(δ, ε)-DIFFERENTIAL IDENTITIES OF $UT_m(F)$

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Abstract. Let δ and ε be the inner derivations of $UT_m(F)$ induced by the unit matrices e_{1m} and e_{mm} respectively. We study the differential polynomial identities of the algebra $UT_m(F)$ under the coupled action of δ and ε. We produce a basis of the differential identities, then we determine the S_n-structure of their proper multilinear spaces and, for the minimal cases $m = 2, 3$, their exact differential codimension sequence.

1. Introduction

Differential polynomial identities are certainly not a brand new topic in PI-theory. Significant contributions to this subject may be dated back to the late 1970's, due to a series of fundamental papers by Kharchenko involving both derivations and automorphisms, but in fact a vast literature on this topics is available (a good source is [1] and its bibliography to this and related topics). In present days, however, new interest is flowing into this subject, mainly because of a new unifying approach towards the several areas related to PI-theory.

As a consequence of the evolution of classic (so to say) PI-theory, almost every special PI-theory has developed a suitable set of tools, techniques and results modeled on those available for the classic case. So, for instance, when dealing with algebras with involutions, superalgebras, or with more general graded algebras, one can properly define universal objects, identities, cocharacters, codimensions and so on, resembling what happens in the ordinary case. It turns out that several results holding for the classic case can be restated for the special ones, although under some suitable, reasonable, assumptions. This is the case, for instance, for the growth behaviour of codimensions of superalgebras, algebras with involution, graded algebras, algebras with derivations. On the contrary, other central results of classical PI-Theory are more eluding. Among them, the most prominent ones are Kemer's Representability Theorem and the finite basis property holding in classic PI-Theory [2]. A generalization of these results has been obtained for algebras graded by a finite group in [3].

Starting from a sparkling intuition of Berele in his influential paper [4] (more precisely, the Remark at page 878), an effort to a unifying approach started, turning around the notion of Hopf-algebra action. This is for instance the case of the relatively recent paper by Gordienko [5]. The Specht property and the Representability theorem have been recently faced within this framework in [6].

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In the case of algebras under the derivation action of a Lie algebra L, the involved Hopf-algebra is the universal enveloping algebra $U(L)$ of L, a more than natural connection. Of course, even when a final unifying theory should be established, it could conceal but not cancel the differences among concrete situations, so differential identities, as well as other types of identities, will still have to be treated and studied according to their specific features, although empowered perhaps with some new profitable idea coming from some other specific situation.

The present paper is based on a very recent joint work with Di Vincenzo [7] which, at the moment, has been just submitted for publication, so it partially serves as an announcement for the results contained in [7] and concerning the description of the differential polynomial identities satisfied by algebras $UT_m(F)$ under the derivation actions of the two-dimensional non abelian Lie algebra, a problem which was in some sense inspired by the recent paper of Giambruno and Rizzo [8].

At the same time, I wanted this paper to be an expository one, hoping to convey the reader the same pleasure I sensed working on this problem. So in writing these notes I chose to present the material in a maybe rather unusual way, that is from an operative point of view rather than from a formal one. The basic definition of differential polynomial identity is therefore given in the next section within this perspective, together with all the necessary background and tools needed to quickly understand the problem and how to attack it, in the same spirit. The subsequent sections are devoted to give the answers, in case the Lie algebra acts faithfully (coupled actions of δ, ε) on $UT_m(F)$ or not (separate actions). Due to the nature of this paper, technical details and proofs have been omitted, but I tried to at least address the reader to the main ideas involved in them. By the way, a couple of statements which were missing on the original paper have been added here, and their proofs are therefore provided within these pages. The last section is instead devoted to present the general topics within a more theoretical setup, in order to confer the objects and tools presented in the preceding sections a more sound and deeper sense.

2. The problem

Let F denote a field of characteristic zero and let A be an associative F-algebra. An F-linear map $d : A \to A$ is called an *F-derivation* if the usual Leibniz rule $(ab)^d = a^d b + ab^d$ holds for all $a, b \in A$. Throughout the paper, we will adopt the exponential notation for derivations, hence derivations will compose from left to right. It is easy to produce concrete derivations on A: for any $a \in A$, just consider the map $[\cdot, a]$ sending $x \in A$ into the Lie product $[x, a] = xa - ax$. It is called the *inner derivation induced by a*. For some relevant algebras, these derivations are actually the only ones available: this is the case for the full matrix algebra $M_m(F)$ and its subalgebra $UT_m(F)$ of upper triangular matrices [9]. In this paper, we are going to deal with the latter one. More precisely, let δ and ε be the inner derivations of $UT_m(F)$ induced by the unit matrices e_{1m} and e_{mm} respectively, that is $\delta = [\cdot, e_{1m}]$ and $\varepsilon = [\cdot, e_{mm}]$. The algebra $UT_m(F)$ is enriched with these derivation actions on it, and we denote U_m this structure, to distinguish it from the simpler algebra structure $UT_m(F)$. Then the identity relations among the elements of $UT_m(F)$ are still valid in U_m, but they are just a part of those holding in U_m: new relations, involving both elements of $UT_m(F)$ and derivations of elements of $UT_m(F)$, appear. For instance, for any $a, b, c \in UT_m(F)$, it holds $a^\varepsilon bc = a^\varepsilon cb$.

These more general identity relations are called *differential* identities, and the basic problem we are going to face is the following:

Determine and describe the differential identities holding in U_m.

Some clarifications are in order: first of all, we need to be more precise on what a differential identity is. Then, we have to agree on what the verbs *determine* and *describe* should mean. About the first point, we are going to pursue a very intuitive approach. It will fit perfectly the operative aspects of our investigations, though is a bit too naïve to be fully satisfying. A more sound and solid approach will be postponed to the last section.

Let us start with a countable set of indeterminates X, and define a new one, namely

$$X^D := \{x^w \mid x \in X,\, w \text{ word in } \delta,\, \varepsilon\}.$$

So, for instance, $x^{\delta\varepsilon\delta} \in X^D$ for all $x \in X$. We will call *letters* the elements of X^D. More precisely, if w is not the empty word, we call x^w a *differential* letter; the letters in X^D which are not differential are substantially indistinguishable from their parent indeterminate, so we identify the letter $x \in X^D$ (corresponding to the empty word) with the indeterminate $x \in X$, and call it an *ordinary* letter. Hence we write $X \subseteq X^D$.

The free associative unitary F-algebra $F\langle X^D\rangle$ generated by X^D inherits a formal derivation action of δ,ε: just define $(x^w)^\alpha := x^{w\alpha}$, for $\alpha \in \{\delta,\varepsilon\}$, on the letters $x^w \in X^D$, and then extend this (right) action to the whole $F\langle X^D\rangle$ by F-linearity and the Leibniz rule. The elements of $F\langle X^D\rangle$ are called differential polynomials; in case $f \in F\langle X^D\rangle$ involves ordinary letters only, it will be called an ordinary polynomial. The natural inclusion $F\langle X\rangle \subseteq F\langle X^D\rangle$ then follows from the definition.

Definition 1. An element $f(x_1,\dots,x_n) \in F\langle X^D\rangle$ is a *differential polynomial identity* of U_m if $f(a_1,\dots,a_n) = 0$ for all $a_1,\dots,a_n \in U_m$. The set of all differential polynomial identities of U_m is denoted by $T_D(U_m)$.

Notice that the writing $f(x_1,\dots,x_n)$ points just to the parent indeterminates of the letters occurring in the polynomial: this is legitimate for the following fact

Lemma 2. *Let A be an F-algebra with derivation actions of δ,ε. Any map $\varphi_0 : X \to A$ uniquely extends to an algebra homomorphism $\varphi : F\langle X^D\rangle \to A$ commuting with the derivation action of δ,ε.*

As a first sign of the naïve nature of our definitions notice that, at the moment, just two algebras with derivation action of δ and ε are available: U_m, from which δ and ε have been constructed, and $F\langle X^D\rangle$ itself. Hence what a generic algebra with (δ,ε)-derivation actions should be is left too vague and subject to criticism, not to mention the definition of $T_D(A)$ for a generic algebra A. Nevertheless, we shall pursue our intuitive perpsective for the moment, and interprete the first statement of the Lemma as a shortcut to mean $A \in \{F\langle X^D\rangle, U_m\}$, while we merely focus on $T_D(U_m)$. Also, a homomorphism $\varphi : F\langle X^D\rangle \to A$ commuting with δ and ε will be called a *D-homomorphism*, for short.

It is worth noticing that the set $T(UT_m(F))$ of usual polynomial identities satisfied by the algebra $UT_m(F)$ coincides with the set of *ordinary* polynomial identities of U_m and is a subset of $T_D(U_m)$, as a consequence of our definitions; this is consistent with the idea that we are adding more general identity relations to the ones related to the mere algebra structure.

The set $T_D(U_m)$ is clearly a two-sided ideal of $F\langle X^D\rangle$, but actually is more than this: it is invariant under all D-endomorphism of $F\langle X^D\rangle$, and is called a T_D-ideal. If $\mathscr{I} \subseteq F\langle X^D\rangle$, it makes sense to consider the least T_D-ideal containing $\mathscr{I}$, and call it the T_D-ideal *generated* by $\mathscr{I}$. So a possible, acceptable answer for *determine the differential polynomial identities of* U_m, is to exhibit a few differential polynomial identities of U_m generating the whole $T_D(U_m)$ as a T_D-ideal.

This also gives a first, very rough, sense to what we may mean by *describing* the T_D-ideal $T_D(U_m)$: in fact, even when a generating set $\mathscr{I}$ is given, it is extremely hard to decide if a random polynomial $f \in F\langle X^D\rangle$ follows from $\mathscr{I}$ (that is if f belongs to the T_D-ideal generated by $\mathscr{I}$). Since F has characteristic zero, a more refined description of $T_D(U_m)$ is provided by its multilinear spaces:

Definition 3. Define $P_n^D := \operatorname{span}_F\langle x_{\sigma(1)}^{w_1}\dots x_{\sigma(n)}^{w_n} \mid \sigma\in S_n,\ w_i \text{ word in } \delta,\varepsilon\rangle$ for all $n \geqslant 1$, and let $P_n^D(U_m)$ denote the factor space $P_n^D/\big(P_n^D \cap T_D(U_m)\big)$.

Any element of P_n^D is called a *multilinear differential polynomial of degree* n, and those in $P_n^D \cap T_D(U_m)$ are named multilinear differential polynomial identities of U_m of degree n. From here on, we shall abbreviate it in *multilinear D-PI.*

The slices $P_n^D \cap T_D(U_m)$ completely characterize $T_D(U_m)$, because their union generates $T_D(U_m)$, by standard Vandermonde argument and multilinearization process. Moreover, each $P_n^D \cap T_D(U_m)$ is actually a submodule of the S_n-module P_n^D, where the (left) action of S_n on the multilinear differential polynomials is the natural one; namely, the one defined on the letters by $\sigma\bullet x_i^w := x_{\sigma(i)}^w$ for all $\sigma\in S_n$. Hence $P_n^D(U_m)$ is a left S_n-module as well, and its S_n-character $\chi_n^D(U_m)$, called the n-th *D-cocharacter* of U_m, indirectly gives a picture of the S_n-structure of $P_n^D\cap T_D(U_m)$. Moreover, the dimension $c_n^D(U_m) := \dim_F P_n^D(U_m)$, called the *$n$-th codimension of* U_m, gives a quantitative measure on how big the slice $P_n^D \cap T_D(U_m)$ is: the greater $c_n^D(U_m)$, the smaller is the space of multilinear D-PI's $P_n^D \cap T_D(U_m)$. A word of caution is due to this proposal: there is no reason, at the moment, to believe that $P_n^D(A)$ is finite dimensional. Indeed, by definition, P_n^D is infinite-dimensional: for instance the set $\{x_1^{\delta^i}x_2\dots x_n \mid i\in\mathbb{N}\}$ is an infinite set of F-independent elements of P_n^D.

Since the multilinear spaces do provide so many useful information on $T_D(U_m)$, both of qualitative and quantitative nature, it is more than agreeable to accept the S_n-cocharacter sequence of U_m as a description of the D-PI of U_m. So our tasks are now operatively clear: to answer the problem, we have to find a small set of D-PI generating $T_D(U_m)$, and give the decomposition of the n-th D-cocharacter $\chi_n^D(U_m)$ into irreducible S_n-characters. The D-codimension sequence $c_n^D(U_m)$, once computed, will give the quantitative description on how big $T_D(U_m)$ is.

These tasks can be made easier if we exploit the fact that U_m is a unitary algebra. In this case, all features of $P_n^D(U_m)$ are encoded in smaller multilinear spaces, consisting of the so-called *proper* multilinear polynomials. There are several ways of presenting the notion of proper polynomials, and the easiest is the following

Definition 4. A polynomial $f\in F\langle X^D\rangle$ is called a *proper* polynomial if $\dfrac{\partial f}{\partial x} = 0$ for all ordinary letters $x\in X$.

A word of caution is needed, also with this definition: we are pointing to the elements of X^D as free generators, so there is no interaction among the formal derivations δ,ε and the usual formal partial derivatives $\frac{\partial}{\partial x}$. Explicitly, $\frac{\partial x^\delta}{\partial x} = 0$, and the same holds for x^ε: x, x^δ and x^ε are different elements among those freely generating $F\langle X^D\rangle$. Actually, talking of *X-proper polynomials* would be more precise. So, in particular, x^δ and x^ε are among the proper polynomials. What is the form of a generic proper polynomial? We need the following

Definition 5. Let $n \geqslant 2$ and let $z_1, \ldots, z_n \in X^D$. The *higher commutator* $[z_1, \ldots, z_n]$ is defined recursively by $[z_1, z_2] = z_1 z_2 - z_2 z_1$ and, for $n \geqslant 3$, $[z_1, z_2, \ldots, z_n] := \big[[z_1, \ldots, z_{n-1}], z_n\big]$. The number n is the *length* of the commutator.

Higher commutators are therefore a (left-normed) generalization of the Lie product among letters of X^D. We may extend the notion to commutators of lengths 0 and 1: a commutator of length 0 is simply an element $a \in F$, while by commutator of length 1 we mean any differential letter. Actually, the proper polynomials are precisely the elements of the unitary subalgebra B^D of $F\langle X^D\rangle$ generated by the commutators of any length. Hence, they are F-linear combinations of products of commutators.

Much more could be said on proper polynomials and their general properties. We address the interested reader to the book [10] for the basic definitions and results. Actually, Drensky re-discovered and gave new life to this class of polynomials, turning them into an amazing and powerful tool employed in several papers. By the way, we just need to focus on a specific type of proper polynomials:

Definition 6. Define $\Gamma_0^D := F$, and for $n \geqslant 1$ let $\Gamma_n^D := P_n^D \cap B^D$. The elements of the set $\Gamma^D := \bigcup_{n\in\mathbb{N}} \Gamma_n^D$ are called *proper multilinear polynomials.*

Proper multilinear polynomials share the same basic property of multilinear polynomials in our settings:

Lemma 7. *$T_D(U_m)$ is generated, as T_D-ideal, by $\Gamma^D \cap T_D(U_m)$.*

Therefore the proper multilinear polynomials in $T_D(U_m)$ completely determine the T_D-ideal, as the multilinear polynomials do. Moreover, since Γ_n^D is an S_n-module as well, the factor space $\Gamma_n^D(U_m) = \Gamma_n^D / \big(\Gamma_n^D \cap T_D(U_m)\big)$ is an S_n-module. Let $\xi_n^D(U_m)$ be its S_n-character (the n-th *proper* differential cocharacter of U_m), and $\gamma_n^D(U_m)$ be its dimension (the n-th *proper* codimension of U_m). Then the cocharacter sequence $\chi_n^D(U_m)$ is simply the so-called Young-derived sequence of $(\xi_n^D(U_m))_{n\in\mathbb{N}}$, that is $\chi_n^D(U_m)$ is obtained from the cocharacters $\xi_0^D(U_m), \ldots, \xi_n^D(U_m)$ via theYoung-Pieri rule (see [11], but also the most comprehensive exposition in [10] illustrating the interplay between proper and ordinary polynomials not only in the multilinear case, but in the more general case of multi-homogeneous one, involving the action of the general linear groups); hence the codimension sequence can be computed from the proper codimension sequence by the simple relation

$$c_n^D(U_m) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k^D(U_m).$$

Therefore, a significatively simpler meaning for the verb *describe* is made available: in order to describe $T_D(U_m)$, it is sufficient to get the decomposition of the proper cocharacters of U_m for all $n \in \mathbb{N}$; the quantitative information on $T_D(U_m)$ are carried from the proper codimension sequence.

A further, last simplification towards this description is possible, by selecting a particular basis for the vector spaces Γ_n^D (see [12]. Proper polynomials with respect to a distinct set of indeterminates were first presented in [13]). Let us fix a total order $\leqslant$ on X^D, such that ordinary letters precede the differential ones.

Definition 8. A higher commutator $[z_1, \ldots, z_n]$ is *normal* if $z_2, \ldots, z_n$ are ordinary letters. Moreover, the normal commutator $[z_1, \ldots, z_n]$ is *standard* if $z_1 > z_2 < \cdots < z_n$.

We include the commutators of lengths 0 and 1 among the normal standard commutators. Of course, if z_1 is a differential letter in the normal commutator $[z_1, z_2, \ldots, z_n]$ then just the order among $z_2, \ldots, z_n$ matters in being standard. The reason for bringing up normal standard commutators is that, as a Corollary of a stronger statement (Proposition 7 in [12]), it holds

Theorem 9. *The elements of Γ_n^D which are products of normal standard commutators constitute an F-basis of Γ_n^D.*

Proof. The products of normal *semi*standard commutators (that is: normal commutators $[z_1, \ldots, z_n]$ such that $z_1 > z_2 \leqslant \cdots \leqslant z_n$) form a basis for the algebra B^D of proper polynomials. Since $\Gamma_n^D = P_n^D \cap B^D$, any polynomial in Γ_n^D is a linear combination of products of normal semistandard commutators but, being multilinear, it is actually a linear combination of products of normal standard commutators. Then, just note that normal standard commutators are in particular semistandard, to get the linear independence. □

3. The coupled actions of δ and ε on $UT_m(F)$

As elements of $End_F(U_m)$, the operators δ and ε satisfy the following relations:

$$\delta^2 = 0, \quad \varepsilon^2 = \varepsilon, \quad \delta\varepsilon = \delta, \quad \varepsilon\delta = 0$$

(recall that their compositions are computed from left to right in our notation). Therefore the following D-PI's are readily available, and depend just upon the selected derivations:

Lemma 10. *The polynomials $x_1^{\delta^2}$, $x_1^{\varepsilon^2} - x_1^{\varepsilon}$, $x_1^{\delta\varepsilon} - x_1^{\delta}$, $x_1^{\varepsilon\delta}$ are in $T_D(U_m)$.*

Notice that they are all in Γ_1^D. Moreover, they cause any differential letter x^w related to a word w of length $\geqslant 2$ to be congruent, modulo $T_D(U_m)$, either to 0 or to a single differential letter $x^{\varepsilon}, x^{\delta}$. Therefore just ordinary letters or the differential letters x^{δ}, x^{ε} need to be considered in the sequel.

The following monomial identities also belong to $T_D(U_m)$:

Corollary 11. *The monomials $x_1^{\delta}x_2^{\varepsilon}$, $x_1^{\varepsilon}x_2^{\delta}$, $x_1^{\delta}x_2^{\delta}$, $x_1^{\varepsilon}x_2^{\varepsilon}$ follow from the identities listed in the previous Lemma. In particular, they are all in $T_D(U_m)$.*

Proof. Let I be the T_D-ideal generated by the polynomials listed in Lemma 10. Then $(x_1x_2)^{\varepsilon\delta} \in I$. Explicitly, one has

$$(x_1x_2)^{\varepsilon\delta} = x_1^{\varepsilon\delta}x_2 + x_1^{\varepsilon}x_2^{\delta} + x_1^{\delta}x_2^{\varepsilon} + x_1x_2^{\varepsilon\delta} \in I. \text{ Hence } x_1^{\varepsilon}x_2^{\delta} + x_1^{\delta}x_2^{\varepsilon} \in I.$$

Replacing x_1 by x_1^{ε} yields $x_1^{\varepsilon^2}x_2^{\delta} + x_1^{\varepsilon\delta}x_2^{\varepsilon} \in I$, so $x_1^{\varepsilon^2}x_2^{\delta} \in I$. By the way, since $x^{\varepsilon^2} \equiv x^{\varepsilon} \pmod{I}$, it follows $x_1^{\varepsilon}x_2^{\delta} \in I$.

The other identities follow easily. Then, since $I \subseteq T_D(U_m)$, they are in particular differential polynomial identities of U_m. □

Hence, any nonvanishing multilinear monomial will involve at most a single differential letter.

There are other basic identities, not depending just on the selected inner derivations but more properly on the interplay of δ and ε with the algebra structure. They are listed in the following

Lemma 12. *Let x, y, x_i, y_i denote distinct elements of X. The following polynomials are all in $T_D(U_m)$:*

(1) $[x_1, x_2]^\delta$;

(2) $x^\delta[x_1, x_2]$, $[x_1, x_2]x^\delta$, $x^\varepsilon[x_1, x_2]$;

(3) $[x_1, y_1] \dots [x_{m-1}, y_{m-1}]x^\varepsilon$;

(4) $[x_1, y_1] \dots [x_m, y_m]$;

(5) $[x_1, y_1] \dots [x_{m-2}, y_{m-2}]\Big([x, y]^\varepsilon - [x, y]\Big)$.

A different, maybe better, way to write the identity (1) is $[x_1^\delta, x_2] + [x_1, x_2^\delta]$. The identity (3) may be considered, in some sense, the ε-analogous of $[x_1, x_2]x^\delta$. The identity (4) is the one generating the whole T-ideal of ordinary polynomial identities of the algebra $UT_m(F)$, as proved in [14] . The last identity of the list is undoubtedly the most remarkable one, and the most difficult to find.

Collecting together the polynomials of Lemma 10 and Lemma 12 we get all the necessary identities we need to generate the whole $T_D(U_m)$. Precisely, it holds

Theorem 13. *Let I be the T_D-ideal generated by the following differential polynomials*

(1) x^{δ^2}, $x^{\varepsilon^2} - x^\varepsilon$, $x^{\delta\varepsilon} - x^\delta$, $x^{\varepsilon\delta}$

(2) $[x_1, x_2]^\delta$

(3) $x^\delta[x_1, x_2]$, $[x_1, x_2]x^\delta$, $x^\varepsilon[x_1, x_2]$

(4) $[x_1, y_1] \dots [x_{m-1}, y_{m-1}]x^\varepsilon$

(5) $[x_1, y_1] \dots [x_m, y_m]$

(6) $[x_1, y_1] \dots [x_{m-2}, y_{m-2}]\Big([x, y]^\varepsilon - [x, y]\Big)$,

where all indeterminates x, y, x_i, y_i belong to X. Then $T_D(U_m) = I$.

It is worth noticing that all these polynomials are multilinear proper polynomials, and are expressed as linear combinations of products of normal standard commutators. The proof of this theorem is quite direct although rather structured, and brings up some objects which turn useful in describing the multilinear spaces, so I am giving the reader a glimpse of the main ideas involved in it.

Of course, $I \subseteq T_D(U_m)$, and in order to prove the reverse inclusion it is enough to compare the proper multilinear parts of the two T_D-ideals. The first thing to do is therefore to exhibit a set of polynomials spanning Γ_n^D modulo I for each $0 < n \in \mathbb{N}$, and then prove that this spanning set is linearly independent modulo $T_D(U_m)$. From this, the fact that $T_D(U_m) \subseteq I$ follows immediately.

The spanning set $\mathscr{S}_n$ we are looking for is partitioned according to the differential letters occurring in its elements, if any. So let us separately construct the sets $\mathscr{S}_n^1, \mathscr{S}_n^\delta, \mathscr{S}_n^\varepsilon$ partitioning $\mathscr{S}_n$.

- $\mathscr{S}_n^1$: take any product $c_1 \dots c_k \in \Gamma_n^D$ of $k < m$ normal standard commutators c_i involving ordinary letters only. Of course $\mathscr{S}_1^1 = \varnothing$, while $\mathscr{S}_2^1 = \{[x_2, x_1]\}$. Notice that $\mathscr{S}_n^1$ is actually an F-basis for $\Gamma_n(UT_m(F))$.
- $\mathscr{S}_n^\delta$: it is a singleton. More precisely, $\mathscr{S}_1^\delta = \{x_1^\delta\}$ and, if $n \geqslant 2$, $\mathscr{S}_n^\delta = \{[x_n^\delta, x_1, \dots, x_{n-1}]\}$.
- $\mathscr{S}_n^\varepsilon$: take any product $c_1 \dots c_k \in \Gamma_n^D$ of $k < m$ normal standard commutators such that
  - $c_1, \dots, c_{k-1}$ involve only ordinary letters. In particular, each of them has length $\geqslant 2$;
  - if $k < m - 1$ then the last commutator c_k is any $[x^\varepsilon, y_1, \dots, y_l]$, that is any normal standard commutator involving the remaining indeterminates
  - if $k = m - 1$ then the last commutator is uniquely determined. More precisely, if $y_1 < y_2 < \cdots < y_l < x$ are the remaining indeterminates, it is $[x^\varepsilon, y_1, \dots, y_l]$.

Our candidate set is therefore $\mathscr{S}_n = \mathscr{S}_n^1 \cup \mathscr{S}_n^\delta \cup \mathscr{S}_n^\varepsilon$, and it is almost easy to see that $\mathscr{S}_n$ in fact spans Γ_n^D modulo I.

To prove that $\mathscr{S}_n$ is linearly independent modulo $T_D(U_m)$ is more tricky. Essentially, we produce a sort of elimination algorithm:

(1) initialize $\mathscr{S} := \mathscr{S}_n$
(2) produce a substitution $\varphi : X \to U_m$ such that $\varphi(w) = 0$ for all $w \in \mathscr{S}$ but a single element w_0
(3) delete w_0 from $\mathscr{S}$ and repeat the previous step until $\mathscr{S} = \varnothing$.

We may now proceed in describing the S_n-structure of the multilinear spaces $\Gamma_n^D(U_m)$. Since we are going to work modulo $T_D(U_m)$, we will simply write f instead of $f + T_D(U_m)$ and so on, in order to keep the notation as simple as possible.

As a byproduct of the preceding proof, $\Gamma_n^D(U_m)$ is finite-dimensional, since it has $\mathscr{S}_n$ as an F-basis. Moreover, setting $\Gamma_n^\alpha(U_m) = F\mathscr{S}_n^\alpha$ for $\alpha \in \{1, \delta, \varepsilon\}$, each $\Gamma_n^\alpha(U_m)$ is an S_n-submodule of $\Gamma_n^D(U_m)$, so we get the decomposition $\Gamma_n^D(U_m) = \Gamma_n^1(U_m) \oplus \Gamma_n^\delta(U_m) \oplus \Gamma_n^\varepsilon(U_m)$ and then consider the three summands separately in order to get the S_n-proper cocharacter of U_m.

Recall that the isomorphism classes of irreducible S_n-modules are in a bijective correspondence with the integer partitions λ of n (which we express by $\lambda \vdash n$). If $\lambda = [\![\lambda_1, \dots, \lambda_k]\!] \vdash n$, the corresponding irreducible character will be denoted by λ as well, thus committing a slight abuse of notation. The character of $\Gamma_n^\alpha(U_m)$ will be denoted $\xi_n^\alpha(U_m)$.

It is easy to see that $\Gamma_0^D(U_m) = F$ and $\Gamma_1^D(U_m) = Fx_1^\delta \oplus Fx_1^\varepsilon$. For $n \geqslant 2$, just $\xi_n^\varepsilon(U_m)$ needs to be investigated. Indeed,

- $\xi_n^\delta(U_m) = [\![n]\!]$ is clear,
- $\xi_n^1(U_m)$ is the proper S_n-cocharacter of the algebra $UT_m(F)$ by [15].

In order to study $\xi_n^\varepsilon(U_m)$, let us denote $(l_1, \dots, l_k) \vDash n$ any weak k-composition of n, that is any sequence of integers $l_1, \dots, l_k \geqslant 0$ such that $l_1 + \cdots + l_k = n$.

Theorem 14. *$\xi_n^\varepsilon(U_2) = [\![n]\!]$ and, if $m \geqslant 3$,*

$$\xi_n^\varepsilon(U_m) = \xi_n^\varepsilon(U_{m-1}) + \sum_{\substack{(\lambda_1,\dots,\lambda_{m-2})\models n\\ \lambda_1,\dots,\lambda_{m-2}\geqslant 2}} \Big([\![\lambda_1 - 1, 1]\!] \otimes \cdots \otimes [\![\lambda_{m-2} - 1, 1]\!]\Big)^{S_n}$$

$$+ \sum_{\substack{(\lambda_1,\dots,\lambda_{m-1})\models n\\ \lambda_1,\dots,\lambda_{m-2}\geqslant 2\\ \lambda_{m-1}\geqslant 1}} \Big([\![\lambda_1 - 1, 1]\!] \otimes \cdots \otimes [\![\lambda_{m-2} - 1, 1]\!] \otimes [\![\lambda_{m-1}]\!]\Big)^{S_n}$$

There is a certain amount of indetermination in both $\xi_n^1(U_m)$ and in $\xi_n^\varepsilon(U_m)$, due to the induced characters involved in their description. The Littlewood-Richardson rule would turn them into a sum of irreducible S_n-characters, but this explicit decomposition, even if possible in principle, could hardly be accepted as a better one. By the way, at least in the small cases $m = 2$ and $m = 3$, they are worth of being computed, to get at least an idea of the general case.

Recall that $\Gamma_0^D(U_m) = F$ and $\Gamma_1^D(U_m) = Fx_1^\delta \oplus Fx_1^\varepsilon$ for all $m \geqslant 2$. Then

Corollary 15. *For any $n \geqslant 2$ it holds $\xi_n^D(U_2) = [\![n-1, 1]\!] + 2[\![n]\!]$. In particular, for all $n \in \mathbb{N}$, it holds $\gamma_n^D(U_2) = n + 1$.*

The differential cocharacter sequence $\chi_n^D(U_2)$ and the differential codimension sequence $c_n^D(U_2)$ follow easily and, of course, coincide with the results in [8]

Corollary 16. *For any $n \geqslant 1$ it holds $\chi_n^D(U_2) = \sum_{\lambda\vdash n} m_\lambda \lambda$, where*

- *$\lambda = [\![n]\!]$ has multiplicity $2n + 1$;*
- *$\lambda = [\![a + b, a]\!]$, with $a > 0$, has multiplicity $3(b + 1)$;*
- *$\lambda = [\![a + b + 1, a + 1, 1]\!]$ has multiplicity $b + 1$.*

In particular, for all $n \in \mathbb{N}$ it holds $c_n^D(U_2) = 2^{n-1}(n + 2)$.

It is interesting to notice that the effective contribution of $\xi_n^\delta(U_2)$ and $\xi_n^\varepsilon(U_2)$ to $\xi_n^D(U_2)$ is limited to the trivial S_n-character $[\![n]\!]$. This however is far from being the general situation, as evidence shows already for U_3:

Corollary 17. *The proper differential cocharacter sequence of U_3 is the following:*

- *$\xi_2^D(U_3) = 2[\![1, 1]\!] \oplus 2[\![2]\!]$,*
- *$\xi_3^D(U_3) = 2[\![3]\!] \oplus 3[\![2, 1]\!] \oplus [\![1, 1, 1]\!]$*

and, for $n \geqslant 4$, $\xi_n^D(U_3) = \sum_{\lambda\vdash n} m_\lambda\lambda$ with multiplicities m_λ determined according to the table

	$\xi_n^1(U_3)$	$\xi_n^\delta(U_3)$	$\xi_n^\varepsilon(U_3)$	$\xi_n^D(U_3)$
$[\![n]\!]$		1	1	2
$[\![n-1, 1]\!]$	1		n	$n+1$
$[\![a+b, a]\!]$ (*if* $a \geqslant 2$)	$b+1$		$b+1$	$2(b+1)$
$[\![n-2, 1, 1]\!]$	$n-3$		$n-2$	$2n-5$
$[\![1+a+b, 1+a, 1]\!]$ (*if* $a \geqslant 1$)	$2(b+1)$		$b+1$	$3(b+1)$
$[\![2+a+b, 2+a, 2]\!]$	$b+1$			$b+1$
$[\![1+a+b, 1+a, 1, 1]\!]$	$b+1$			$b+1$

In particular, $\gamma_0^D(U_3) = 1$, $\gamma_1^D(U_3) = 2$, $\gamma_2^D(U_3) = 4$, $\gamma_3^D(U_3) = 9$ and, for $n \geqslant 4$, $\gamma_n^D(U_3) = n(n-3)2^{n-2} + 3n$.

It is evident that, even in this small case, the main contribution to $\xi_n^D(U_3)$ comes from the ordinary proper cocharacter $\xi_n(UT_3(F))$ but the contribution due to $\xi_n^\varepsilon(U_3)$ is very close to it, while $\xi_n^\delta(U_3) = [\![n]\!]$.

The explicit decomposition of the n-th differential cocharacter of U_3 would already result in an awkward list of partitions and multiplicities, so it is hard to conceive it as a better description of $T_D(U_3)$ than the one provided through proper characters. It is however interesting to compute the differential codimension sequence:

Corollary 18. *It holds $c_0^D(U_3) = 1$ and, for $n \geqslant 1$,*

$$c_n^D(U_3) = n(n-4)3^{n-2} + 3n2^{n-1} + 1.$$

4. The separate actions of δ and ε on $UT_m(F)$

We are going to consider the action of the single derivations δ and ε on the identities of $UT_m(F)$. Denote U_m^δ and U_m^ε these two structures, respectively. The considerations made for U_m may be replied in each case, and we want to determine and describe the differential identities of U_m^δ and U_m^ε. It is now easy to get the following results

Theorem 19. *$T_D(U_m^\delta)$ is generated by the following differential polynomials:*

- x^{δ^2}
- $[x_1, y_1] \dots [x_m, y_m]$
- $x^\delta[x_1, x_2], [x_1, x_2]x^\delta, [x_1, x_2]^\delta$

where all indeterminates belong to X.

Theorem 20. *$T_D(U_m^\varepsilon)$ is generated by the following differential polynomials*

- $x^{\varepsilon^2} - x^\varepsilon$
- $[x_1, y_1] \dots [x_m, y_m]$
- $x^\varepsilon[x_1, x_2]$, $[x_1, y_1] \dots [x_{m-1}, y_{m-1}]x^\varepsilon$
- $[x_1, y_1] \dots [x_{m-2}, y_{m-2}]\Big([x, y]^\varepsilon - [x, y]\Big)$

where all indeterminates belong to X.

Also the proper cocharacters and codimensions follow easily; actually, they can be read off from the proper cocharacter decomposition of U_m, and summarized in

- $\xi_n^D(U_m^\delta) = \xi_n(UT_m(F)) + [\![n]\!]$ for all $n \geqslant 1$, so a bit more than the usual proper cocharacter of $UT_m(F)$;
- $\xi_n^D(U_m^\varepsilon) = \xi_n^D(U_m) - [\![n]\!]$ for all $n \geqslant 1$, so a bit less than the other extreme, the differential proper cocharacter of U_m.

This is hardly surprising. Informally speaking, in fact, the two chosen derivations have extreme, opposite, features: δ is a nilpotent transformation of class 2 while ε is an idempotent transformation. These differences are concealed by the case $m = 2$ (the algebra is too small), but emerge already in case $m = 3$.

We record here the codimension sequences for these small cases:

Corollary 21. *The proper codimension sequence and the codimension sequence of U_2^δ and U_2^ε are the following:*

- $\gamma_n^D(U_2^\delta) = n = \gamma_n^D(U_2^\varepsilon)$ *(for $n \geqslant 1$),*

- $c_n^D(U_2^\delta) = n2^{n-1} + 1 = c_n^D(U_2^\varepsilon)$ *(for* $n \in \mathbb{N}$*).*

The proper codimension sequence and the codimension sequence of U_3^δ *and* U_3^ε *are the following:*

- $\gamma_n^D(U_3^\delta) = 2^{n-2}(n-1)(n-4) + 2n - 1$ *(for* $n \geqslant 4$*),*
- $\gamma_n^D(U_3^\varepsilon) = 2^{n-2}n(n-3) + 3n - 1$ *(for* $n \geqslant 4$*),*
- $c_n^D(U_3^\delta) = 3^{n-2}(n^2 - 7n + 9) + 2^n(n-1) + \frac{1}{6}(2n^3 - 3n^2 + n + 6)$ *(for* $n \geqslant 1$*),*
- $c_n^D(U_3^\varepsilon) = 3^{n-2}n(n-4) + 2^{n-1}(3n-2) + 2$ *(for* $n \geqslant 1$*).*

In particular we get back the sequences $\chi_n^D(U_2^\varepsilon)$ and $c_n^D(U_2^\varepsilon)$ computed in [8].

5. Behind the scenes

It is high time we gave a more precise and sound grounding to the notions employed so far. Let us start by recalling that the set $Der(A)$ of all derivations on A is a Lie algebra laying inside $End_F(A)$. If L is any Lie algebra, we say that L acts on A by derivation if A is a Lie L-module. It amounts to say that there is a Lie-homomorphism from L to $Der(A)$. By a fundamental property of the universal enveloping algebra $U(L)$ of the Lie algebra L, this is equivalent to say that A is turned into an $U(L)$-module (in our settings, a *right* $U(L)$-module). We summarize these facts in the following

Definition 22. Let L be a Lie algebra over F and let A be an associative F-algebra. We say that A is an *L-algebra*, or that *L acts on A by derivations*, if A is a $U(L)$-module.

One can define a universal object in the class of L-algebras: start by a countable set of indeterminates X, and consider the F-vector space $V := FX \otimes U(L)$. Then the tensor algebra of V, denoted $F\langle X \mid L\rangle$, is an associative, unitary F-algebra, spanned by the (tensor) products of the simple tensors $x \otimes w$ for $x \in X$ and $w \in U(L)$. The regular right action of $U(L)$ on the simple tensors $x \otimes w$, defined by $(x \otimes w) \bullet u := x \otimes wu$, turns $F\langle X \mid L\rangle$ into a right $U(L)$-module, therefore induces a derivation action of L on the tensor algebra and turns it into an L-algebra. Moreover, if A is any L-algebra, any map $\varphi_0 : X \to A$ uniquely defines an algebra homomorphism from $F\langle X \mid L\rangle$ commuting with the derivation action of L (which we call an L-homomorphism), due to the general properties of the tensor algebra of a vector space and to the defining right action of $U(L)$ on it. It is therefore natural to define $T_L(A)$ as the intersection of all the kernels of L-homomorphisms from $F\langle X \mid L\rangle$ to A.

In order to keep the notation under control, it is a good idea to write x^u to denote the simple tensor $x \otimes u$ for $x \in X$ and $u \in U(L)$; if $u = 1$ (the unit element of $U(L)$) one identifies $x \otimes 1$ with x. Hence, for the "critical" (for one's own understanding) case of $V^{\otimes 2}$, the spanning tensors $(x \otimes u) \otimes (y \otimes v)$, with $x, y \in X$ and $u, v \in U(L)$, can be written in the simpler form $x^u y^v$; moreover, the action of L (which is canonically embedded in $U(L)$ by the Poincaré-Birkhoff-Witt Theorem) can be written in the usual form $(x^u y^v)^a = x^{ua}y^v + x^u y^{va}$, that is the Leibniz rule.

Example 23. Let $L = Fa$ be the one-dimensional Lie algebra, spanned by the basis element a. Then its universal enveloping algebra is the (infinite dimensional) polynomial algebra $F[a]$, and $F\langle X \mid L\rangle$ is the noncommutative associative unitary F-algebra generated by the (countable) set $\{x^{a^i} \mid x \in X,\, i \in \mathbb{N}\}$. Each indeterminate x^{a^i} is just a simpler writing for the simple tensor $x \otimes a^i \in FX \otimes U(L)$.

So, when we considered U_m^δ and U_m^ε, what we really did was to choose a derivation $\alpha \in Der(UT_m(F))$, and fix a Lie homomorphism $\varphi : L = F\alpha \to End_F(UT_m(F))$. This uniquely defines an algebra homomorphism $\varphi^* : U(L) \to End_F(UT_m(F))$, turning $UT_m(F)$ into a right $U(L)$-module. Moreover, the algebra we intuitively produced as $F\langle X^D\rangle$ is nothing more than the free algebra $F\langle X \mid L\rangle$, with the once formal letters x^w now becoming the generators $x \otimes w$ of the tensor algebra $F\langle X \mid L\rangle$.

A natural question, at this point, arises: it does not matter if $\alpha = \delta$ or $\alpha = \varepsilon$, because the Lie algebras $F\delta$ and $F\varepsilon$ are isomorphic, being one dimensional. Therefore in both cases $U(L)$ is the same algebra, up to isomorphisms. What makes the differences among them? It is, of course, the kernel of the action: $U(L)$ acts on $UT_m(F)$ in both cases, but the in case $\alpha = \delta$ the kernel is the two-sided ideal generated by the generator α^2, in the other case it is the ideal generated by $\alpha^2 - \alpha$. These relations affect the differential polynomial identities of U_m^δ and U_m^ε, and correspond precisely to the identities x^{δ^2} and $x^{\varepsilon^2} - x^\varepsilon$, respectively.

When we considered the coupled action of δ and ε, that is U_m, a similar process was in action: this time, the Lie algebra L acting on $UT_m(F)$ is a two-dimensional Lie algebra and, since δ and ε do not commute, it must be the two-dimensional non commutative Lie algebra (sometimes named the two-dimensional *metabelian* Lie algebra). It is well known that one can choose a linear basis $\{a, b\}$ in L such that $[ab] = a$ (this is the true Lie product in L, so we are writing it without the separating comma), and the obvious map carrying a and b in δ and ε rispectively provides a faithful representation of L. Once again, this uniquely defines an algebra homomorphism from $U(L)$ to $End_F(UT_m(F))$, thus turning $UT_m(F)$ into a right $U(L)$-module, whose kernel is the two-sided ideal generated by the elements $a^2, b^2 - b, ab - a, ba$, from which the differential identities x^{δ^2}, $x^{\varepsilon^2} - x^\varepsilon$, $x^{\delta\varepsilon} - x^\delta$ and $x^{\varepsilon\delta}$, respectively, arise.

This also explains why the differential identities of U_m^δ and U_m^ε resemble so closely the ones of U_m involving separately the δ- and ε-letters: any map $\varphi : L \to End_F(UT_m(F))$ such that $[\varphi(a), \varphi(b)] = \varphi(a)$ uniquely defines a Lie homomorphism of $L = Fa \oplus Fb$ in $End_F(UT_m)$. The map φ involved in forming U_m of course does the job, but the same do the maps φ_δ sending $a \to 0$ and $b \to \delta$ and φ_ε carrying a in 0 and b in ε. In this case, the differences with U_m^δ and U_m^ε are little more than formal, and depend on adding the generator x^a to the kernel of the $U(L)$ action (that is, respectively, to add the differential identity x^ε or x^δ). Of course, choosing φ as the zero map still yields a Lie homomorphisms. In this case L acts trivially on $UT_m(F)$, and the ideal of differential identities coincides with the usual T-ideal of $UT_m(F)$ (that is: formally x^ε and x^δ are among the differential identities).

Another natural question is the following: how tightly the differential polynomial identities depend upon the Lie algebra L? The answer is: very weakly. An easy example has been already provided by the one-dimensional algebra $L = Fa$. Indeed, U_m^δ and U_m^ε have very different differential identities. One can think that this is due just to the different types of δ and ε: a nilpotent versus an idempotent transformation. This is true, but it is not the only reason. Let us consider the following example: let $\eta := [\cdot, -e_{11}] = [e_{11}, \cdot]$ be the inner derivation induced on $UT_m(F)$ by the matrix $-e_{11}$, and let θ be the map defined by $\theta(a) = \eta$. This of course defines a Lie homomorphism from the one-dimensional Lie algebra $L = Fa$ in $End_F(UT_m(F))$. Since $\eta^2 = \eta$, the $U(L)$ action on $UT_m(F)$ is exactly the same as the one we got assigning $a \to \varepsilon$ (both the kernels are generated by $a^2 - a$). By the way, the differential polynomial identities satisfied by U_m^η differ from the ones of U_m^ε. For instance, the basic identity $x^a[x_1, y_1]$ in the latter (where the differential letter x^a means x^ε) is no longer holding in the former, where it is replaced by $[x_1, y_1]x^a$ (so the differential letter changes side). Thus, the only direct part played by L in determining the differential polynomial identities of $UT_m(F)$ is limited to the identities arising from the kernel of the $U(L)$-action, but the relations among the selected derivations and the algebra structure play a decisive role in determining the actual differential identities of the algebra.

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