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Blow – up result for a semilinear wave equation with a nonlinear memory term

Wenhui Chen and Alessandro Palmieri

Abstract In this note, we study the blow – up dynamic of a semilinear Cauchy problem for the wave equation with a nonlinear memory term. More precisely, we consider as memory term the Riemann – Liouville fractional integral of order $1-\gamma$ of the p power of the solution, where $\gamma \in (0,1)$. We prove two blow – up results by using an iteration argument. In the subcritical case we show the blow – up in finite time of the space average of a local in time solution, under certain integral sign assumptions for the initial data. In the result for the limit case, we refine this approach by considering a weighted average of a local solution instead and applying the so – called slicing method.

Keywords Semilinear wave equation, nonlinear memory term, Riemann – Liouville fractional integral, generalized Strauss exponent, blow – up, iteration argument

AMS Classification (2010) Primary: 35B44, 35L05, 35L71 ; Secondary: 26A33 , 35B33

1 Introduction

In this paper, we investigate the blow – up dynamic for local in time solutions to the semilinear wave equation with the Riemann – Liouville fractional integral of order $1-\gamma$ of the p power of the solution as nonlinear term

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$$
\begin{cases} u_{tt} - \Delta u = N_{\gamma,p}(u) & x \in \mathbb{R}^n, \ t \in (0,T), \\ u(0,x) = \varepsilon u_0(x) & x \in \mathbb{R}^n, \\ u_t(0,x) = \varepsilon u_1(x) & x \in \mathbb{R}^n, \end{cases} \tag{1}
$$

where

$$
N_{\gamma,p}(u)(t,x) \doteq c_\gamma \int_0^t (t-s)^{-\gamma} |u(s,x)|^p \, \mathrm{d}s, \qquad c_\gamma \doteq 1/\Gamma(1-\gamma), \tag{2}
$$

and $p > 1$, $\gamma \in (0,1)$, $\varepsilon > 0$ is a parameter describing the size of initial data and Γ denotes the Euler integral of the second kind.

For the sake of brevity we shall refer hereafter to the nonlinearity $N_{\gamma,p}(u)$ in (2) as nonlinear memory term and, vice versa, whenever we mention in what follows a nonlinear memory term we mean the nonlinearity in (2).

Over the last decade several papers have been devoted to the study of semilinear evolution model with the nonlinear term of memory type as in (2). In the pioneering paper [4] the authors determine the critical exponent for the semilinear heat equation with nonlinear memory term. Afterwards, this kind of result has been generalized for fractional (either in space or in time) heat equations [21, 13, 39] and for weakly coupled system of heat equations [12, 26, 37].

Another evolution equation, which has already been studied with nonlinear memory term on the right – hand side, is the classical damped wave equation (cf. [10, 2, 7, 3]). Moreover, we recall that the structural damped wave equation and the beam equation have been investigated in the case of a nonlinear memory term in [6] and [8], respectively.

Finally, we mention that the semilinear wave equation with nonlinar memory term has been considered in the case of bounded domains in [11] and in the case of initial – boundary value problem (and in space dimension 1) in [22]. So far, up to the knowledge of the authors, no satisfactory result has been obtain for the semilinear wave equation with nonlinear memory term in the whole space. For this reason, we shall determine two blow – up results for the Cauchy problem (1).

By a slight abuse of terminology, we shall refer to the two different cases in which we are able to prove the blow – up of the solution as to the subcritical case and to the critical case, respectively.

Recalling that

$$
\lim_{\gamma \to 1^-} c_\gamma s_+^{-\gamma} = \delta_0(s) \quad \text{in the sense of distributions, where } s_+^{-\gamma} \doteq \begin{cases} s^{-\gamma} & \text{if } s > 0, \\ 0 & \text{if } s < 0, \end{cases}
$$

it would be suitable to find in the blow – up results an upper bound $p_0(n,\gamma)$ for the exponent p in (2) that satisfies formally

$$
\lim_{\gamma \to 1^-} p_0(n,\gamma) = p_{\mathrm{Str}}(n), \tag{3}
$$

where $p_{\mathrm{Str}}(n)$ denotes the Strauss exponent, i.e. the critical exponent for the semilinear wave equation with power nonlinearity $|u|^p$, whose analytic expression can be derived from the quadratic equation $\frac{n-1}{2}p^2 - \frac{n+1}{2}p - 1 = 0$ for $n \geqslant 2$ (in the one spatial dimensional case, we put $p_{\mathrm{Str}}(1) = \infty$). For the formulation and proof of Strauss' conjecture on the critical exponent for the semilinear wave equation with power nonlinearity we refer to classical works [19, 20, 32, 31, 30, 15, 16, 43, 25, 14, 35, 18, 38, 44] (moreover, for the sharp lifespan estimates in the subcritical and critical case we quote [24, 40, 41, 42, 9, 34, 23, 45, 33, 17]).

Let us introduce the following quadratic equation:

$$\frac{n-1}{2}p^2 - \left(\frac{n+1}{2} + 1 - \gamma\right)p - 1 = 0, \tag{4}$$

where $\gamma \in (0,1)$ and $p > 1$. Then, for any $n \geqslant 2$ we denote by $p_0(n,\gamma)$ the positive root of the above equation, that is,

$$p_0(n,\gamma) \doteq \frac{n + 3 - 2\gamma + \sqrt{n^2 + (14 - 4\gamma)n + 4\gamma(\gamma - 3) + 1}}{2(n-1)}.$$

Moreover, for $n = 1$ we set formally $p_0(1,\gamma) = \infty$ for any $\gamma \in (0,1)$. This exponent $p_0(n,\gamma)$ is the upper bound for p, below which we shall prove the blow – up results. Let us point out explicitly that according to this choice of $p_0(n,\gamma)$, the formal limit relation (3) is always fulfilled.

Therefore, goal of this paper is to show the blow – up in finite time of local in time solutions to (1) in the case $1 < p \leqslant p_0(n,\gamma)$, provided that the initial data satisfy certain integral sign assumptions and regardless of the size of the Cauchy data. Our approach is a quite standard one; in fact, we will study the blow – up dynamic of the spatial average of a local in time solution by determining a sequence of lower bound estimates for this time – dependent functional via an iteration procedure. Let us stress that in the critical case (that is, for $p = p_0(n,\gamma)$ and $n \geqslant 2$), this standard approach with the spatial average is no longer successful and it has to be refined by working with a weighted space average instead. More specifically, we shall employ the approach recently introduced in [36]. As byproducts of the iteration arguments we will obtain upper bound estimates for the lifespan of the solution.

1.1 Main results

Before stating the main results, we introduce the notion of energy solutions to the Cauchy problem (1) that we are going to use in our results.

Definition 1 Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We say that

$$u \in \mathscr{C}\big([0,T), H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0,T), L^2(\mathbb{R}^n)\big) \text{ such that } N_{\gamma,p}(u) \in L^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)$$

is an energy solution of (1) on $[0,T)$ if u fulfills $u(0,\cdot) = \varepsilon u_0$ in $H^1(\mathbb{R}^n)$ and the integral relation

$$\begin{aligned}
&\int_{\mathbb{R}^n} \partial_t u(t,x)\,\psi(t,x)\,\mathrm{d}x - \varepsilon \int_{\mathbb{R}^n} u_1(x)\,\psi(0,x)\,\mathrm{d}x \\
&\quad + \int_0^t \int_{\mathbb{R}^n} \left(\nabla u(s,x)\cdot\nabla\psi(s,x) - \partial_t u(s,x)\,\psi_s(s,x)\right)\mathrm{d}x\,\mathrm{d}s \\
&= c_\gamma \int_0^t \int_{\mathbb{R}^n} \psi(s,x) \int_0^s (s-\tau)^{-\gamma} |u(\tau,x)|^p\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{5}$$

for any $\psi \in \mathscr{C}_0^\infty([0,T)\times\mathbb{R}^n)$ and any $t \in [0,T)$.

After a further step of integration by parts in (5), one has

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\psi(t,x)\,\partial_t u(t,x) - \psi_s(t,x)\,u(t,x)\right)\mathrm{d}x - \varepsilon \int_{\mathbb{R}^n} \left(\psi(0,x)\,u_1(x) - \psi_s(0,x)\,u_0(x)\right)\mathrm{d}x \\
&\quad + \int_0^t \int_{\mathbb{R}^n} \left(\psi_{ss}(s,x) - \Delta\psi(s,x)\right) u(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&= c_\gamma \int_0^t \int_{\mathbb{R}^n} \psi(s,x) \int_0^s (s-\tau)^{-\gamma} |u(\tau,x)|^p\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{6}$$

for any $\psi \in \mathscr{C}_0^\infty([0,T)\times\mathbb{R}^n)$ and any $t \in [0,T)$.

Let us state now our first result in the subcritical case.

Theorem 1 *Let us consider* $p > 1$ *such that*

$$\begin{cases} p < \infty & \text{if } n = 1, \\ p < p_0(n,\gamma) & \text{if } n \geqslant 2. \end{cases}$$

Let $u_0 \in H^1(\mathbb{R}^n)$ *and* $u_1 \in L^2(\mathbb{R}^n)$ *be nonnegative and compactly supported functions with supports contained in* B_R *for some* $R > 0$ *such that* u_0 *is not identically zero. Let*

$$u \in \mathscr{C}\big([0,T), H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0,T), L^2(\mathbb{R}^n)\big) \quad \text{such that } N_{\gamma,p}(u) \in L^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)$$

be an energy solution on $[0,T)$ *to* (1) *according to Definition 1 with lifespan* $T = T(\varepsilon)$ *such that*

$$\operatorname{supp} u(t,\cdot) \subset B_{R+t} \quad \text{for any } t \in (0,T). \tag{7}$$

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0,u_1,n,p,\gamma,R)$ *such that for any* $\varepsilon \in (0,\varepsilon_0]$ *the energy solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan*

$$T(\varepsilon) \leqslant C\varepsilon^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}}$$

holds, where C is a positive constant independent of ε *and*

$$\Upsilon(p,n,\gamma) \doteq 2 + (n+1+2(1-\gamma))p - (n-1)p^2. \tag{8}$$

In the next result, we examine the critical case.

Theorem 2 *Let $n \geqslant 2$ and $p = p_0(n,\gamma)$. Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ be nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$. Let*

$$u \in \mathscr{C}\big([0,T), H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0,T), L^2(\mathbb{R}^n)\big) \quad \text{such that } N_{\gamma,p}(u) \in L^1_{\text{loc}}([0,T)\times\mathbb{R}^n)$$

be an energy solution on $[0,T)$ to (1) according to Definition 1 with lifespan $T = T(\varepsilon)$ and satisfying (7). Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0,u_1,n,p,\gamma,R)$ such that for any $\varepsilon \in (0,\varepsilon_0]$ the energy solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan

$$T(\varepsilon) \leqslant \exp\Big(C\varepsilon^{-p(p-1)}\Big)$$

holds, where C is a positive constant independent of ε.

Notation

We give some notations to be used in this paper. We write $f \lesssim g$ when there exists a positive constant C such that $f \leqslant Cg$. We denote $g \lesssim f \lesssim g$ by $f \approx g$. Moreover, B_R denotes the ball around the origin with radius R in $\mathbb{R}^n$.

2 Subcritical case: Proof of Theorem 1

Let us introduce the time – dependent functional

$$U(t) \doteq \int_{\mathbb{R}^n} u(t,x)\,\mathrm{d}x.$$

We can choose ψ such that $\psi = 1$ over $\{(s,x) \in [0,t]\times\mathbb{R}^n : |x| \leqslant R+s\}$. Then, using this test function in (5), it results

$$\int_{\mathbb{R}^n} u_t(t,x)\,\mathrm{d}x - \varepsilon\int_{\mathbb{R}^n} u_1(x)\,\mathrm{d}x = c_\gamma \int_0^t\int_{\mathbb{R}^n}\int_0^s (s-\tau)^{-\gamma}|u(\tau,x)|^p\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}s,$$

that is,

$$U'(t) = U'(0) + c_\gamma \int_0^t\int_{\mathbb{R}^n}\int_0^s (s-\tau)^{-\gamma}|u(\tau,x)|^p\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}s. \tag{9}$$

Hence, integrating the above relation over $[0,t]$, we get

$$\begin{aligned}U(t) &= U(0) + U'(0)\,t + c_\gamma \int_0^t \int_0^s \int_{\mathbb{R}^n} \int_0^\tau (\tau-\sigma)^{-\gamma} |u(\sigma,x)|^p \,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s \\ &\geqslant c_\gamma \int_0^t \int_0^s \int_0^\tau (\tau-\sigma)^{-\gamma} \int_{\mathbb{R}^n} |u(\sigma,x)|^p \,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}s \geqslant 0,\end{aligned}$$

where the nonnegativity of u_0 and u_1 is applied.
The use of Hölder's inequality, as well as (7), implies

$$\int_{\mathbb{R}^n} |u(\sigma,x)|^p \,\mathrm{d}x \geqslant C(R+\sigma)^{-n(p-1)}(U(\sigma))^p,$$

which leads to

$$U(t) \geqslant Cc_\gamma \int_0^t \int_0^s \int_0^\tau (\tau-\sigma)^{-\gamma}(R+\sigma)^{-n(p-1)}(U(\sigma))^p \,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}s. \tag{10}$$

Our proof of Theorem 1 is based on an iteration procedure which provides us a sequence of lower bounds for the functional U. This sequence of lower bounds will be determined iteratively by applying the iteration frame (10).

With the aim of deriving a first lower bound estimate for functional $U(t)$, we follow [38] and we introduce the function

$$\Phi(x) \doteq \begin{cases} \mathrm{e}^x + \mathrm{e}^{-x} & \text{if } n=1, \\ \int_{\mathbb{S}^{n-1}} \mathrm{e}^{x\cdot\omega} \mathrm{d}\sigma_\omega & \text{if } n \geqslant 2. \end{cases} \tag{11}$$

The function Φ is a positive smooth function and satisfies the remarkable properties

$$\begin{aligned} &\Delta\Phi = \Phi, \\ &\Phi(x) \sim |x|^{-\frac{n-1}{2}} \mathrm{e}^x \qquad \text{as } |x| \to \infty. \end{aligned}$$

If we introduce the function with separate variables $\Psi = \Psi(t,x) = \mathrm{e}^{-t}\Phi(x)$, clearly, the function Ψ is a solution to the wave equation $\Psi_{tt} - \Delta\Psi = 0$.

Furthermore, we introduce the auxiliary functional

$$U_0(t) \doteq \int_{\mathbb{R}^n} u(t,x)\Psi(t,x)\,\mathrm{d}x.$$

Differentiating with respect to t the equation (9), we obtain

$$U''(t) = c_\gamma \int_0^t (t-s)^{-\gamma} \int_{\mathbb{R}^n} |u(s,x)|^p \,\mathrm{d}x\,\mathrm{d}s.$$

Therefore, by applying Hölder's inequality to $U_0(s)$, one finds

$$\int_{\mathbb{R}^n} |u(s,x)|^p \,\mathrm{d}x \geqslant |U_0(s)|^p \left(\int_{B_{R+s}} |\Psi(s,x)|^{\frac{p}{p-1}} \,\mathrm{d}x \right)^{-(p-1)}. \tag{12}$$

So, if we determine a lower bound estimate for $U_0(s)$, then, the previous inequality provides a lower bound for $\int_{\mathbb{R}^n} |u(s,x)|^p \, \mathrm{d}x$ in turn.

According to [38] the time – dependent functional U_0 satisfies

$$U_0(t) \geqslant \tfrac{\varepsilon}{2}(1-\mathrm{e}^{-2t}) \int_{\mathbb{R}^n} (u_0(x)+u_1(x))\, \Phi(x) \, \mathrm{d}x + \varepsilon \, \mathrm{e}^{-2t} \int_{\mathbb{R}^n} u_0(x) \Phi(x) \, \mathrm{d}x \geqslant \widetilde{C}\, \varepsilon$$

for any $t \geqslant 0$ with a suitable constant $\widetilde{C} > 0$ depending on u_0 and u_1, where we applied our assumption that u_0 is nonnegative and not identically 0.

Indeed, in [38, Lemma 2.2] the only condition on the nonlinearity that is actually used is the nonnegativity, which holds trivially also for our nonlinear memory term. For a detailed proof of the lower bound estimate for U_0 see also [27, Lemma 4.3.6], for example.

Additionally, by the asymptotic behavior of Ψ, it is known that the inequality

$$\int_{B_{R+s}} |\Psi(s,x)|^{\frac{p}{p-1}} \, \mathrm{d}x \leqslant \widetilde{K}(R+s)^{(n-1)(1-p'/2)}$$

holds for some positive constant $\widetilde{K} > 0$, cf. [38, Estimate (2.5)]. So, from (12) we have

$$\int_{\mathbb{R}^n} |u(s,x)|^p \, \mathrm{d}x \geqslant C_0 \varepsilon^p (R+s)^{n-1-\frac{n-1}{2}p} \qquad \text{for any } s \geqslant 0, \tag{13}$$

where $C_0 = \widetilde{C}^p \widetilde{K}^{1-p}$, and, consequently,

$$\begin{aligned} U''(t) &\geqslant C_0 c_\gamma \varepsilon^p \int_0^t (t-s)^{-\gamma} (R+s)^{(n-1)(1-p/2)} \, \mathrm{d}s \\ &\geqslant C_0 c_\gamma \varepsilon^p \int_0^t (t-s)^{-\gamma} (R+s)^{-\frac{n-1}{2}p} s^{n-1} \, \mathrm{d}s \\ &\geqslant \frac{C_0 c_\gamma \varepsilon^p}{n} (R+t)^{-\frac{n-1}{2}p} t^{n-\gamma} \end{aligned}$$

for any $t \geqslant 0$. By integrating the above inequality twice, we get for U the lower bound estimate

$$\begin{aligned} U(t) &\geqslant U(0) + U'(0)\, t + \frac{C_0 c_\gamma \varepsilon^p}{n} \int_0^t \int_0^s (R+\tau)^{-\frac{n-1}{2}p} \tau^{n-\gamma} \, \mathrm{d}\tau \, \mathrm{d}s \\ &\geqslant \frac{C_0 c_\gamma \varepsilon^p}{n(n-\gamma+1)(n-\gamma+2)} (R+t)^{-\frac{n-1}{2}p} t^{n+2-\gamma} \end{aligned}$$

for any $t \geqslant 0$. In other words, we have

$$U(t) \geqslant K_0 (R+t)^{-\alpha_0} t^{\beta_0} \quad \text{for any } t \geqslant 0, \tag{14}$$

where the multiplicative constant is defined by

$$K_0 \doteq \frac{C_0 c_\gamma \varepsilon^p}{n(n-\gamma+1)(n-\gamma+2)}$$

and the exponents are

$$\alpha_0 \doteq \frac{n-1}{2}p \quad \text{and} \quad \beta_0 \doteq n+2-\gamma.$$

In the next step, we will derive a sequence of lower bounds of U by using the iteration frame (10). To be specific, we will show that

$$U(t) \geqslant K_j (R+t)^{-\alpha_j} t^{\beta_j} \quad \text{for any } t \geqslant 0, \tag{15}$$

where $\{K_j\}_{j\in\mathbb{N}}$, $\{\alpha_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative real numbers that will be specified later.

Obviously, we already proved (15) for $j=0$. Therefore, in order to prove (15) for all $j \in \mathbb{N}$ by using an inductive argument, it remains to show the induction step.

Plugging (15) in the iteration frame (10), we derive

$$\begin{aligned} U(t) &\geqslant C c_\gamma K_j^p \int_0^t \int_0^s \int_0^\tau (\tau-\sigma)^{-\gamma} (R+\sigma)^{-n(p-1)-p\alpha_j} \sigma^{p\beta_j} \, \mathrm{d}\sigma \, \mathrm{d}\tau \, \mathrm{d}s \\ &\geqslant C c_\gamma K_j^p (R+t)^{-n(p-1)-p\alpha_j} t^{-\gamma} \int_0^t \int_0^s \int_0^\tau \sigma^{p\beta_j} \, \mathrm{d}\sigma \, \mathrm{d}\tau \, \mathrm{d}s \\ &\geqslant \frac{C c_\gamma K_j^p}{(p\beta_j+1)(p\beta_j+2)(p\beta_j+3)} (R+t)^{-n(p-1)-p\alpha_j} t^{p\beta_j+3-\gamma} \end{aligned}$$

for all $t \geqslant 0$. Thus, we showed (15) for $j+1$, provided that the recursive relations

$$K_{j+1} \doteq \frac{C c_\gamma K_j^p}{(p\beta_j+1)(p\beta_j+2)(p\beta_j+3)}, \ \alpha_{j+1} \doteq n(p-1)+p\alpha_j, \ \beta_{j+1} \doteq p\beta_j + 3 - \gamma$$

are satisfied.

For what follows it is useful to determine a suitable estimate from below of K_j. For this purpose, we have to determine first the explicit representation for α_j and β_j. From the relation $\alpha_j = n(p-1)+p\alpha_{j-1}$ and $\beta_j = p\beta_{j-1}+3-\gamma$, we deduce

$$\alpha_j = p^j \alpha_0 + n(p-1)\left(1+p+\cdots+p^{j-1}\right) = (\alpha_0+n)p^j - n, \tag{16}$$

$$\beta_j = p^j \beta_0 + (3-\gamma)\left(1+p+\cdots+p^{j-1}\right) = \left(\tfrac{\gamma-3}{1-p}+\beta_0\right)p^j - \tfrac{\gamma-3}{1-p}. \tag{17}$$

Thus,

$$\begin{aligned} (p\beta_{j-1}+1)(p\beta_{j-1}+2)(p\beta_{j-1}+3) &\leqslant (p\beta_{j-1}+2)^3 = (\beta_j+\gamma-1)^3 \\ &\leqslant \beta_j^3 \leqslant \left(\tfrac{\gamma-3}{1-p}+\beta_0\right)^3 p^{3j}, \end{aligned}$$

where we used $\gamma \in (0,1)$. It follows that

$$K_j \geqslant \underbrace{\tfrac{C}{\Gamma(1-\gamma)}\left(\tfrac{\gamma-3}{1-p}+\beta_0\right)^{-3}}_{\doteq D} p^{-3j} K_{j-1}^p = D\, p^{-3j} K_{j-1}^p \quad \text{for any } j \in \mathbb{N}.$$

Applying the logarithmic function to both sides of the inequality $K_j \geqslant D p^{-3j} K_{j-1}^p$ and using iteratively the resulting inequality, we derive

$$\begin{aligned} \log K_j &\geqslant p^j \log K_0 - 3\left(\sum_{k=0}^{j-1}(j-k)p^k\right)\log p + \left(\sum_{k=0}^{j-1} p^k\right)\log D \\ &\geqslant p^j\left(\log K_0 - \frac{3p\log p}{(p-1)^2} + \frac{\log D}{p-1}\right) + \frac{3j\log p}{p-1} + \frac{3p\log p}{(p-1)^2} - \frac{\log D}{p-1} \end{aligned}$$

for any $j \in \mathbb{N}$, where the identity

$$\sum_{k=0}^{j-1}(j-k)p^k = \frac{1}{p-1}\left(\frac{p^{j+1}-p}{p-1} - j\right) \tag{18}$$

is used. Let $j = j_0(n,\gamma,p) \in \mathbb{N}$ be the smallest nonnegative integer such that

$$j_0 \geqslant \frac{\log D}{3\log p} - \frac{p}{p-1}.$$

Therefore, for any $j \geqslant j_0$ the inequality holds

$$\begin{aligned} \log K_j &\geqslant p^j\left(\log K_0 - \frac{3p\log p}{(p-1)^2} + \frac{\log D}{p-1}\right) \\ &= p^j \log\left(p^{-3p/(p-1)^2} D^{1/(p-1)} K_0\right) = p^j \log\left(E_0 \varepsilon^p\right) \end{aligned} \tag{19}$$

for a suitable constant $E_0 = E_0(n,\gamma,p) > 0$.

If we combine with (15), (16), (17) and (19), we get

$$\begin{aligned} U(t) &\geqslant \exp\left(p^j \log(E_0\varepsilon^p)\right)(R+t)^{-\alpha_j} t^{\beta_j} \\ &= \exp\left(p^j\left(\log(E_0\varepsilon^p) - (\alpha_0+n)\log(R+t) + \left(\tfrac{\gamma-3}{1-p}+\beta_0\right)\log t\right)\right)(R+t)^n t^{\frac{3-\gamma}{1-p}} \end{aligned}$$

for any $j \geqslant j_0$ and any $t \geqslant 0$.

Finally, since for $t \geqslant R$ it holds $\log(t+R) \leqslant \log(2t)$, from the previous inequality we have

$$U(t) \geqslant \exp\left(p^j \log\left(E_0 \varepsilon^p 2^{-(\alpha_0+n)} t^{\frac{\gamma-3}{1-p}+\beta_0-(\alpha_0+n)}\right)\right)(R+t)^n t^{\frac{3-\gamma}{1-p}} \tag{20}$$

for any $j \geqslant j_0$. The exponent of t in the exponential term in the last inequality is

$$\tfrac{\gamma-3}{1-p}+\beta_0-(\alpha_0+n)=\tfrac{1}{2(p-1)}\left(2+(n+3-2\gamma)p-(n-1)p^2\right)=\tfrac{\Upsilon(p,n,\gamma)}{2(p-1)},$$

where $\Upsilon(p,n,\gamma)$ is defined in (8). So, for $p>1$ when $n=1$ and $1<p<p_0(n,\gamma)$ when $n\geqslant 2$, the exponent for t in the exponential term of (20) is positive. Let us fix $\varepsilon_0=\varepsilon_0(u_0,u_1,n,p,\gamma,R)>0$ such that

$$\varepsilon_0^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}}\geqslant E_1R,\quad\text{where } E_1\doteq\left(2^{-(\alpha_0+n)}E_0\right)^{\frac{2(p-1)}{\Upsilon(p,n,\gamma)}}.$$

Thus, for any $\varepsilon\in(0,\varepsilon_0]$ and $t>E_1^{-1}\varepsilon^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}}\geqslant R$, it holds

$$\log\left(\varepsilon^p\,2^{-(\alpha_0+n)}E_0\,t^{\frac{\Upsilon(p,n,\gamma)}{2(p-1)}}\right)>0.$$

Consequently, for any $\varepsilon\in(0,\varepsilon_0]$ and any $t>E_1\varepsilon^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}}$ letting $j\to\infty$ in (20) we may observe that the lower bound for $U(t)$ blows up. So, U may not be finite for this t as well. This proves that u is not globally in time defined and, in particular, the lifespan of the local (in time) of u can be estimated by

$$T(\varepsilon)\lesssim\varepsilon^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}}.$$

All in all, the proof of Theorem 1 is complete.

3 Critical case: Proof of Theorem 2

3.1 Auxiliary functions

Let us recall the definition of a pair of auxiliary functions from [36], which are necessary in order to introduce the time – dependent functional that will be considered for the iteration argument in the critical case $p=p_0(n,\gamma)$.

Let $r>-1$ be a real parameter. Then, we introduce the functions

$$\xi_r(t,x)\doteq\int_0^{\lambda_0}\mathrm{e}^{-\lambda(t+R)}\cosh(\lambda t)\,\Phi(\lambda x)\,\lambda^r\,\mathrm{d}\lambda, \tag{21}$$

$$\eta_r(t,s,x)\doteq\int_0^{\lambda_0}\mathrm{e}^{-\lambda(t+R)}\frac{\sinh(\lambda(t-s))}{\lambda(t-s)}\,\Phi(\lambda x)\,\lambda^r\,\mathrm{d}\lambda, \tag{22}$$

where λ_0 is a fixed positive parameter and Φ is defined by (11).

Some useful properties of ξ_r and η_r are stated in the following lemma, whose proof can be found in [36, Lemma 3.1].

Lemma 1 *Let $n\geqslant 2$ and $\lambda_0>0$. Then, the following properties hold:*

(i) *if $r>-1$, $|x|\leqslant R$ and $t\geqslant 0$, then,*

$$\xi_r(t,x) \geqslant A_0,$$
$$\eta_r(t,0,x) \geqslant B_0\langle t\rangle^{-1};$$

(ii) *if* $r > -1$, $|x| \leqslant s + R$ *and* $t > s \geqslant 0$, *then,*

$$\eta_r(t,s,x) \geqslant B_1\langle t\rangle^{-1}\langle s\rangle^{-r};$$

(iii) *if* $r > \frac{n-3}{2}$, $|x| \leqslant t + R$ *and* $t > 0$, *then,*

$$\eta_r(t,t,x) \leqslant B_2\langle t\rangle^{-\frac{n-1}{2}}\langle t-|x|\rangle^{\frac{n-3}{2}-r}.$$

Here A_0 *and* B_k, *with* $k = 0, 1, 2$, *are positive constants depending only on* λ_0, r *and* R *and we denote* $\langle y\rangle \doteq 3 + |y|$.

Remark 1 Although in [36] the previous lemma is stated by assuming $r > 0$ in (i) and (ii), the proof provided in that paper holds true for any $r > -1$ as well.

Proposition 1 *Let* $n \geqslant 2$ *and* $r > -1$. *Assume that* $u_0 \in H^1(\mathbb{R}^n)$ *and* $u_1 \in L^2(\mathbb{R}^n)$ *are nonnegative, nontrivial and compactly supported in* B_R *functions. Let* u *be an energy solution to* (1) *on* $[0,T)$ *according to Definition 1 satisfying* (7)*. Then, the following integral identity holds:*

$$\int_{\mathbb{R}^n} u(t,x)\,\eta_r(t,t,x)\,\mathrm{d}x = \varepsilon\int_{\mathbb{R}^n} u_0(x)\,\xi_r(t,x)\,\mathrm{d}x + \varepsilon t\int_{\mathbb{R}^n} u_1(x)\,\eta_r(t,0,x)\,\mathrm{d}x$$
$$+ c_\gamma\int_0^t (t-s)\int_0^s (s-\sigma)^{-\gamma}\int_{\mathbb{R}^n} |u(\sigma,x)|^p\,\eta_r(t,s,x)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}s, \quad (23)$$

for any $t \in (0,T)$, *where* ξ_r *and* η_r *are defined in* (21) *and* (22)*, respectively.*

Proof According to (7) $u(t,\cdot)$ has compact support contained in B_{R+t} for any $t \geqslant 0$. Therefore, we may employ (6) for a noncompactly supported test function. So, we choose as test function

$$\psi = \psi(s,x) = \lambda^{-1}\sinh(\lambda(t-s))\Phi(\lambda x),$$

where Φ is defined by (11). As Φ is an eigenfunction of the Laplace operator and the function $y(t,s;\lambda) = \lambda^{-1}\sinh(\lambda(t-s))$ solves the parameter dependent ODE

$$(\partial_s^2 - \lambda^2)y(t,s;\lambda) = 0$$

with final conditions $y(t,t;\lambda) = 0$ and $\partial_s y(t,t;\lambda) = -1$, we get that ψ solves the free wave equation $\psi_{ss} - \Delta\psi = 0$ and satisfies

$$\begin{aligned} &\psi(t,x) = 0, && \psi(0,x) = \lambda^{-1}\sinh(\lambda t)\Phi(\lambda x),\\ &\psi_s(t,x) = -\Phi(\lambda x), && \psi_s(0,x) = -\cosh(\lambda t)\Phi(\lambda x). \end{aligned}$$

Let us prove (23). Employing in (6) the above defined ψ and its properties, we get

$$\int_{\mathbb{R}^n} u(t,x)\,\Phi(\lambda x)\,\mathrm{d}x = \varepsilon\cosh(\lambda t)\int_{\mathbb{R}^n} u_0(x)\,\Phi(\lambda x)\,\mathrm{d}x + \varepsilon\frac{\sinh(\lambda t)}{\lambda}\int_{\mathbb{R}^n} u_1(x)\,\Phi(\lambda x)\,\mathrm{d}x$$
$$+ c_\gamma\int_0^t \frac{\sinh(\lambda(t-s))}{\lambda}\int_{\mathbb{R}^n}\int_0^s (s-\sigma)^{-\gamma}|u(\sigma,x)|^p\,\mathrm{d}\sigma\,\Phi(\lambda x)\,\mathrm{d}x\,\mathrm{d}s.$$

Multiplying both sides of the last equality by $\mathrm{e}^{-\lambda(t+R)}\lambda^r$, integrating with respect to λ over $[0,\lambda_0]$ and applying Tonelli's theorem, we get finally (23). ◻

3.2 Iteration frame and first lower bound estimate

Hereafter until the end of Section 3, we shall assume that u_0, u_1 satisfy the assumptions from the statement of Theorem 2. Let u be an energy solution of (1) on $[0,T)$. We introduce the following time – dependent functional:

$$\mathcal{U}(t) \doteq \int_{\mathbb{R}^n} u(t,x)\,\eta_r(t,t,x)\,\mathrm{d}x, \tag{24}$$

where

$$r \doteq \frac{n-1}{2} - \frac{1}{p}.$$

From Proposition 1 it follows immediately the positiveness of the functional $\mathcal{U}$.

The next step is to derive an integral inequalities involving $\mathcal{U}$ both in the left and in the right – hand side, which will set the iteration frame for the iteration procedure.

Proposition 2 *Let $\mathcal{U}$ be the functional defined by* (24). *Then, there exist positive constants C depending on $n, p, \gamma, \lambda_0, R$ such that the estimate*

$$\mathcal{U}(t) \geqslant C\langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}}\int_0^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{(n-1)(1-\frac{p}{2})}\frac{(\mathcal{U}(\sigma))^p}{(\log\langle\sigma\rangle)^{(p-1)}}\,\mathrm{d}\sigma\,\mathrm{d}s \tag{25}$$

holds for any $t \geqslant 0$.

Proof For the proof of this proposition we follow the main ideas of Proposition 4.2 in [36]. Applying Hölder's inequality and the support property for $u(\sigma,\cdot)$, we obtain

$$\mathcal{U}(\sigma) \leqslant \left(\int_{\mathbb{R}^n} |u(\sigma,x)|^p\eta_r(t,s,x)\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_{\sigma+R}} \frac{\eta_r(\sigma,\sigma,x)^{p'}}{\eta_r(t,s,x)^{\frac{p'}{p}}}\,\mathrm{d}x\right)^{\frac{1}{p'}}. \tag{26}$$

We begin with the estimate of the second factor on the right hand side in the last inequality.

By (ii) and (iii) in Lemma 1 (note that, according to our choice of r, both $r > \frac{n-3}{2}$ and $r > -1$ are always fulfilled), since $|x| \leqslant \sigma + R$ implies $|x| \leqslant s + R$ for any $\sigma \in [0,s]$, we obtain

$$
\begin{aligned}
\int_{B_{\sigma+R}} \frac{\eta_r(\sigma,\sigma,x)^{p'}}{\eta_r(t,s,x)^{\frac{p'}{p}}}\,\mathrm{d}x &\lesssim \langle t\rangle^{\frac{p'}{p}} \langle s\rangle^{\frac{p'}{p}r} \langle\sigma\rangle^{-\frac{n-1}{2}p'} \int_{B_{\sigma+R}} \langle\sigma-|x|\rangle^{(\frac{n-3}{2}-r)p'}\,\mathrm{d}x \\
&\lesssim \langle t\rangle^{\frac{1}{p-1}} \langle s\rangle^{\frac{r}{p-1}} \langle\sigma\rangle^{-\frac{n-1}{2}p'} \int_{B_{\sigma+R}} \langle\sigma-|x|\rangle^{-1}\,\mathrm{d}x \\
&\lesssim \langle t\rangle^{\frac{1}{p-1}} \langle s\rangle^{\frac{r}{p-1}} \langle\sigma\rangle^{-\frac{n-1}{2}p'+n-1} \log\langle\sigma\rangle,
\end{aligned}
$$

where in the second step we used the definition of r. Combining (23), (26) and the previous estimate, we find

$$
\begin{aligned}
\mathcal{U}(t) &\gtrsim \int_0^t (t-s) \int_0^s (s-\sigma)^{-\gamma} \int_{\mathbb{R}^n} |u(\sigma,x)|^p\, \eta_r(t,s,x)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\gtrsim \int_0^t (t-s) \int_0^s (s-\sigma)^{-\gamma} \langle t\rangle^{-1} \langle s\rangle^{-r} \langle\sigma\rangle^{\frac{n-1}{2}p-(n-1)(p-1)} \frac{(\mathcal{U}(\sigma))^p}{(\log\langle\sigma\rangle)^{(p-1)}}\,\mathrm{d}\sigma\,\mathrm{d}s
\end{aligned}
$$

which is exactly (25). □

Proposition 3 *Let us assume* $p = p_0(n,\gamma)$. *Let* $\mathcal{U}$ *be the functional defined by* (24). *Then, there exist a positive constant* M *depending on* $n, p, \gamma, \lambda_0, R, u_0, u_1$ *such that*

$$
\mathcal{U}(t) \geqslant M\varepsilon^p \log(2t/3) \tag{27}
$$

holds for any $t \geqslant 3/2$.

Proof We start by noticing that (13) may be rewritten as

$$
\int_{\mathbb{R}^n} |u(\sigma,x)|^p\,\mathrm{d}x \geqslant C_0\varepsilon^p \langle\sigma\rangle^{n-1-\frac{n-1}{2}p} \qquad \text{for any } \sigma \geqslant 1, \tag{28}
$$

up to a modification of the multiplicative constant. By using (23), Lemma 1 (ii) and (28), we get

$$
\begin{aligned}
\mathcal{U}(t) &\gtrsim \int_0^t (t-s) \int_0^s (s-\sigma)^{-\gamma} \int_{\mathbb{R}^n} |u(\sigma,x)|^p\, \eta_r(t,s,x)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\gtrsim \langle t\rangle^{-1} \int_0^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}} \int_0^s (s-\sigma)^{-\gamma} \int_{\mathbb{R}^n} |u(\sigma,x)|^p\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\gtrsim \varepsilon^p \langle t\rangle^{-1} \int_1^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}} \int_1^s (s-\sigma)^{-\gamma} \langle\sigma\rangle^{n-1-\frac{n-1}{2}p}\,\mathrm{d}\sigma\,\mathrm{d}s.
\end{aligned}
$$

Therefore, for $t \geqslant 1$ by shrinking the domain of integration we find

$$\begin{aligned}\mathcal{U}(t) &\gtrsim \varepsilon^p \langle t\rangle^{-1}\int_1^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}}\int_{s/2}^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{n-1-\frac{n-1}{2}p}\,\mathrm{d}\sigma\,\mathrm{d}s\\
&\gtrsim \varepsilon^p \langle t\rangle^{-1}\int_1^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}+n-1-\frac{n-1}{2}p}s^{1-\gamma}\,\mathrm{d}s\\
&\gtrsim \varepsilon^p \langle t\rangle^{-1}\int_1^t (t-s)\langle s\rangle^{-\frac{n-1}{2}p+\frac{n-1}{2}+1-\gamma+\frac{1}{p}}\,\mathrm{d}s.\end{aligned}$$

Since $p = p_0(n,\gamma)$, from (4) we get

$$-\tfrac{n-1}{2}p+\tfrac{n-1}{2}+1-\gamma+\tfrac{1}{p} = -1. \tag{29}$$

Hence, for $t \geqslant 3/2$ it follows

$$\begin{aligned}\mathcal{U}(t) &\gtrsim \varepsilon^p \langle t\rangle^{-1}\int_1^t (t-s)\langle s\rangle^{-1}\,\mathrm{d}s \gtrsim \varepsilon^p \langle t\rangle^{-1}\int_1^t \frac{t-s}{s}\,\mathrm{d}s = \varepsilon^p \langle t\rangle^{-1}\int_1^t \log s\,\mathrm{d}s\\
&\gtrsim \varepsilon^p (3t)^{-1}\int_{2t/3}^t \log s\,\mathrm{d}s \gtrsim \varepsilon^p \log(2t/3).\end{aligned}$$

This completes the proof. □

In this subsection we determined the iteration frame (25) for the functional $\mathcal{U}$ and a first lower bound estimate (27) for $\mathcal{U}$ containing a logarithmic factors. In the next subsection we are going to prove a sequence of lower bound estimates for $\mathcal{U}$ by using the so – called slicing procedure, which has been introduced for the first time in [1]. More specifically, we will follow the main ideas of [28, 29, 5] concerning the slicing procedure.

3.3 Iteration argument via slicing method

Let us introduce the sequence $\{\ell_j\}_{j\in\mathbb{N}}$, where $\ell_j \doteq 2-2^{-(j+1)}$. The goal is to prove the following sequence of lower bound estimates for the functional $\mathcal{U}$

$$\mathcal{U}(t) \geqslant M_j(\log\langle t\rangle)^{-b_j}\left(\log\left(\frac{t}{\ell_{2j}}\right)\right)^{a_j} \qquad \text{for } t \geqslant \ell_{2j} \text{ and for any } j\in\mathbb{N}, \tag{30}$$

where $\{M_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative real numbers that we shall determine recursively throughout the iteration procedure. For $j=0$ we have already shown that (30) is true thanks to Proposition 3 with

$$M_0 \doteq M\varepsilon^p, \quad a_0 \doteq 1 \quad \text{and} \quad b_0 \doteq 0.$$

We are going to prove the validity of (30) for any $j\in\mathbb{N}$ by using an inductive proof. As we have already pointed out the validity of the base case, it remains to prove the inductive step. Let us assume that (30) holds for $j \geqslant 1$, we want to prove it now for $j+1$. Plugging (30) for j in (25), one finds

$$\begin{aligned}\mathcal{U}(t) &\geqslant CM_j^p\langle t\rangle^{-1}\int_{\ell_{2j}}^t (t-s)\langle s\rangle^{-r}\int_{\ell_{2j}}^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{(n-1)(1-\frac{p}{2})}\frac{\left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_jp}}{(\log\langle\sigma\rangle)^{(p-1)+b_jp}}\,\mathrm{d}\sigma\,\mathrm{d}s\\ &\geqslant CM_j^p\,(\log\langle t\rangle)^{-(p-1)-b_jp}\,\langle t\rangle^{-1}\\ &\qquad\times\int_{\ell_{2j}}^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}-\frac{n-1}{2}p}\int_{\ell_{2j}}^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{n-1}\left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_jp}\mathrm{d}\sigma\,\mathrm{d}s\end{aligned}$$

for $t\geqslant\ell_{2j+2}$. For $s\geqslant\ell_{2j+1}$, the σ – integral in the last line can be estimated in the following way:

$$\begin{aligned}&\int_{\ell_{2j}}^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{n-1}\left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_jp}\,\mathrm{d}\sigma\\ &\qquad\geqslant\int_{\frac{\ell_{2j}s}{\ell_{2j+1}}}^s (s-\sigma)^{-\gamma}\sigma^{n-1}\left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_jp}\,\mathrm{d}\sigma\\ &\qquad\geqslant\left(\frac{\ell_{2j}}{\ell_{2j+1}}\right)^{n-1}s^{n-1}\left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_jp}\int_{\frac{\ell_{2j}s}{\ell_{2j+1}}}^s (s-\sigma)^{-\gamma}\,\mathrm{d}\sigma\\ &\qquad\geqslant\frac{1}{1-\gamma}\left(\frac{\ell_{2j}}{\ell_{2j+1}}\right)^{n-1}\left(1-\frac{\ell_{2j}}{\ell_{2j+1}}\right)^{1-\gamma}s^{n-\gamma}\left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_jp}.\end{aligned}$$

Using the inequalities $2\ell_{2j}>\ell_{2j+1}$ and $1-\ell_{2j}/\ell_{2j+1}>2^{-(2j+3)}$ and the estimate $4s\geqslant\langle s\rangle$ for any $s\geqslant 1$, it follows

$$\begin{aligned}&\int_{\ell_{2j}}^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{n-1}\left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_jp}\,\mathrm{d}\sigma\\ &\qquad\geqslant\frac{1}{1-\gamma}2^{-2(1-\gamma)j-3n-2+5\gamma}\langle s\rangle^{n-\gamma}\left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_jp}.\end{aligned}$$

So, combining the lower bound estimate for the σ – integral with the lower bound estimate for $\mathcal{U}(t)$ and using again (29), for $t\geqslant\ell_{2j+2}$ it holds

$$\begin{aligned}\mathcal{U}(t) &\geqslant\widehat{C}\,2^{-2(1-\gamma)j}M_j^p\,(\log\langle t\rangle)^{-(p-1)-b_jp}\,\langle t\rangle^{-1}\\ &\qquad\times\int_{\ell_{2j+1}}^t (t-s)\langle s\rangle^{-\frac{n-1}{2}p+\frac{n-1}{2}+1-\gamma+\frac{1}{p}}\left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_jp}\mathrm{d}s\\ &\geqslant\widehat{C}\,2^{-2(1-\gamma)j}M_j^p\,(\log\langle t\rangle)^{-(p-1)-b_jp}\,\langle t\rangle^{-1}\int_{\ell_{2j+1}}^t (t-s)\langle s\rangle^{-1}\left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_jp}\mathrm{d}s\\ &\geqslant 2^{-2}\widehat{C}\,2^{-2(1-\gamma)j}M_j^p\,(\log\langle t\rangle)^{-(p-1)-b_jp}\,\langle t\rangle^{-1}\int_{\ell_{2j+1}}^t \frac{t-s}{s}\left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_jp}\mathrm{d}s,\end{aligned}$$

where $\widehat{C}\doteq C(1-\gamma)^{-1}2^{-3n-2+5\gamma}$. Integration by parts and a further shrinking of the domain of integration lead to

$$\begin{aligned}\mathcal{U}(t) &\geqslant \frac{2^{-2}\widehat{C}\,M_j^p}{2^{2(1-\gamma)j}(a_jp+1)}\,(\log\langle t\rangle)^{-(p-1)-b_jp}\,\langle t\rangle^{-1}\int_{\ell_{2j+1}}^{t}\left(\log\left(\tfrac{s}{\ell_{2j+1}}\right)\right)^{a_jp+1}\mathrm{d}s\\ &\geqslant \frac{2^{-2}\widehat{C}\,M_j^p}{2^{2(1-\gamma)j}(a_jp+1)}\,(\log\langle t\rangle)^{-(p-1)-b_jp}\,\langle t\rangle^{-1}\int_{\frac{\ell_{2j+1}t}{\ell_{2j+2}}}^{t}\left(\log\left(\tfrac{s}{\ell_{2j+1}}\right)\right)^{a_jp+1}\mathrm{d}s\\ &\geqslant \frac{2^{-2}\widehat{C}\,M_j^p}{2^{2(1-\gamma)j}(a_jp+1)}\left(1-\tfrac{\ell_{2j+1}}{\ell_{2j+2}}\right)(\log\langle t\rangle)^{-(p-1)-b_jp}\,\langle t\rangle^{-1}t\left(\log\left(\tfrac{t}{\ell_{2j+2}}\right)\right)^{a_jp+1}\\ &\geqslant 2^{-8}\widehat{C}\,(a_jp+1)^{-1}2^{-2(2-\gamma)j}M_j^p\,(\log\langle t\rangle)^{-(p-1)-b_jp}\left(\log\left(\tfrac{t}{\ell_{2j+2}}\right)\right)^{a_jp+1}\end{aligned}$$

for $t \geqslant \ell_{2j+2}$. Also, we proved (30) for $j+1$ provided that

$$M_{j+1} \doteq 2^{-8}\widehat{C}\,(a_jp+1)^{-1}2^{-2(2-\gamma)j}M_j^p,\quad a_{j+1} \doteq a_jp+1,\quad b_{j+1} \doteq (p-1)+b_jp.$$

Next we determine a suitable lower bound for the term M_j. For this purpose, we provide the explicit representations of the exponents a_j and b_j. By using recursively the relations $a_j = 1+pa_{j-1}$ and $b_j = (p-1)+pb_{j-1}$ and the initial exponents $a_0 = 1$, $b_0 = 0$, we get

$$a_j = a_0p^j + \sum_{k=0}^{j-1} p^k = \tfrac{p^{j+1}-1}{p-1} \quad\text{and}\quad b_j = p^jb_0 + (p-1)\sum_{k=0}^{j-1} p^k = p^j - 1. \tag{31}$$

In particular, $a_{j-1}p+1 = a_j \leqslant p^{j+1}/(p-1)$ implies that

$$M_j \geqslant \widehat{D}\,(2^{2(2-\gamma)}p)^{-j}M_{j-1}^p \tag{32}$$

for any $j \geqslant 1$, where $\widehat{D} \doteq 2^{-8+2(2-\gamma)}\widehat{C}(p-1)/p$. Applying the logarithmic function to both sides of (32) and using iteratively the resulting inequality, we obtain

$$\begin{aligned}\log M_j &\geqslant p\log M_{j-1} - j\log\left(2^{2(2-\gamma)}p\right) + \log\widehat{D}\\ &\geqslant p^j\log M_0 - \left(\sum_{k=0}^{j-1}(j-k)p^k\right)\log\left(2^{2(2-\gamma)}p\right) + \left(\sum_{k=0}^{j-1}p^k\right)\log\widehat{D}\\ &= p^j\left(\log M_0 - \frac{p\log\left(2^{2(2-\gamma)}p\right)}{(p-1)^2} + \frac{\log\widehat{D}}{p-1}\right)\\ &\qquad + \left(\frac{j}{p-1} + \frac{p}{(p-1)^2}\right)\log\left(2^{2(2-\gamma)}p\right) - \frac{\log\widehat{D}}{p-1},\end{aligned}$$

where we used the identity (18). Let us define $j_1 = j_1(n,p,\gamma)$ as the smallest nonnegative integer such that

$$j_1 \geqslant \frac{\log\widehat{D}}{\log\left(2^{2(2-\gamma)}p\right)} - \frac{p}{p-1}.$$

Then, for any $j \geqslant j_1$ we may estimate

$$\log M_j \geqslant p^j \left(\log M_0 - \frac{p \log\left(2^{2(2-\gamma)} p\right)}{(p-1)^2} + \frac{\log \widehat{D}}{p-1} \right) = p^j \log(L_0 \varepsilon^p), \tag{33}$$

where $L_0 \doteq M\left(2^{2(2-\gamma)}p\right)^{-p/(p-1)^2} \widehat{D}^{1/(p-1)}$.

Combining (30), (31) and (33), we arrive at

$$\begin{aligned} \mathcal{U}(t) &\geqslant \exp\left(p^j \log(L_0\varepsilon^p)\right) (\log\langle t\rangle)^{-p^j+1} (\log(t/2))^{(p^{j+1}-1)/(p-1)} \\ &= \exp\left(p^j \log\left(L_0\varepsilon^p (\log\langle t\rangle)^{-1} (\log(t/2))^{p/(p-1)}\right)\right) \log\langle t\rangle\, (\log(t/2))^{-1/(p-1)} \end{aligned}$$

for $t \geqslant 2$ and any $j \geqslant j_1$.

For $t \geqslant 4$ the inequalities

$$\log\langle t\rangle \leqslant \log(2t) \leqslant 2\log t \quad \text{and} \quad \log(t/2) \geqslant 2^{-1}\log t$$

hold true, so,

$$\mathcal{U}(t) \geqslant \exp\left(p^j \log\left(2^{-(2p-1)/(p-1)} L_0 \varepsilon^p (\log t)^{1/(p-1)}\right)\right) \log\langle t\rangle\, (\log(t/2))^{-1/(p-1)} \tag{34}$$

for $t \geqslant 4$ and any $j \geqslant j_1$. Let us denote $J(t,\varepsilon) \doteq 2^{-(2p-1)/(p-1)} L_0 \varepsilon^p (\log t)^{1/(p-1)}$. We can choose $\varepsilon_0 = \varepsilon_0(n, p, \gamma, \lambda_0, R, u_0, u_1)$ sufficiently small so that

$$\exp\left(2^{1-2p} L_0^{1-p} \varepsilon_0^{-p(p-1)}\right) \geqslant 4.$$

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp\left(2^{2p-1} L_0^{1-p} \varepsilon^{-p(p-1)}\right)$ we get $t \geqslant 4$ and $J(t,\varepsilon) > 1$. Therefore, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp\left(2^{2p-1} L_0^{1-p} \varepsilon^{-p(p-1)}\right)$ taking the limit as $j \to \infty$ in (34) we see that the lower bound for $\mathcal{U}(t)$ blows up; so, $\mathcal{U}(t)$ may not be finite. Thus, we proved that $\mathcal{U}(t)$ blows up in finite time and, furthermore, we have shown the upper bound estimate for the lifespan

$$T(\varepsilon) \leqslant \exp\left(2^{2p-1} L_0^{1-p} \varepsilon^{-p(p-1)}\right).$$

This completes the proof of Theorem 2.

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