equation (2.12) that we determined in this section will be employed in Section 2.3 to introduce a second time-dependent functional with the purpose to establish a first lower bound estimate for U_0.

2.2. Derivation of the iteration frame

In this section we are going to determine the iteration frame for the functional $U_0 = U_0(t)$ defined in (2.11). Let us choose as test function $\psi = \psi(s,x)$ in the integral relation (1.7) such that $\psi = 1$ on the forward cone $\{(s,x) \in [1,t] \times \mathbb{R}^n : |x| \leqslant R + A_k(s)\}$. Then,

$$\int_{\mathbb{R}^n} \partial_t u(t,x)\,\mathrm{d}x - \varepsilon \int_{\mathbb{R}^n} u_1(x)\,\mathrm{d}x + \int_1^t \int_{\mathbb{R}^n} \mu s^{-1} \partial_t u(s,x)\,\mathrm{d}x\,\mathrm{d}s = \int_1^t \int_{\mathbb{R}^n} |u(s,x)|^p\,\mathrm{d}x\,\mathrm{d}s$$

which can be rewritten as

$$U_0'(t) - U_0'(1) + \int_1^t \mu s^{-1} U_0'(s)\,\mathrm{d}s = \int_1^t \int_{\mathbb{R}^n} |u(s,x)|^p\,\mathrm{d}x\,\mathrm{d}s.$$

Differentiating the last identity with respect to t, we get

$$U_0''(t) + \mu t^{-1} U_0'(t) = \int_{\mathbb{R}^n} |u(t,x)|^p\,\mathrm{d}x.$$

Multiplying the previous equation by t^μ, it follows

$$t^\mu U_0''(t) + \mu t^{\mu-1} U_0'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(t^\mu U_0'(t)\big) = t^\mu \int_{\mathbb{R}^n} |u(t,x)|^p\,\mathrm{d}x.$$

Integrating this relation over $[1,t]$, multiplying the resulting equation by $t^{-\mu}$ and then integrating over $[1,t]$ again, we find

$$U_0(t) = U_0(1) + U_0'(1) \int_1^t \tau^{-\mu}\,\mathrm{d}\tau + \int_1^t \tau^{-\mu} \int_1^\tau s^\mu \int_{\mathbb{R}^n} |u(s,x)|^p\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau. \tag{2.20}$$

On the one hand, from (2.20) we derive the lower bound estimate

$$U_0(t) \gtrsim \varepsilon, \tag{2.21}$$

where the unexpressed positive multiplicative constant depends on u_0, u_1 due to the nonnegativeness of nontrivial u_0, u_1 and $U^{(j)}(1) = \varepsilon \int_{\mathbb{R}^n} u_j(x)\mathrm{d}x$ for $j \in \{0,1\}$. On the other hand, we obtain the estimate

$$\begin{aligned} U_0(t) &\geqslant \int_1^t \tau^{-\mu} \int_1^\tau s^\mu \int_{\mathbb{R}^n} |u(s,x)|^p \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tau \\ &\gtrsim \int_1^t \tau^{-\mu} \int_1^\tau s^\mu (R + A_k(s))^{-n(p-1)} (U_0(s))^p \,\mathrm{d}s\,\mathrm{d}\tau, \end{aligned} \tag{2.22}$$

where in the second step we applied Jensen's inequality and the support property for $u(s,\cdot)$. Therefore, we proved the following iteration frame for U_0

$$U_0(t) \geqslant C \int_1^t \tau^{-\mu} \int_1^\tau s^{\mu-(1-k)n(p-1)} (U_0(s))^p \,\mathrm{d}s\,\mathrm{d}\tau \tag{2.23}$$

for a suitable positive constant $C = C(n,p,k)$ and for $t \geqslant 1$. In Section 2.2 we will employ (2.23) to derive iteratively a sequence of lower bound estimates for U_0. However, we shall first derive in Section 2.3 another lower bound estimate for U_0 that will provide, together with (2.21), the starting point for the iteration procedure.

2.3. First lower bound estimate for the functional

Let $\Psi = \Psi(t,x)$ be the function defined by (2.18). Since this function is smooth and positive, by applying the integral relation (1.8) to Ψ and using the fact that Ψ solves the adjoint equation (2.12), we get

$$\begin{aligned} 0 &\leqslant \int_1^t \int_{\mathbb{R}^n} |u(s,x)|^p \Psi(s,x) \,\mathrm{d}x\,\mathrm{d}s \\ &= \int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\,\mathrm{d}x - \int_{\mathbb{R}^n} u(t,x)\Psi_s(t,x)\,\mathrm{d}x + \mu\, t^{-1} \int_{\mathbb{R}^n} u(t,x)\Psi(t,x)\,\mathrm{d}x \\ &\quad - \varepsilon \int_{\mathbb{R}^n} (\varrho(1)u_1(x) + (\mu\varrho(1) - \varrho'(1))u_0(x))\,\varphi(x)\,\mathrm{d}x. \end{aligned}$$

If we introduce the auxiliary functional

$$U_1(t) \doteq \int_{\mathbb{R}^n} u(t,x)\Psi(t,x)\,\mathrm{d}x, \tag{2.24}$$

then, from the last estimate we have

$$U_1'(t) - \frac{2\varrho'(t)}{\varrho(t)}\,U_1(t) + \mu\, t^{-1} U_1(t) \geqslant \varepsilon \int_{\mathbb{R}^n} \big(\varrho(1)u_1(x) + (\mu\varrho(1) - \varrho'(1))u_0(x)\big)\varphi(x)\,\mathrm{d}x, \tag{2.25}$$

where we applied the relation

$$U_1'(t) = \int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\,\mathrm{d}x + \int_{\mathbb{R}^n} u(t,x)\Psi_s(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \partial_t u(t,x)\Psi(t,x)\,\mathrm{d}x + \frac{\varrho'(t)}{\varrho(t)} U_1(t).$$

Let compute more explicitly the term on the right-hand side of (2.25) and show its positiveness. By using the recursive identity

$$\mathrm{K}_\gamma'(z) = -\mathrm{K}_{\gamma+1}(z) + \frac{\gamma}{z}\,\mathrm{K}_\gamma(z)$$

for the derivative of the modified Bessel function of the second kind and $\gamma = \frac{\mu-1}{2(1-k)}$, it follows

$$\begin{aligned} \varrho'(t) &= \tfrac{1+\mu}{2}\, t^{\frac{\mu-1}{2}} \mathrm{K}_\gamma\big(\phi_k(t)\big) + t^{\frac{1+\mu}{2}-k} \mathrm{K}_\gamma'\big(\phi_k(t)\big) \\ &= \tfrac{1+\mu}{2}\, t^{\frac{\mu-1}{2}} \mathrm{K}_\gamma\big(\phi_k(t)\big) + t^{\frac{1+\mu}{2}-k} \Big(- \mathrm{K}_{\gamma+1}\big(\phi_k(t)\big) + \tfrac{\mu-1}{2}\, t^{-1+k} \mathrm{K}_\gamma\big(\phi_k(t)\big)\Big) \\ &= \mu\, t^{\frac{\mu-1}{2}} \mathrm{K}_\gamma\big(\phi_k(t)\big) - t^{\frac{1+\mu}{2}-k} \mathrm{K}_{\gamma+1}\big(\phi_k(t)\big). \end{aligned}$$

In particular, the following relations hold

$$\mu\varrho(1) - \varrho'(1) = \mathrm{K}_{\gamma+1}\big(\phi_k(1)\big) > 0, \qquad \varrho(1) = \mathrm{K}_\gamma\big(\phi_k(1)\big) > 0,$$

so that we may rewrite (2.25) as

$$U_1'(t)-\frac{2\varrho'(t)}{\varrho(t)}\,U_1(t)+\mu\,t^{-1}U_1(t)\geqslant\varepsilon\underbrace{\int_{\mathbb{R}^n}\big(\mathrm{K}_\gamma\big(\phi_k(1)\big)u_1(x)+\mathrm{K}_{\gamma+1}\big(\phi_k(1)\big)u_0(x)\big)\varphi(x)\,\mathrm{d}x}_{\doteq I_{k,\mu}[u_0,u_1]}. \tag{2.26}$$

Multiplying (2.26) by $t^\mu/\varrho^2(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^\mu}{\varrho^2(t)}\,U_1(t)\right)=\frac{t^\mu}{\varrho^2(t)}U_1'(t)-\frac{2\varrho'(t)}{\varrho^3(t)}\,t^\mu U_1(t)+\mu\,t^{\mu-1}\frac{1}{\varrho^2(t)}U_1(t)\geqslant\varepsilon I_{k,\mu}[u_0,u_1]\frac{t^\mu}{\varrho^2(t)}.$$

Integrating the previous inequality over $[1,t]$ and using the sign assumption on u_0, we get

$$\begin{aligned}U_1(t)&\geqslant\frac{\varrho^2(t)t^{-\mu}}{\varrho^2(1)}U_1(1)+\varepsilon I_{k,\mu}[u_0,u_1]\,\frac{\varrho^2(t)}{t^\mu}\int_1^t\frac{s^\mu}{\varrho^2(s)}\,\mathrm{d}s\\&\geqslant\varepsilon I_{k,\mu}[u_0,u_1]\,\frac{\varrho^2(t)}{t^\mu}\int_1^t\frac{s^\mu}{\varrho^2(s)}\,\mathrm{d}s.\end{aligned}$$

Thanks to (2.19), there exists $T_0=T_0(k,\mu)>1$ such that

$$U_1(t)\gtrsim\varepsilon I_{k,\mu}[u_0,u_1]\,t^k\mathrm{e}^{-2\phi_k(t)}\int_{T_0}^t s^{-k}\mathrm{e}^{2\phi_k(s)}\,\mathrm{d}s$$

for $t\geqslant T_0$. Consequently, for $t\geqslant 2T_0$, shrinking the domain of integration in the last inequality, we have

$$\begin{aligned}U_1(t)&\gtrsim\varepsilon I_{k,\mu}[u_0,u_1]\,t^k\mathrm{e}^{-2\phi_k(t)}\int_{t/2}^t s^{-k}\mathrm{e}^{2\phi_k(s)}\,\mathrm{d}s=2^{-1}\varepsilon I_{k,\mu}[u_0,u_1]\,t^k\mathrm{e}^{-2\phi_k(t)}\left(\mathrm{e}^{2\phi_k(t)}-\mathrm{e}^{2\phi_k(\frac{t}{2})}\right)\\&=2^{-1}\varepsilon I_{k,\mu}[u_0,u_1]\,t^k\left(1-\mathrm{e}^{2\phi_k(\frac{t}{2})-2\phi_k(t)}\right)=2^{-1}\varepsilon I_{k,\mu}[u_0,u_1]\,t^k\left(1-\mathrm{e}^{-\frac{2}{1-k}(1-2^{k-1})t^{1-k}}\right)\\&\geqslant 2^{-1}\varepsilon I_{k,\mu}[u_0,u_1]\,t^k\left(1-\mathrm{e}^{-\frac{2}{1-k}(2^{1-k}-1)T_0^{1-k}}\right)\gtrsim\varepsilon t^k.\end{aligned}\tag{2.27}$$

By repeating exactly the same computations as in [30, Section 3] (which are completely independent of the amplitude function A_k), we obtain

$$\int_{B_{R+A_k(t)}}(\Psi(t,x))^{p'}\mathrm{d}x=(\varrho(t))^{p'}\int_{B_{R+A_k(t)}}(\varphi(x))^{p'}\mathrm{d}x\lesssim(\varrho(t))^{p'}\mathrm{e}^{p'(R+A_k(t))}(R+A_k(t))^{n-1-\frac{n-1}{2}p'}.$$

Therefore, by using (2.19), for $t\geqslant T_0$ we get

$$\begin{aligned}\int_{B_{R+A_k(t)}}(\Psi(t,x))^{p'}\mathrm{d}x&\lesssim\mathrm{e}^{p'(R-\phi_k(1))}t^{\frac{k+\mu}{2}p'}(R+A_k(t))^{n-1-\frac{n-1}{2}p'}\\&\lesssim t^{(1-k)(n-1)+\left[\frac{k+\mu}{2}-(1-k)\frac{n-1}{2}\right]p'}.\end{aligned}\tag{2.28}$$

Then, combining Hölder's inequality, (2.27) and (2.28), it follows

$$\begin{aligned}\int_{\mathbb{R}^n}|u(t,x)|^p\mathrm{d}x&\geqslant(U_1(t))^p\left(\int_{B_{R+A_k(t)}}(\Psi(t,x))^{p'}\mathrm{d}x\right)^{-(p-1)}\\&\gtrsim\varepsilon^p t^{kp-(1-k)(n-1)(p-1)+\left[(1-k)\frac{n-1}{2}-\frac{k+\mu}{2}\right]p}\\&\gtrsim\varepsilon^p t^{(1-k)(n-1)+\frac{k}{2}p-\left((1-k)\frac{n-1}{2}+\frac{\mu}{2}\right)p}\end{aligned}\tag{2.29}$$

for $t \geqslant T_1 \doteq 2T_0$. Finally, plugging (2.29) in (2.20), for $t \geqslant T_1$ it holds

$$\begin{aligned} U_0(t) &\geqslant \int_{T_1}^t \tau^{-\mu} \int_{T_1}^\tau s^\mu \int_{\mathbb{R}^n} |u(s,x)|^p \, dx \, ds \, d\tau \gtrsim \varepsilon^p \int_{T_1}^t \tau^{-\mu} \int_{T_1}^\tau s^{\mu+(1-k)(n-1)+\frac{k}{2}p-\left((1-k)\frac{n-1}{2}+\frac{\mu}{2}\right)p} \, ds \, d\tau \\ &\gtrsim \varepsilon^p t^{-\left((1-k)\frac{n-1}{2}+\frac{\mu}{2}\right)p-\mu} \int_{T_1}^t \int_{T_1}^\tau (s-T_1)^{\mu+(1-k)(n-1)+\frac{k}{2}p} \, ds \, d\tau \\ &\gtrsim \varepsilon^p t^{-\left((1-k)\frac{n-1}{2}+\frac{\mu}{2}\right)p-\mu} (t-T_1)^{\mu+(1-k)(n-1)+\frac{k}{2}p+2}. \end{aligned}$$

Summarizing we proved the lower bound estimate for the functional U_0

$$U_0(t) \geqslant K\varepsilon^p t^{-a_0}(t-T_1)^{b_0} \tag{2.30}$$

for $t \geqslant T_1$, where $K = K(n,k,\mu,p,R,u_0,u_1)$ is a suitable positive constant and

$$a_0 \doteq \left((1-k)\tfrac{n-1}{2} + \tfrac{\mu}{2}\right)p + \mu, \qquad b_0 \doteq \mu + (1-k)(n-1) + \tfrac{k}{2}p + 2. \tag{2.31}$$

2.4. Iteration argument

In this section we will use the iteration frame (2.23) to prove that U_0 blows up in finite time under the assumptions of Theorem 1.3. More precisely, we are going to prove the sequence of lower bound estimates

$$U_0(t) \geqslant D_j t^{-a_j}(t-T_1)^{b_j} \tag{2.32}$$

for $t \geqslant T_1$, where $\{D_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative real numbers that will be determined iteratively during the proof.

Clearly, for $j=0$ the estimate in (2.32) is nothing but (2.30) with $D_0 = K\varepsilon^p$ and a_0, b_0 defined by (2.31). We will prove (2.32) for $j \geqslant 1$ iteratively. Let us assume the validity of (2.32) for some j. We prove now its validity for $j+1$ too.

Plugging (2.32) into (2.23), for $t \geqslant T_1$ we get

$$\begin{aligned} U_0(t) &\geqslant C \int_{T_1}^t \tau^{-\mu} \int_{T_1}^\tau s^{\mu-(1-k)n(p-1)} (U_0(s))^p \, ds \, d\tau \\ &\geqslant C D_j^p \int_{T_1}^t \tau^{-\mu} \int_{T_1}^\tau s^{\mu-(1-k)n(p-1)-a_j p} (s-T_1)^{b_j p} \, ds \, d\tau \\ &\geqslant C D_j^p t^{-(1-k)n(p-1)-\mu-a_j p} \int_{T_1}^t \int_{T_1}^\tau (s-T_1)^{\mu+b_j p} \, ds \, d\tau \\ &= \frac{C D_j^p}{(1+\mu+b_j p)(2+\mu+b_j p)} t^{-(1-k)n(p-1)-\mu-a_j p} (t-T_1)^{2+\mu+b_j p}, \end{aligned}$$

which is exactly (2.32) for $j+1$ provided that

$$D_{j+1} \doteq \frac{C D_j^p}{(1+\mu+b_j p)(2+\mu+b_j p)}, \tag{2.33}$$

$$a_{j+1} \doteq \underbrace{(1-k)n(p-1)+\mu}_{\doteq \alpha} + p a_j, \quad b_{j+1} \doteq \underbrace{2+\mu}_{\doteq \beta} + p b_j. \tag{2.34}$$

Employing recursively (2.34), we may express explicitly a_j and b_j as follows

$$a_j = \alpha + p a_{j-1} = \cdots = \alpha \sum_{k=0}^{j-1} p^k + a_0 p^j = \left(\tfrac{\alpha}{p-1} + a_0\right) p^j - \tfrac{\alpha}{p-1}, \tag{2.35}$$

$$b_j = \beta + p b_{j-1} = \cdots = \beta \sum_{k=0}^{j-1} p^k + b_0 p^j = \left(\tfrac{\beta}{p-1} + b_0\right) p^j - \tfrac{\beta}{p-1}. \tag{2.36}$$

Combining (2.34) and (2.36), we find

$$b_j = 2+\mu+pb_{j-1} < \left(\tfrac{\beta}{p-1}+b_0\right)p^j,$$

that implies, in turn,

$$D_j \geqslant \frac{CD_{j-1}^p}{(2+\mu+pb_{j-1})^2} = \frac{CD_{j-1}^p}{b_j^2} \geqslant \underbrace{\frac{C}{\left(\frac{\beta}{p-1}+b_0\right)^2}}_{\doteq \widetilde{C}} D_{j-1}^p p^{-2j} = \widetilde{C} D_{j-1}^p p^{-2j}.$$

Applying the logarithmic function to both sides of the last inequality and using the resulting inequality iteratively, we get

$$\begin{aligned} \log D_j &\geqslant p\log D_{j-1} - 2j\log p + \log\widetilde{C} \\ &\geqslant p^2 \log D_{j-2} - 2(j+(j-1)p)\log p + (1+p)\log\widetilde{C} \\ &\geqslant \cdots \geqslant p^j \log D_0 - 2\log p \sum_{k=0}^{j-1}(j-k)p^k + \log\widetilde{C}\sum_{k=0}^{j-1}p^k. \end{aligned}$$

Using the well-known formulas

$$\sum_{k=0}^{j-1}(j-k)p^k = \frac{1}{p-1}\left(\frac{p^{j+1}-p}{p-1}-j\right) \qquad \text{and} \qquad \sum_{k=0}^{j-1}p^k = \frac{p^j-1}{p-1}, \tag{2.37}$$

we obtain

$$\begin{aligned} \log D_j &\geqslant p^j \log D_0 - \frac{2\log p}{p-1}\left(\frac{p^{j+1}-p}{p-1}-j\right) + (p^j-1)\frac{\log\widetilde{C}}{p-1} \\ &= p^j\left(\log D_0 - \frac{2p\log p}{(p-1)^2} + \frac{\log\widetilde{C}}{p-1}\right) + \frac{2j\log p}{p-1} + \frac{2p\log p}{(p-1)^2} - \frac{\log\widetilde{C}}{p-1}. \end{aligned}$$

Let us denote by $j_0 = j_0(n,p,k,\mu) \in \mathbb{N}$ the smallest integer greater than $\frac{\log\widetilde{C}}{2\log p} - \frac{p}{p-1}$. Then, for any $j \geqslant j_0$ we have

$$\log D_j \geqslant p^j\left(\log D_0 - \frac{2p\log p}{(p-1)^2} + \frac{\log\widetilde{C}}{p-1}\right) = p^j \log\left(Kp^{-(2p)/(p-1)^2}\widetilde{C}^{1/(p-1)}\varepsilon^p\right) = p^j\log\left(E_0\varepsilon^p\right), \tag{2.38}$$

where $E_0 \doteq Kp^{-(2p)/(p-1)^2}\widetilde{C}^{1/(p-1)}$. Combining (2.32), (2.35), (2.36) and (2.38), for $j \geqslant j_0$ and $t \geqslant T_1$ it holds

$$\begin{aligned} U_0(t) &\geqslant \exp\left(p^j\log\left(E_0\varepsilon^p\right)\right) t^{-a_j}(t-T_1)^{b_j} \\ &= \exp\left(p^j\left(\log\left(E_0\varepsilon^p\right) - \left(\tfrac{\alpha}{p-1}+a_0\right)\log t + \left(\tfrac{\beta}{p-1}+b_0\right)\log(t-T_1)\right)\right) t^{\alpha/(p-1)}(t-T_1)^{-\beta/(p-1)}. \end{aligned}$$

For $t \geqslant 2T_1$, we have $\log(t-T_1) \geqslant \log(t/2)$, so for $j \geqslant j_0$

$$\begin{aligned} U_0(t) &\geqslant \exp\left(p^j\left(\log\left(E_0\varepsilon^p\right) + \left(\tfrac{\beta-\alpha}{p-1}+b_0-a_0\right)\log t - \left(\tfrac{\beta}{p-1}+b_0\right)\log 2\right)\right) t^{\alpha/(p-1)}(t-T_1)^{-\beta/(p-1)} \\ &= \exp\left(p^j\left(\log\left(2^{-b_0-\beta/(p-1)}E_0\varepsilon^p t^{\frac{\theta(n,k,\mu,p)}{p-1}}\right)\right)\right) t^{\alpha/(p-1)}(t-T_1)^{-\beta/(p-1)}, \end{aligned} \tag{2.39}$$

where for the exponent of t in the last equality we used

$$\begin{aligned} \tfrac{\beta-\alpha}{p-1}+b_0-a_0 &= \tfrac{2}{p-1}-(1-k)n+(1-k)(n-1)+\tfrac{k}{2}p+2-\big((1-k)\tfrac{n-1}{2}+\tfrac{\mu}{2}\big)\,p \\ &= \tfrac{2p}{p-1}-(1-k)-\Big((1-k)\tfrac{n-1}{2}+\tfrac{\mu-k}{2}\Big)\,p \\ &= \tfrac{1}{p-1}\Big\{1-k+\Big((1-k)\tfrac{n+1}{2}+\tfrac{\mu+3k}{2}\Big)\,p-\Big((1-k)\tfrac{n-1}{2}+\tfrac{\mu-k}{2}\Big)\,p^2\Big\} \\ &= \tfrac{\theta(n,k,\mu,p)}{p-1}. \end{aligned} \tag{2.40}$$

Note that $\theta(n,k,\mu,p)$ is a positive quantity for $p<p_0\big(k,n+\frac{\mu}{1-k}\big)$. Let us fix $\varepsilon_0>0$ sufficiently small so that

$$\varepsilon_0^{-\frac{p(p-1)}{\theta(n,k,\mu,p)}} \geqslant 2^{1-\frac{b_0(p-1)+\beta}{\theta(n,k,\mu,p)}}\, E_0^{\frac{p-1}{\theta(n,k,\mu,p)}}\, T_1.$$

Then, for any $\varepsilon\in(0,\varepsilon_0]$ and for $t\geqslant 2^{(b_0(p-1)+\beta)/\theta(n,k,\mu,p)}E_0^{-(p-1)/\theta(n,k,\mu,p)}\varepsilon^{-\frac{p(p-1)}{\theta(n,k,\mu,p)}}$ it results

$$t\geqslant 2T_1 \qquad \text{and} \qquad 2^{-b_0-\beta/(p-1)}E_0\varepsilon^p t^{\frac{\theta(n,k,\mu,p)}{p-1}}>1,$$

also, letting $j\to\infty$ in (2.39) it turns out that $U_0(t)$ blows up. Consequently, we proved the blowing-up of U_0 in finite time for any $\varepsilon\in(0,\varepsilon_0]$ whenever $p<p_0\big(k,n+\frac{\mu}{1-k}\big)$ and, moreover, as byproduct we found the upper bound estimate for the lifespan $T(\varepsilon)\lesssim\varepsilon^{-\frac{p(p-1)}{\theta(n,k,\mu,p)}}$ as well.

So far we applied only the lower bound estimate in (2.30) for U_0. Nevertheless, we also proved another lower bound estimate for U_0, namely, (2.21). Using (2.21) instead of (2.30), the initial values for the parameters in (2.32) are $a_0=b_0=0$ and $D_0\approx\varepsilon$. Repeating the computations analogously as in the previous case and using

$$\log D_j\geqslant p^j\log(E_1\varepsilon)$$

for $j\geqslant j_1$, where j_1 is a suitable nonnegative integer and E_1 is a suitable positive constant, in place of (2.38) and

$$\tfrac{\beta-\alpha}{p-1}+b_0-a_0=\tfrac{2}{p-1}-(1-k)n$$

instead of (2.40), we obtain immediately the blow-up of U_0 in finite time for $p<p_1(k,n)$ and the corresponding upper bound estimate for the lifespan in (1.9).

3. Critical case: part I

In order to study the critical case $p=p_0\big(k,n+\frac{\mu}{1-k}\big)$, we will follow an approach which is based on the technique introduced in [38] and subsequently applied to different frameworks in [39, 31, 32, 23, 3, 4, 29].

From (2.39) it is clear that we can no longer employ U_0 as functional to study the blow-up dynamic. Therefore, we need to sharpen the choice of the functional. More precisely, we are going to consider a weighted space average of a local in time solution to (1.4). Hence, the blow-up result will be proved by applying the so-called *slicing procedure* in an iteration argument to show a sequence of lower bound estimates for the above mentioned functional. Throughout this section we work under the assumptions of Theorem 1.4.

The section is organized as follows: in Section 3.1 we determine a pair of auxiliary functions which have a fundamental role in the definition of the time-dependent functional and in the determination of the iteration frame, while in Section 3.2 we establish some fundamental properties for these functions; finally, in Section 3.3 we determine the iteration frame for the weighted space average whose dynamic provides the blow-up result.

3.1. Auxiliary functions

In this section, we introduce two auxiliary functions (see ξ_q and η_q below). These auxiliary functions represent a generalization of the solution to the classical free wave equation given in [41] and are defined by using the remarkable function φ introduced in [40], that we have already used in the section for the subcritical case (the definition of this function is given in (2.13)).

According to our purpose of introducing the auxiliary functions, we begin by determining the solutions $y_j = y_j(t,s;\lambda,k,\mu)$, $j\in\{0,1\}$ of the non-autonomous, parameter-dependent, ordinary Cauchy problems

$$\begin{cases}\partial_t^2 y_j(t,s;\lambda,k,\mu)-\lambda^2 t^{-2k}y_j(t,s;\lambda,k,\mu)+\mu\, t^{-1}y_j(t,s;\lambda,k,\mu)=0, & t>s,\\ y_j(s,s;\lambda,k,\mu)=\delta_{0j}, \\ \partial_t y_j(s,s;\lambda,k,\mu)=\delta_{1j}, \end{cases} \tag{3.41}$$

where δ_{ij} denotes the Kronecker delta, $s\geqslant 1$ is the initial time and $\lambda>0$ is a real parameter. To find a system of independent solutions to

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-\lambda^2t^{-2k}y+\mu\, t^{-1}\frac{\mathrm{d}y}{\mathrm{d}t}=0 \tag{3.42}$$

we start by performing the change of variable $\tau=\tau(t;\lambda,k)\doteq\lambda\phi_k(t)$. By the straightforward relations

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\lambda t^{-k}\frac{\mathrm{d}y}{\mathrm{d}\tau},\qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2}=\lambda^2t^{-2k}\frac{\mathrm{d}^2y}{\mathrm{d}\tau^2}-\lambda k t^{-k-1}\frac{\mathrm{d}y}{\mathrm{d}\tau},$$

it follows that y solves (3.42) if and only if

$$\tau\frac{\mathrm{d}^2y}{\mathrm{d}\tau^2}+\frac{\mu-k}{1-k}\frac{\mathrm{d}y}{\mathrm{d}\tau}-\tau y=0. \tag{3.43}$$

Carrying out the transformation $y(\tau)=\tau^\nu w(\tau)$ with $\nu=\nu(k,\mu)\doteq\frac{1-\mu}{2(1-k)}$, it turns out that y solves (3.43) if and only if w solves the modified Bessel equation of order ν

$$\tau^2\frac{\mathrm{d}^2w}{\mathrm{d}\tau^2}+\tau\frac{\mathrm{d}w}{\mathrm{d}\tau}-\left(\nu^2+\tau^2\right)w=0. \tag{3.44}$$

Employing the modified Bessel function of first and second kind of order ν, denoted, respectively, by $\mathrm{I}_\nu(\tau)$ and $\mathrm{K}_\nu(\tau)$, as independent solutions to (3.44), then, we obtain

$$V_0(t;\lambda,k,\mu)\doteq\tau^\nu\mathrm{I}_\nu(\tau)=(\lambda\phi_k(t))^\nu\mathrm{I}_\nu(\lambda\phi_k(t)),$$
$$V_1(t;\lambda,k,\mu)\doteq\tau^\nu\mathrm{K}_\nu(\tau)=(\lambda\phi_k(t))^\nu\mathrm{K}_\nu(\lambda\phi_k(t))$$

as basis for the space of solutions to (3.42).

Proposition 3.1. *The functions*

$$y_0(t,s;\lambda,k,\mu)\doteq\lambda\,\phi_k(s)\,s^{\frac{\mu-1}{2}}t^{\frac{1-\mu}{2}}\left[\mathrm{I}_{\nu-1}(\lambda\phi_k(s))\,\mathrm{K}_\nu(\lambda\phi_k(t))+\mathrm{K}_{\nu-1}(\lambda\phi_k(s))\,\mathrm{I}_\nu(\lambda\phi_k(t))\right], \tag{3.45}$$

$$y_1(t,s;\lambda,k,\mu)\doteq(1-k)^{-1}s^{\frac{1+\mu}{2}}t^{\frac{1-\mu}{2}}\left[\mathrm{K}_\nu(\lambda\phi_k(s))\,\mathrm{I}_\nu(\lambda\phi_k(t))-\mathrm{I}_\nu(\lambda\phi_k(s))\,\mathrm{K}_\nu(\lambda\phi_k(t))\right], \tag{3.46}$$

solve the Cauchy problems (3.41) *for* $j=0$ *and* $j=1$*, respectively, where* $\nu=\frac{1-\mu}{2(1-k)}$ *and* $\mathrm{I}_\nu,\mathrm{K}_\nu$ *denote the modified Bessel function of order* ν *of the first and second kind, respectively.*

Proof. Since we proved that V_0,V_1 form a system of independent solutions to (3.42), we may express the solutions to (3.41) as linear combinations of V_0,V_1 in the following way

$$y_j(t,s;\lambda,k,\mu)=a_j(s;\lambda,k,\mu)V_0(t;\lambda,k,\mu)+b_j(s;\lambda,k,\mu)V_1(t;\lambda,k,\mu) \tag{3.47}$$

for suitable coefficients $a_j(s;\lambda,k,\mu),b_j(s;\lambda,k,\mu)$, with $j\in\{0,1\}$.

We can describe the initial conditions $\partial_t^i y_j(s,s;\lambda,k)=\delta_{ij}$ through the system

$$\begin{pmatrix} V_0(s;\lambda,k,\mu) & V_1(s;\lambda,k,\mu)\\ \partial_tV_0(s;\lambda,k,\mu) & \partial_tV_1(s;\lambda,k,\mu)\end{pmatrix}\begin{pmatrix} a_0(s;\lambda,k,\mu) & a_1(s;\lambda,k,\mu)\\ b_0(s;\lambda,k,\mu) & b_1(s;\lambda,k,\mu)\end{pmatrix}=I,$$

where I denotes the identity matrix. Also, to determine the coefficients in (3.47), we calculate the inverse matrix

$$\begin{pmatrix} V_0(s;\lambda,k,\mu) & V_1(s;\lambda,k,\mu) \\ \partial_t V_0(s;\lambda,k,\mu) & \partial_t V_1(s;\lambda,k,\mu) \end{pmatrix}^{-1} = (\mathscr{W}(V_0,V_1)(s;\lambda,k,\mu))^{-1} \begin{pmatrix} \partial_t V_1(s;\lambda,k,\mu) & -V_1(s;\lambda,k,\mu) \\ -\partial_t V_0(s;\lambda,k,\mu) & V_0(s;\lambda,k,\mu) \end{pmatrix}, \tag{3.48}$$

where $\mathscr{W}(V_0,V_1)$ denotes the Wronskian of V_0, V_1. Next, we compute explicitly the function $\mathscr{W}(V_0,V_1)$. Thanks to

$$\begin{aligned} \partial_t V_0(t;\lambda,k,\mu) &= \nu(\lambda\phi_k(t))^{\nu-1}\lambda\phi_k'(t)\,\mathrm{I}_\nu(\lambda\phi_k(t)) + (\lambda\phi_k(t))^{\nu}\,\mathrm{I}_\nu'(\lambda\phi_k(t))\lambda\phi_k'(t), \\ \partial_t V_1(t;\lambda,k,\mu) &= \nu(\lambda\phi_k(t))^{\nu-1}\lambda\phi_k'(t)\,\mathrm{K}_\nu(\lambda\phi_k(t)) + (\lambda\phi_k(t))^{\nu}\,\mathrm{K}_\nu'(\lambda\phi_k(t))\lambda\phi_k'(t), \end{aligned}$$

recalling $\phi_k'(t) = t^{-k}$ and $2\nu - 1 = \frac{k-\mu}{1-k}$, we can express $\mathscr{W}(V_0,V_1)$ as follows:

$$\begin{aligned} \mathscr{W}(V_0,V_1)(t;\lambda,k,\mu) &= (\lambda\phi_k(t))^{2\nu}(\lambda\phi_k'(t))\big\{\mathrm{K}_\nu'(\lambda\phi_k(t))\,\mathrm{I}_\nu(\lambda\phi_k(t)) - \mathrm{I}_\nu'(\lambda\phi_k(t))\,\mathrm{K}_\nu(\lambda\phi_k(t))\big\} \\ &= (\lambda\phi_k(t))^{2\nu}(\lambda\phi_k'(t))\mathscr{W}(\mathrm{I}_\nu,\mathrm{K}_\nu)(\lambda\phi_k(t)) = -(\lambda\phi_k(t))^{2\nu-1}(\lambda\phi_k'(t)) \\ &= -\lambda^{2\nu}(\phi_k(t))^{2\nu-1}\phi_k'(t) = -c_{k,\mu}^{-1}\lambda^{2\nu}t^{-\mu}, \end{aligned}$$

where $c_{k,\mu} \doteq (1-k)^{\frac{k-\mu}{1-k}}$ and in the third equality we used the value of the Wronskian of $\mathrm{I}_\nu, \mathrm{K}_\nu$

$$\mathscr{W}(\mathrm{I}_\nu,\mathrm{K}_\nu)(z) = \mathrm{I}_\nu(z)\frac{\partial\,\mathrm{K}_\nu}{\partial z}(z) - \mathrm{K}_\nu(z)\frac{\partial\,\mathrm{I}_\nu}{\partial z}(z) = -\frac{1}{z}.$$

Plugging the previously determined representation of $\mathscr{W}(V_0,V_1)$ in (3.48), we have

$$\begin{pmatrix} a_0(s;\lambda,k,\mu) & a_1(s;\lambda,k,\mu) \\ b_0(s;\lambda,k,\mu) & b_1(s;\lambda,k,\mu) \end{pmatrix} = c_{k,\mu}\lambda^{-2\nu}s^{\mu} \begin{pmatrix} -\partial_t V_1(s;\lambda,k,\mu) & V_1(s;\lambda,k,\mu) \\ \partial_t V_0(s;\lambda,k,\mu) & -V_0(s;\lambda,k,\mu) \end{pmatrix}.$$

Let us begin by showing (3.45). Using the above representation of $a_0(s;\lambda,k,\mu), b_0(s;\lambda,k,\mu)$ in (3.47), we find

$$\begin{aligned} y_0(t,s;\lambda,k,\mu) &= c_{k,\mu}\lambda^{-2\nu}s^{\mu}\big\{\partial_t V_0(s;\lambda,k,\mu)V_1(t;\lambda,k,\mu) - \partial_t V_1(s;\lambda,k,\mu)V_0(t;\lambda,k,\mu)\big\} \\ &= c_{k,\mu}\,\nu\, s^{\mu}\phi_k'(s)(\phi_k(s))^{\nu-1}(\phi_k(t))^{\nu}\big\{\mathrm{I}_\nu(\lambda\phi_k(s))\,\mathrm{K}_\nu(\lambda\phi_k(t)) - \mathrm{K}_\nu(\lambda\phi_k(s))\,\mathrm{I}_\nu(\lambda\phi_k(t))\big\} \\ &\quad + c_{k,\mu}\,\lambda\, s^{\mu}\phi_k'(s)(\phi_k(s))^{\nu}(\phi_k(t))^{\nu}\big\{\mathrm{I}_\nu'(\lambda\phi_k(s))\,\mathrm{K}_\nu(\lambda\phi_k(t)) - \mathrm{K}_\nu'(\lambda\phi_k(s))\,\mathrm{I}_\nu(\lambda\phi_k(t))\big\}. \end{aligned}$$

Using the following recursive relations for the derivatives of the modified Bessel functions

$$\begin{aligned} \frac{\partial\,\mathrm{I}_\nu}{\partial z}(z) &= -\frac{\nu}{z}\,\mathrm{I}_\nu(z) + \mathrm{I}_{\nu-1}(z), \\ \frac{\partial\,\mathrm{K}_\nu}{\partial z}(z) &= -\frac{\nu}{z}\,\mathrm{K}_\nu(z) - \mathrm{K}_{\nu-1}(z), \end{aligned}$$

there is a cancellation in the last relation, so, we arrive at

$$y_0(t,s;\lambda,k,\mu) = c_{k,\mu}\,\lambda\, s^{\mu}\phi_k'(s)(\phi_k(s)\phi_k(t))^{\nu}\big\{\mathrm{I}_{\nu-1}(\lambda\phi_k(s))\,\mathrm{K}_\nu(\lambda\phi_k(t)) + \mathrm{K}_{\nu-1}(\lambda\phi_k(s))\,\mathrm{I}_\nu(\lambda\phi_k(t))\big\}. \tag{3.49}$$

Thanks to

$$c_{k,\mu}\, s^{\mu}\phi_k'(s)(\phi_k(s)\phi_k(t))^{\nu} = (1-k)^{-1}s^{\mu-k}(st)^{\frac{1-\mu}{2}} = \phi_k(s)s^{\frac{\mu-1}{2}}t^{\frac{1-\mu}{2}},$$

from (3.49) it follows immediately (3.45). Let us show now the representation for y_1. Plugging the above determined expressions for $a_1(s;\lambda,k,\mu), b_1(s;\lambda,k,\mu)$ in (3.47), we get

$$\begin{aligned} y_1(t,s;\lambda,k,\mu) &= c_{k,\mu}\lambda^{-2\nu}s^{\mu}\big\{V_1(s;\lambda,k,\mu)V_0(t;\lambda,k,\mu) - V_0(s;\lambda,k,\mu)V_1(t;\lambda,k,\mu)\big\} \\ &= c_{k,\mu}\lambda^{-2\nu}s^{\mu}(\lambda\phi_k(s))^{\nu}(\lambda\phi_k(t))^{\nu}\big\{\mathrm{K}_\nu(\lambda\phi_k(s))\,\mathrm{I}_\nu(\lambda\phi_k(t)) - \mathrm{I}_\nu(\lambda\phi_k(s))\,\mathrm{K}_\nu(\lambda\phi_k(t))\big\} \\ &= c_{k,\mu}s^{\mu}(\phi_k(s)\phi_k(t))^{\nu}\big\{\mathrm{K}_\nu(\lambda\phi_k(s))\,\mathrm{I}_\nu(\lambda\phi_k(t)) - \mathrm{I}_\nu(\lambda\phi_k(s))\,\mathrm{K}_\nu(\lambda\phi_k(t))\big\}. \end{aligned} \tag{3.50}$$

Hence, due to $c_{k,\mu}s^{\mu}(\phi_k(s)\phi_k(t))^{\nu} = (1-k)^{-1}s^{\frac{1+\mu}{2}}t^{\frac{1-\mu}{2}}$, from (3.50) it results (3.46). The proof is complete. □

Lemma 3.2. *Let y_0, y_1 be the functions defined in* (3.45) *and* (3.46)*, respectively. Then, the following identities are satisfied for any $t \geqslant s \geqslant 1$*

$$\frac{\partial y_1}{\partial s}(t,s;\lambda,k,\mu) = -y_0(t,s;\lambda,k,\mu) + \mu s^{-1}y_1(t,s;\lambda,k,\mu), \tag{3.51}$$

$$\frac{\partial^2 y_1}{\partial s^2}(t,s;\lambda,k,\mu) - \lambda^2 s^{-2k}y_1(t,s;\lambda,k,\mu) - \mu s^{-1}\frac{\partial y_1}{\partial s}(t,s;\lambda,k,\mu) + \mu s^{-2}y_1(t,s;\lambda,k,\mu) = 0. \tag{3.52}$$

Remark 3.3. *As the operator $\partial_s^2 - \lambda^2 s^{-2k} - \mu s^{-1}\partial_s + \mu s^{-2}$ is the formal adjoint of $\partial_t^2 - \lambda^2 t^{-2k} + \mu t^{-1}\partial_t$, in particular,* (3.51) *and* (3.52) *tell us that y_1 solves also the adjoint problem to* (3.42) *with final conditions* $(0,-1)$.

Proof. Let us introduce the pair of independent solutions to (3.42)

$$\begin{aligned} z_0(t;\lambda,k,\mu) &\doteq y_0(t,1;\lambda,k,\mu), \\ z_1(t;\lambda,k,\mu) &\doteq y_1(t,1;\lambda,k,\mu). \end{aligned}$$

Since the Wronskian $\mathscr{W}(z_0,z_1)(t;\lambda,k,\mu)$ solves the differential equation $\mathscr{W}'(z_0,z_1) = -\mu t^{-1}\mathscr{W}(z_0,z_1)$ with initial condition $\mathscr{W}(z_0,z_1)(1;\lambda,k,\mu) = 1$, then, $\mathscr{W}(z_0,z_1)(t;\lambda,k,\mu) = t^{-\mu}$. Therefore, repeating similar computations as in the proof of Proposition 3.1, we may show the representations

$$\begin{aligned} y_0(t,s;\lambda,k,\mu) &= s^{\mu}\left\{z_1'(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu) - z_0'(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu)\right\}, \\ y_1(t,s;\lambda,k,\mu) &= s^{\mu}\left\{z_0(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\}. \end{aligned}$$

Let us prove (3.51). Differentiating the second one of the previous representations with respect to s, we find

$$\begin{aligned} \frac{\partial y_1}{\partial s}(t,s;\lambda,k) &= \mu s^{\mu-1}\left\{z_0(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\} \\ &\quad + s^{\mu}\left\{z_0'(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1'(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\} \\ &= \mu s^{-1}y_1(t,s;\lambda,k,\mu) - y_0(t,s;\lambda,k,\mu). \end{aligned}$$

On the other hand, due to the fact that z_0, z_1 satisfy (3.42), then,

$$\begin{aligned} \frac{\partial^2 y_1}{\partial s^2}(t,s;\lambda,k) &= s^{\mu}\left\{z_0''(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1''(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\} \\ &\quad + 2\mu s^{\mu-1}\left\{z_0'(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1'(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\} \\ &\quad + \mu(\mu-1)s^{\mu-2}\left\{z_0(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\} \\ &= s^{\mu}\Big\{\left[\lambda^2 s^{-2k}z_0(s;\lambda,k,\mu) - \mu s^{-1}z_0'(s;\lambda,k,\mu)\right]z_1(t;\lambda,k,\mu) \\ &\qquad - \left[\lambda^2 s^{-2k}z_1(s;\lambda,k,\mu) - \mu s^{-1}z_1'(s;\lambda,k,\mu)\right]z_0(t;\lambda,k,\mu)\Big\} \\ &\quad + 2\mu s^{\mu-1}\left\{z_0'(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1'(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\} \\ &\quad + \mu(\mu-1)s^{\mu-2}\left\{z_0(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\} \\ &= \lambda^2 s^{-2k}s^{\mu}\Big\{z_0(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\Big\} \\ &\quad + \mu s^{\mu-1}\left\{z_0'(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1'(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\} \\ &\quad + \mu(\mu-1)s^{\mu-2}\left\{z_0(s;\lambda,k,\mu)z_1(t;\lambda,k,\mu) - z_1(s;\lambda,k,\mu)z_0(t;\lambda,k,\mu)\right\} \\ &= \lambda^2 s^{-2k}y_1(t,s;\lambda,k,\mu) - \mu s^{-1}y_0(t,s;\lambda,k,\mu) + \mu(\mu-1)s^{-2}y_1(t,s;\lambda,k,\mu). \end{aligned}$$

Applying (3.51), from the last chain of equalities we get

$$\begin{aligned}\frac{\partial^2 y_1}{\partial s^2}(t,s;\lambda,k) &= \lambda^2 s^{-2k} y_1(t,s;\lambda,k,\mu) + \mu s^{-1}\left(\frac{\partial y_1}{\partial s}(t,s;\lambda,k) - \mu s^{-1} y_1(t,s;\lambda,k,\mu)\right) \\ &\quad + \mu(\mu-1)s^{-2} y_1(t,s;\lambda,k,\mu) \\ &= \lambda^2 s^{-2k} y_1(t,s;\lambda,k,\mu) + \mu s^{-1}\frac{\partial y_1}{\partial s}(t,s;\lambda,k) - \mu s^{-2} y_1(t,s;\lambda,k,\mu).\end{aligned}$$

Thus, we proved (3.52) too. This completes the proof. □

Proposition 3.4. *Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ be functions such that* $\operatorname{supp} u_j \subset B_R$ *for $j = 0, 1$ and for some $R > 0$ and let $\lambda > 0$ be a parameter. Let u be a local in time energy solution to* (1.4) *on $[1, T)$ according to Definition 1.1. Then, the following integral identity is satisfied for any $t \in [1, T)$*

$$\begin{aligned}\int_{\mathbb{R}^n} u(t,x)\varphi_\lambda(x)\,\mathrm{d}x &= \varepsilon\, y_0(t,1;\lambda,k)\int_{\mathbb{R}^n} u_0(x)\varphi_\lambda(x)\,\mathrm{d}x + \varepsilon\, y_1(t,1;\lambda,k)\int_{\mathbb{R}^n} u_1(x)\varphi_\lambda(x)\,\mathrm{d}x \\ &\quad + \int_1^t y_1(t,s;\lambda,k)\int_{\mathbb{R}^n} |u(s,x)|^p \varphi_\lambda(x)\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{3.53}$$

where $\varphi_\lambda(x) \doteq \varphi(\lambda x)$ and φ is defined by (2.13).

Proof. Assuming u_0, u_1 compactly supported, we can consider a test function $\psi \in \mathscr{C}^\infty([1,T)\times\mathbb{R}^n)$ in Definition 1.1 according to Remark 1.2. Hence, we take $\psi(s,x) = y_1(t,s;\lambda,k,\mu)\varphi_\lambda(x)$ (here t, λ can be treated as fixed parameters). Consequently, ψ satisfies

$$\begin{aligned}\psi(t,x) &= y_1(t,t;\lambda,k,\mu)\varphi_\lambda(x) = 0, \qquad \psi(1,x) = y_1(t,1;\lambda,k,\mu)\varphi_\lambda(x), \\ \psi_s(t,x) &= \partial_s y_1(t,t;\lambda,k,\mu)\varphi_\lambda(x) = \left(\mu t^{-1} y_1(t,t;\lambda,k,\mu) - y_0(t,t;\lambda,k,\mu)\right)\varphi_\lambda(x) = -\varphi_\lambda(x), \\ \psi_s(1,x) &= \partial_s y_1(t,1;\lambda,k,\mu)\varphi_\lambda(x) = \left(\mu y_1(t,1;\lambda,k,\mu) - y_0(t,1;\lambda,k,\mu)\right)\varphi_\lambda(x),\end{aligned}$$

and

$$\begin{aligned}\psi_{ss}(s,x) - s^{-2k}\Delta\psi(s,x) - \mu\partial_s(s^{-1}\psi(s,x)) &= \left(\partial_s^2 - \lambda^2 s^{-2k} - \mu s^{-1}\partial_s + \mu s^{-2}\right) y_1(t,s;\lambda,k,\mu)\varphi_\lambda(x) \\ &= 0,\end{aligned}$$

where we used (3.51), (3.52) and the property $\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$. Then, employing the above defined ψ in (1.8), we find immediately (3.53). This completes the proof. □

Proposition 3.5. *Let y_0, y_1 be the functions defined in* (3.45) *and* (3.46)*, respectively. Then, the following estimates are satisfied for any $t \geqslant s \geqslant 1$*

$$y_0(t,s;\lambda,k,\mu) \geqslant s^{\frac{\mu-k}{2}} t^{\frac{k-\mu}{2}} \cosh\left(\lambda(\phi_k(t) - \phi_k(s))\right) \qquad \text{if } \mu \in [2-k, \infty), \tag{3.54}$$

$$y_1(t,s;\lambda,k,\mu) \geqslant s^{\frac{\mu+k}{2}} t^{\frac{k-\mu}{2}} \frac{\sinh\left(\lambda(\phi_k(t) - \phi_k(s))\right)}{\lambda} \qquad \text{if } \mu \in [0,k]\cup[2-k,\infty). \tag{3.55}$$

Proof. The proof of the inequalities (3.54) and (3.55) is based on the following minimum type principle: *let $w = w(t,s;\lambda,k,\mu)$ be a solution of the Cauchy problem*

$$\begin{cases}\partial_t^2 w - \lambda^2 t^{-2k} w + \mu\, t^{-1}\partial_t w = h, & \text{for } t > s \geqslant 1, \\ w(s) = \tilde{w}_0,\ \partial_t w(s) = \tilde{w}_1, \end{cases} \tag{3.56}$$

where $h = h(t,s;\lambda,k,\mu)$ is a continuous function; if $h \geqslant 0$ and $\tilde{w}_0 = \tilde{w}_1 = 0$ (i.e. w is a supersolution *of the homogeneous problem with trivial initial conditions), then, $w(t,s;\lambda,k,\mu) \geqslant 0$ for any $t > s$.*

In order to prove this minimum principle, we apply the continuous dependence on initial conditions (note that for $t \geqslant 1$ the functions t^{-2k} and μt^{-1} are smooth). Indeed, if we denote by w_ϵ the solution to (3.56) with $\tilde{w}_0 = \epsilon > 0$ and $\tilde{w}_1 = 0$, then, w_ϵ solves the integral equation

$$w_\epsilon(t,s;\lambda,k,\mu) = \epsilon + \int_s^t \tau^{-\mu}\int_s^\tau \sigma^\mu\left(\lambda^2\sigma^{-2k} w_\epsilon(\sigma,s;\lambda,k,\mu) + h(\sigma,s;\lambda,k,\mu)\right)\mathrm{d}\sigma\,\mathrm{d}\tau.$$

By contradiction, one can prove easily that $w_\epsilon(t,s;\lambda,k,\mu)>0$ for any $t>s$. Hence, by the continuous dependence on initial data, letting $\epsilon\to 0$, we find that $w(t,s;\lambda,k,\mu)\geqslant 0$ for any $t>s$.

Let us prove the validity of (3.55). Denoting by $w_1=w_1(t,s;\lambda,k,\mu)$ the function on the right-hand side of (3.55), we find immediately $w_1(s,s;\lambda,k,\mu)=0$ and $\partial_t w_1(s,s;\lambda,k,\mu)=1$. Moreover,

$$\begin{aligned}
\partial_t^2 w_1(t,s;&\lambda,k,\mu)\\
&=\lambda^{-1}s^{\frac{k+\mu}{2}}t^{\frac{k-\mu}{2}}\Big[\tfrac{k-\mu}{2}\left(\tfrac{k-\mu}{2}-1\right)t^{-2}\sinh\big(\lambda(\phi_k(t)-\phi_k(s))\big)\\
&\qquad\qquad+(k-\mu)\,t^{-1}\cosh\big(\lambda(\phi_k(t)-\phi_k(s))\big)\lambda\phi_k'(t)\\
&\qquad\qquad+\sinh\big(\lambda(\phi_k(t)-\phi_k(s))\big)(\lambda\phi_k'(t))^2+\cosh\big(\lambda(\phi_k(t)-\phi_k(s))\big)\lambda\phi_k''(t)\Big]\\
&=\left[\tfrac{k-\mu}{2}\left(\tfrac{k-\mu}{2}-1\right)t^{-2}+\lambda^2t^{-2k}\right]w_1(t,s;\lambda,k,\mu)-\mu s^{\frac{k+\mu}{2}}t^{-1-\frac{k+\mu}{2}}\cosh\big(\lambda(\phi_k(t)-\phi_k(s))\big)
\end{aligned}$$

and

$$\begin{aligned}
\partial_t w_1(t,s;\lambda,k,\mu)&=\lambda^{-1}s^{\frac{k+\mu}{2}}t^{\frac{k-\mu}{2}}\Big[\tfrac{k-\mu}{2}\,t^{-1}\sinh\big(\lambda(\phi_k(t)-\phi_k(s))\big)+\lambda t^{-k}\cosh\big(\lambda(\phi_k(t)-\phi_k(s))\big)\Big]\\
&=\tfrac{k-\mu}{2}\,t^{-1}w_1(t,s;\lambda,k,\mu)+s^{\frac{k+\mu}{2}}t^{-\frac{k+\mu}{2}}\cosh\big(\lambda(\phi_k(t)-\phi_k(s))\big)
\end{aligned}$$

imply that

$$\begin{aligned}
\partial_t^2 w_1(t,s;\lambda,k,\mu)-\lambda^2t^{-2k}w_1(t,s;\lambda,k,\mu)+\mu\, t^{-1}\partial_t w_1(t,s;\lambda,k,\mu)&=\tfrac{k-\mu}{2}\left(\tfrac{k+\mu}{2}-1\right)w_1(t,s;\lambda,k,\mu)\\
&\leqslant 0,
\end{aligned}$$

where in the last step we employ the assumption $\mu\notin(k,2-k)$ to guarantee that the multiplicative constant is negative. Therefore, y_1-w_1 is a supersolution of (3.56) with $h=0$ and $\tilde w_0=\tilde w_1=0$. Thus, applying the minimum principle we have that $(y_1-w_1)(t,s;\lambda,k)\geqslant 0$ for any $t>s$, that is, we showed (3.55).

In a completely analogous way, one can prove (3.54), repeating the previous argument based on the minimum principle with $w_0(t,s;\lambda,k,\mu)\doteq s^{\frac{\mu-k}{2}}t^{\frac{k-\mu}{2}}\cosh\big(\lambda(\phi_k(t)-\phi_k(s))\big)$ in place of $w_1(t,s;\lambda,k,\mu)$ and y_0 in place of y_1, respectively. However, in order to guarantee that $w_0(s,s;\lambda,k,\mu)=1$ and $\partial_t w_0(s,s;\lambda,k,\mu)\leqslant 0$, we are forced to require $\mu\geqslant k$, which provides, together with the condition $\mu\notin(k,2-k)$ that is necessary to ensure that w_0 is actually a subsolution of the homogeneous equation, the range for μ in (3.54). □

Remark 3.6. *Although* (3.54) *might be restrictive from the viewpoint of the range for μ in the statement of Theorem 1.4, we can actually overcome this difficulty by showing a transformation which allows to link the case $\mu\in[0,k]$ to the case $\mu\in[2-k,2]$, when a lower bound estimate for y_0 is available. Indeed, if we perform the transformation $v=v(t,x)\doteq t^{\mu-1}u(t,x)$, then, u is a solution to* (1.4) *if and only if v solves*

$$\begin{cases}v_{tt}-t^{-2k}\Delta v+(2-\mu)\,t^{-1}v_t=t^{(1-\mu)(p-1)}|v|^p & x\in\mathbb{R}^n,\ t\in(1,T),\\ v(1,x)=\varepsilon u_0(x) & x\in\mathbb{R}^n,\\ u_t(1,x)=\varepsilon u_1(x)+\varepsilon(1-\mu)u_0(x) & x\in\mathbb{R}^n.\end{cases}\tag{3.57}$$

Let us point out that in (3.57) *a time-dependent factor which decays with polynomial order appears in the nonlinear term on the right-hand side. Therefore, we will reduce the case $\mu\in[0,k]$ to the case $\mu\geqslant 2-k$, up to the time-dependent factor $t^{(1-\mu)(p-1)}$ in the nonlinearity.*

We can introduce now for $t\geqslant s\geqslant 1$ and $x\in\mathbb{R}^n$ the definition of the following *auxiliary function*

$$\xi_q(t,s,x;k,\mu)\doteq\int_0^{\lambda_0}\mathrm{e}^{-\lambda(A_k(t)+R)}y_0(t,s;\lambda,k,\mu)\,\varphi_\lambda(x)\lambda^q\,\mathrm{d}\lambda,\tag{3.58}$$

$$\eta_q(t,s,x;k,\mu)\doteq\int_0^{\lambda_0}\mathrm{e}^{-\lambda(A_k(t)+R)}\frac{y_1(t,s;\lambda,k,\mu)}{\phi_k(t)-\phi_k(s)}\,\varphi_\lambda(x)\lambda^q\,\mathrm{d}\lambda,\tag{3.59}$$

where $q > -1$, $\lambda_0 > 0$ is a fixed parameter and A_k is defined by (1.6).

Combining Proposition 3.4 and (3.58) and (3.59), we establish a fundamental equality, whose role will be crucial in the next sections in order to prove the blow-up result.

Corollary 3.7. *Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ such that $\operatorname{supp} u_j \subset B_R$ for $j = 0,1$ and for some $R > 0$. Let u be a local in time energy solution to (1.4) on $[1,T)$ according to Definition 1.1. Let $q > -1$ and let $\xi_q(t,s,x;k), \eta_q(t,s,x;k)$ be the functions defined by (3.58) and (3.59), respectively. Then,*

$$\begin{aligned}\int_{\mathbb{R}^n} u(t,x)\, \xi_q(t,t,x;k,\mu)\,\mathrm{d}x = \varepsilon \int_{\mathbb{R}^n} u_0(x)\, \xi_q(t,1,x;k,\mu)\,\mathrm{d}x \\ + \varepsilon\,(\phi_k(t) - \phi_k(1)) \int_{\mathbb{R}^n} u_1(x)\, \eta_q(t,1,x;k,\mu)\,\mathrm{d}x \\ + \int_1^t (\phi_k(t) - \phi_k(s)) \int_{\mathbb{R}^n} |u(s,x)|^p \eta_q(t,s,x;k,\mu)\,\mathrm{d}x\,\mathrm{d}s \qquad (3.60)\end{aligned}$$

for any $t \in [1,T)$.

Proof. Multiplying both sides of (3.53) by $\mathrm{e}^{-\lambda(A_k(t)+R)}\lambda^q$, integrating with respect to λ over $[0,\lambda_0]$ and applying Fubini's theorem, we get easily (3.60). □

3.2. Properties of the auxiliary functions

In this section, we establish lower and upper bound estimates for the auxiliary functions ξ_q, η_q under suitable assumptions on q. In the lower bound estimates, we may restrict our considerations to the case $\mu \geqslant 2 - k$ thanks to Remark 3.6, even though the estimate for η_q that will be proved thanks to (3.55) clearly would be true also for $\mu \in [0,k]$.

Lemma 3.8. *Let $n \geqslant 1$, $k \in [0,1)$, $\mu \geqslant 2 - k$ and $\lambda_0 > 0$. If we assume $q > -1$, then, for $t \geqslant s \geqslant 1$ and $|x| \leqslant A_k(s) + R$ the following lower bound estimates are satisfied:*

$$\xi_q(t,s,x;k,\mu) \geqslant B_0 s^{\frac{\mu-k}{2}} t^{\frac{k-\mu}{2}} \langle A_k(s)\rangle^{-q-1}; \qquad (3.61)$$

$$\eta_q(t,s,x;k,\mu) \geqslant B_1 s^{\frac{\mu+k}{2}} t^{\frac{k-\mu}{2}} \langle A_k(t)\rangle^{-1} \langle A_k(s)\rangle^{-q}. \qquad (3.62)$$

Here B_0, B_1 are positive constants depending only on λ_0, q, R, k and we employ the notation $\langle y\rangle \doteq 3+|y|$.

Proof. We adapt the main ideas in the proof of Lemma 3.1 in [38] to our model. Since

$$\langle |x|\rangle^{-\frac{n-1}{2}} \mathrm{e}^{|x|} \lesssim \varphi(x) \lesssim \langle |x|\rangle^{-\frac{n-1}{2}} \mathrm{e}^{|x|} \qquad (3.63)$$

holds for any $x \in \mathbb{R}^n$, there exists a constant $B = B(\lambda_0, R, k) > 0$ independent of λ and s such that

$$B \leqslant \inf_{\lambda \in \left[\frac{\lambda_0}{\langle A_k(s)\rangle}, \frac{2\lambda_0}{\langle A_k(s)\rangle}\right]} \inf_{|x| \leqslant A_k(s)+R} \mathrm{e}^{-\lambda(A_k(s)+R)} \varphi_\lambda(x).$$

Let us begin by proving (3.61). Using the lower bound estimate in (3.54), shrinking the domain of integration in (3.58) to $\left[\frac{\lambda_0}{\langle A_k(s)\rangle}, \frac{2\lambda_0}{\langle A_k(s)\rangle}\right]$ and applying the previous inequality, we arrive at

$$\begin{aligned}\xi_q(t,s,x;k,\mu) &\geqslant s^{\frac{\mu-k}{2}} t^{\frac{k-\mu}{2}} \int_{\lambda_0/\langle A_k(s)\rangle}^{2\lambda_0/\langle A_k(s)\rangle} \mathrm{e}^{-\lambda(A_k(t)-A_k(s))} \cosh\big(\lambda(\phi_k(t)-\phi_k(s))\big) \mathrm{e}^{-\lambda(A_k(s)+R)} \varphi_\lambda(x) \lambda^q\,\mathrm{d}\lambda \\ &\geqslant B s^{\frac{\mu-k}{2}} t^{\frac{k-\mu}{2}} \int_{\lambda_0/\langle A_k(s)\rangle}^{2\lambda_0/\langle A_k(s)\rangle} \mathrm{e}^{-\lambda(A_k(t)-A_k(s))} \cosh\big(\lambda(\phi_k(t)-\phi_k(s))\big) \lambda^q\,\mathrm{d}\lambda \\ &= \tfrac{B}{2} s^{\frac{\mu-k}{2}} t^{\frac{k-\mu}{2}} \int_{\lambda_0/\langle A_k(s)\rangle}^{2\lambda_0/\langle A_k(s)\rangle} \left(1 + \mathrm{e}^{-2\lambda(\phi_k(t)-\phi_k(s))}\right) \lambda^q\,\mathrm{d}\lambda \\ &\geqslant \tfrac{B}{2} s^{\frac{\mu-k}{2}} t^{\frac{k-\mu}{2}} \int_{\lambda_0/\langle A_k(s)\rangle}^{2\lambda_0/\langle A_k(s)\rangle} \lambda^q\,\mathrm{d}\lambda = s^{\frac{\mu-k}{2}} t^{\frac{k-\mu}{2}} \frac{B(2^{q+1}-1)\lambda_0^{q+1}}{2(q+1)} \langle A_k(s)\rangle^{-q-1}.\end{aligned}$$

Repeating similar steps as before, thanks to (3.55) we obtain

$$
\begin{aligned}
\eta_q(t,s,x;k,\mu) &\geqslant s^{\frac{\mu+k}{2}} t^{\frac{k-\mu}{2}} \int_{\lambda_0/\langle A_k(s)\rangle}^{2\lambda_0/\langle A_k(s)\rangle} \mathrm{e}^{-\lambda(A_k(t)-A_k(s))} \frac{\sinh\big(\lambda(\phi_k(t)-\phi_k(s))\big)}{\lambda(\phi_k(t)-\phi_k(s))} \mathrm{e}^{-\lambda(A_k(s)+R)} \varphi_\lambda(x)\lambda^q \mathrm{d}\lambda \\
&\geqslant \tfrac{B}{2} s^{\frac{\mu+k}{2}} t^{\frac{k-\mu}{2}} \int_{\lambda_0/\langle A_k(s)\rangle}^{2\lambda_0/\langle A_k(s)\rangle} \frac{1-\mathrm{e}^{-2\lambda(\phi_k(t)-\phi_k(s))}}{\phi_k(t)-\phi_k(s)} \lambda^{q-1}\,\mathrm{d}\lambda \\
&\geqslant \tfrac{B}{2} s^{\frac{\mu+k}{2}} t^{\frac{k-\mu}{2}} \frac{1-\mathrm{e}^{-2\lambda_0\frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s)\rangle}}}{\phi_k(t)-\phi_k(s)} \int_{\lambda_0/\langle A_k(s)\rangle}^{2\lambda_0/\langle A_k(s)\rangle} \lambda^{q-1}\,\mathrm{d}\lambda \\
&= \frac{B(2^q-1)\lambda_0^q}{2q} s^{\frac{\mu+k}{2}} t^{\frac{k-\mu}{2}} \langle A_k(s)\rangle^{-q} \frac{1-\mathrm{e}^{-2\lambda_0\frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s)\rangle}}}{\phi_k(t)-\phi_k(s)},
\end{aligned}
$$

with obvious modifications in the case $q=0$. The previous inequality implies (3.62), provided that we show the validity of the inequality

$$\frac{1-\mathrm{e}^{-2\lambda_0\frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s)\rangle}}}{\phi_k(t)-\phi_k(s)} \gtrsim \langle A_k(t)\rangle^{-1}.$$

Hence, we need to prove this inequality. For $\phi_k(t)-\phi_k(s) \geqslant \frac{1}{2\lambda_0}\langle A_k(s)\rangle$, it holds

$$1-\mathrm{e}^{-2\lambda_0\frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s)\rangle}} \geqslant 1-\mathrm{e}^{-1}$$

and, consequently,

$$\frac{1-\mathrm{e}^{-2\lambda_0\frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s)\rangle}}}{\phi_k(t)-\phi_k(s)} \gtrsim \big(\phi_k(t)-\phi_k(s)\big)^{-1} \geqslant A_k(t)^{-1} \geqslant \langle A_k(t)\rangle^{-1}.$$

On the other hand, when $\phi_k(t)-\phi_k(s) \leqslant \frac{1}{2\lambda_0}\langle A_k(s)\rangle$, using the estimate $1-\mathrm{e}^{-\sigma} \geqslant \sigma/2$ for $\sigma\in[0,1]$, we get easily

$$\frac{1-\mathrm{e}^{-2\lambda_0\frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s)\rangle}}}{\phi_k(t)-\phi_k(s)} \geqslant \frac{\lambda_0}{\langle A_k(s)\rangle} \geqslant \frac{\lambda_0}{\langle A_k(t)\rangle}.$$

Therefore, the proof of (3.62) is completed. □

Next we prove an upper bound estimate in the special case $s=t$.

Lemma 3.9. *Let $n\geqslant 1$, $k\in[0,1)$, $\mu\geqslant 0$ and $\lambda_0>0$. If we assume $q>(n-3)/2$, then, for $t\geqslant 1$ and $|x|\leqslant A_k(t)+R$ the following upper bound estimate holds:*

$$\xi_q(t,t,x;k,\mu) \leqslant B_2\langle A_k(t)\rangle^{-\frac{n-1}{2}} \langle A_k(t)-|x|\rangle^{\frac{n-3}{2}-q}. \tag{3.64}$$

Here B_2 is a positive constant depending only on λ_0,q,R,k and $\langle y\rangle$ denotes the same function as in the statement of Lemma 3.8.

Proof. Due to the representation

$$\xi_q(t,t,x;k,\mu) = \int_0^{\lambda_0} \mathrm{e}^{-\lambda(A_k(t)+R)}\varphi_\lambda(x)\lambda^q\,\mathrm{d}\lambda,$$

the proof is exactly the same as in [29, Lemma 2.7]. □

3.3. Derivation of the iteration frame

In this section, we define the time-dependent functional whose dynamic is studied in order to prove the blow-up result. Then, we derive the iteration frame for this functional and a first lower bound estimate of logarithmic type.

For $t \geqslant 1$ we introduce the functional

$$\mathcal{U}(t) \doteq t^{\frac{\mu-k}{2}} \int_{\mathbb{R}^n} u(t,x)\, \xi_q(t,t,x;k,\mu)\, \mathrm{d}x \tag{3.65}$$

for some $q > (n-3)/2$.

From (3.60), (3.61) and (3.62), it follows

$$\mathcal{U}(t) \gtrsim B_0 \varepsilon \int_{\mathbb{R}^n} u_0(x)\, \mathrm{d}x + B_1 \varepsilon \frac{A_k(t)}{\langle A_k(t)\rangle} \int_{\mathbb{R}^n} u_1(x)\, \mathrm{d}x.$$

If we assume both u_0, u_1 nonnegative and nontrivial, then, we find that

$$\mathcal{U}(t) \gtrsim \varepsilon \tag{3.66}$$

for any $t \in [1,T)$, where the unexpressed multiplicative constant depends on u_0, u_1. In the next proposition, we derive the iteration frame for the functional $\mathcal{U}$ for a given value of q.

Proposition 3.10. *Let $n \geqslant 1$, $k \in [0,1)$ and $\mu \in [0,k] \cup [2-k,\infty)$. Let us consider $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ such that* $\operatorname{supp} u_j \subset B_R$ *for $j=0,1$ and for some $R>0$ and let u be a local in time energy solution to* (1.4) *on* $[1,T)$ *according to Definition 1.1. If $\mathcal{U}$ is defined by* (3.65) *with $q = (n-1)/2 - 1/p$, then, there exists a positive constant $C = C(n,p,R,k,\mu)$ such that*

$$\mathcal{U}(t) \geqslant C \langle A_k(t)\rangle^{-1} \int_1^t \frac{\phi_k(t)-\phi_k(s)}{s} \big(\log\langle A_k(s)\rangle\big)^{-(p-1)} (\mathcal{U}(s))^p\, \mathrm{d}s \tag{3.67}$$

for any $t \in (1,T)$.

Proof. By (3.65), applying Hölder's inequality we find

$$s^{\frac{k-\mu}{2}} \mathcal{U}(s) \leqslant \left(\int_{\mathbb{R}^n} |u(s,x)|^p \eta_q(t,s,x;k,\mu)\, \mathrm{d}x \right)^{1/p} \left(\int_{B_{R+A_k(s)}} \frac{\big(\xi_q(s,s,x;k,\mu)\big)^{p'}}{\big(\eta_q(t,s,x;k,\mu)\big)^{p'/p}}\, \mathrm{d}x \right)^{1/p'}.$$

Hence,

$$\int_{\mathbb{R}^n} |u(s,x)|^p \eta_q(t,s,x;k,\mu)\, \mathrm{d}x \geqslant \big(s^{\frac{k-\mu}{2}} \mathcal{U}(s)\big)^p \left(\int_{B_{R+A_k(s)}} \frac{\big(\xi_q(s,s,x;k,\mu)\big)^{p/(p-1)}}{\big(\eta_q(t,s,x;k,\mu)\big)^{1/(p-1)}}\, \mathrm{d}x \right)^{-(p-1)}. \tag{3.68}$$

Let us determine an upper bound for the integral on the right-hand side of (3.68). By using (3.64) and (3.62), we obtain

$$\begin{aligned}
&\int_{B_{R+A_k(s)}} \frac{\big(\xi_q(s,s,x;k,\mu)\big)^{p/(p-1)}}{\big(\eta_q(t,s,x;k,\mu)\big)^{1/(p-1)}}\, \mathrm{d}x \\
&\quad \leqslant B_1^{-\frac{1}{p-1}} B_2^{\frac{p}{p-1}} s^{-\frac{\mu+k}{2(p-1)}} t^{-\frac{k-\mu}{2(p-1)}} \langle A_k(s)\rangle^{-\frac{n-1}{2}\frac{p}{p-1}+\frac{q}{p-1}} \langle A_k(t)\rangle^{\frac{1}{p-1}} \int_{B_{R+A_k(s)}} \langle A_k(s)-|x|\rangle^{(\frac{n-3}{2}-q)\frac{p}{p-1}} \mathrm{d}x \\
&\quad \leqslant B_1^{-\frac{1}{p-1}} B_2^{\frac{p}{p-1}} s^{-\frac{\mu+k}{2(p-1)}} t^{\frac{\mu-k}{2(p-1)}} \langle A_k(t)\rangle^{\frac{1}{p-1}} \langle A_k(s)\rangle^{\frac{1}{p-1}(-\frac{n-1}{2}p+\frac{n-1}{2}-\frac{1}{p})} \int_{B_{R+A_k(s)}} \langle A_k(s)-|x|\rangle^{-1} \mathrm{d}x \\
&\quad \leqslant B_1^{-\frac{1}{p-1}} B_2^{\frac{p}{p-1}} s^{-\frac{\mu+k}{2(p-1)}} t^{\frac{\mu-k}{2(p-1)}} \langle A_k(t)\rangle^{\frac{1}{p-1}} \langle A_k(s)\rangle^{\frac{1}{p-1}(-\frac{n-1}{2}p+\frac{n-1}{2}-\frac{1}{p})+n-1} \log\langle A_k(s)\rangle,
\end{aligned}$$

where in the second inequality we used value of q to get exactly -1 as power for the function in the integral. Consequently, from (3.68) we have

$$\begin{aligned}\int_{\mathbb{R}^n} &|u(s,x)|^p \eta_q(t,s,x;k,\mu)\,\mathrm{d}x \\ &\gtrsim \big(s^{\frac{k-\mu}{2}}\mathcal{U}(s)\big)^p s^{\frac{\mu+k}{2}} t^{\frac{k-\mu}{2}} \langle A_k(t)\rangle^{-1}\langle A_k(s)\rangle^{-\frac{n-1}{2}(p-1)+\frac{1}{p}}\big(\log\langle A_k(s)\rangle\big)^{-(p-1)} \\ &\gtrsim t^{\frac{k-\mu}{2}}\langle A_k(t)\rangle^{-1} s^{\frac{k}{2}(p+1)+\frac{\mu}{2}(1-p)}\langle A_k(s)\rangle^{-\frac{n-1}{2}(p-1)+\frac{1}{p}}\big(\log\langle A_k(s)\rangle\big)^{-(p-1)}\big(\mathcal{U}(s)\big)^p.\end{aligned}$$

Combining the previous lower bound estimate and (3.60), we arrive at

$$\begin{aligned}\mathcal{U}(t) &\geqslant t^{\frac{\mu-k}{2}}\int_1^t (\phi_k(t)-\phi_k(s))\int_{\mathbb{R}^n}|u(s,x)|^p\eta_q(t,s,x;k,\mu)\,\mathrm{d}x\,\mathrm{d}s \\ &\gtrsim \langle A_k(t)\rangle^{-1}\int_1^t (\phi_k(t)-\phi_k(s))\, s^{\frac{k}{2}(p+1)+\frac{\mu}{2}(1-p)}\langle A_k(s)\rangle^{-\frac{n-1}{2}(p-1)+\frac{1}{p}}\frac{\big(\mathcal{U}(s)\big)^p}{\big(\log\langle A_k(s)\rangle\big)^{(p-1)}}\,\mathrm{d}s \\ &\gtrsim \langle A_k(t)\rangle^{-1}\int_1^t (\phi_k(t)-\phi_k(s))\langle A_k(s)\rangle^{\frac{k(p+1)}{2(1-k)}-\frac{\mu(p-1)}{2(1-k)}-\frac{n-1}{2}(p-1)+\frac{1}{p}}\frac{\big(\mathcal{U}(s)\big)^p}{\big(\log\langle A_k(s)\rangle\big)^{(p-1)}}\,\mathrm{d}s \\ &\gtrsim \langle A_k(t)\rangle^{-1}\int_1^t (\phi_k(t)-\phi_k(s))\langle A_k(s)\rangle^{-\left(\frac{n-1}{2}+\frac{\mu-k}{2(1-k)}\right)p+\left(\frac{n-1}{2}+\frac{\mu+k}{2(1-k)}\right)+\frac{1}{p}}\frac{\big(\mathcal{U}(s)\big)^p}{\big(\log\langle A_k(s)\rangle\big)^{(p-1)}}\,\mathrm{d}s,\end{aligned}$$

where in third step we used $s=(1-k)^{\frac{1}{1-k}}(A_k(s)+\phi_k(1))^{\frac{1}{1-k}}\approx\langle A_k(s)\rangle^{\frac{1}{1-k}}$ for $s\geqslant 1$. Since $p=p_0\big(k,n+\frac{\mu}{1-k}\big)$ from (1.5) it follows

$$-\left(\tfrac{n-1}{2}+\tfrac{\mu-k}{2(1-k)}\right)p+\left(\tfrac{n-1}{2}+\tfrac{\mu+k}{2(1-k)}\right)+\tfrac{1}{p}=-1-\tfrac{k}{1-k}=-\tfrac{1}{1-k}, \tag{3.69}$$

then, plugging (3.69) in the above lower bound estimate for $\mathcal{U}(t)$ it yields

$$\begin{aligned}\mathcal{U}(t) &\gtrsim \langle A_k(t)\rangle^{-1}\int_1^t (\phi_k(t)-\phi_k(s))\langle A_k(s)\rangle^{-\frac{1}{1-k}}\big(\log\langle A_k(s)\rangle\big)^{-(p-1)}\big(\mathcal{U}(s)\big)^p\,\mathrm{d}s \\ &\gtrsim \langle A_k(t)\rangle^{-1}\int_1^t \frac{\phi_k(t)-\phi_k(s)}{s}\big(\log\langle A_k(s)\rangle\big)^{-(p-1)}\big(\mathcal{U}(s)\big)^p\,\mathrm{d}s,\end{aligned}$$

which is exactly (3.67). Therefore, the proof is completed. □

Lemma 3.11. *Let $n\geqslant 1$, $k\in[0,1)$ and $\mu\in[0,k]\cup[2-k,\infty)$. Let us consider $u_0\in H^1(\mathbb{R}^n)$ and $u_1\in L^2(\mathbb{R}^n)$ such that* $\operatorname{supp} u_j\subset B_R$ *for $j=0,1$ and for some $R>0$ and let u be a local in time energy solution to* (1.4) *on $[1,T)$ according to Definition 1.1. Then, there exists a positive constant $K=K(u_0,u_1,n,p,R,k,\mu)$ such that the lower bound estimate*

$$\int_{\mathbb{R}^n}|u(t,x)|^p\,\mathrm{d}x\geqslant K\varepsilon^p\langle A_k(t)\rangle^{(n-1)(1-\frac{p}{2})+\frac{(k-\mu)p}{2(1-k)}} \tag{3.70}$$

holds for any $t\in(1,T)$.

Proof. We modify of the proof of Lemma 5.1 in [38] accordingly to our model. Let us fix $q>(n-3)/2+1/p'$. Combining (3.65), (3.66) and Hölder's inequality, it results

$$\begin{aligned}\varepsilon t^{\frac{k-\mu}{2}} \lesssim t^{\frac{k-\mu}{2}}\mathcal{U}(t) &= \int_{\mathbb{R}^n} u(t,x)\,\xi_q(t,t,x;k,\mu)\,\mathrm{d}x \\ &\leqslant \left(\int_{\mathbb{R}^n}|u(t,x)|^p\,\mathrm{d}x\right)^{1/p}\left(\int_{B_{R+A_k(t)}}\big(\xi_q(t,t,x;k,\mu)\big)^{p'}\mathrm{d}x\right)^{1/p'}.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^n} |u(t,x)|^p \,\mathrm{d}x \gtrsim \varepsilon^p t^{\frac{k-\mu}{2}p} \left(\int_{B_{R+A_k(t)}} \big(\xi_q(t,t,x;k,\mu)\big)^{p'} \mathrm{d}x \right)^{-(p-1)}. \tag{3.71}$$

Let us determine an upper bound estimates for the integral of $\xi_q(t,t,x;k,\mu)^{p'}$. By using (3.64), we have

$$\begin{aligned}
\int_{B_{R+A_k(t)}} \big(\xi_q(t,t,x;k,\mu)\big)^{p'} \mathrm{d}x &\lesssim \langle A_k(t)\rangle^{-\frac{n-1}{2}p'} \int_{B_{R+A_k(t)}} \langle A_k(t)-|x|\rangle^{(n-3)p'/2-p'q} \,\mathrm{d}x \\
&\lesssim \langle A_k(t)\rangle^{-\frac{n-1}{2}p'} \int_0^{R+A_k(t)} r^{n-1} \langle A_k(t)-r\rangle^{(n-3)p'/2-p'q} \,\mathrm{d}r \\
&\lesssim \langle A_k(t)\rangle^{-\frac{n-1}{2}p'+n-1} \int_0^{R+A_k(t)} \langle A_k(t)-r\rangle^{(n-3)p'/2-p'q} \,\mathrm{d}r.
\end{aligned}$$

Performing the change of variable $A_k(t)-r=\varrho$, one gets

$$\begin{aligned}
\int_{B_{R+A_k(t)}} \big(\xi_q(t,t,x;k,\mu)\big)^{p'} \mathrm{d}x &\lesssim \langle A_k(t)\rangle^{-\frac{n-1}{2}p'+n-1} \int_{-R}^{A_k(t)} (3+|\varrho|)^{(n-3)p'/2-p'q} \,\mathrm{d}\varrho \\
&\lesssim \langle A_k(t)\rangle^{-\frac{n-1}{2}p'+n-1}
\end{aligned}$$

because of $(n-3)p'/2-p'q<-1$. If we combine this upper bound estimates for the integral of $\xi_q(t,t,x;k,\mu)^{p'}$, the inequality (3.71) and we employ $t \approx \langle A_k(t)\rangle^{\frac{1}{1-k}}$ for $t \geqslant 1$, then, we arrive at (3.70). This completes the proof. □

In Proposition 3.10, we derive the iteration frame for $\mathcal{U}$. In the next result, we shall prove a first lower bound estimate of logarithmic type for $\mathcal{U}$, as base case for the iteration argument.

Proposition 3.12. *Let $n \geqslant 1$, $k \in [0,1)$ and $\mu \in [0,k] \cup [2-k,\infty)$. Let us consider $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ such that $\operatorname{supp} u_j \subset B_R$ for $j=0,1$ and for some $R>0$ and let u be a local in time energy solution to (1.4) on $[1,T)$ according to Definition 1.1. Let $\mathcal{U}$ be defined by (3.65) with $q=(n-1)/2-1/p$. Then, for $t \geqslant 3/2$ the functional $\mathcal{U}(t)$ fulfills*

$$\mathcal{U}(t) \geqslant M\varepsilon^p \log\left(\tfrac{2t}{3}\right), \tag{3.72}$$

where the positive constant M depends on u_0,u_1,n,p,R,k,μ.

Proof. From (3.60) it results

$$\mathcal{U}(t) \geqslant t^{\frac{\mu-k}{2}} \int_1^t (\phi_k(t)-\phi_k(s)) \int_{\mathbb{R}^n} |u(s,x)|^p \eta_q(t,s,x;k,\mu)\,\mathrm{d}x\,\mathrm{d}s.$$

Consequently, applying (3.62) first and then (3.70), we find

$$\begin{aligned}
\mathcal{U}(t) &\geqslant B_1 \langle A_k(t)\rangle^{-1} \int_1^t (\phi_k(t)-\phi_k(s))\, s^{\frac{\mu+k}{2}} \langle A_k(s)\rangle^{-q} \int_{\mathbb{R}^n} |u(s,x)|^p \,\mathrm{d}x\,\mathrm{d}s \\
&\geqslant B_1 K \varepsilon^p \langle A_k(t)\rangle^{-1} \int_1^t (\phi_k(t)-\phi_k(s))\, s^{\frac{\mu+k}{2}} \langle A_k(s)\rangle^{-q+(n-1)(1-\frac{p}{2})+\frac{(k-\mu)p}{2(1-k)}} \,\mathrm{d}s \\
&\gtrsim \varepsilon^p \langle A_k(t)\rangle^{-1} \int_1^t (\phi_k(t)-\phi_k(s)) \langle A_k(s)\rangle^{\frac{\mu+k}{2(1-k)}-\frac{n-1}{2}+\frac{1}{p}+(n-1)(1-\frac{p}{2})+\frac{(k-\mu)p}{2(1-k)}} \,\mathrm{d}s \\
&\gtrsim \varepsilon^p \langle A_k(t)\rangle^{-1} \int_1^t (\phi_k(t)-\phi_k(s)) \langle A_k(s)\rangle^{-\left(\frac{n-1}{2}+\frac{\mu-k}{2(1-k)}\right)p+\left(\frac{n-1}{2}+\frac{\mu+k}{2(1-k)}\right)+\frac{1}{p}} \,\mathrm{d}s \\
&\gtrsim \varepsilon^p \langle A_k(t)\rangle^{-1} \int_1^t (\phi_k(t)-\phi_k(s)) \langle A_k(s)\rangle^{-\frac{1}{1-k}} \,\mathrm{d}s \gtrsim \varepsilon^p \langle A_k(t)\rangle^{-1} \int_1^t \frac{\phi_k(t)-\phi_k(s)}{s} \,\mathrm{d}s.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_1^t \frac{\phi_k(t)-\phi_k(s)}{s}\,\mathrm{d}s &= \big(\phi_k(t)-\phi_k(s)\big)\log s\,\Big|_{s=1}^{s=t} + \int_1^t \phi_k'(s)\log s\,\mathrm{d}s \\ &= \int_1^t s^{-k}\log s\,\mathrm{d}s \geqslant t^{-k}\int_1^t \log s\,\mathrm{d}s.\end{aligned}$$

Consequently, for $t\geqslant 3/2$

$$\begin{aligned}\mathcal{U}(t) &\gtrsim \varepsilon^p\langle A_k(t)\rangle^{-1}t^{-k}\int_1^t \log s\,\mathrm{d}s \geqslant \varepsilon^p\langle A_k(t)\rangle^{-1}t^{-k}\int_{2t/3}^t \log s\,\mathrm{d}s \geqslant (1/3)\varepsilon^p\langle A_k(t)\rangle^{-1}t^{1-k}\log(2t/3) \\ &\gtrsim \varepsilon^p\log(2t/3),\end{aligned}$$

where in the last line we applied $t\approx\langle A_k(t)\rangle^{\frac{1}{1-k}}$ for $t\geqslant 1$. Thus, the proof is over. □

In order to conclude the proof of Theorem 1.4 it remains to use an iteration argument together with a slicing procedure for the domain of integration. This procedure consists in determining a sequence of lower bound estimates for $\mathcal{U}(t)$ (indexes by $j\in\mathbb{N}$) and, then, proving that $\mathcal{U}(t)$ may not be finite for t over a certain ε-dependent threshold by taking the limit as $j\to\infty$. Since the iteration frame (3.67) and the first lower bound estimate (3.72) are formally identical to those in [29, Section 2.3] (of course, for different values of the critical exponent p), the iteration argument can be rewritten verbatim as in [29, Section 2.4].

Finally, we show how the previous steps can be adapted to the treatment of the case $\mu\in[0,k]$. According to Remark 3.6 , through the transformation $v(t,x)=t^{\mu-1}u(t,x)$, we may consider the transformed semilinear Cauchy problem (3.57) for v. Note that $v_0\doteq u_0$ and $v_1\doteq u_1+(1-\mu)u_0$ satisfies the same assumptions for u_0 and u_1 in the statement of Theorem 1.4 in this case (nonnegativeness and nontriviality, compactly supported and belongingness to the energy space $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$). Of course, we may introduce the auxiliary function $\xi_q(t,s,x;k,2-\mu),\xi_q(t,s,x;k,2-\mu)$ as in (3.58), (3.59) replacing μ by $2-\mu$. In Corollary 3.7, nevertheless, we have to replace the fundamental identity (3.53) by

$$\begin{aligned}\int_{\mathbb{R}^n} v(t,x)\,\xi_q(t,t,x;k,2-\mu)\,\mathrm{d}x &= \varepsilon\int_{\mathbb{R}^n} v_0(x)\,\xi_q(t,1,x;k,2-\mu)\,\mathrm{d}x \\ &\quad+\varepsilon\,A_k(t)\int_{\mathbb{R}^n} v_1(x)\,\eta_q(t,s,x;k,2-\mu)\,\mathrm{d}x \\ &\quad+\int_1^t(\phi_k(t)-\phi_k(s))s^{(1-\mu)(p-1)}\int_{\mathbb{R}^n}|v(s,x)|^p\eta_q(t,s,x;k,2-\mu)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

As we have already pointed out in Remark 3.6, the estimates in (3.54) and (3.55) hold true in this case with $2-\mu$ instead of μ (we recall that this was the actual reason to consider the transformed problem in place of the original one). Moreover, also the lower bound estimate in (3.70) is valid for v, provided that we replace μ by $2-\mu$. Accordingly to what we have just remarked, the suitable time-dependent functional to study for the transformed problem is

$$\mathcal{V}(t)\doteq t^{1-\frac{\mu+k}{2}}\int_{\mathbb{R}^n} v(t,x)\,\xi_q(t,t,x;k,2-\mu)\,\mathrm{d}x.$$

In fact, $\mathcal{V}$ satisfies $\mathcal{V}(t)\gtrsim\varepsilon$ for $t\in[1,T)$ and, furthermore, it is possible to derive for $\mathcal{V}$ completely analogous iteration frame and first logarithmic lower bound, respectively, as the ones for $\mathcal{U}$ in (3.67) and (3.72), respectively. We point out that both for the iteration frame and for the first logarithmic lower bound estimate the time-dependent factor $t^{(1-\mu)(p-1)}$ in the nonlinearity compensates the modifications due to the replacement of μ by $2-\mu$ in the proofs of Propositions 3.10 and 3.12.

4. Critical case: part II

In Section 2, we derived the upper bound for the lifespan in the subcritical case, whereas in Section 3 we studied the critical case $p = p_0\big(k, n + \frac{\mu}{1-k}\big)$. It remains to consider the critical case $p = p_1(k, n)$, that is, when $\mu \geqslant \mu_0(k, n)$. In this section, we are going to prove Theorem 1.5. In this critical case, our approach will be based on a basic iteration argument combined with the slicing procedure introduced for the first time in the paper [1]. The parameters characterizing the slicing procedure are given by the sequence $\{\ell_j\}_{j\in\mathbb{N}}$, where $\ell_j \doteq 2 - 2^{-(j+1)}$.

As time-depending functional we consider the same one studied in Section 2, namely, U_0 defined in (2.11). Hence, since $p = p_1(k, n)$ is equivalent to the condition

$$(1-k)n(p-1) = 2, \tag{4.73}$$

we can rewrite (2.23) as

$$U_0(t) \geqslant C \int_1^t \tau^{-\mu} \int_1^\tau s^{\mu-2} (U_0(s))^p \, \mathrm{d}s \, \mathrm{d}\tau \tag{4.74}$$

for any $t \in (1, T)$ and for a suitable positive constant $C > 0$. Let us underline that (4.74) will be the iteration frame in the iteration procedure for the critical case $p = p_1(k, n)$.

We know that $U_0(t) \geqslant K\varepsilon$ for any $t \in (1, T)$ and for a suitable positive constant K, provided that u_0, u_1 are nonnegative, nontrivial and compactly supported (cf. the estimate in (2.21)). Thus,

$$\begin{aligned} U_0(t) &\geqslant CK^p\varepsilon^p \int_1^t \tau^{-\mu} \int_1^\tau s^{\mu-2} \, \mathrm{d}s \, \mathrm{d}\tau \geqslant CK^p\varepsilon^p \int_1^t \tau^{-\mu-2} \int_1^\tau (s-1)^{\mu} \, \mathrm{d}s \, \mathrm{d}\tau \\ &= \frac{CK^p\varepsilon^p}{\mu+1} \int_1^t \tau^{-\mu-2} (\tau-1)^{\mu+1} \, \mathrm{d}\tau \geqslant \frac{CK^p\varepsilon^p}{\mu+1} \int_{\ell_0}^t \tau^{-\mu-2} (\tau-1)^{\mu+1} \, \mathrm{d}\tau \\ &\geqslant \frac{CK^p\varepsilon^p}{3^{\mu+1}(\mu+1)} \int_{\ell_0}^t \tau^{-1} \, \mathrm{d}\tau \geqslant \frac{CK^p\varepsilon^p}{3^{\mu+1}(\mu+1)} \log\left(\frac{t}{\ell_0}\right) \end{aligned} \tag{4.75}$$

for $t \geqslant \ell_0 = 3/2$, where we used $\tau \leqslant 3(\tau-1)$ for $\tau \geqslant \ell_0$ in the second last step.

Therefore, by using recursively (4.74), we prove now the sequence of lower bound estimates

$$U_0(t) \geqslant K_j \left(\log\left(\frac{t}{\ell_j}\right)\right)^{\sigma_j} \qquad \text{for } t \geqslant \ell_j \tag{4.76}$$

for any $j \in \mathbb{N}$, where $\{K_j\}_{j\in\mathbb{N}}$, $\{\sigma_j\}_{j\in\mathbb{N}}$ are sequences of positive reals that we determine afterwards in the inductive step.

Clearly (4.76) for $j = 0$ holds true thanks to (4.75), provided that $K_0 = (CK^p\varepsilon^p)/(3^{\mu+1}(\mu+1))$ and $\sigma_0 = 1$. Next we show the validity of (4.76) by using an inductive argument. Assuming that (4.76) is satisfied for some $j \geqslant 0$, we prove (4.76) for $j + 1$. According to this purpose, we plug (4.76) in (4.74), so, after shrinking the domain of integration, we get

$$U_0(t) \geqslant CK_j^p \int_{\ell_j}^t \tau^{-\mu} \int_{\ell_j}^\tau s^{\mu-2} \left(\log\left(\tfrac{s}{\ell_j}\right)\right)^{\sigma_j p} \mathrm{d}s \, \mathrm{d}\tau$$

for $t \geqslant \ell_{j+1}$. If we shrink the domain of integration to $[(\ell_j/\ell_{j+1})\tau, \tau]$ in the s-integral (this operation is possible for $\tau \geqslant \ell_{j+1}$), we find

$$\begin{aligned} U_0(t) &\geqslant CK_j^p \int_{\ell_{j+1}}^t \tau^{-\mu-2} \int_{\frac{\ell_j \tau}{\ell_{j+1}}}^{\tau} s^\mu \left(\log\left(\tfrac{s}{\ell_j}\right)\right)^{\sigma_j p} \mathrm{d}s\,\mathrm{d}\tau \\ &\geqslant CK_j^p \int_{\ell_{j+1}}^t \tau^{-\mu-2} \left(\log\left(\tfrac{\tau}{\ell_{j+1}}\right)\right)^{\sigma_j p} \int_{\frac{\ell_j \tau}{\ell_{j+1}}}^{\tau} \left(s - \tfrac{\ell_j}{\ell_{j+1}}\tau\right)^\mu \mathrm{d}s\,\mathrm{d}\tau \\ &= CK_j^p(\mu+1)^{-1}\left(1-\tfrac{\ell_j}{\ell_{j+1}}\right)^{\mu+1} \int_{\ell_{j+1}}^t \tau^{-1}\left(\log\left(\tfrac{\tau}{\ell_{j+1}}\right)\right)^{\sigma_j p} \mathrm{d}\tau \\ &\geqslant 2^{-(j+3)(\mu+1)} CK_j^p(\mu+1)^{-1}(1+p\sigma_j)^{-1}\left(\log\left(\tfrac{t}{\ell_{j+1}}\right)\right)^{\sigma_j p+1} \end{aligned}$$

for $t \geqslant \ell_{j+1}$, where in the last step we applied the inequality $1-\ell_j/\ell_{j+1} > 2^{-(j+3)}$. Hence, we proved (4.76) for $j+1$ provided that

$$K_{j+1} \doteq 2^{-(j+3)(\mu+1)} C(\mu+1)^{-1}(1+p\sigma_j)^{-1} K_j^p \quad \text{and} \quad \sigma_{j+1} \doteq 1+\sigma_j p.$$

Let us establish a suitable lower bound for K_j. Using iteratively the relation $\sigma_j = 1+p\sigma_{j-1}$ and the initial exponent $\sigma_0 = 1$, we have

$$\sigma_j = \sigma_0 p^j + \sum_{k=0}^{j-1} p^k = \tfrac{p^{j+1}-1}{p-1}. \tag{4.77}$$

In particular, the inequality $\sigma_{j-1}p+1 = \sigma_j \leqslant p^{j+1}/(p-1)$ yields

$$K_j \geqslant L\left(2^{\mu+1}p\right)^{-j} K_{j-1}^p \tag{4.78}$$

for any $j \geqslant 1$, where $L \doteq 2^{-2(\mu+1)}C(\mu+1)^{-1}(p-1)/p$. Applying the logarithmic function to both sides of (4.78) and using the resulting inequality iteratively, we obtain

$$\begin{aligned} \log K_j &\geqslant p\log K_{j-1} - j\log\left(2^{\mu+1}p\right) + \log L \\ &\geqslant \ldots \geqslant p^j \log K_0 - \left(\sum_{k=0}^{j-1}(j-k)p^k\right)\log\left(2^{\mu+1}p\right) + \left(\sum_{k=0}^{j-1}p^k\right)\log L \\ &= p^j\left(\log\left(\frac{CK^p\varepsilon^p}{3^{\mu+1}(\mu+1)}\right) - \frac{p\log\left(2^{\mu+1}p\right)}{(p-1)^2} + \frac{\log L}{p-1}\right) + \left(\frac{j}{p-1}+\frac{p}{(p-1)^2}\right)\log\left(2^{\mu+1}p\right) - \frac{\log L}{p-1}, \end{aligned}$$

where we applied again the identities in (2.37). Let us define $j_2 = j_2(n,p,k,\mu)$ as the smallest nonnegative integer such that

$$j_2 \geqslant \frac{\log L}{\log\left(2^{\mu+1}p\right)} - \frac{p}{p-1}.$$

Consequently, for any $j \geqslant j_2$ the following estimate holds

$$\log K_j \geqslant p^j\left(\log\left(\frac{CK^p\varepsilon^p}{3^{\mu+1}(\mu+1)}\right) - \frac{p\log\left(2^{\mu+1}p\right)}{(p-1)^2} + \frac{\log L}{p-1}\right) = p^j\log(N\varepsilon^p), \tag{4.79}$$

where $N \doteq 3^{-(\mu+1)}CK^p(\mu+1)^{-1}\left(2^{\mu+1}p\right)^{-p/(p-1)^2}L^{1/(p-1)}$.

Combining (4.76), (4.77) and (4.79), we arrive at

$$\begin{aligned} U_0(t) &\geqslant \exp\left(p^j\log(N\varepsilon^p)\right)\left(\log\left(\tfrac{t}{\ell_j}\right)\right)^{\sigma_j} \\ &\geqslant \exp\left(p^j\log(N\varepsilon^p)\right)\left(\tfrac{1}{2}\log t\right)^{(p^{j+1}-1)/(p-1)} \\ &= \exp\left(p^j\log\left(2^{-p/(p-1)}N\varepsilon^p\left(\log t\right)^{p/(p-1)}\right)\right)\left(\tfrac{1}{2}\log t\right)^{-1/(p-1)} \end{aligned}$$

for $t \geqslant 4$ and for any $j \geqslant j_2$, where we employed the inequality $\log(t/\ell_j) \geqslant \log(t/2) \geqslant (1/2)\log t$ for $t \geqslant 4$. Introducing the notation $H(t,\varepsilon) \doteq 2^{-p/(p-1)} N\varepsilon^p (\log t)^{p/(p-1)}$, the previous estimate may be rewritten as

$$U_0(t) \geqslant \exp\left(p^j \log H(t,\varepsilon)\right)\left(\tfrac{1}{2}\log t\right)^{-1/(p-1)} \tag{4.80}$$

for $t \geqslant 4$ and any $j \geqslant j_2$.

If we fix $\varepsilon_0 = \varepsilon_0(n,p,k,\mu,R,u_0,u_1)$ such that

$$\exp\left(2N^{-(1-p)/p}\varepsilon_0^{-(p-1)}\right) \geqslant 4,$$

then, for any $\varepsilon \in (0,\varepsilon_0]$ and for $t > \exp\left(2N^{-(1-p)/p}\varepsilon^{-(p-1)}\right)$ we have $t \geqslant 4$ and $H(t,\varepsilon) > 1$. Therefore, for any $\varepsilon \in (0,\varepsilon_0]$ and for $t > \exp\left(2N^{-(1-p)/p}\varepsilon^{-(p-1)}\right)$ letting $j \to \infty$ in (4.80) we see that the lower bound for $U_0(t)$ blows up and, consequently, $U_0(t)$ may not be finite as well. Summarizing, we proved that U_0 blows up in finite time and, moreover, we showed the upper bound estimate for the lifespan

$$T(\varepsilon) \leqslant \exp\left(2N^{-(1-p)/p}\varepsilon^{-(p-1)}\right).$$

Hence the proof of Theorem 1.5 in the critical case $p = p_1(k,n)$ is complete.

5. Final remarks

According to the results we obtained in Theorems 1.3, 1.4 and 1.5 it is quite natural to conjecture that

$$\max\left\{p_0\left(k, n + \tfrac{\mu}{1-k}\right), p_1(k,n)\right\}$$

is the critical exponent for the semilinear Cauchy problem (1.4), although the global existence of small data solutions is completely open in the supercritical case. Furthermore, this exponent is consistent with other models studied in the literature.

In the flat case $k = 0$, this exponent coincide with $\max\{p_{\mathrm{Str}}(n+\mu), p_{\mathrm{Fuj}}(n)\}$ which in many subcases has been showed to be optimal in the case of semilinear wave equation with time-dependent scale-invariant damping, see [5, 8, 7, 24, 18, 36, 30, 33, 26, 27, 6] and references therein for further details.

On the other hand, in the undamped case $\mu = 0$ (that is, for the semilinear wave equation with speed of propagation t^{-k}) the exponent $\max\{p_0(k,n), p_1(k,n)\}$ is consistent with the result for the generalized semilinear Tricomi equation (i.e., the semilinear wave equation with speed of propagation t^ℓ, where $\ell > 0$) obtained in the recent works [15, 16, 17, 23].

Clearly, in the very special case $\mu = 0$ and $k = 0$, our result is nothing but a blow-up result for the classical semilinear wave equation for exponents below $p_{\mathrm{Str}}(n)$, which is well-known to be optimal (for a detailed historical overview on Strauss' conjecture and a complete list of references we address the reader to the introduction of the paper [35]).

As we have already explained in the introduction, for $\mu = 2$ and $k = 2/3$ the equation in (1.4) is the semilinear wave equation in the Einstein-de Sitter spacetime. In particular, our result is a natural generalization of the results in [12, 29].

Furthermore, we underline explicitly the fact that the exponent $p_0\left(k, n+\frac{\mu}{1-k}\right)$ for (1.4) is obtained by the corresponding exponent in the not damped case $\mu = 0$ via a formal shift in the dimension of magnitude $\frac{\mu}{1-k}$. This phenomenon is due to the threshold nature of the time-dependent coefficient of the damping term and it has been widely observed in the special case $k = 0$ not only for the semilinear Cauchy problem with power nonlinearity but also with nonlinarity of derivative type $|u_t|^p$ (see [34, 13]) or weakly coupled system (see [2, 28, 34, 14]).

Finally, we have to point out that after the completion of the final version of this work, we found out the existence of the paper [37], where the same model is considered. We stress that the approach we used in the critical case is completely different, and that we slightly improved their result, by removing the assumption on the size of the support of the Cauchy data (cf. [37, Theorem 2.3]), even though we might not cover the full range $\mu \in [0, \mu_0(k,n)]$ in the critical case due to the assumption $\mu \notin (k, 2-k)$.

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