

GRADED MONOMIAL IDENTITIES AND ALMOST NON-DEGENERATE GRADINGS ON MATRICES

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ABSTRACT. Let F be a field of characteristic zero, G be a group and $M_n(F)$ be the algebra of matrices of size n with entries from F with a G -grading. Bahturin and Drensky proved that if the G grading on $M_n(F)$ is elementary and the neutral component of $M_n(F)$ is commutative, then the graded identities of $M_n(F)$ follow from three basic types of identities and monomial identities of length ≥ 2 bounded by a function $f(n)$ of n . In this paper we prove the best upper bound is $f(n) = n$. More generally, we prove that all the graded monomial identities of an elementary G -grading on $M_n(F)$ follow from those of degree at most n . We also study gradings which satisfy no graded multilinear monomial identities but the trivial ones, which we call almost non-degenerate gradings. The description of non-degenerate elementary gradings on matrix algebras is reduced to the description of non-degenerate elementary gradings on matrix algebras that have commutative neutral component. We provide necessary conditions so that the grading on $M_n(F)$ is almost non-degenerate and we apply the results on monomial identities to describe all almost non-degenerate \mathbb{Z} -gradings on $M_n(F)$ for $n \leq 5$.

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1. INTRODUCTION

A fundamental problem in the theory of algebras with polynomial identities is finding a set of generators (or basis), as a verbal ideal (or T -ideal), for the ideal of identities of any given algebra. For associative

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algebras over a field of characteristic zero Kemer (see [18],[19]) proved that there always exists a finite set of generators but he did not give any algorithm to determine such a set. A finite basis for the ideal of identities is known for commutative algebras, whereas for matrix algebras over an infinite field F it is known only for the algebra of 2×2 matrices, see [14], [22]. If F is a field of positive characteristic $p > 2$ we have a description of a finite basis of identities for $M_2(F)$ (see [20]). We note that some further partial results for $M_2(F)$ in the case of fields of characteristic 2 were obtained in [13] and [21] but it is still unknown if the ideal of identities of $M_2(F)$ is finitely generated or not in this case. Hence finding out explicit sets of generators of T -ideals is in general an interesting problem. In this frame some "weaker" identities arise such as the graded polynomial identities for graded algebras. This kind of identities started to be studied in the theory developed by Kemer and plays an important role in the study of ordinary polynomial identities. In particular the T -ideal of any associative algebra can be obtained manipulating (via the Grassmann envelope) the ideal of \mathbb{Z}_2 -graded identities of a finite dimensional \mathbb{Z}_2 -graded algebra. Moreover two algebras graded by the same group satisfying the same graded identities must satisfy the same ordinary identities. A fundamental result by Bahturin, Giambruno and Riley (see [5]) shows that if an algebra is graded by a finite group and the component associated to the neutral element of the group is a PI-algebra, then the whole algebra is also a PI-algebra. All of this justifies the interest in studying graded identities.

A grading on a matrix algebra is said to be elementary if the elementary matrices are homogeneous in the grading. Group gradings on matrix algebras are described in terms of elementary gradings and division gradings, i.e., gradings in which every non-zero homogeneous element is invertible. If the group is cyclic or torsion-free, then every grading on a matrix algebra is an elementary grading, see the paper [6] by Bahturin and Zaicev and the references therein. From now on every field is assumed to be of characteristic 0. A basis for the graded identities of $M_2(F)$ with the canonical \mathbb{Z}_2 -grading, and related algebras, was determined by Di Vincenzo in [11]. Later Vasilovsky provided a basis for the graded identities of the algebra of $n \times n$ matrices with the canonical gradings by the groups \mathbb{Z}_n and \mathbb{Z} , see [24], [25]. In [4] Bahturin and Drensky described a basis for the identities of the algebra $M_n(F)$ of $n \times n$ matrices with an elementary grading such that the neutral component is commutative. In particular, they proved that every graded identity follows from three types of identities and monomial identities of length (the usual degree of the monomial) bounded by a

function $f(n)$ of n . In [8, Theorem 3.1] a similar result is obtained for every subalgebra of $M_n(F)$ having a basis consisting of elementary matrices with the canonical (or Vasilovsky) \mathbb{Z}_n -grading. The authors proved that in the ideal of graded identities we only need monomial identities of degree up to $2n - 1$. It was conjectured in [8, Conjecture 5.1] that the best upper bound is n . In [10] the authors extend such result, and the corresponding conjecture, for arbitrary groups and for infinite fields of arbitrary characteristic. In this paper we prove the last conjecture for $M_n(F)$. More precisely we prove that the graded monomial identities of $M_n(F)$ with an elementary grading by a group G follow from those of degree up to n , see Theorem 3.5 in the sequel.

The canonical \mathbb{Z} -grading on $M_n(F)$ has an interesting property: the monomial identities are consequence of the most *trivial* ones, i.e., the indeterminates with homogeneous degrees that lie in the complement of the support of the grading. A natural question is to consider the G -gradings on the matrix algebra $M_n(F)$ satisfying the same property, i.e., that each of its graded multilinear monomial identities follow from the trivial ones. We shall call such gradings *almost non-degenerate*. This notion is related to the so called *non-degenerate gradings*, (see [2, Definition 1.1]). Such gradings are defined as gradings $R = \bigoplus_{g \in G} R_g$ such that given an n -tuple (g_1, \dots, g_n) of elements of G , we have $R_{g_1} \cdots R_{g_n} \neq 0$. Using the terminology of this paper, to be non-degenerate means R does not satisfy any graded multilinear monomial identity. As examples, the canonical (or Vasilovsky) \mathbb{Z}_n -grading on $M_n(F)$ is non-degenerate, whereas the canonical \mathbb{Z} -grading on $M_n(F)$ does not satisfy this property. Indeed it is clear from the definition that if a finite dimensional graded algebra graded by a group G has a non-degenerate G -grading, then G is finite. Of course the notions of non-degenerate grading and almost non-degenerate grading are strictly related. Notice that if A is an algebra with an almost non-degenerate grading, then the grading on A is non-degenerate if and only if its support coincides with the grading group. Hence if an algebra with an almost non-degenerate fails to be non-degenerate then it admits a trivial monomial identity.

A relation between the grading group G and the exponent of a PI-algebra W ($\exp(W)$) that admits a non-degenerate grading is given in [2]: there exists an abelian subgroup U of G with $[G : U] \leq \exp(W)^K$, where K is a constant not depending neither on W nor G . The graded simple finite-dimensional algebras with a non-degenerate grading are easy to describe: a graded simple algebra A has a non-degenerate grading if and only if the grading is strong, i.e., $A_g A_h = A_{gh}$ for every $g, h \in G$, see [2, Lemma 3.3]. This motivates the following problem.

Problem 1.1. *Given a group G determine the G -graded simple finite-dimensional algebras which are almost non-degenerate.*

The above problem **for elementary gradings on matrix algebras** is reduced, using Proposition 3.2, to the problem of describing the elementary gradings on matrix algebras with commutative neutral component that are non-degenerate. The polynomial identities of such gradings on $M_n(F)$ are completely described in [4]. However even for such gradings and $G = \mathbb{Z}$ the problem above is not trivial. In the last section we use Theorem 3.5 to solve Problem 1.1 for the \mathbb{Z} -graded matrix algebras $M_n(F)$, where $n \leq 5$ (see Theorems 4.12 and 4.13 in the text) and F is a field of characteristic 0. **It is worth mentioning that a similar problem was considered in [12, Theorem 21] for other verbally prime algebras.**

2. PRELIMINARIES

All fields we refer to are assumed to be of characteristic zero and all algebras we consider are associative and unitary.

Let G be a group with neutral element 1_G and let F be a field. If A is an F -algebra, the decomposition

$$\Gamma : A = \bigoplus_{g \in G} A_g$$

of A as a direct sum of subspaces indexed by the elements of the group G is called a G -grading on A if $A_g A_h \subseteq A_{gh}$ for every $g, h \in G$. The subspaces A_g , $g \in G$, are called *homogeneous components* of the grading. An algebra A with a fixed G -grading is said to be G -graded. If $0 \neq a \in A_g$ we say that a is *homogeneous of degree g* and we write $\deg a = g$. We define the *support* of Γ as

$$\text{Supp}(\Gamma) := \{g \in G \mid A_g \neq 0\}.$$

An ideal I of A is *graded* by the group G if $I = \bigoplus_{g \in G} (I \cap A_g)$. A graded algebra A is called *graded simple* if $A^2 \neq 0$ and the only graded ideals of A are the zero ideal and A . Moreover a graded algebra is a *graded division algebra* if every non-zero homogeneous element is invertible.

In order to compare two different gradings we need the notions of isomorphic, weak isomorphic and equivalent gradings.

Let $\Gamma_1 : A = \bigoplus_{g \in G} A_g$ and $\Gamma_2 : B = \bigoplus_{h \in H} B_h$ be a G -grading on A and an H -grading on B respectively. Let $\varphi : A \rightarrow B$ be an isomorphism of algebras. We say φ is an *isomorphism* of the gradings Γ_1 and Γ_2 if $G = H$, and for each $g \in G$, $\varphi(A_g) = B_g$. If there exists an isomorphism

of groups $\alpha : G \rightarrow H$ such that $\varphi(A_g) = B_{\alpha(g)}$ for all $g \in G$, then we say φ is a *weak isomorphism* of the gradings Γ_1 and Γ_2 . We say that φ is an *equivalence of the gradings* Γ_1 and Γ_2 if there exists a bijection $\alpha : \text{Supp}(\Gamma_1) \rightarrow \text{Supp}(\Gamma_2)$ such that $\varphi(A_g) = B_{\alpha(g)}$ for every $g \in G$.

Note that weakly isomorphic gradings are equivalent, but the converse, however, does not hold. For example, the \mathbb{Z} -gradings on $M_n(F)$ induced by $(0, 1, \dots, n-1)$ and $(0, d, \dots, d(n-1))$, for $d > 1$, are equivalent but are not weakly isomorphic. Nevertheless, for non-degenerate gradings those equivalence relations coincide by [8, Corollary 3.6].

For the above notions, we follow the terminology in [15]. Different terminology is also used in the literature, see [16, Section 2.7] or [17, Remark 1.3].

Another important notion we are going to use is that of a coarsening of a grading. Let $\Gamma : A = \bigoplus_{g \in G} A_g$ and $\Gamma' : A = \bigoplus_{h \in H} A'_h$ be a G -grading and an H -grading on A respectively. We say Γ' is a *coarsening* of Γ if for every $g \in G$ there exists $h \in H$ such that $A_g \subseteq A'_h$.

Let us list some useful examples of gradings. Any algebra can be endowed with a *trivial* G -grading, where G is any group, if we set $A_{1_G} = A$ and for each $g \neq 1_G$ $A_g = 0$. This is called *trivial grading* on A . Let E be the algebra generated by a countable infinite set $\{e_1, e_2, \dots\}$ with the condition $e_i e_j = -e_j e_i$ for all $i, j \geq 1$. This algebra is called *Grassmann algebra* and it can be seen its basis as a vector space is given by $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$, where $\mathcal{B}_0 = \{e_{i_1} \cdots e_{i_k} \mid i_1 < \cdots < i_k, k \text{ is even}\} \cup \{1\}$ and $\mathcal{B}_1 = \{e_{i_1} \cdots e_{i_k} \mid i_1 < \cdots < i_k, k \text{ is odd}\}$. We denote by E_i the subspace of E generated by \mathcal{B}_i . Then it is easily seen that E is \mathbb{Z}_2 -graded by $E = E_0 \oplus E_1$ which is called *canonical* \mathbb{Z}_2 -grading of E .

A grading on a matrix algebra $R = M_n(F)$ is called *elementary* if the elementary matrices e_{ij} are homogeneous in the grading. Given an n -tuple $\bar{g} = (g_1, \dots, g_n)$ of elements of G we set $R_g = \text{span}\{e_{pq} \mid g_p^{-1} g_q = g\}$, then $R = \bigoplus_{g \in G} R_g$ is an elementary G -grading on R . Moreover, it was proved in [9] that every elementary grading on $M_n(F)$ can be defined this way. The importance of such gradings is that over an algebraically closed field of characteristic 0 every grading on R by a finite group is obtained by a certain tensor product construction from an elementary grading and a fine grading (see [6]).

We will refer to the \mathbb{Z} -grading on $M_n(F)$ induced by $(0, 1, \dots, n-1)$ as the *canonical* \mathbb{Z} -grading on $M_n(F)$. Analogously the \mathbb{Z}_n -grading on $M_n(F)$ induced by $(\bar{0}, \bar{1}, \dots, \overline{n-1})$ is called *canonical* \mathbb{Z}_n -grading on $M_n(F)$.

In order to give the analogous definition of a polynomial identity for graded algebras, we define a free object in the class of G -graded

algebras. Let $\{X^g \mid g \in G\}$ be a family of disjoint countable sets of variables. Set $X = \bigcup_{g \in G} X^g$ and denote by $F\langle X|G \rangle$ the free associative algebra freely generated by the set X over F . If $x \in X^g$, we set $\deg x = g$. For a monomial $m = x_{i_1}x_{i_2} \cdots x_{i_k}$ we set $\deg m = \deg x_{i_1} \cdot \deg x_{i_2} \cdots \deg x_{i_k}$. Given $g \in G$ we denote by $F\langle X|G \rangle_g$ the subspace of $F\langle X|G \rangle$ spanned by the set $\{m = x_{i_1}x_{i_2} \cdots x_{i_k} \mid \deg m = g\}$. Notice that $F\langle X|G \rangle_g F\langle X|G \rangle_h \subseteq F\langle X|G \rangle_{gh}$ for all $g, h \in G$. Thus

$$F\langle X|G \rangle = \bigoplus_{g \in G} F\langle X|G \rangle_g$$

is a G -graded algebra. We refer to the elements of $F\langle X|G \rangle$ as *graded polynomials*.

An ideal I of $F\langle X|G \rangle$ is said to be a T_G -ideal if it is invariant under all F -endomorphisms $\varphi : F\langle X|G \rangle \rightarrow F\langle X|G \rangle$ such that $\varphi(F\langle X|G \rangle_g) \subseteq F\langle X|G \rangle_g$ for all $g \in G$. A subset $S \subseteq I$ is a *basis* for the T_G -ideal I if I is the intersection of all the T_G -ideals of $F\langle X|G \rangle$ containing S .

If A is a G -graded algebra, a polynomial $f(x_1, \dots, x_n) \in F\langle X|G \rangle$ is said to be a *graded polynomial identity* of A if $f(a_1, a_2, \dots, a_n) = 0$ for all $a_1, a_2, \dots, a_n \in \bigcup_{g \in G} A_g$ such that $a_k \in A_{\deg x_k}$, $k = 1, \dots, n$. If A satisfies a non-trivial graded polynomial identity, A is said to be a *(G -)graded PI-algebra*. We denote by $T_G(A)$ the set of all graded polynomial identities of A , it is clear that $T_G(A)$ is a T_G -ideal of $F\langle X|G \rangle$. We remark that if the group G is finite, then for every G -graded algebra A the T_G -ideal $T_G(A)$ admits a finite basis, see [1], [23]. In the case the group G is the trivial one we shall refer to ordinary polynomials (or simply polynomials), polynomial identities, T -ideals, etc.

Let R be the matrix algebra $M_n(F)$ with an elementary grading induced by the n -tuple $\bar{g} = (g_1, \dots, g_n) \in G^n$. Since for an elementary grading the trivial component contains the subalgebra of diagonal matrices, it is clear that the neutral component R_{1_G} is commutative if and only if the entries of \bar{g} are pairwise distinct. In this case the following are graded polynomial identities for R .

- (1) $x_1x_2 - x_2x_1 = 0, \deg x_1 = \deg x_2 = 1_G$
- (2) $x_1x_2x_3 - x_3x_2x_1 = 0, \deg x_1 = \deg x_2^{-1} = \deg x_3 \neq 1_G$
- (3) $x_1 = 0, R_{\deg x_1} = 0.$

In the identities of (2), we may consider $R_{\deg x_3} \neq 0$ and $R_{\deg x_2} \neq 0$.

If S is a subalgebra of R admitting a linear basis consisting of elementary matrices, then $S = \bigoplus_{g \in G} S_g$, where $S_g = S \cap R_g$, is a G -grading on S .

In Theorem 3.7 of [10], the authors prove the next result.

Theorem 2.1. *Let F be an infinite field and let G be a group. Let us consider $M_n(F)$ with the elementary G -grading induced by the n -tuple $\bar{g} = (g_1, \dots, g_n) \in G^n$, where the elements g_1, \dots, g_n are pairwise different. If S is a subalgebra of $M_n(F)$ generated by elementary matrices e_{ij} , then a basis of the graded polynomial identities of S consists of (1) – (3) and a finite number of graded monomial identities of degree p where $2 \leq p \leq 2n - 1$.*

Determining a finite set of monomial identities as in the previous theorem is not a trivial task and some examples can be found in [4], [8]. In [4] the authors proved the graded monomial identities of $M_n(F)$ are bounded by a function $f(n)$ whereas in [8] and in [10] it is conjectured that every graded monomial identity of the subalgebra S as in the previous theorem follows from the graded monomial identities of degree at most n . In the next section we are going to prove such conjecture for $S = M_n(F)$. We start off with the following definition.

Definition 2.2. [12, Definition 17] *Let R be a G -graded algebra, let I_0 be the T_G -ideal generated by the set $\{x_1 \mid \deg(x_1) \notin \text{Supp}(R)\}$. A monomial $M \in T_G(R)$ is a trivial graded monomial identity for R if $M \in I_0$, otherwise we say that M is a non-trivial graded monomial identity for R .*

We recall that the G -grading $\Gamma : R = \bigoplus_{g \in G} R_g$ on the algebra R is called *non-degenerate* if for every tuple $(g_1, \dots, g_r) \in G^r$ we have $R_{g_1} \cdots R_{g_r} \neq 0$, see [2, Definition 1.1]. Equivalently the grading Γ is non-degenerate if it admits no monomial identity and in this case $\text{Supp}(\Gamma) = G$. The grading Γ is called *strong* if $R_g R_h = R_{gh}$. It is clear that if Γ is strong and $\text{Supp}(\Gamma) = G$, then it is also non-degenerate. The canonical \mathbb{Z}_n -grading on $M_n(F)$ is a strong grading and, in particular, it is also a non-degenerate grading. The canonical \mathbb{Z} -grading on $M_n(F)$ is not a non-degenerate grading because it satisfies the identities x_1 , where $|\deg x_1| \geq n$. This grading, however, satisfies no non-trivial graded monomial identities (see [24] for the description of its graded identities). With this example in mind, we introduce here the concept of almost non-degenerate grading.

Definition 2.3. *The G -grading $R = \bigoplus_{g \in G} R_g$ on the algebra R is almost non-degenerate if it satisfies no non-trivial multilinear graded monomial identities.*

In Section 4 we consider the problem of determining the almost non-degenerate gradings on matrix algebras. A more specific problem is to determine the almost non-degenerate \mathbb{Z} -gradings on $M_n(F)$. Even this is a non-trivial task, and we provide necessary conditions so that certain

gradings on $M_n(F)$ are almost non-degenerate and we completely solve this problem for $n \leq 5$.

Let n be a positive integer $\bar{g} = (g_1, \dots, g_n)$ be a tuple of elements of the group G and let $D = \bigoplus_{g \in G} D_g$ be a G -graded algebra. Then it is easy to verify that the algebra $M_n(F) \otimes D$ may be endowed with a G -grading such that the element $e_{ij} \otimes d$ is homogeneous of degree $g_i^{-1} h g_j$ for every $1 \leq i, j \leq n$ and every $d \in D_h$. We refer to this as the grading on $M_n(F) \otimes D$ induced by \bar{g} . If $D = \bigoplus_{g \in G} D_g$ is a division grading then $M_n(F) \otimes D$ is a graded simple algebra. The next result implies every finite-dimensional graded simple algebra is obtained in this way, it is a direct consequence of some of the results in [6], [7] and [15, Corollary 2.12].

Theorem 2.4. *Let G be a group and let R be a G -graded algebra. If R is a finite-dimensional graded simple algebra, then there exist a positive integer n , an n -tuple \bar{g} of elements of G and a graded division algebra D such that R is isomorphic to $M_n(F) \otimes D$ with the grading induced by \bar{g} .*

3. MONOMIAL IDENTITIES IN MATRIX ALGEBRAS

In this section we prove that all multilinear graded monomial identities of the full matrix algebra of size n follow from those of degree n provided the grading is elementary.

First, we mention that for elementary gradings a graded monomial identity is a consequence of a suitable graded multilinear monomial identity.

Remark 3.1. *A monomial $M = x_{i_1} \cdots x_{i_m}$ is a graded monomial identity for a n elementary G -grading A on a matrix algebra if and only if the multilinear **graded** monomial $\widetilde{M} = x_1 \cdots x_m$, where $\deg_G x_j = \deg_G x_{i_j}$, is a graded identity for A . Hence an elementary G -grading on a matrix algebra is almost non-degenerate if and only if it satisfies no non-trivial monomial identity.*

Proof. Let $h_i = \deg_G x_{i_i}$, $i = 1, \dots, m$ and for each $h \in G$, let

$$M_h = \sum_{g_i^{-1} g_j = h} e_{ij}.$$

For $1 \leq i, j \leq n$ denote by $k_{i,j}$ the number of tuples of elementary matrices $(e_{i_1, j_1}, \dots, e_{i_m, j_m})$ such that $\deg_G e_{i_t, j_t} = h_t$ for $t = 1, \dots, m$ and $e_{i_1, j_1} \cdots e_{i_m, j_m} = e_{ij}$. Then we have

$$(4) \quad M_{h_1} \cdots M_{h_m} = \sum_{i,j} k_{i,j} e_{ij}.$$

Clearly if \tilde{M} is an identity for A then M is also an identity for A . Now assume that $M \in T_G(A)$. In this case $M_{h_1} \cdots M_{h_m} = 0$, hence it follows from (4) that $k_{ij} = 0$ for every $1 \leq i, j \leq n$. Assume that \tilde{M} is not an identity for A , hence there exist a tuple of elementary matrices $(e_{i_1, j_1}, \dots, e_{i_m, j_m})$ such that $\deg_G e_{i_t, j_t} = h_t$ for $t = 1, \dots, m$ and $e_{i_1, j_1} \cdots e_{i_m, j_m} \neq 0$. As a consequence $k_{i_1, j_m} \neq 0$ which is a contradiction since $k_{ij} = 0$ for every $1 \leq i, j \leq n$. Hence we conclude that $\tilde{M} \in T_G(A)$. \square

The above remark is not true in general. For instance, let us consider the Grassmann algebra E with its canonical \mathbb{Z}_2 -grading. Of course E satisfies the monomial identity z^2 , where z is an indeterminate of degree $\bar{1}$, whereas it does not satisfy any multilinear monomial identity.

Let us show now the next fact. If a matrix algebra is graded by a group G and this G -grading is elementary and induced by an n -tuple (g_1, \dots, g_n) of elements of G , then it is not a restriction for our purposes to consider g_1, \dots, g_n to be pairwise distinct. This is the content of the following result.

Proposition 3.2. *Let A be the matrix algebra $M_n(F)$ with the elementary grading induced by (g_1, \dots, g_n) . Let h_1, \dots, h_k be pairwise different elements of the group G such that $\{g_1, \dots, g_n\} = \{h_1, \dots, h_k\}$. Let A_0 be the algebra $M_k(F)$ with the elementary grading induced by (h_1, \dots, h_k) . Then $M = x_1 \dots x_m$ is a graded multilinear monomial identity for A if and only if it is a graded multilinear monomial identity for A_0 . In particular, the algebra A has an almost non-degenerate grading if and only if A_0 has an almost non-degenerate grading.*

Proof. Note that $\text{supp } A = \text{supp } A_0$. The result now follows if we prove that a multilinear monomial is an identity for A if and only if it is an identity for A_0 . Let $M = x_1 \cdots x_m$ be a monomial in the free G -graded algebra. Note that A_0 is isomorphic to a suitable subalgebra of A . Hence if M is a graded identity for A , then M is a graded identity for A_0 . Now we assume that M is a graded identity for A_0 . We may assume, up to isomorphism, that the entries of the n -tuple (g_1, \dots, g_n) lie in the set $\{h_1, \dots, h_k\}$ and that the first q_1 entries are h_1 , the next q_2 entries are h_2 and so on. Let $d_i = q_1 + \cdots + q_i$ for $i = 1, \dots, k$. Let $J_1 = \{1, \dots, d_1\}$ and $J_i = \{d_{i-1} + 1, \dots, d_i\}$ for $i = 2, \dots, k$. Note that $\cup_l J_l$ is a partition of $\{1, \dots, n\}$. Then $g_i = h_l$ if $i \in J_l$. For each i in $\{1, \dots, n\}$ there exists a unique l such that $i \in J_l$ and we denote by i' the smallest integer in J_l . Now let $e_{i_1, j_1}, \dots, e_{i_m, j_m} \in A$ such that $\deg_G e_{i_t, j_t} = \deg_G x_t$ for each $t = 1, \dots, m$ and let r be the result of the

substitution of x_t by e_{i_t, j_t} on M . Then

$$r = e_{i_1, j_1} \cdots e_{i_m, j_m}.$$

Now let

$$r' := e_{i'_1, j'_1} \cdots e_{i'_m, j'_m}.$$

If $r \neq 0$, then $r' \neq 0$. On the other hand, r' is the result of an admissible substitution for M by elements of A' which is the subalgebra of A generated by the elementary matrices $e_{i, j}$, where $i, j \in \{1', \dots, n'\}$. As a consequence, M is not a graded identity for A' . However A' is isomorphic to A_0 that is a contradiction. Thus $r = 0$ and M is a graded identity for A . \square

Until the end of this section R will denote the algebra $M_n(F)$ with the elementary grading induced by an n -tuple $\bar{g} = (g_1, \dots, g_n)$ of pairwise distinct elements of G . For every $g \in \text{Supp}(R)$ let $\{e_{i_1, j_1}, \dots, e_{i_l, j_l}\}$ be the basis of R_g consisting of elementary matrices. Let $I_g = \{i_1, \dots, i_l\}$, $J_g = \{j_1, \dots, j_l\}$. Since the entries of \bar{g} are pairwise distinct given $i \in I_g$ there exists a unique $j \in J_g$ such that $e_{ij} \in R_g$. This defines an injective map $\hat{g} : I_g \rightarrow J_g$, this a map was introduced in [10].

We start off with the next result about graded multilinear monomial identities, which is a key step in the proof of the main theorem of this section.

Lemma 3.3. *Let G be a group and let R be the elementary G -grading on $M_n(F)$ induced by the n -tuple (g_1, \dots, g_n) of pairwise distinct elements of G . For $h \in \text{Supp}(R)$, let us denote*

$$M_h := \sum_{g_k^{-1}g_l=h} e_{kl}.$$

Let h_1, \dots, h_k be elements of $\text{Supp}(R)$. If $h_1 \cdots h_k = 1_G$, then the number of non-zero lines in $M_1 = M_{h_1}M_{h_2} \cdots M_{h_k}$ and $M_2 = M_{h_2} \cdots M_{h_k}$ is the same.

Proof. Let i_1, \dots, i_r be the non-zero lines of M_1 which are, of course, also non-zero lines of M_{h_1} . Then $j_1 = \hat{h}_1(i_1), \dots, j_r = \hat{h}_1(i_r)$ are non-zero lines of M_2 . The claim follows once we prove j_1, \dots, j_r are exactly the non-zero lines of M_2 .

Since M_{h_1} is homogeneous of degree h_1 , we have M_2 is homogeneous of degree h_1^{-1} . If the j -th line of M_2 is non-zero, then there exists i such that e_{ji} has degree h_1^{-1} . Of course $\deg(e_{ij}) = h_1$ and $j = \hat{h}_1(i)$, where i is a non-zero line of M_1 , since $e_{ij}M_2 \neq 0$ and the result follows. \square

As an easy consequence of the previous result we get the next result.

Lemma 3.4. *Let $m = x_1 \cdots x_k$ be a graded monomial identity of homogeneous degree 1_G of R , where for every $i = 1, \dots, k$ we have $\deg(x_i) \in \text{Supp}(R)$. Then m is a consequence of the graded monomial identity $m' = x_2 \cdots x_k$.*

Now we are ready for the main result of the section.

Theorem 3.5. *Let G be a group and let R be the algebra $M_n(F)$ endowed with an elementary G -grading. If $m = x_1 \cdots x_k$ is a graded monomial identity for R and $k > n$, then m is a consequence of a graded monomial identity of R of degree at most n .*

Proof. From Proposition 3.2 we may consider R an elementary grading induced by the tuple (g_1, \dots, g_n) of pairwise distinct elements of G . Suppose $k > n$. If $x_1 \cdots x_n$ is a graded monomial identity for R we are done. Assume that $x_1 \cdots x_n$ is not a graded identity for R . In this case there are indexes $i_1, j_1, \dots, i_n, j_n$ such that $\deg e_{i_r, j_r} = h_r$ for all $r \in \{1, \dots, n\}$ and $e_{i_1, j_1} \cdots e_{i_n, j_n} \neq 0$. Of course, $j_r = i_{r+1}$, for all $r < n$. Defining $i_{n+1} := j_n$, since $i_r \in \{1, \dots, n\}$, for all r , at least two among the indexes i_1, \dots, i_{n+1} are equal. Let i_s and $i_{t+1} = j_t$ be such indexes. Then $\deg x_{i_s} \cdots x_{i_t} = 1_G$, i.e., $x_1 \cdots x_k$ has a submonomial $m' = x_{i_s} \cdots x_{i_t}$ of degree 1_G .

Suppose first $s = 1$, i.e., m' is at the beginning of the monomial m . Then Lemma 3.4 shows that m is a consequence of $x_2 \cdots x_k$.

Suppose now $s > 1$. We claim the monomial

$$x_1 \cdots x_{s-2} y x_{s+1} \cdots x_k,$$

is a graded monomial identity for R , where $\deg x_i = h_i \in G$, for all i and $\deg y = h_{s-1} h_s$. For this, it is enough to show that the non-zero lines of $M_{h_{s-1}} M_{h_s} \cdots M_{h_t}$ and $M_{h_{s-1} h_s} M_{h_{s+1}} \cdots M_{h_t}$ are the same. To prove this claim notice that any non-zero line of the former product is a non-zero line of the latter.

Now let i be a non-zero line of $M_{h_{s-1} h_s} M_{h_{s+1}} \cdots M_{h_t}$. As before, there are matrices e_{i_r, j_r} for $r \in \{s+1, \dots, t\}$ such that $\deg e_{i_r, j_r} = h_r$ and $e_{i, j} e_{i_{s+1}, j_{s+1}} \cdots e_{i_t, j_t} \neq 0$, where $\deg e_{i, j} = h_{s-1} h_s$. Since $h_s \cdots h_t = 1_G$, we have $\deg e_{j_t, i_{s+1}} = h_s$.

Comparing degrees, one gets $\deg e_{i, j_t} = h_{s-1}$. Hence,

$$e_{i, j_t} e_{j_t, i_{s+1}} e_{i_{s+1}, j_{s+1}} \cdots e_{i_t, j_t} \neq 0,$$

which means i is a non-zero line of $M_{h_{s-1}} M_{h_s} M_{h_{s+1}} \cdots M_{h_t}$.

In order to complete the proof we have to argue by induction on $k \geq n+1$. If $k = n+1$ the discussion above shows m is a consequence of a graded monomial of degree n and we are done. Suppose the result is true for $k-1 \geq n+1$ and let us prove it for $k \geq n+2$. As above,

m is a consequence of a graded identity of degree strictly less than k that is, by induction, a consequence of a graded monomial of degree less than or equal to n and now the proof is complete. \square

We remark that the proof of the above result uses the hypothesis that R is a grading on $M_n(F)$, hence for an arbitrary subalgebra of $M_n(F)$ generated by elementary matrices the conjecture in [10] is still open.

The next example shows that the bound $f(n) = n$ is sharp.

Example 3.6. *Let us consider $R = M_n(F)$ with the \mathbb{Z}_{n+1} -grading given by $(\overline{0}, \overline{1}, \dots, \overline{n-1})$. In this case, $R_{\overline{1}}^n = 0$ provides a graded monomial identity of length n which does not follow from a graded monomial identity of lower degree. In fact, R does not satisfy any monomial identity of degree less than n , indeed notice that if $g \neq 1_G$ then the component of R_g is $n-1$ -dimensional, so if we consider the graded monomial $x_{g_1} \dots x_{g_k}$, we get the linear span of its image (considered as a function of R^k to R) is a vector subspace of $R_{g_1 \dots g_k}$ of dimension at least $n-k$.*

In the next section we consider almost non-degenerate gradings on matrix algebras.

4. ALMOST NON-DEGENERATE GRADINGS

Let G be a group. In this section we consider the problem of determining the n -tuples of elements of G inducing almost non-degenerate elementary gradings on $M_n(F)$. We recall that almost non-degenerate gradings are gradings which satisfy no non-trivial **multilinear** graded monomial identities.

Throughout this section R will denote the matrix algebra $M_n(F)$ over an algebraically closed field of characteristic zero. Moreover we will use the word *length* instead of *degree* of a monomial.

In the next paragraphs we would like to show why it is sufficient to consider the hypothesis of the neutral component being commutative, starting with we the following characterization result.

Due to the fact the ground field is algebraically closed, then $D_{1_G} = F$ is commutative and the neutral component of the algebra A_0 above is commutative which makes the hypothesis of the neutral component being commutative consistent at the light of the characterization of the graded algebras given by Bahturin and Zaicev in [6].

Now we start the study of the almost non-degenerate gradings on $M_n(F)$. By Lemma 3.2, we can restrict our attention to elementary gradings on $M_n(F)$ generated by n -tuples of pairwise distinct elements.

Remark 4.1. *The elementary gradings on $M_n(F)$ induced by the tuples (g_1, \dots, g_n) and (h_1, \dots, h_n) in G^n are isomorphic if and only if there exists a permutation $\pi \in S_n$ and $g \in G$ such that $h_i = g_{\pi(i)}g$, for $i = 1, \dots, n$ (see [3] and [15, Corollary 2.12]).*

As we have already mentioned, a direct consequence of the main result of [24] is the following.

Proposition 4.2. *The canonical \mathbb{Z} -grading on $M_n(F)$ is an almost non-degenerate grading.*

In this section we determine sufficient conditions so that a G -grading on $M_n(F)$, where G is a linearly ordered abelian group, turns out to be almost non-degenerate. We recall a *linearly ordered group* is a group equipped with a total order that is translation-invariant. In particular, linearly ordered groups are always torsion-free. Since we are working with abelian groups, we will use additive notation for the group G . In particular, the neutral element will be denoted by 0.

The proof of the next lemma is easy and will be omitted.

Lemma 4.3. *Let G be a linearly ordered abelian group, let n be a positive integer and let $0 = g_1 < g_2 < \dots < g_n \in G$ such that $g_j - g_i \in \{g_1, \dots, g_n\}$, for each i and $j \in \{1, \dots, n\}$ with $i < j$. Then for all $k \in \{1, \dots, n\}$ we have $g_k = (k - 1)g_2$.*

Remark 4.4. *Any grading by a torsion-free group on a matrix algebra is an elementary grading (see [15, Corollary 2.22]). In particular a grading by a linearly ordered group G on a matrix algebra is an elementary grading.*

The next proposition gives a characterization of G -gradings on matrix algebras that are equivalent to the canonical \mathbb{Z} -grading, where G is a linearly ordered abelian group.

Proposition 4.5. *Let G be a linearly ordered abelian group and let*

$$\Gamma : M_n(F) = R = \bigoplus_{g \in G} R_g$$

be a grading on $M_n(F)$ such that the neutral component is commutative. Then the following statements are equivalent:

- (a) Γ is equivalent to the canonical \mathbb{Z} -grading of $M_n(F)$.
- (b) The support of the grading has exactly $2n - 1$ elements.

- (c) *There exists $g > 0$ in G such that Γ is the elementary grading induced by $(0, g, 2g, \dots, (n-1)g)$.*
 (d) *There exists $g \in G$ such that $\dim R_g = n - 1$.*

Proof. By Remark 4.4 we get Γ is an elementary grading. The neutral component is commutative, hence Γ is induced by an n -tuple of pairwise distinct elements of G . As a consequence of Remark 4.1 we may assume, without loss of generality, that Γ is an elementary grading induced by an n -tuple $(g_1, \dots, g_n) \in G^n$ such that $0 = g_1 < g_2 < \dots < g_n$. Note that $\text{Supp}(\Gamma)$ has at least $2n - 1$ elements, indeed the $2n - 1$ elements $e_{11}, e_{12}, \dots, e_{1n}, e_{21}, \dots, e_{n1}$ have pairwise distinct degrees.

(a) \Rightarrow (b) It is immediate since the cardinality of the support is invariant under equivalence of gradings.

(b) \Rightarrow (c) Now assume that $\text{Supp}(\Gamma)$ has $2n - 1$ elements, then

$$\text{Supp}(\Gamma) = \{0, g_2, \dots, g_n, -g_2, \dots, -g_n\}.$$

In this case, for each $i < j$, $g_j - g_i$ is one among g_2, \dots, g_n . By Lemma 4.3, for each k , $g_k = (k - 1)g_2$.

(c) \Rightarrow (d) Notice that the set $e_{12}, e_{23}, \dots, e_{n-1,n}$ is a linear basis of the subspace R_g , hence $\dim R_g = n - 1$.

(d) \Rightarrow (c) Up to an isomorphism of gradings, by Remark 4.1, we may assume the gradings is induced by the n -tuple (g_1, \dots, g_n) with $0 = g_1 < g_2 < \dots < g_n$. Let $g \neq 0$ be an element of G such that $\dim R_g = n - 1$. Since $\dim R_g = \dim R_{-g}$, we may assume $g > 0$. If $\deg e_{ij} > 0$, then $i < n$, therefore there exist i_1, \dots, i_{n-1} such that the elementary matrices with degree g are $e_{1i_1}, \dots, e_{n-1, i_{n-1}}$. For $1 \leq k < l \leq n - 1$ the equality $g_k - g_{i_k} = g_l - g_{i_l}$ implies that $g_{i_k} < g_{i_l}$. Therefore $1 < i_1 < \dots < i_{n-1} \leq n$. Thus $i_t = t + 1$ for $t = 1, \dots, n - 1$. Hence $g = g_2 - g_1 = g_3 - g_2 = \dots = g_n - g_{n-1}$ and for each $k \in \{1, \dots, n\}$, $g_k = (k - 1)g_2$.

(c) \Rightarrow (a) It is immediate that the map induced by $1 \in \mathbb{Z} \mapsto g \in G$ gives an equivalence between the canonical \mathbb{Z} -grading of R and the grading Γ . \square

Notice the hypothesis of G being linearly ordered cannot be removed in Proposition 4.5 as we show in the next example.

Remark 4.6. *Let R be endowed with the \mathbb{Z}_{2n-1} -grading induced by the n -tuple $(\overline{0}, \overline{1}, \dots, \overline{n-1})$. We have $R_{\overline{n-1}} = \langle E_{1n} \rangle$. In particular, $R_{\overline{n-1}}^2 = 0$, and this yields a non-trivial monomial identity, since $\overline{2(n-1)} = \overline{-1}$*

and $R_{-\bar{1}} \neq 0$. The grading R is equivalent to the canonical \mathbb{Z} -grading on $M_n(F)$, however it satisfies a non-trivial monomial identity.

The next result shows some characterizations of the canonical \mathbb{Z} -grading on R . This will be helpful in order to determine the almost non-degenerate gradings on R .

Theorem 4.7. *If G is a linearly ordered abelian group and Γ is a G -grading on R such that $|\text{Supp}(\Gamma)| = 2n - 1$, then Γ is an almost non-degenerate grading.*

Proof. Proposition 4.5 implies that there exists $h > 0$ in G such that Γ is the elementary grading on $M_n(F)$ induced by $(0, h, 2h, \dots, (n-1)h)$. Therefore $\text{Supp}(\Gamma) \subseteq H$, where H is the subgroup of G generated by h . Let A denote $M_n(F)$ with the canonical \mathbb{Z} -grading. The map $\varphi : z \mapsto zh$ is an isomorphism from \mathbb{Z} to H such that $A_z = R_{\varphi(z)}$ for every $z \in \mathbb{Z}$. This implies that the gradings on R and A , viewed as gradings by the groups \mathbb{Z} and H respectively, are weakly isomorphic. Proposition 4.2 implies that the H -grading on R is also almost non-degenerate. This gives that Γ is an almost non-degenerate grading. \square

A natural question now arises. Does the converse of Proposition 4.7 hold? More precisely, if G is a linearly ordered abelian group and the G -grading Γ on $M_n(F)$ is almost non-degenerate does it follow that Γ is equivalent to the canonical \mathbb{Z} -grading on $M_n(F)$?

The answer to the above question is negative and counterexamples exist only for $n \geq 4$, as we can see in what follows.

Proposition 4.8. *Let G be a linearly ordered group and Γ be an almost non-degenerate elementary G -grading on R , $n \leq 3$, such that the neutral component is commutative. Then Γ is equivalent to the canonical \mathbb{Z} -grading on R .*

Proof. Let us consider $n = 2$, then for any non-trivial G -grading there exists $g \in G$ such that

$$R_0 = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}, R_g = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}, R_{-g} = \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}.$$

It is clear that this grading is equivalent to the canonical \mathbb{Z} -grading on R . If $n = 3$, by Remark 4.4 and Remark 4.1 we may assume that Γ is the elementary grading induced by the triple $\bar{g} = (0, g_1, g_2)$ of elements of G , where $0 < g_1 < g_2$. In this case the positive elements in $\text{Supp}(\Gamma)$

are $\{g_1, g_2, g_2 - g_1\}$. If $g_2 \neq 2g_1$ we have

$$R_{g_1} = \begin{pmatrix} 0 & F & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R_{g_2} = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R_{g_2 - g_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}.$$

As a consequence, it satisfies a non-trivial monomial identity of length 2: $x_1x_2 = 0$, if $\deg x_1 = g_2 - g_1$ and $\deg x_2 = g_1$. This contradicts the fact that Γ is almost non-generate. Therefore we get $g_2 = 2g_1$ which leads us to state the grading on R is equivalent to the canonical \mathbb{Z} -grading on R . \square

We now introduce a grading on R which will play an important role in the characterization of its almost non-degenerate gradings.

Let n be a positive integer and denote by e_i the element of $\mathbb{Z}^{\lfloor \frac{n}{2} \rfloor}$ whose i -th entry is equal to 1 and the remaining entries are 0. We consider the $(n-1)$ -tuple $\bar{d} = (d_1, \dots, d_{n-1})$ where $d_i = e_i$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $d_i = d_{n-i}$ for $\lfloor \frac{n}{2} \rfloor < i \leq n-1$. Now let $\bar{g} = (g_1, \dots, g_n)$ be the n -tuple of elements of $\mathbb{Z}^{\lfloor \frac{n}{2} \rfloor}$ such that $g_1 = 0$ and $g_{i+1} - g_i = d_i$ for $i = 1, \dots, n$. Let

$$(5) \quad R = \bigoplus_{g \in \mathbb{Z}^{\lfloor \frac{n}{2} \rfloor}} R_g$$

be the elementary $\mathbb{Z}^{\lfloor \frac{n}{2} \rfloor}$ -grading on R induced by \bar{g} . Then we get the following.

Proposition 4.9. *Let G be an abelian group without elements of order 2. Let us consider R with an elementary grading induced by a tuple (g_1, \dots, g_n) of pairwise distinct elements of G . If there exists $g \in G$ such that $\dim R_g = 1$ and R satisfies no non-trivial multilinear identity of length 2, then this grading is isomorphic to a coarsening of the $\mathbb{Z}^{\lfloor \frac{n}{2} \rfloor}$ -grading in (5).*

Proof. Let $R_g := \text{span}\{e_{ij}\}$ be a one dimensional component of the grading and let $\bar{h} = (h_1, \dots, h_n)$, where $h_t = g_t - g_i$ for $t = 1, \dots, n$.

We claim that for every $t \in \{1, \dots, n\}$ there exists an index $l(t)$ such that $e_{l(t)j} \in R_{h_t}$. Otherwise there exists t_0 such that $e_{lj} \notin R_{h_{t_0}}$ for $l = 1, \dots, n$ or, equivalently, $e_{jl} \notin R_{-h_{t_0}}$, for $l = 1, \dots, n$. This implies that $m = x_1x_2$, with $\deg x_1 = g$ and $\deg x_2 = -h_{t_0}$, is a graded monomial identity for R . Since $\deg m = g - h_{t_0} = g_j - g_{t_0} \in \text{Supp}(R)$, then m is a non-trivial monomial identity, that is a contradiction. Thus we obtain a map $t \mapsto l(t)$, where $l(t)$ is the index such that $h_j - h_{l(t)} = g_j - g_{l(t)} = h_t$. This map is injective and therefore a permutation in

S_n . Hence we conclude that $h_j - h_t \in \{h_1, \dots, h_n\}$, for $t \in \{1, \dots, n\}$. Thus we have a permutation $\alpha \in S_n$ so that

$$(6) \quad h_j - h_t = h_{\alpha(t)},$$

for $t = 1, \dots, n$. Since the group G is abelian we conclude that $\alpha^2 = 1$, hence α is the product of disjoint transpositions. Since G has no elements of order 2, the homomorphism $g \mapsto 2g$ is injective and α has at most one fixed point. We may consider, up to isomorphism, R being graded by the tuple $(h_{\beta(1)}, \dots, h_{\beta(n)})$, for some $\beta \in S_n$. The effect of this in (6) is to change α for $\beta^{-1}\alpha\beta$. Indeed, let $h'_t = h_{\beta(t)}$, for $t = 1, \dots, n$ and let us rewrite (6) for the tuple (h'_1, \dots, h'_n) . Let $j' = \beta^{-1}(j)$, then

$$h'_{j'} - h'_t = h_j - h_{\beta(t)} = h_{\alpha\beta(t)} = h'_{\beta^{-1}\alpha\beta(t)}.$$

Since α has at most one fixed point, we can choose, without loss of generality, α being of the following form

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}.$$

In this case, (6) implies that $h_t + h_{n+1-t} = h_j$ for $t = 1, \dots, n$. Now let $d_t = h_{t+1} - h_t$ for $t = 1, \dots, n-1$. As a consequence we have

$$d_{n-t} - d_t = h_{n-t+1} - h_{n-t} - (h_{t+1} - h_t) = h_{n-t+1} + h_t - (h_{n-t} + h_{t+1}) = 0,$$

and $d_t = d_{n-t}$, which means the grading is a coarsening of the grading in (5). \square

Remark 4.10. *The converse of the above proposition does not hold and the grading in Remark 4.6 provides a counter-example since it is isomorphic to a coarsening of the grading in (5), has a 1-dimensional component but satisfies a monomial identities of length 2, for $n > 2$.*

Looking back at the hypothesis of Proposition 4.9, it is natural to ask whether or not the fact that R satisfies a non-trivial graded monomial identity, implies R satisfies a non-trivial monomial identity of length 2. The answer to this question is negative and Example 3.6 furnishes a counterexample. In fact, the graded algebra in Example 3.6 satisfies a non-trivial monomial identity of length n but does not satisfies any non-trivial monomial of length 2.

Remark 4.11. *Let us specialize Proposition 4.9 for G being a linearly ordered abelian group and $R = M_n(F)$ with a grading induced by an n -tuple (g_1, \dots, g_n) of pairwise distinct elements of G . Up to isomorphism of gradings, we may consider $0 = g_1 < g_2 < \dots < g_n$. In particular, R_{g_n} plays the role of the 1-dimensional component of R and $R_{g_n} =$*

$\text{span}\{e_{1n}\}$. In the notation of the proof of Proposition 4.9, $i = 1$, $j = n$ and since $g_n - g_1 > g_n - g_2 > \cdots > g_n - g_n$, and all of them lie in $\{g_1, \dots, g_t\}$, we have $g_n - g_t = g_{n-t+1}$. In this case, this grading is a coarsening of the grading in (5), defined by an $n-1$ -tuple (d_1, \dots, d_{n-1}) , with $d_t > 0$, for all t . More precisely $d_t = g_{t+1} - g_t$.

Now we are ready to state our first main result of the section linking graded identities of $M_4(F)$ and almost non-degenerate gradings.

Theorem 4.12. *Let G be a linearly ordered abelian group. Let R denote the algebra $M_4(F)$ with a G -grading such that the neutral component is commutative. Then the T_G -ideal $T_G(R)$ is generated as a T_G -ideal by the identities (1) – (3) if and only if there exist $a, b > 0$ in G , with $a \neq 2b$, $2a \neq b$, such that R is isomorphic to the elementary grading on $M_4(F)$ induced by $(0, a, a + b, 2a + b)$.*

Proof. We assume that $T_G(R)$ is generated by the identities (1) – (3). Proposition 4.9 and Remark 4.11 imply that there exist $a, b > 0$ in G such that R is isomorphic to the elementary grading induced by $(0, a, a + b, 2a + b)$. If $a = 2b$ or $2a = b$, then $R_{2b} \neq 0$ and $R_b = \langle e_{23} \rangle$, hence $x_1 x_2$, with $\deg x_1 = \deg x_2 = b$, is a non-trivial monomial identity, which is a contradiction. Therefore we get $a \neq 2b$ and $2a \neq b$.

Now assume R is endowed with the elementary grading induced by $(0, a, a + b, 2a + b)$ with $a, b > 0$, $a \neq 2b$ and $2a \neq b$. If $a = b$, then by Theorem 4.7 we have $T_G(R)$ is generated by (1) – (3). Now assume that $a \neq b$; in this case the non-zero homogeneous components of R are

$$\begin{aligned} R_0 &= \langle e_{11}, e_{22}, e_{33}, e_{44} \rangle, R_a = \langle e_{12}, e_{34} \rangle, R_b = \langle e_{23} \rangle, \\ R_{a+b} &= \langle e_{13}, e_{24} \rangle, R_{2a+b} = \langle e_{14} \rangle, R_{-a} = \langle e_{21}, e_{43} \rangle, \\ R_{-b} &= \langle e_{32} \rangle, R_{-a-b} = \langle e_{31}, e_{42} \rangle, R_{-2a-b} = \langle e_{41} \rangle. \end{aligned}$$

By Theorem 3.5, we may consider only monomials of length up to 4. Let us assume first that there exists a monomial $M = x_1 x_2 x_3 x_4$ of length 4, with $\deg x_i = g_i \in G$, that is not a consequence of monomials of length less than 4. Note that if $R_{g_i} R_{g_{i+1}} = R_{g_i + g_{i+1}}$ for some $i \in \{1, 2, 3\}$, then the monomial N of degree 3 obtained by replacing $x_i x_{i+1}$ in M by a variable of degree $g_i + g_{i+1}$ is an identity for R . It is clear that M is a consequence of N , which is a contradiction. If M has a submonomial of degree 0, the same argument in the proof of Proposition 3.5 implies that M is a consequence of a monomial identity of length 3, and we may assume that M does not have

a submonomial of degree zero. In particular, each variable has non-zero degree. Direct verification shows that $\dim(R_g R_h) = 1$ whenever $0 \neq g, h \in \text{Supp}(R)$ and $0 \neq g + h \in \text{Supp}(R)$. Hence we get $\dim R_{g_i} R_{g_{i+1}} \leq 1$ for $i = 1, 2, 3$. If $R_{g_i} R_{g_{i+1}} = 0$ for some $i \in \{1, 2, 3\}$, then M is a consequence of a monomial identity of degree 2, which is a contradiction. Therefore we have $\dim R_{g_i} R_{g_{i+1}} = 1$ for $i = 1, 2, 3$. The monomials $x_1 x_2 x_3$, $x_2 x_3 x_4$ are not identities for R and, as a consequence, $\dim R_{g_1} R_{g_2} R_{g_3} = \dim R_{g_2} R_{g_3} R_{g_4} = 1$. This implies that there exist uniquely determined tuples (i_1, i_2, i_3, i_4) and (i'_2, i'_3, i'_4, i_5) such that $e_{i_1 i_2}, e_{i_2 i_3}, e_{i_3 i_4}$ have degree g_1, g_2, g_3 , respectively and $e_{i'_2 i'_3}, e_{i'_3 i'_4}, e_{i'_4 i_5}$ have degree g_2, g_3, g_4 , respectively. Note that $e_{i_2 i_4} = e_{i_2 i_3} e_{i_3 i_4}$ and $e_{i'_2 i'_4} = e_{i'_2 i'_3} e_{i'_3 i'_4}$ lie in $R_{g_2} R_{g_3}$. Since $\dim(R_{g_2} R_{g_3}) = 1$ we obtain $i_2 = i'_2$, $i_3 = i'_3$ and $i_4 = i'_4$. It means the matrices $e_{i_1 i_2}, e_{i_2 i_3}, e_{i_3 i_4}, e_{i_4 i_5}$ have degree g_1, g_2, g_3, g_4 , respectively. Then $e_{i_1 i_2} e_{i_2 i_3} e_{i_3 i_4} e_{i_4 i_5} = e_{i_1 i_5} \neq 0$, which is a contradiction, since M is a graded identity for R . Hence, every non-trivial identity of length 4 is a consequence of identities of shorter length.

We suppose now $m = x_1 x_2 x_3$ is a monomial identity of degree 3, with $\deg x_i = g_i$, for $i = 1, 2, 3$. First we observe that if $\dim R_{g_2} = 1$, then m is a consequence of one of the monomial identities $x_{g_1} x_{g_2}$ or $x_{g_2} x_{g_3}$, or $\dim R_{g_1} R_{g_2} = \dim R_{g_2} R_{g_3} = 1$ and, in this case, m is not an identity. Hence, it remains to consider the case $\dim R_{g_2} = 2$. We must have $\dim R_{g_3} = 2$ or $\dim R_{g_1} = 2$, otherwise, by direct verification, m is a consequence of a monomial of length 2. In the cases above, we may verify that $R_{g_2} R_{g_3} = R_{g_2+g_3}$ or $R_{g_1} R_{g_2} = R_{g_1+g_2}$. In any case, m is a consequence of a monomial identity of length 2.

Finally we consider the case of monomial identities of degree 2. In this case, we just need to observe that they are consequences of $x_g = 0$, with $g \notin \text{Supp}(\Gamma)$. One can check that, since $a, b > 0$, this happens if and only if $a \neq b$, $a \neq 2b$ and $b \neq 2a$. \square

Now we are going to show an analogous result about $M_5(F)$.

Theorem 4.13. *Let G be a linearly ordered abelian group. Let R denote the algebra $M_5(F)$ with a G -grading such that the neutral component is commutative. The T_G -ideal $T_G(R)$ is generated, as a T_G -ideal, by the identities (1) – (3) if and only if there exist $a, b > 0$ in G , with $a \neq 2b$, $b \neq 2a$, $a \neq 3b$ and $a \neq 4b$, such that R is isomorphic to the elementary grading on $M_5(F)$ induced by $\bar{g} = (0, a, a+b, a+2b, 2a+2b)$.*

Proof. We assume that $T_G(R)$ is generated by the identities (1) – (3). Proposition 4.9 and Remark 4.11 imply that there exist $a, b > 0$ in G such that R is isomorphic to the elementary grading induced by

$(0, a, a + b, a + 2b, 2a + 2b)$. Now we prove that $a \neq 2b$, $b \neq 2a$, $a \neq 3b$ and $a \neq 4b$. If $a = b$ then these inequalities are clearly true, now assume that $a \neq b$. If $a = 2b$, then $2a = a + 2b \in \text{Supp}(R)$ and x_1x_2 , with $\deg x_1 = \deg x_2 = a$, is a non-trivial graded monomial identity. If $2a = b$, then $2a = b \in \text{Supp}(R)$ and x_1x_2 , with $\deg x_1 = \deg x_2 = a$, is a non-trivial graded monomial identity. If $a = 3b$, then $a - b = 2b \in \text{Supp}(R)$ and $x_1x_2 = 0$, with $\deg(x_1) = a$ and $\deg(x_2) = -b$, is a non-trivial graded monomial identity again. Finally, if $a = 4b$, then $4b \in \text{Supp}(R)$ and $x_1x_2 = 0$, with $\deg(x_1) = 2b$ and $\deg(x_2) = 2b$, is a non-trivial graded monomial identity. Since by assumption, R is generated by (1) – (3), it does not satisfy non-trivial graded monomial identities and this implies that $a \neq 2b$, $b \neq 2a$, $a \neq 3b$ and $a \neq 4b$.

Now assume the elementary grading on R is induced by $(0, a, a + b, a + 2b, 2a + 2b)$ with $a, b > 0$ in G and $a \neq 2b$. If $a = b$, then Theorem 4.7 implies that $T_G(R)$ is generated by (1) – (3). Now assume $a \neq b$. In this case the non-zero homogeneous components of R are

$$\begin{aligned} R_0 &= \langle e_{11}, e_{22}, e_{33}, e_{44}, e_{55} \rangle, R_a = \langle e_{12}, e_{45} \rangle, R_{a+b} = \langle e_{13}, e_{35} \rangle, \\ R_{a+2b} &= \langle e_{14}, e_{25} \rangle, R_{2a+2b} = \langle e_{15} \rangle, R_b = \langle e_{23}, e_{34} \rangle, R_{2b} = \langle e_{24} \rangle \\ R_{-a} &= \langle e_{21}, e_{54} \rangle, R_{-a-b} = \langle e_{31}, e_{53} \rangle, R_{-a-2b} = \langle e_{41}, e_{52} \rangle, \\ R_{-2a-2b} &= \langle e_{51} \rangle, R_{-b} = \langle e_{32}, e_{43} \rangle, R_{-2b} = \langle e_{42} \rangle \end{aligned}$$

Direct calculations show that $\dim(R_g R_h) = 1$ whenever $0 \neq g, h \in \text{Supp}(R)$ and $0 \neq g + h \in \text{Supp}(R)$. Again, by Theorem 3.5, we may consider only monomials of length up to 5. Following the same arguments in the proof of Theorem 4.12, we obtain any monomial identity of length 3, 4, or 5 is a consequence of a monomial identity of shorter length.

Finally, we consider the case of monomial identities of degree 2. By direct computations we obtain that, since $a, b > 0$, the graded monomial identities of length 2 are consequences of $x_g = 0$, with $g \notin \text{Supp}(\Gamma)$ if $a \neq 2b$ and $b \neq 2a$, $a \neq 3b$ and $a \neq 4b$ and we are done. \square

The problem of the existence of non-trivial monomial identities has the following combinatorial interpretation. Let (g_1, \dots, g_n) be a tuple of integers and consider the set

$$S := \{g_i - g_j, 1 \leq i, j \leq n\}.$$

For any $k \in S$ let X_k be the matrix with 1 in the positions (i, j) such that $g_j - g_i = k$ and 0 elsewhere. We consider the complete directed graph G with vertices $\{v_1, \dots, v_n\}$ (i.e. a graph with arrows $v_i \rightarrow v_j$ for every $1 \leq i, j \leq n$) and arrows $v_i \rightarrow v_j$ labeled by $g_i - g_j$. Then for $k_1, \dots, k_t \in S$, the (i, j) -th entry of $X_{k_1} \cdots X_{k_t}$ is the number of paths

from v_i to v_j going through t vertices (not necessarily distinct) such that the sequence of numbers in the arrows of the path is (k_1, \dots, k_t) . We now consider the following "elementary" related problem. We say (g_1, \dots, g_n) is a *good sequence* of integers if $X_{k_1} \cdots X_{k_t} = 0$ implies that there exist $1 \leq i \leq j \leq t$ such that $k_i + \cdots + k_j \notin S$ or one of the products $X_{k_1} \cdots X_{k_{i-1}}$, $X_{k_2} \cdots X_{k_t}$ is zero.

Problem 4.14. *Determine all good sequences of integers.*

5. COMPETING INTERESTS DECLARATION

The author(s) declare none.

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