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Nontrivial solutions for a class of gradient–type quasilinear elliptic systems *

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Abstract

The aim of this paper is investigating the existence of weak bounded solutions of the gradient–type quasilinear elliptic system

$$(P) \qquad \begin{cases} -\mathrm{div}(a_i(x,u_i,\nabla u_i)) + A_{i,t}(x,u_i,\nabla u_i) = G_i(x,\mathbf{u}) & \text{in } \Omega \\ \qquad\qquad \text{for } \ i \in \{1,\dots,m\}, & \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $m \geq 2$ and $\mathbf{u} = (u_1,\dots,u_m)$, where $\Omega \subset \mathbb{R}^N$ is an open bounded domain and some functions $A_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $i \in \{1,\dots,m\}$, and $G : \Omega \times \mathbb{R}^m \to \mathbb{R}$ exist such that $a_i(x,t,\xi) = \nabla_\xi A_i(x,t,\xi)$, $A_{i,t}(x,t,\xi) = \frac{\partial A_i}{\partial t}(x,t,\xi)$ and $G_i(x,\mathbf{u}) = \frac{\partial G}{\partial u_i}(x,\mathbf{u})$.

We prove that, under suitable hypotheses, the functional $\mathcal{J}$ related to problem (P) is $\mathcal{C}^1$ on a "good" Banach space X and satisfies the weak Cerami–Palais–Smale condition. Then, generalized versions of the Mountain Pass Theorems allow us to prove the existence of at least one critical point and, if $\mathcal{J}$ is even, of infinitely many ones, too.

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1 Introduction

In this paper we look for weak bounded solutions of the following class of gradient–type quasilinear elliptic systems

$$\begin{cases} -\mathrm{div}(a_i(x,u_i,\nabla u_i)) + A_{i,t}(x,u_i,\nabla u_i) = G_i(x,\mathbf{u}) & \text{in } \Omega \\ \qquad\qquad \text{for } \ i \in \{1,\dots,m\}, & \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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with $m \geq 2$ and $\mathbf{u} = (u_1, \dots, u_m)$, where $\Omega \subset \mathbb{R}^N$ is an open bounded domain and some functions $A_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $i \in \{1, \dots, m\}$, and $G : \Omega \times \mathbb{R}^m \to \mathbb{R}$ exist such that

$$A_{i,t}(x,t,\xi) = \frac{\partial A_i}{\partial t}(x,t,\xi),\ a_i(x,t,\xi) = \left(\frac{\partial A_i}{\partial \xi_1}(x,t,\xi), \dots, \frac{\partial A_i}{\partial \xi_N}(x,t,\xi)\right) \tag{1.2}$$

if $1 \leq i \leq m$ and $\nabla_{\mathbf{u}} G(x,\mathbf{u}) = (G_1(x,\mathbf{u}), \dots, G_m(x,\mathbf{u})$, i.e.,

$$G_i(x,\mathbf{u}) = \frac{\partial G}{\partial u_i}(x,\mathbf{u}) \qquad \text{if } 1 \leq i \leq m. \tag{1.3}$$

A special model of system (1.1) is obtained if $A_i(x,t,\xi) = \frac{1}{p_i}\bar{A}_i(x,t)|\xi|^{p_i}$, $p_i > 1$, so (1.1) reduces to problem

$$\begin{cases} -\mathrm{div}(\bar{A}_i(x,u_i)|\nabla u_i|^{p_i-2}\nabla u_i) + \frac{1}{p_i}\bar{A}_{i,t}(x,u_i)|\nabla u_i|^{p_i} = G_i(x,\mathbf{u}) & \text{in } \Omega \\ \qquad\qquad \text{for } \ i \in \{1,\dots,m\}, & \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\bar{A}_{i,t}(x,t) = \frac{\partial \bar{A}_i}{\partial t}(x,t)$, which has been studied in [12] if $m = 2$, and generalizes the classical gradient-type $(p_1, \dots, p_m)$–Laplacian system

$$\begin{cases} -\Delta_{p_i} u_i = G_i(x,\mathbf{u}) & \text{in } \Omega, \text{ for } i \in \{1,\dots,m\}, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

which has been widely analized in the past (see, e.g., [4, 13, 19, 21]).

Many variants of system (1.1) have been studied by using several theories such as the fixed point index, the cohomological index, the sub–super solutions methods, the bifurcation theory and also some non–variational techniques (see, e.g., [2, 4, 6, 13, 14, 15, 18, 21] and references therein).

On the contrary, here we use the variational approach introduced in [7, 8], and already applied to systems in [12], so that, under suitable hypotheses, finding solutions of problem (1.1) turns into searching critical points of the functional

$$\mathcal{J}(\mathbf{u}) \ = \ \sum_{i=1}^{m} \int_{\Omega} A_i(x,u_i,\nabla u_i)dx \ - \ \int_{\Omega} G(x,\mathbf{u})dx \tag{1.4}$$

in a suitable Banach space X obtained as product of the intersection spaces between Sobolev spaces and $L^\infty(\Omega)$ (for more details, see Section 3).

Differently from the problem in [12], here we deal with a more general system. In spite of it, a regularity result on $\mathcal{J}$ in X can be proved under basic assumptions on $G(x,\mathbf{u})$ and suitable growth conditions on the $\mathcal{C}^1$–Carathéodory function $A_i(x,t,\xi)$ and its partial derivatives, $i \in \{1,\dots,m\}$, but pointing out that it has to growth with power $p_i > 1$ with respect to ξ (see Proposition 3.5). Since, in general, $\mathcal{J}$ satisfies neither the Palais–Smale condition nor its classical Cerami variant (see [10, Example 4.3]), following the lead of ideas developed in [8], which exploit the interaction between two different norms on X, we inquire whether $\mathcal{J}$ verifies the weak Cerami–Palais–Smale condition (see Definition 2.1) so to apply the abstract theorems in [9]. To this aim, by considering not the interplay just between $A_i(x,t,\xi)$ and the partial derivative $A_{i,t}(x,t,\xi)$ as in [12], but the interaction among them and $a_i(x,t,\xi)$ (see the hypotheses at the beginning of Section 4), together with an Ambrosetti–Rabinowitz type condition and suitable subcritical growth assumptions on $G(x,\mathbf{u})$, we have that the weak Cerami–Palais–Smale condition holds (see Proposition 4.8). Then, suitable requirements on the behavior of $G(x,\mathbf{u})$ in a neighborhood of the origin, respectively at infinity, allow us to state an existence result, respectively a multiplicity one if $\mathcal{J}$ is even by means of a "good" decomposition of the Sobolev spaces $W_0^{1,p_i}(\Omega)$ as given in [8, Section 5].

Anyway, in order to not weigh this introduction down with too many details, we prefer to specify each hypothesis when required and to state our main results at the beginning of Section 5 (see Theorems 5.1 and 5.2). Note that, here, we look for bounded solutions of problem (1.1), so we introduce some subcritical growth hypotheses on $G(x,\mathbf{u})$ which are stronger than the usual ones (compare (3.8) with [4, Theorem 3]).

This paper is organized as follows. In Section 2 we introduce the abstract tools and, in particular, the weak Cerami–Palais–Smale condition and some related existence and multiplicity results which generalize classical Mountain Pass Theorems. In Section 3, we introduce the variational setting and give the first assumptions in order to prove the variational principle required by problem (1.1). Then, in Section 4 we prove that functional $\mathcal{J}$ satisfies the weak Cerami–Palais–Smale condition and, finally, in Section 5, our main results are stated and proved.

2 Abstract tools

We denote $\mathbb{N} = \{1,2,\dots\}$ and, throughout this section, we assume that:

- $(X, \|\cdot\|_X)$ is a Banach space with dual $(X', \|\cdot\|_{X'})$;
- $(W, \|\cdot\|_W)$ is a Banach space such that $X \hookrightarrow W$ continuously, i.e. $X \subset W$ and a constant $\sigma_0 > 0$ exists such that
$$\|y\|_W \ \le\ \sigma_0\ \|y\|_X \qquad \text{for all } y \in X;$$
- $J : \mathcal{D} \subset W \to \mathbb{R}$ and $J \in C^1(X,\mathbb{R})$ with $X \subset \mathcal{D}$.

Anyway, in order to avoid any ambiguity and simplify, when possible, the notation, from now on by X we denote the space equipped with its given norm $\|\cdot\|_X$ while, if the norm $\|\cdot\|_W$ is involved, we write it explicitly.

For simplicity, taking $\beta \in \mathbb{R}$, we say that a sequence $(y_n)_n \subset X$ is a *Cerami–Palais–Smale sequence at level* β, briefly $(CPS)_\beta$*–sequence*, if

$$\lim_{n\to+\infty} J(y_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(y_n)\|_{X'}(1+\|y_n\|_X) = 0.$$

As $(CPS)_\beta$–sequences may exist which are unbounded in $\|\cdot\|_X$ but converge with respect to $\|\cdot\|_W$ (in our setting with $m = 1$, see [10, Example 4.3]), we have to weaken the classical Cerami–Palais–Smale condition in a suitable way according to the ideas already developed in previous papers (see, e.g., [7, 8, 9]).

Definition 2.1. The functional J satisfies the *weak Cerami–Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ *condition*, if for every $(CPS)_\beta$–sequence $(y_n)_n$, a point $y \in X$ exists, such that *(i)* $\lim_{n\to+\infty} \|y_n - y\|_W = 0$ (up to subsequences),

(ii) $J(y) = \beta, \ dJ(y) = 0$.

If J satisfies the $(wCPS)_\beta$ condition at each level $\beta \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I.

Since the $(wCPS)_\beta$ condition allows one to prove a Deformation Lemma (see [9, Lemma 2.3]), then the following generalization of the Mountain Pass Theorem can be stated (see [9, Theorem 1.7] and compare it with the classical statement [20, Theorem 2.2]).

Theorem 2.2. *Let $J \in C^1(X,\mathbb{R})$ be such that $J(0) = 0$ and the $(wCPS)$ condition holds in $\mathbb{R}_+$. Moreover, assume that there exist some constants R_0, $\varrho_0 > 0$, and a point $e \in X$ such that*

(i) $y \in X, \quad \|y\|_W = R_0 \quad \Longrightarrow \quad J(y) \geq \varrho_0$;

(ii) $\|e\|_W > R_0$ *and* $J(e) < \varrho_0$.

Then, J has a Mountain Pass critical point $y \in X$ such that $J(y) \geq \varrho_0$.

Furthermore, with the stronger assumption that J is even, also the symmetric Mountain Pass Theorem can be generalized as follows (see [11, Theorem 2.4] or also [9, Theorem 1.8] and compare it with [20, Theorem 9.12] and [3, Theorem 2.4]).

Theorem 2.3. *Let $J \in C^1(X,\mathbb{R})$ be an even functional such that $J(0) = 0$ and the $(wCPS)$ condition holds in $\mathbb{R}_+$. Moreover, assume that $\varrho > 0$ exists so that:*

$(\mathcal{H}_\varrho)$ *three closed subsets V_ϱ, Y_ϱ and $\mathcal{M}_\varrho$ of X and a constant $R_\varrho > 0$ exist which satisfy the following conditions:*

(i) *V_ϱ and Y_ϱ are subspaces of X such that*

$$V_\varrho + Y_\varrho = X, \qquad \operatorname{codim} Y_\varrho \ < \ \dim V_\varrho \ < \ +\infty;$$

(ii) *$\mathcal{M}_\varrho = \partial\mathcal{N}$, where $\mathcal{N} \subset X$ is a neighborhood of the origin which is symmetric and bounded with respect to $\|\cdot\|_W$;*

(iii) $y \in \mathcal{M}_\varrho \cap Y_\varrho \quad \Longrightarrow \quad J(y) \geq \varrho$;

(iv) $y \in V_\varrho, \quad \|y\|_X \geq R_\varrho \quad \Longrightarrow \quad J(y) \leq 0$.

Then, if we put

$$\beta_\varrho \ = \ \inf_{\gamma\in\Gamma_\varrho} \sup_{y\in V_\varrho} J(\gamma(y)),$$

with

$$\Gamma_\varrho = \{\gamma : X \to X : \gamma \text{ odd homeomeorphism such that } \gamma(y) = y \text{ if } y \in V_\varrho \text{ with } \|y\|_X \geq R_\varrho\},$$

functional J possesses at least a pair of symmetric critical points in X with corresponding critical level β_ϱ which belongs to $[\varrho, \varrho_1]$, where $\varrho_1 \geq \sup_{y\in V_\varrho} J(y) > \varrho$.

If we can apply infinitely many times Theorem 2.3, then the following multiplicity abstract result can be stated.

Corollary 2.4. *Let $J \in C^1(X, \mathbb{R})$ be an even functional such that $J(0) = 0$, the $(wCPS)$ condition holds in $\mathbb{R}_+$ and a sequence $(\varrho_n)_n \subset]0, +\infty[$ exists such that $\varrho_n \nearrow +\infty$ and assumption $(\mathcal{H}_{\varrho_n})$ holds for all $n \in \mathbb{N}$.*
Then, functional J possesses a sequence of critical points $(u_{k_n})_n \subset X$ such that $J(u_{k_n}) \nearrow +\infty$ as $n \nearrow +\infty$.

3 Variational setting and first properties

From now on, let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, $N \geq 2$, and $m \in \mathbb{N}$ such that $m \geq 2$, so we denote by:

- $\mathbf{u} = (u_1, \dots, u_m)$, $\mathbf{u}_n = (u_1^n, \dots, u_m^n)$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^m$;
- $\{\mathbf{e}_j : 1 \leq j \leq m\}$ the standard basis of the Euclidean space $\mathbb{R}^m$, i.e., $\mathbf{e}_j$ has components $e_i^j = \delta_i^j$;
- $L^r(\Omega) = L^r(\Omega, \mathbb{R})$, $1 \leq r < +\infty$, the classical Lebesgue space with norm $|u|_r = \left(\int_\Omega |u|^r dx\right)^{1/r}$;
- $L^\infty(\Omega) = L^\infty(\Omega, \mathbb{R})$ the space of Lebesgue–measurable essentially bounded functions with norm $|u|_\infty = \operatorname{ess\,sup}_\Omega |u|$;
- $W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega, \mathbb{R})$ the classical Sobolev space equipped with norm $\|u\|_{W_0^{1,p}} = |\nabla u|_p$ if $1 \leq p < +\infty$;
- $\operatorname{meas}(D)$ the usual Lebesgue measure of a measurable set D in $\mathbb{R}^N$.

For simplicity, here and in the following we denote by $|\cdot|$ the standard norm on any Euclidean space, as the dimension of the considered vector is clear and no ambiguity occurs. Moreover, for short, we replace

$$\sum_{i=1}^{m} \quad \text{with} \quad \sum_{i} \qquad \text{and} \qquad \sum_{\substack{j=1\\ j\neq i}}^{m} \quad \text{with} \quad \sum_{j\neq i}.$$

Definition 3.1. A function $f : \Omega \times \mathbb{R}^l \to \mathbb{R}$, $l \in \mathbb{N}$, is a $\mathcal{C}^k$–Carathéodory function, $k \in \mathbb{N} \cup \{0\}$, if

- $f(\cdot, \omega) : x \in \Omega \mapsto f(x, \omega) \in \mathbb{R}$ is measurable for all $\omega \in \mathbb{R}^l$,
- $f(x, \cdot) : \omega \in \mathbb{R}^l \mapsto f(x, \omega) \in \mathbb{R}$ is $\mathcal{C}^k$ for a.e. $x \in \Omega$.

For each $i \in \{1, \dots, m\}$, let $A_i : (x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto A_i(x, t, \xi) \in \mathbb{R}$ be a given function such that the following conditions hold:

(h_0) $A_i(x, t, \xi)$ is a $\mathcal{C}^1$–Carathéodory function with partial derivatives

$$A_{i,t}(x, t, \xi) \quad \text{and} \quad a_i(x, t, \xi)$$

as in (1.2);

(h_1) a power $p_i > 1$ and some positive continuous functions Φ_0^i, ϕ_0^i, Φ_1^i, ϕ_1^i, Φ_2^i, $\phi_2^i : \mathbb{R} \to \mathbb{R}$ exist such that

$$
\begin{aligned}
|A_i(x,t,\xi)| &\leq \Phi_0^i(t) + \phi_0^i(t)|\xi|^{p_i}, && (3.1)\\
|A_{i,t}(x,t,\xi)| &\leq \Phi_1^i(t) + \phi_1^i(t)|\xi|^{p_i}, && (3.2)\\
|a_i(x,t,\xi)| &\leq \Phi_2^i(t) + \phi_2^i(t)|\xi|^{p_i-1}, && (3.3)
\end{aligned}
$$

for a.e. $x \in \Omega$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

So, taking $p_i > 1$ as in (h_1), we consider the related Sobolev space

$$W_i = W_0^{1,p_i}(\Omega) \quad \text{with norm } \|\cdot\|_{W_i} = \|\cdot\|_{W_0^{1,p_i}}.$$

From the Sobolev Embedding Theorem, W_i is continuously embedded in $L^r(\Omega)$ for any $r \in [1, p_i^*]$ with $p_i^* = \frac{Np_i}{N-p_i}$ if $N > p_i$, or $r \in [1,+\infty[$ with $p_i^* = +\infty$ if $p_i \geq N$, i.e., for such an r a positive constant $\tau_{i,r}$ exists such that

$$|u|_r \leq \tau_{i,r}\|u\|_{W_i} \quad \text{for all } u \in W_i. \tag{3.4}$$

For simplicity, we put

$$\frac{1}{p_i^*} = 0 \quad \text{if} \quad p_i^* = +\infty. \tag{3.5}$$

Now, assume that a function $G : (x,\mathbf{u}) \in \Omega \times \mathbb{R}^m \mapsto G(x,\mathbf{u}) \in \mathbb{R}$ exists such that

(g_0) $G(x,\mathbf{u})$ is a $\mathcal{C}^1$–Caratheodory function with partial derivatives $G_i(x,\mathbf{u})$ as in (1.3) such that

$$G(\cdot,\mathbf{0}) \in L^\infty(\Omega)$$

and

$$G_i(x,\mathbf{0}) = 0 \quad \text{for a.e. } x \in \Omega, \text{ for each } i \in \{1,\dots,m\};$$

(g_1) for every i, $j \in \{1,\dots,m\}$, $j \neq i$, some real numbers $q_i \geq 1$, $s_{i,j} \geq 0$ and a constant $\sigma > 0$ exist such that

$$|G_i(x,\mathbf{u})| \leq \sigma\left(1 + |u_i|^{q_i-1} + \sum_{j\neq i}|u_j|^{s_{i,j}}\right) \tag{3.6}$$

for a.e. $x \in \Omega$ and all $\mathbf{u} \in \mathbb{R}^m$, with

$$1 \leq q_i < p_i^* \tag{3.7}$$

and

$$0 \leq s_{i,j} < \frac{p_i}{N}\left(1 - \frac{1}{p_i^*}\right)p_j^*. \tag{3.8}$$

Remark 3.2. For a.e. $x \in \Omega$ and all $\mathbf{u} \in \mathbb{R}^m$, condition (g_0) together with Mean Value Theorem implies that $t \in]0,1[$ exists such that

$$|G(x,\mathbf{u})| \leq |G(\cdot,\mathbf{0})|_\infty + \sum_i |G_i(x,t\mathbf{u})||u_i|,$$

then from (3.6) it follows that

$$|G(x,\mathbf{u})| \ \le\ \sigma_1 \sum_i \left(1 + |u_i|^{q_i} + \sum_{j\ne i} |u_i||u_j|^{s_{i,j}}\right) \tag{3.9}$$

for a.e. $x \in \Omega$, all $\mathbf{u} \in \mathbb{R}^m$, with $\sigma_1 > 0$ which depends on σ, m and $|G(\cdot,\mathbf{0})|_\infty$.

In order to investigate the existence of weak solutions of the nonlinear problem (1.1) as critical points of $\mathcal{J}$ defined as in (1.4), we have to introduce the "right" Banach space. To this aim, the notation introduced for the abstract setting in Section 2 is referred to

$$W = W_1 \times \cdots \times W_m \tag{3.10}$$

with norm

$$\|\mathbf{u}\|_W = \|(u_1,\dots,u_m)\|_W = \sum_i \|u_i\|_{W_i} \quad \text{if } \mathbf{u} \in W, \tag{3.11}$$

while the Banach space $(X, \|\cdot\|_X)$ is defined as

$$X = X_1 \times \cdots \times X_m \tag{3.12}$$

with norm

$$\|\mathbf{u}\|_X = \|(u_1,\dots,u_m)\|_X = \sum_i \|u_i\|_{X_i} \quad \text{if } \mathbf{u} \in X,$$

where, for any $i \in \{1,\dots,m\}$, it is

$$X_i := W_i \cap L^\infty(\Omega)$$

equipped with norm

$$\|u\|_{X_i} = \|u\|_{W_i} + |u|_\infty \qquad \text{if } u \in X_i. \tag{3.13}$$

We note that, setting

$$L = L^\infty(\Omega) \times \cdots \times L^\infty(\Omega)$$

with norm

$$\|\mathbf{u}\|_L = \|(u_1,\dots,u_m)\|_L = \sum_i |u_i|_\infty \quad \text{if } \mathbf{u} \in L,$$

we have that X in (3.12) can also be written as

$$X = W \cap L \qquad \text{with} \qquad \|\mathbf{u}\|_X = \|\mathbf{u}\|_W + \|\mathbf{u}\|_L.$$

For every $i \in \{1,\dots,m\}$, we have that $(W_i, \|\cdot\|_{W_i})$ is a reflexive Banach space and, by definition, it is $X_i \hookrightarrow W_i$ and $X_i \hookrightarrow L^\infty(\Omega)$ with continuous embeddings. Thus, also $(W, \|\cdot\|_W)$ is a reflexive Banach space and, obvioulsy, $X \hookrightarrow W$ and $X \hookrightarrow L$ with continuous embeddings, too.

Remark 3.3. If $i \in \{1,\dots,m\}$ is such that $p_i > N$, then $X_i = W_i$, as $W_i \hookrightarrow L^\infty(\Omega)$. So, in general, if an $i \in \{1,\dots,m\}$ exists such that $p_i \le N$ then $X \ne W$, but if for all $i \in \{1,\dots,m\}$ it is $p_i > N$, then $X = W$ and the classical Mountain Pass Theorems in [1] can be used, if required.

If conditions (h_0)–(h_1), (g_0) and (3.6) hold, by (3.9) and direct computations it follows that $\mathcal{J}(\mathbf{u})$ in (1.4) is well defined for all $\mathbf{u} \in X$. Moreover, taking any $\mathbf{u}, \mathbf{v} \in X$, the Gâteaux differential of functional $\mathcal{J}$ in $\mathbf{u}$ along the direction $\mathbf{v}$ is well defined as

$$d\mathcal{J}(\mathbf{u})[\mathbf{v}] = \sum_i \Big(\int_\Omega a_i(x, u_i, \nabla u_i) \cdot \nabla v_i \, dx + \int_\Omega A_{i,t}(x, u_i, \nabla u_i) v_i \, dx - \int_\Omega G_i(x, \mathbf{u}) v_i \, dx \Big). \tag{3.14}$$

We note that, since $\mathbf{u}, \mathbf{v} \in X$ imply that $\mathbf{u}, \mathbf{v} \in L$, no critical growth upper bound on the powers q_i and $s_{i,j}$ is required in order to have $d\mathcal{J}(\mathbf{u})[\mathbf{v}] \in \mathbb{R}$.

For smplicity, for every $i \in \{1, \dots, m\}$ we introduce the i–th partial derivative of $\mathcal{J}$ in $\mathbf{u} \in X$ as

$$\frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}) : v \in X_i \mapsto \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u})[v] = d\mathcal{J}(\mathbf{u})[v \mathbf{e}_i] \in \mathbb{R},$$

where from (3.14) it follows that

$$\frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u})[v] = \int_\Omega a_i(x, u_i, \nabla u_i) \cdot \nabla v \, dx + \int_\Omega A_{i,t}(x, u_i, \nabla u_i) v \, dx - \int_\Omega G_i(x, \mathbf{u}) v \, dx. \tag{3.15}$$

Remark 3.4. Taking $\mathbf{u} \in X$, since $d\mathcal{J}(\mathbf{u}) \in X'$, then

$$\frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}) \in X_i' \qquad \text{for all } i \in \{1, \dots, m\}$$

and

$$d\mathcal{J}(\mathbf{u})[\mathbf{v}] = \sum_i \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u})[v_i] \qquad \text{for all } \mathbf{v} = (v_1, \dots, v_m) \in X. \tag{3.16}$$

Moreover, direct computations imply that

$$\left\| \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}) \right\|_{X_i'} \leq \| d\mathcal{J}(\mathbf{u}) \|_{X'} \qquad \text{for all } i \in \{1, \dots, m\} \tag{3.17}$$

and

$$\| d\mathcal{J}(\mathbf{u}) \|_{X'} \leq \sum_i \left\| \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}) \right\|_{X_i'}. \tag{3.18}$$

Clearly, we have that

$$d\mathcal{J}(\mathbf{u}) = 0 \text{ in } X \quad \iff \quad \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}) = 0 \text{ in } X_i \quad \text{for all } i \in \{1, \dots, m\}.$$

Now, we can state a regularity result.

Proposition 3.5. *Suppose that conditions (h_0)–(h_1), (g_0) and (3.6) hold. Let $(\mathbf{u}_n)_n \subset X$ and $\mathbf{u} \in X$ be such that*

$$\mathbf{u}_n \to \mathbf{u} \text{ strongly in } W, \qquad \mathbf{u}_n \to \mathbf{u} \text{ a.e. in } \Omega \quad \text{if } n \to +\infty. \tag{3.19}$$

If $M > 0$ exists such that

$$\|\mathbf{u}_n\|_L \le M \quad \text{for all } n \in \mathbb{N}, \tag{3.20}$$

then

$$\mathcal{J}(\mathbf{u}_n) \to \mathcal{J}(\mathbf{u}) \quad \text{and} \quad \|d\mathcal{J}(\mathbf{u_n}) - d\mathcal{J}(\mathbf{u})\|_{X'} \to 0 \quad \text{as } n \to +\infty.$$

Hence, $\mathcal{J}$ is a C^1 functional on X with Fréchet differential defined as in (3.14).

Proof. Let $(\mathbf{u}_n)_n \subset X$ and $\mathbf{u} \in X$ be such that (3.19) and (3.20) hold. Taking any $i \in \{1, \dots, m\}$, from hypotheses (h_0)–(h_1), (3.19) and (3.20), by reasoning as in the proof of [8, Proposition 3.1], it follows that the "partial" functional

$$\mathcal{A}_i : u \in X_i \mapsto \int_\Omega A_i(x, u, \nabla u) dx \in \mathbb{R}$$

is such that

$$\mathcal{A}_i(u_i^n) \to \mathcal{A}_i(u_i) \quad \text{as } n \to +\infty.$$

Moreover, by means of Dominated Convergence Theorem, conditions (g_0), (3.9), (3.19) and (3.20) imply that

$$\int_\Omega G(x, \mathbf{u}_n) dx \longrightarrow \int_\Omega G(x, \mathbf{u}) dx.$$

Thus, summing up, we have that

$$\mathcal{J}(\mathbf{u}_n) \to \mathcal{J}(\mathbf{u}) \quad \text{as } n \to +\infty.$$

Now, we observe that (3.16) gives

$$\|d\mathcal{J}(\mathbf{u}_n) - d\mathcal{J}(\mathbf{u})\|_{X'} \le \sum_i \left\| \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n) - \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}) \right\|_{X_i'},$$

so, in order to prove that $\|d\mathcal{J}(\mathbf{u_n}) - d\mathcal{J}(\mathbf{u})\|_{X'} \to 0$, it is enough to verify that

$$\left\| \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n) - \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}) \right\|_{X_i'} \to 0 \quad \text{for all } i \in \{1, \dots, m\}. \tag{3.21}$$

To this aim, fixing any $i \in \{1, \dots, m\}$ and taking $v \in X_i$ such that $\|v\|_{X_i} \le 1$, from (3.15) we have that

$$\begin{aligned} \left| \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n)[v] - \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u})[v] \right| &\le \int_\Omega |a_i(x, u_i^n, \nabla u_i^n) - a_i(x, u_i, \nabla u_i)||\nabla v| dx \\ &+ \int_\Omega |A_{i,t}(x, u_i^n, \nabla u_i^n) - A_{i,t}(x, u_i, \nabla u_i)| dx + \int_\Omega |G_i(x, \mathbf{u}_n) - G_i(x, \mathbf{u})| dx, \end{aligned}$$

where, by reasoning as in [8, Proposition 3.1], it can be proved that

$$\int_\Omega |a_i(x, u_i^n, \nabla u_i^n) - a_i(x, u_i, \nabla u_i)||\nabla v|dx \to 0 \quad \text{uniformly with respect to } v$$

and

$$\int_\Omega |A_{i,t}(x, u_i^n, \nabla u_i^n) - A_{i,t}(x, u_i, \nabla u_i)|dx \to 0.$$

Moreover, (3.19), (3.20), hypotheses (g_0), (3.6) and, again, Dominated Convergence Theorem, imply that

$$\int_\Omega |G_i(x, \mathbf{u}_n) - G_i(x, \mathbf{u})|dx \to 0.$$

Hence, summing up, we conclude that

$$\left|\frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n)[v] - \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u})[v]\right| \to 0 \quad \text{uniformly with respect to } v \in X_i,\ \|v\|_{X_i} \leq 1,$$

and, by the arbitrariness of $i \in \{1, \ldots, m\}$, it follows that (3.21) is satisfied, too. □ □

4 The weak Cerami–Palais–Smale condition

In order to prove some more properties of functional $\mathcal{J} : X \to \mathbb{R}$ defined as in (1.4), we require that not only (h_0)–(h_1) hold but also $R \geq 1$ exists such that for each $i \in \{1, \ldots, m\}$ function $A_i(x, t, \xi)$ and its partial derivatives in (1.2) satisfy the following conditions:

(h_2) a constant $\lambda > 0$ exists such that

$$a_i(x, t, \xi) \cdot \xi \geq \lambda |\xi|^{p_i} \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N$$

with $p_i > 1$ as in (h_1);

(h_3) some constants η_1, $\eta_2 > 0$ exist such that

$$A_i(x, t, \xi) \leq \eta_1 a_i(x, t, \xi) \cdot \xi \quad \text{a.e. in } \Omega \text{ if } |(t, \xi)| \geq R, \tag{4.1}$$

$$\sup_{|(t,\xi)| \leq R} |A_i(x, t, \xi)| \leq \eta_2 \quad \text{a.e. in } \Omega; \tag{4.2}$$

(h_4) a constant $\mu_1 > 0$ exists such that

$$a_i(x, t, \xi) \cdot \xi + A_{i,t}(x, t, \xi)t \geq \mu_1 a_i(x, t, \xi) \cdot \xi \quad \text{a.e. in } \Omega \text{ if } |(t, \xi)| \geq R;$$

(h_5) taking $p_i > 1$ as in (h_1), some positive constants $\theta_i, \mu_2 > 0$ exist such that

$$A_i(x, t, \xi) - \theta_i a_i(x, t, \xi) \cdot \xi - \theta_i A_{i,t}(x, t, \xi)t \geq \mu_2 a_i(x, t, \xi) \cdot \xi$$

a.e. in Ω if $|(t, \xi)| \geq R$, with

$$\theta_i < \frac{1}{p_i}; \tag{4.3}$$

(h_6) for all $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$, it is

$$[a_i(x,t,\xi) - a_i(x,t,\xi')] \cdot [\xi - \xi'] > 0 \quad \text{a.e. in } \Omega, \quad \text{for all } t \in \mathbb{R}.$$

Moreover, let us assume that function $G(x, \mathbf{u})$ satisfies not only hypotheses (g_0)–(g_1) but also the following Ambrosetti–Rabinowitz type condition:

(g_2) taking θ_i as in (h_5) for all $i \in \{1, \ldots, m\}$, we have that

$$0 < G(x, \mathbf{u}) \leq \sum_i \theta_i G_i(x, \mathbf{u}) u_i \quad \text{if } |\mathbf{u}| \geq R, \; \mathbf{u} = (u_1, \ldots, u_m),$$

for a.e. $x \in \Omega$, with $R > 0$ as in the previous set of hypotheses (h_3)–(h_5).

Remark 4.1. We note that hypothesis (h_4) is satisfied also if $t = 0$ and $|\xi| \geq R$, then from (h_2) we have $\mu_1 \leq 1$. Furthermore, hypotheses (h_4) and (h_5) give

$$A_i(x,t,\xi) \geq (\theta_i \mu_1 + \mu_2)\, a_i(x,t,\xi) \cdot \xi \quad \text{a.e. in } \Omega \;\; \text{if } |(t,\xi)| \geq R, \tag{4.4}$$

whence, from (h_2) we have that

$$A_i(x,t,\xi) \geq (\theta_i \mu_1 + \mu_2) \lambda |\xi|^{p_i} \geq 0 \quad \text{a.e. in } \Omega \;\; \text{if } |(t,\xi)| \geq R. \tag{4.5}$$

Summing up, from (4.5) and assumption (4.2) it follows that a positive constant η_3 exists such that

$$A_i(x,t,\xi) \geq (\theta_i \mu_1 + \mu_2) \lambda |\xi|^{p_i} - \eta_3 \quad \text{a.e. in } \Omega \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{4.6}$$

Remark 4.2. Taking $i \in \{1, \ldots, m\}$, we note that (3.1) in (h_1) is not required as hypothesis if (h_2)–(h_5) and (3.3) hold. In fact, in these assumptions not only (4.6) is satisfied but also direct computations imply that

$$A_i(x,t,\xi) \leq \eta_1 \Phi_2^i(t) + \eta_2 + \eta_1 (\Phi_2^i(t) + \phi_2^i(t)) |\xi|^{p_i}$$

a.e. in Ω, for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Hence, ($h_1$) can be replaced by the weaker condition

(h_1') assumptions (3.2) and (3.3) hold.

Furthermore, taking $t = 0$ and $|\xi| \geq R$ in both (4.1) and (h_5), from (h_2) it follows $\eta_1 \geq \mu_2 + \theta_i$ so, without loss of generality, we can assume that μ_2 is so that

$$\eta_1 - \mu_2 - \theta_i > 0.$$

Thus, in order to give better growth conditions on function $A_i(x,t,\xi)$, hypotheses (4.1) and (h_5) give

$$\left(\frac{\eta_1 - \mu_2 - \theta_i}{\eta_1 \theta_i} \right) A_i(x,t,\xi) \geq A_{i,t}(x,t,\xi) t \quad \text{a.e. in } \Omega \;\; \text{if } |(t,\xi)| \geq R. \tag{4.7}$$

Hence, (4.5), (4.7) with hypotheses (3.3), (4.1) and direct computations imply that $\eta_4 > 0$ exists such that

$$A_i(x,t,\xi) \leq \eta_4 |t|^{\frac{\eta_1 - \mu_2 - \theta_i}{\eta_1 \theta_i}} |\xi|^{p_i} \quad \text{a.e. in } \Omega \;\; \text{if } |t| \geq 1, \; |\xi| \geq R,$$

then (4.4) gives

$$a_i(x,t,\xi) \cdot \xi \leq \frac{\eta_4}{\theta_i \mu_1 + \mu_2} |t|^{\frac{\eta_1 - \mu_2 - \theta_i}{\eta_1 \theta_i}} |\xi|^{p_i} \quad \text{a.e. in } \Omega \;\; \text{if } |t| \geq 1, \; |\xi| \geq R.$$

Remark 4.3. Taking $i \in \{1, \dots, m\}$, from (g_2) we have that

$$0 < G(x, u\,\mathbf{e}_i) \leq \theta_i G_i(x, u\,\mathbf{e}_i)u \quad \text{ for a.e. } x \in \Omega, \ \text{ if } \ u \in \mathbb{R}, \, |u| \geq R.$$

Hence, (g_0) and direct computations imply that $h_i \in L^\infty(\Omega)$, $h_i(x) > 0$ for a.a. $x \in \Omega$, exists such that

$$G(x, u\,\mathbf{e}_i) \geq h_i(x)|u|^{\frac{1}{\theta_i}} \quad \text{ for a.a. } x \in \Omega, \ \text{ if } \ u \in \mathbb{R}, \ |u| \geq R. \tag{4.8}$$

Thus, if also (3.6) holds, from (3.9), (4.3) and (4.8) not only we obtain that

$$1 < p_i < \frac{1}{\theta_i} \leq q_i \tag{4.9}$$

but also

$$G(x, u\,\mathbf{e}_i) \geq h_i(x)|u|^{\frac{1}{\theta_i}} - \sigma_i \quad \text{ for a.a. } x \in \Omega, \ \text{ all } u \in \mathbb{R} \tag{4.10}$$

for a suitable $\sigma_i > 0$.

Example 4.4. Let us consider

$$G(x, \mathbf{u}) = \sum_i c_i |u_i|^{q_i} + c_* \prod_{i=1}^m |u_i|^{\gamma_i},$$

where we assume that

$$q_i > 1, \quad \gamma_i > 1 \qquad \text{for all } i \in \{1, \dots, m\}.$$

Then, (g_0) is verified and for any $i \in \{1, \dots, m\}$ we have

$$G_i(x, \mathbf{u}) = c_i q_i |u_i|^{q_i - 2} u_i + c_* \gamma_i |u_i|^{\gamma_i - 2} u_i \prod_{\substack{j=1 \\ j \neq i}}^m |u_j|^{\gamma_j} \quad \text{ for a.e. } x \in \Omega.$$

If, in addition, we suppose that

$$\gamma_i < q_i \qquad \text{for all } i \in \{1, \dots, m\},$$

then the generalized Young inequality and direct computations allow us to conclude that also (3.6) holds by taking

$$s_{i,j} = (m-1)\gamma_j \frac{q_i - 1}{q_i - \gamma_i} \qquad \text{for all } i,\, j \in \{1, \dots, m\},\, j \neq i.$$

Finally, if we have that

$$\sum_i \frac{\gamma_i}{q_i} \geq 1,$$

then hypothesis (g_2) is verified, too, by taking each $\theta_i \geq \frac{1}{q_i}$.

Up to now, no upper bound is required for the growth of the nonlinear term $G(x, \mathbf{u})$. Anyway, the subcritical assumptions (3.7) and (3.8) are required for proving the weak Cerami–Palais–Smale condition.

Remark 4.5. Let i, $j \in \{1, \ldots, m\}$, $j \neq i$, be such that $s_{i,j}$ verifies condition (3.8). Then, $\tilde{q}_i > 0$ exists such that

$$1 < \tilde{q}_i < p_i^*, \quad 0 \leq \widetilde{s}_{i,j} := \frac{s_{i,j}\tilde{q}_i}{\tilde{q}_i - 1} < p_j^*. \tag{4.11}$$

In fact, if both $p_i < N$ and $p_j < N$, then (3.8) implies that

$$p_i p_j^* - N s_{i,j} > 0 \quad \text{and} \quad 1 \leq \frac{p_j^*}{p_j^* - s_{i,j}} \leq \frac{p_i p_j^*}{p_i p_j^* - N s_{i,j}} < p_i^*,$$

hence, $\tilde{q}_i$ exists such that

$$\frac{p_i p_j^*}{p_i p_j^* - N s_{i,j}} < \tilde{q}_i < p_i^* \tag{4.12}$$

so that both estimates in (4.11) hold. On the other hand, if $p_i \geq N$, respectively $p_j \geq N$, with $p_i^* = +\infty$, respectively $p_j^* = +\infty$, and (3.5) imply that the conditions in (4.11) are less restrictive and the existence of $\tilde{q}_i$ and $\widetilde{s}_{i,j}$ is easier to prove.

Remark 4.6. Assume that (g_0) and (g_1) hold. If for every i, $j \in \{1, \ldots, m\}$, $j \neq i$, by using the same notations in Remark 4.5, from (4.11) and Young inequality it follows that

$$|u_i|\ |u_j|^{s_{i,j}} \ \leq\ \frac{|u_i|^{\tilde{q}_i}}{\tilde{q}_i} + \frac{\tilde{q}_i - 1}{\tilde{q}_i}|u_j|^{\widetilde{s}_{i,j}} \leq\ |u_i|^{\tilde{q}_i} + |u_j|^{\widetilde{s}_{i,j}} \quad \text{for all } \mathbf{u} \in \mathbb{R}^m. \tag{4.13}$$

Thus, from (3.9) and (4.13) we obtain that

$$|G(x, \mathbf{u})| \leq \sigma_1 \sum_i \Big(1 + |u_i|^{q_i} + \sum_{j \neq i}(|u_i|^{\tilde{q}_i} + |u_j|^{\widetilde{s}_{i,j}})\Big) \quad \text{for all } \mathbf{u} \in \mathbb{R}^m.$$

Hence, setting

$$\overline{q}_i = \max\{q_i, \tilde{q}_i, \widetilde{s}_{j,i} :\ 1 \leq j \leq m,\ j \neq i\} \quad \text{for any } i \in \{1, \ldots, m\}, \tag{4.14}$$

it follows that

$$|G(x, \mathbf{u})| \leq \sigma_0 \sum_i (1 + |u_i|^{\overline{q}_i}) \quad \text{for all } \mathbf{u} \in \mathbb{R}^m \tag{4.15}$$

for a suitable constant $\sigma_0 \geq \sigma_1$.
At last, if (g_2) is verified too, then from (4.9) and (4.11) we obtain that

$$1 < p_i < \frac{1}{\theta_i} \leq \overline{q}_i < p_i^* \qquad \text{for all } i \in \{1, \ldots, m\}. \tag{4.16}$$

Now, we show that the $(wCPS)$– condition holds. To this aim, the following boundedness result is required (for its proof, see [16, Theorem II.5.1]).

Lemma 4.7. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and consider $u \in W_0^{1,p}(\Omega)$ with $p \leq N$. Suppose that $\gamma > 0$ and $k_0 \in \mathbb{N}$ exist such that*

$$\int_{\Omega_k^+} |\nabla u|^p dx \ \leq\ \gamma \left(\int_{\Omega_k^+} (u - k)^r dx\right)^{\frac{p}{r}} + \gamma \sum_{l=1}^{\nu} k^{\alpha_l} [\text{meas}(\Omega_k^+)]^{1 - \frac{p}{N} + \varepsilon_l}$$

for all $k \geq k_0$*, with* $\Omega_k^+ := \{x \in \Omega : u(x) > k\}$ *and* $r, \nu, \alpha_l, \varepsilon_l$ *are positive constants such that*

$$1 \leq r < p^*, \qquad \varepsilon_l > 0, \qquad p \leq \alpha_l < \varepsilon_l p^* + p.$$

Then, $\operatorname{ess\,sup}_{\Omega} u$ *is bounded from above by a positive constant which can be chosen so that it depends only on* $\operatorname{meas}(\Omega), N, p, \gamma, k_0, r, \nu, \varepsilon_l, \alpha_l, |u|_{p^*}$ *(eventually,* $|u|_q$ *for some* $q > r$ *if* $p^* = +\infty$*).*

Proposition 4.8. *Assume that hypotheses* (h_0)*–*(h_6) *and* (g_0)*–*(g_2) *hold. Then functional* $\mathcal{J}$ *in* (1.4) *satisfies condition* $(wCPS)$ *in* $\mathbb{R}$*.*

Proof. Taking $\beta \in \mathbb{R}$, let $(\mathbf{u}_n)_n \subset X$, with $\mathbf{u}_n = (u_1^n, \dots, u_m^n)$, be a sequence such that

$$\mathcal{J}(\mathbf{u}_n) \to \beta \quad \text{and} \quad \|d\mathcal{J}(\mathbf{u}_n)\|_{X'} (1 + \|\mathbf{u}_n\|_X) \to 0. \tag{4.17}$$

We have to prove that $\mathbf{u} = (u_1, \dots, u_m) \in X$ exists such that

- (i) $\mathbf{u}_n \to \mathbf{u}$ strongly in W;
- (ii) $\mathcal{J}(\mathbf{u}) = \beta$, $d\mathcal{J}(\mathbf{u}) = 0$.

To this aim, our proof is divided in the following steps:
1. $(\mathbf{u}_n)_n$ is bounded in W; hence, up to subsequences, $\mathbf{u} \in W$ exists such that

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ weakly in } W, \tag{4.18}$$

i.e., $u_i^n \rightharpoonup u_i$ weakly in W_i for all $i \in \{1, \dots, m\}$,

$$u_i^n \to u_i \text{ strongly in } L^{r_i}(\Omega) \text{ if } r_i \in [1, p_i^*[, \quad \text{for all } i \in \{1, \dots, m\}, \tag{4.19}$$

$$\mathbf{u}_n \to \mathbf{u} \text{ a.e. in } \Omega; \tag{4.20}$$

2. $\mathbf{u} \in L$, then $\mathbf{u} \in X$;
3. for any $k > 0$, let $T_k : \mathbb{R} \to \mathbb{R}$ and $\mathcal{T}_k : \mathbb{R}^m \to \mathbb{R}^m$ be defined as

$$T_k t := \begin{cases} t & \text{if } |t| \leq k \\ k\frac{t}{|t|} & \text{if } |t| > k \end{cases}$$

and

$$\mathcal{T}_k(t_1, \dots, t_m) := (T_k t_1, \dots, T_k t_m), \tag{4.21}$$

then, taking any $k \geq \max\{\|\mathbf{u}\|_L, R\} + 1$ (with $R \geq 1$ as in our set of hypotheses), we have

$$\|d\mathcal{J}(\mathcal{T}_k \mathbf{u}_n)\|_{X'} \to 0 \tag{4.22}$$

and

$$\mathcal{J}(\mathcal{T}_k \mathbf{u}_n) \to \beta; \tag{4.23}$$

4. $\|\mathcal{T}_k \mathbf{u}_n - \mathbf{u}\|_W \to 0$, then (i) holds;
5. (ii) is satisfied.

For simplicity, here and in the following we will use the notation $(\varepsilon_n)_n$ for any infinitesimal sequence depending only on sequence $(\mathbf{u}_n)_n$, $(\varepsilon_{k,n})_n$ for any infinitesimal sequence depending not only

on $(\mathbf{u}_n)_n$ but also on a fixed integer k. Moreover, b_l will denote any strictly positive constant independent of n.
Step 1. Firstly, we observe that (3.17) and (4.17) imply that

$$\frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n)[u_i^n] = \varepsilon_n \quad \text{for all } i \in \{1, \dots, m\}. \tag{4.24}$$

Thus, taking θ_i, $i \in \{1, \dots, m\}$, as in hypotheses (h_5) and (g_2), and fixing $n \in \mathbb{N}$, from (1.4), (3.15), (4.17) and (4.24) we have that

$$\begin{aligned}\beta + \varepsilon_n &= \mathcal{J}(\mathbf{u}_n) - \sum_i \theta_i \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n)[u_i^n] \\ &= \sum_i \int_\Omega \big(A_i(x, u_i^n, \nabla u_i^n) - \theta_i a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla u_i^n - \theta_i A_{i,t}(x, u_i^n, \nabla u_i^n) u_i^n\big) dx \\ &\quad + \int_\Omega \Big(\sum_i \big(\theta_i G_i(x, \mathbf{u}_n) u_i^n\big) - G(x, \mathbf{u}_n)\Big) dx.\end{aligned}$$

Now, fix $i \in \{1, \dots, m\}$ and set

$$\Omega_{n,R}^i = \{x \in \Omega : \, |(u_i^n(x), \nabla u_i^n(x))| > R\}.$$

Then, we have that

$$\int_{\Omega \setminus \Omega_{n,R}^i} |\nabla u_i^n|^{p_i} dx \le R^{p_i} \operatorname{meas}(\Omega). \tag{4.25}$$

On the other hand, hypothesis (h_1) implies that

$$\begin{aligned}\sum_i \int_{\Omega \setminus \Omega_{n,R}^i} &|A_i(x, u_i^n, \nabla u_i^n) - \theta_i a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla u_i^n \\ &\qquad - \theta_i A_{i,t}(x, u_i^n, \nabla u_i^n) u_i^n| dx \le b_1,\end{aligned}$$

while from assumptions (h_2) and (h_5) it follows that

$$\begin{aligned}\sum_i \int_{\Omega_{n,R}^i} &\big(A_i(x, u_i^n, \nabla u_i^n) u_i^n - \theta_i a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla u_i^n - \theta_i A_{i,t}(x, u_i^n, \nabla u_i^n) u_i^n\big) dx \\ &\ge \mu_2 \sum_i \int_{\Omega_{n,R}^i} a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla u_i^n dx \ge \mu_2 \lambda \sum_i \int_{\Omega_{n,R}^i} |\nabla u_i^n|^{p_i} dx.\end{aligned}$$

Moreover, from hypotheses (3.6), (g_2) with (3.9) and direct computations we have that

$$\int_\Omega \Big(\Big(\sum_i \theta_i G_i(x, \mathbf{u}_n) u_i^n\Big) - G(x, \mathbf{u}_n)\Big) dx \ge -b_2.$$

Thus, summing up, from the previous estimates and (4.25), we obtain that

$$\beta + \varepsilon_n \ge \mu_2 \lambda \sum_i \int_{\Omega_{n,R}^i} |\nabla u_i^n|^{p_i} dx - b_3 \ge \mu_2 \lambda \sum_i \|u_i^n\|_{W_i}^{p_i} - b_4.$$

Hence, $(\mathbf{u}_n)_n$ is bounded in W and so $\mathbf{u} \in W$ exists such that, up to subsequences, (4.18)–(4.20) hold.

Step 2. In order to prove that $\mathbf{u} \in L$, arguing by contradiction, we assume that $i \in \{1, \dots, m\}$ exists such that $p_i < N$ (the proof if $p_i = N$ is simpler) and $u_i \notin L^\infty(\Omega)$, then either

$$\operatorname*{ess\,sup}_{\Omega} u_i = +\infty \tag{4.26}$$

or

$$\operatorname*{ess\,sup}_{\Omega}(-u_i) = +\infty. \tag{4.27}$$

If (4.26) holds, then for any fixed $k \in \mathbb{N}$ we have that

$$\text{meas}(\Omega_k^{i,+}) > 0, \quad \text{with } \Omega_k^{i,+} = \{x \in \Omega : \ u_i(x) > k\}. \tag{4.28}$$

Defining $R_k^+ : \mathbb{R} \to \mathbb{R}$ as

$$R_k^+ t := \begin{cases} 0 & \text{if } t \le k \\ t - k & \text{if } t > k \end{cases}, \tag{4.29}$$

from (4.18) it follows that

$$R_k^+ u_i^n \rightharpoonup R_k^+ u_i \quad \text{weakly in } W_i,$$

whence the sequentially weakly lower semicontinuity of $\|\cdot\|_{W_i}$ gives

$$\int_{\Omega_k^{i,+}} |\nabla u_i|^{p_i} dx \le \liminf_n \int_{\Omega_{n,k}^{i,+}} |\nabla u_i^n|^{p_i} dx, \tag{4.30}$$

with $\Omega_{n,k}^{i,+} = \{x \in \Omega : \ u_i^n(x) > k\}$. On the other hand, from (3.17), (4.17) and definition (4.29) we have that

$$\frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n)[R_k^+ u_i^n] \to 0,$$

so, from (4.28) an integer $n_k \in \mathbb{N}$ exists such that

$$\frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n)[R_k^+ u_i^n] < \text{meas}(\Omega_k^{i,+}) \quad \text{for all } n \ge n_k. \tag{4.31}$$

Now, taking $k > R$ (R as in our set of hypotheses) and $n \in \mathbb{N}$, as $\mu_1 \le 1$ (see Remark 4.1) from (3.15), (h_2), (h_4) and direct calculations it follows that

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n)[R_k^+ u_i^n] &= \int_{\Omega_{n,k}^{i,+}} \Big(1 - \frac{k}{u_i^n}\Big)(a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla u_i^n + A_{i,t}(x, u_i^n, \nabla u_i^n) u_i^n) dx \\ &\quad + \int_{\Omega_{n,k}^{i,+}} \frac{k}{u_i^n} \ a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla u_i^n dx - \int_\Omega G_i(x, \mathbf{u}_n) R_k^+ u_i^n dx \\ &\ge \mu_1 \int_{\Omega_{n,k}^{i,+}} a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla u_i^n dx - \int_\Omega G_i(x, \mathbf{u}_n) R_k^+ u_i^n dx \\ &\ge \mu_1 \lambda \int_{\Omega_{n,k}^{i,+}} |\nabla u_i^n|^{p_i} dx - \int_\Omega G_i(x, \mathbf{u}_n) R_k^+ u_i^n dx, \end{aligned}$$

which, together with (4.31), implies that

$$\mu_1\lambda\int_{\Omega^{i,+}_{n,k}}|\nabla u_i^n|^{p_i}dx \le \mathrm{meas}(\Omega^{i,+}_k)+\int_\Omega G_i(x,\mathbf{u}_n)R^+_k u_i^n dx \tag{4.32}$$

for all $n\ge n_k$. We note that

$$\int_\Omega G_i(x,\mathbf{u}_n)R^+_k u_i^n dx \longrightarrow \int_\Omega G_i(x,\mathbf{u})R^+_k u_i dx. \tag{4.33}$$

In fact, from (g_0) and (4.20) we obtain that

$$G_i(x,\mathbf{u}_n)R^+_k u_i^n \to G_i(x,\mathbf{u})R^+_k u_i \quad \text{a.e. in } \Omega,$$

while, since (3.8) implies that suitable exponents can be choosen so that (4.11) holds, from (3.6), (4.13) and also (3.7), (4.11), (4.19), [5, Theorem 4.9] it follows that a function $h\in L^1(\Omega)$ exists such that

$$|G_i(x,\mathbf{u}_n)R^+_k u_i^n| \le \sigma\Big(|u_i^n|+|u_i^n|^{q_i}+\sum_{j\ne i}(|u_i^n|^{\tilde{q}_i}+|u_j^n|^{\widetilde{s}_{i,j}})\Big)\le h(x) \quad \text{a.e. in } \Omega,$$

then Dominated Convergence Theorem applies.
Thus, summing up, from (4.30), (4.32), (4.33) and, again, (3.6) and (4.13) we have that (3.8) and direct computations imply that

$$\int_{\Omega^{i,+}_k}|\nabla u_i|^{p_i}dx \le b_5\Big(\mathrm{meas}(\Omega^{i,+}_k)+\int_{\Omega^{i,+}_k}|u_i|^{\overline{q}_i}dx+\sum_{j\ne i}\int_{\Omega^{i,+}_k}|u_j|^{\widetilde{s}_{i,j}}dx\Big), \tag{4.34}$$

with $\overline{q}_i$ as in (4.14) and $\widetilde{s}_{i,j}$ as in (4.11).
At last, Hölder inequality and (4.11) imply that

$$\int_{\Omega^{i,+}_k}|u_j|^{\widetilde{s}_{i,j}}dx \le |u_j|^{\widetilde{s}_{i,j}}_{p_j^*}[\mathrm{meas}(\Omega^{i,+}_k)]^{1-\frac{\widetilde{s}_{i,j}}{p_j^*}} \quad \text{for each } j\ne i,$$

while (4.16) and direct computations give

$$\int_{\Omega^{i,+}_k}|u_i|^{\overline{q}_i}dx \le 2^{\overline{q}_i-1}|u_i|^{\overline{q}_i-p_i}_{\overline{q}_i}\Big(\int_{\Omega^{i,+}_k}(u_i-k)^{\overline{q}_i}dx\Big)^{\frac{p_i}{\overline{q}_i}}+2^{\overline{q}_i-1}k^{\overline{q}_i}\mathrm{meas}(\Omega^{i,+}_k),$$

so from (4.34), Sobolev Embedding Theorems and direct computations we obtain that

$$\begin{aligned}\int_{\Omega^{i,+}_k}|\nabla u_i|^{p_i}dx &\le b_6\Big(\int_{\Omega^{i,+}_k}(u_i-k)^{\overline{q}_i}dx\Big)^{\frac{p_i}{\overline{q}_i}}\\ &\quad + b_6\sum_j k^{\alpha_j}[\mathrm{meas}(\Omega^{i,+}_k)]^{1-\frac{p_i}{N}+\varepsilon_j},\end{aligned} \tag{4.35}$$

with $b_6=b_6(\|\mathbf{u}\|_W)>0$, where we set

$$\alpha_j=\begin{cases}\overline{q}_i & \text{if } j=i\\ 0 & \text{if } j\ne i\end{cases}, \qquad \varepsilon_j=\begin{cases}\frac{p_i}{N} & \text{if } j=i\\ \frac{p_i}{N}-\frac{\widetilde{s}_{i,j}}{p_j^*} & \text{if } j\ne i\end{cases}.$$

From (4.12) it follows that $\varepsilon_j > 0$ for each $j \neq i$, so (4.35) with (4.16) (here, $\varepsilon_i p_i^* + p_i = p_i^*$) allows us to apply Lemma 4.7 and u_i is essentially bounded from above in contradiction with (4.26). Similar arguments make us to rule out also (4.27); hence, it has to be $u_i \in L^\infty(\Omega)$.
Step 3. In order to prove this statement, we extend the main arguments used in the corresponding *Step 3* in the proof of [12, Proposition 4.6] to our setting but introducing some technical changes as in [8, Proposition 4.6]. Anyway, for the sake of completeness, here we give the main tools. Taking $k \geq \max\{\|\mathbf{u}\|_L, R\} + 1$, we define $R_k : \mathbb{R} \to \mathbb{R}$ and $\mathcal{R}_k : \mathbb{R}^m \to \mathbb{R}^m$ as

$$R_k t = t - T_k t = \begin{cases} 0 & \text{if } |t| \leq k \\ t - k\frac{t}{|t|} & \text{if } |t| > k \end{cases},$$

$$\mathcal{R}_k(t_1, \dots, t_m) = (R_k t_1, \dots, R_k t_m),$$

and denote

$$\Omega^i_{n,k} := \{x \in \Omega : |u_i^n(x)| > k\} \qquad \text{for any } n \in \mathbb{N},\ i \in \{1, \dots, m\}.$$

By definition, it follows that

$$\|\mathcal{T}_k \mathbf{u}_n\|_X \leq \|\mathbf{u}_n\|_X \quad \text{and} \quad \|\mathcal{R}_k \mathbf{u}_n\|_X \leq \|\mathbf{u}_n\|_X \quad \text{for all } n \in \mathbb{N}. \tag{4.36}$$

Moreover, we have that $\mathcal{T}_k \mathbf{u} = \mathbf{u}$ and $\mathcal{R}_k \mathbf{u} = \mathbf{0}$ a.e. in Ω, so from (4.18)–(4.20), in particular, it follows that

$$\mathcal{T}_k \mathbf{u}_n \to \mathbf{u} \ \text{ a.e. in } \Omega, \tag{4.37}$$

$$\mathcal{R}_k \mathbf{u}_n \to \mathbf{0} \quad \text{in } L^{r_1}(\Omega) \times \cdots \times L^{r_m}(\Omega) \tag{4.38}$$

for any $(r_1, \dots, r_m) \in [1, p_1^*[\times \cdots \times [1, p_m^*[$,

$$\text{meas}(\Omega^i_{n,k}) \to 0 \quad \text{for all } i \in \{1, \dots, m\}. \tag{4.39}$$

Fixing any $i \in \{1, \dots, m\}$, from (3.17), (4.17) and (4.36) we have that

$$\left\| \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n) \right\|_{X_i'} \|R_k u_i^n\|_{X_i} \to 0. \tag{4.40}$$

Then, reasoning as in the previous *Step 2* but replacing $R_k^+ u_i$ with $R_k u_i$, from (4.40) it follows that

$$\begin{aligned} \varepsilon_n + \int_\Omega G_i(x, \mathbf{u}_n) R_k u_i^n dx &= \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n)[R_k u_i^n] + \int_\Omega G_i(x, \mathbf{u}_n) R_k u_i^n dx \\ &\geq \mu_1 \int_{\Omega^i_{n,k}} a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla u_i^n dx \\ &\geq \lambda \mu_1 \int_{\Omega^i_{n,k}} |\nabla u_i^n|^{p_i} dx. \end{aligned} \tag{4.41}$$

Since arguments similar to those ones used for proving (4.33) apply, from (4.38) we obtain that

$$\int_\Omega G_i(x, \mathbf{u}_n) R_k u_i^n dx \longrightarrow 0,$$

then (4.41) implies both

$$\int_{\Omega^i_{n,k}} |\nabla u_i^n|^{p_i} dx \longrightarrow 0, \quad \text{i.e.} \quad \|R_k u_i^n\|_{W_i} \to 0, \tag{4.42}$$

and

$$\int_{\Omega^i_{n,k}} a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla u_i^n dx \longrightarrow 0. \tag{4.43}$$

By means of (3.18), if we prove that

$$\left\| \frac{\partial \mathcal{J}}{\partial u_i}(\mathcal{T}_k \mathbf{u}_n) \right\|_{X_i'} \to 0 \quad \text{for all } i \in \{1, \dots, m\},$$

then (4.22) holds. To this aim, fixing any $i \in \{1, \dots, m\}$, we take $v \in X_i$ such that $\|v\|_{X_i} = 1$. Direct computations imply that

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial u_i}(\mathcal{T}_k \mathbf{u}_n)[v] \ &= \frac{\partial \mathcal{J}}{\partial u_i}(\mathbf{u}_n)[v] \\ &\quad - \int_{\Omega^i_{n,k}} (a_i(x, u_i^n, \nabla u_i^n) \cdot \nabla v + A_{i,t}(x, u_i^n, \nabla u_i^n) v)\, dx \\ &\quad + \int_{\Omega^i_{n,k}} (a_i(x, T_k u_i^n, 0) \cdot \nabla v + A_{i,t}(x, T_k u_i^n, 0) v)\, dx \\ &\quad + \int_{\Omega} (G_i(x, \mathbf{u}_n) - G_i(x, \mathcal{T}_k \mathbf{u}_n)) v \ dx. \end{aligned} \tag{4.44}$$

We observe that (h_1) with $\xi = 0$ and $|T_k u_i^n| \le k$ for all $n \in \mathbb{N}$ imply the boundness of both $A_{i,t}(x, T_k u_i^n, 0)$ and $a_i(x, T_k u_i^n, 0)$ in set $\Omega^i_{n,k}$; hence, from (4.39) and direct computations so to "erase" v from the limit, we obtain that

$$\int_{\Omega^i_{n,k}} (a_i(x, T_k u_i^n, 0) \cdot \nabla v + A_{i,t}(x, T_k u_i^n, 0) v)\, dx \to 0 \tag{4.45}$$

uniformly with respect to v. On the other hand, Dominated Convergence Theorem implies that

$$\int_{\Omega} |G_i(x, \mathbf{u}_n) - G_i(x, \mathcal{T}_k \mathbf{u}_n)| dx \to 0. \tag{4.46}$$

In fact, we have that

$$\begin{aligned} \int_{\Omega} |G_i(x, \mathbf{u}_n) - G_i(x, \mathcal{T}_k \mathbf{u}_n)| dx \le & \int_{\Omega} |G_i(x, \mathbf{u}_n) - G_i(x, \mathbf{u})| dx \\ & + \int_{\Omega} |G_i(x, \mathcal{T}_k \mathbf{u}_n) - G_i(x, \mathbf{u})| dx, \end{aligned}$$

where (g_0) together with (4.20), respectively (4.37), give

$$G_i(x, \mathbf{u}_n) \to G_i(x, \mathbf{u}) \quad \text{and} \quad G_i(x, \mathcal{T}_k \mathbf{u}_n) \to G_i(x, \mathbf{u}) \quad \text{a.e. in } \Omega,$$

and from (3.6), Young inequality, (3.7), (4.11), (4.19) and [5, Theorem 4.9] we obtain that

$$|G_i(x,\mathbf{u}_n)| \le b_7 \left(1 + |u_i^n|^{q_i} + \sum_{j\neq i} |u_j^n|^{\widetilde{s}_{i,j}}\right) \le \bar{h}(x) \quad \text{ a.e. in } \Omega$$

for a suitable $\bar{h} \in L^1(\Omega)$, while, again, (3.6) and the boundedness of $(\mathcal{T}_k\mathbf{u}_n)_n$ in L give

$$|G_i(x,\mathcal{T}_k\mathbf{u}_n)| \le b_8 \quad \text{ a.e. in } \Omega,$$

so Dominated Convergence Theorem applies and (4.46) holds.
Thus, summing up, from (4.44)–(4.46) we obtain that

$$\begin{aligned}\left|\frac{\partial \mathcal{J}}{\partial u_i}(\mathcal{T}_k\mathbf{u}_n)[v]\right| &\le \varepsilon_{k,n} \\ &\quad + \left|\int_{\Omega^i_{n,k}} (a_i(x,u_i^n,\nabla u_i^n)\cdot\nabla v + A_{i,t}(x,u_i^n,\nabla u_i^n)v)dx\right|,\end{aligned} \tag{4.47}$$

where $\varepsilon_{k,n}$ is independent of v. At last, the estimate of the last integral in (4.47) can be obtained as in the corresponding *Step 3* in the proof of [8, Proposition 4.6] but testing $\frac{\partial \mathcal{J}}{\partial u_i}(\mathcal{T}_k\mathbf{u}_n)$ on the new test functions $vR_k^+u_i^n$ and $vR_k^-u_i^n$, with $R_k^- : \mathbb{R} \to \mathbb{R}$ such that

$$R_k^- t = \begin{cases} 0 & \text{if } t \ge -k \\ t+k & \text{if } t < -k \end{cases}.$$

Then, (4.22) holds.
Finally, we have to prove (4.23). From (1.4), (4.21) and direct computations, we have that

$$\begin{aligned}\mathcal{J}(\mathcal{T}_k\mathbf{u}_n) = \mathcal{J}(\mathbf{u}_n) &- \sum_i \int_{\Omega^i_{n,k}} (A_i(x,u_i^n,\nabla u_i^n) - A_i(x,T_k u_i^n,0))dx \\ &+ \int_\Omega (G(x,\mathbf{u}_n) - G(x,\mathcal{T}_k\mathbf{u}_n))\,dx,\end{aligned}$$

where (4.1), (4.5), (4.43), respectively the boundness of $A_i(x,T_k u_i^n,0)$ and (4.39), imply that

$$\int_{\Omega^i_{n,k}} A_i(x,u_i^n,\nabla u_i^n)\,dx \longrightarrow 0 \quad \text{and} \quad \int_{\Omega^i_{n,k}} A_i(x,T_k u_i^n,0)\,dx \longrightarrow 0$$

for all $i \in \{1,\dots,m\}$. On the other hand, from (4.15), (4.16), (4.19), (4.20), (4.37) and, again, Dominated Convergence Theorem, we have that

$$\int_\Omega (G(x,\mathbf{u}_n) - G(x,\mathcal{T}_k\mathbf{u}_n))\,dx \longrightarrow 0.$$

Thus, (4.23) follows from (4.17).
Step 4. For each $i \in \{1,\dots,m\}$, by using the same arguments as in *Step 4* in the proof of [8, Proposition 4.6] but applied to the partial derivative $\frac{\partial \mathcal{J}}{\partial u_i}(\mathcal{T}_k\mathbf{u}_n)$, we prove that

$$\|T_k u_i^n - u_i\|_{W_i} \to 0. \tag{4.48}$$

So, condition (i) follows from (4.42) and (4.48).
Step 5. By applying Proposition 3.5 to the uniformly bounded sequence $(\mathcal{T}_k\mathbf{u}_n)_n$, from (i) and (4.37) we have that

$$\mathcal{J}(\mathcal{T}_k\mathbf{u}_n) \to \mathcal{J}(\mathbf{u}) \quad \text{and} \quad \|d\mathcal{J}(\mathcal{T}_k\mathbf{u}_n) - d\mathcal{J}(\mathbf{u})\|_{X'} \to 0,$$

which, together with (4.22) and (4.23), implies (ii). □ □

5 Main results

In order to prove a multiplicity result, for each $i \in \{1, \dots, m\}$ we make use of the decomposition of W_i as introduced in [8, Section 5]. More precisely, it is known that the first eigenvalue of $-\Delta_{p_i}$ in W_i is given by

$$\lambda_{i,1} := \inf_{u\in W_i\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p_i} dx}{\int_\Omega |u|^{p_i} dx}, \tag{5.1}$$

it is simple, strictly positive and isolated and has a unique eigenfunction $\varphi_{i,1}$ such that

$$\varphi_{i,1} > 0 \text{ a.e. in } \Omega, \quad \varphi_{i,1} \in L^\infty(\Omega) \quad \text{and} \quad |\varphi_{i,1}|_{p_i} = 1 \tag{5.2}$$

(see, e.g., [17]). Then, a sequence of positive real numbers $(\lambda_{i,n})_n$ and a corresponding sequence of pseudo–eigenfunctions $(\psi_{i,n})_n \subset W_i$ exist such that

$$0 < \lambda_{i,1} < \lambda_{i,2} \le \dots \le \lambda_{i,n} \le \dots, \quad \text{with } \lambda_{i,n} \nearrow +\infty \ \text{ as } n \to +\infty, \tag{5.3}$$

and for all $n \in \mathbb{N}$ we have that $\psi_{i,n} \in L^\infty(\Omega)$, hence $\psi_{i,n} \in X_i$. Moreover, taking $k \in \mathbb{N}$ and

$$V_{i,k} := \operatorname{span}\{\psi_{i,1}, \dots, \psi_{i,k}\},$$

a suitable (infinite dimensional) topological complement $Y_{i,k}$ can be found in W_i such that

$$W_i = V_{i,k} \oplus Y_{i,k}$$

and the following inequality holds:

$$\lambda_{i,k+1} \int_\Omega |w|^{p_i} dx \ \le \ \int_\Omega |\nabla w|^{p_i} dx \quad \text{for all } w \in Y_{i,k} \tag{5.4}$$

(cf. [8, Proposition 5.4]). Hence, defining $Y_k^{X_i} := Y_{i,k} \cap L^\infty(\Omega) \subset X_i$, from the boundedness of each $\psi_{i,n}$ we have that $V_{i,k}$ is a subspace of X_i, too, and then

$$X_i = V_{i,k} \oplus Y_k^{X_i},$$

with

$$\dim V_{i,k} = \operatorname{codim} Y_k^{X_i} = k.$$

Thus, by means of (3.10) we have that

$$W = (V_{1,k} \times \dots \times V_{m,k}) \oplus (Y_{1,k} \times \dots \times Y_{m,k})$$

and then from (3.12) it follows that

$$X = V_k \oplus Y_k^X$$

with

$$\begin{aligned} &V_k = V_{1,k} \times \cdots \times V_{m,k}, \qquad Y_k^X = Y_k^{X_1} \times \cdots \times Y_k^{X_m},\\ &\text{where } \dim V_k = \operatorname{codim} Y_k^X < +\infty. \end{aligned} \tag{5.5}$$

Now, we are able to state our main results.

Theorem 5.1. *For each $i \in \{1,\dots,m\}$ let $p_i > 1$ and assume that $A_i(x,t,\xi)$ satisfies hypotheses (h_0)–(h_6). Moreover, suppose that a given function $G(x,\mathbf{u})$ verifies (g_0)–(g_2). If, in addition, $\mu_3 > 0$ exists such that*

(h_7) *for every $i \in \{1,\dots,m\}$ it is*

$$A_i(x,t,\xi) \geq \mu_3|\xi|^{p_i} \quad \text{a.e. in } \Omega, \quad \text{for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N;$$

(g_3) *taking $\lambda_1^* := \min\{\lambda_{i,1}:\ 1 \leq i \leq m\}$, with $\lambda_{i,1}$ as in (5.1), assume that*

$$\limsup_{\mathbf{u}\to\mathbf{0}} \frac{G(x,\mathbf{u})}{\sum_i |u_i|^{p_i}} < \mu_3\lambda_1^* \quad \text{uniformly a.e. in } \Omega;$$

then functional $\mathcal{J}$ in (1.4) possesses at least one nontrivial critical point in X; hence, problem (1.1) admits a nontrivial weak bounded solution.

Theorem 5.2. *For each $i \in \{1,\dots,m\}$ let $p_i > 1$ and assume that $A_i(x,t,\xi)$ and $G(x,\mathbf{u})$ satisfy hypotheses (h_0)–(h_6), (g_0)–(g_2). If we also suppose that*

(h_8) *for each $i \in \{1,\dots,m\}$ function $A_i(x,\cdot,\cdot)$ is even in $\mathbb{R}\times\mathbb{R}^N$ for a.e. $x \in \Omega$;*

(g_4) $\displaystyle\liminf_{|\mathbf{u}|\to+\infty} \frac{G(x,\mathbf{u})}{\sum_i |u_i|^{\frac{1}{\theta_i}}} > 0$ *uniformly a.e. in Ω;*

(g_5) *$G(x,\cdot)$ is even in $\mathbb{R}^m$ for a.e. $x \in \Omega$;*

then functional $\mathcal{J}$ in (1.4) has an unbounded sequence of critical points $(\mathbf{u}_n)_n \subset X$ such that $\mathcal{J}(\mathbf{u}_n) \nearrow +\infty$; hence, problem (1.1) admits infinitely many distinct weak bounded solutions.

From now on, assume that for each $i \in \{1,\dots,m\}$ function $A_i(x,t,\xi)$ satisfies (h_0)–(h_6) while $G(x,\mathbf{u})$ verifies (g_0)–(g_2). Moreover, let $1 < p_i < N$ for each $i \in \{1,\dots,m\}$ (otherwise, the proof is simpler) and, for simplicity, suppose that

$$\int_\Omega A_i(x,0,\mathbf{0}_N)dx = 0 \quad \text{for all } i \in \{1,\dots,m\}, \tag{5.6}$$

with $\mathbf{0}_N = (0,\dots,0) \in \mathbb{R}^N$, and

$$\int_\Omega G(x,\mathbf{0})dx = 0 \tag{5.7}$$

(otherwise, we replace $\mathcal{J}(\mathbf{u})$ with $\mathcal{J}(\mathbf{u}) - \sum_i \int_\Omega A_i(x,0,\mathbf{0}_N)dx + \int_\Omega G(x,\mathbf{0})dx$ which has the same differential on X).

Firstly, we state the following preliminary result (for the proof, see [8, Proposition 6.5]).

Proposition 5.3. *Let $i \in \{1,\dots,m\}$ be fixed. Then, for any $(t,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $|(t,\xi)| \ge R$ we have that*

$$A_i(x,st,s\xi) \le s^{\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)} A_i(x,t,\xi) \quad \text{a.e. in } \Omega, \text{ for all } s \ge 1,$$

with R, θ_i, μ_2 and η_1 as in our set of hypotheses. Moreover, some constants b_1^*, $b_2^* > 0$ exist, independent of i, such that for all $(t,\xi) \in \mathbb{R}\times\mathbb{R}^N$ it is

$$|A_i(x,t,\xi)| \le b_1^*\left(1+|t|^{\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)}\right) + b_2^*\left(1+|t|^{\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)-p_i}\right)|\xi|^{p_i} \tag{5.8}$$

a.e. in Ω, with $\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)-p_i > 0$ (without loss of generality, as we can take, a priori, either μ_2 small enough or η_1 large enough).

Now, we are able to prove our main results.

of Theorem 5.1. We note that, being $\mu_3\lambda_1^* > 0$, then from hypothesis (g_3) a constant $\overline{\lambda} \in \mathbb{R}$ exists such that

$$\overline{\lambda} > 0 \quad \text{and} \quad \limsup_{\mathbf{u}\to\mathbf{0}} \frac{G(x,\mathbf{u})}{\sum_i |u_i|^{p_i}} < \overline{\lambda} < \mu_3\lambda_1^* \quad \text{uniformly a.e. in } \Omega. \tag{5.9}$$

Thus, a radius $\rho^* > 0$ exists such that

$$G(x,\mathbf{u}) \;\le\; \overline{\lambda}\sum_i |u_i|^{p_i} \quad \text{for a.e. } x\in\Omega \;\text{ if }\; |\mathbf{u}| \le \rho^*. \tag{5.10}$$

Now, let $\mathbf{u} \in \mathbb{R}^m$ be such that $|\mathbf{u}| > \rho^*$, $\mathbf{u} = (u_1,\dots,u_m)$. Then, an integer $i^* \in \{1,\dots,m\}$ exists such that $|u_{i^*}| \ge \frac{\rho^*}{m}$; hence, $1 \le \left(\frac{m}{\rho^*}\right)^{\overline{q}_{i^*}} |u_{i^*}|^{\overline{q}_{i^*}}$. Thus, from (4.15) direct computations allow us to prove the existence of a constant $\sigma^* > 0$ such that

$$|G(x,\mathbf{u})| \le \sigma^* \sum_i |u_i|^{\overline{q}_i} \quad \text{for a.e. } x\in\Omega \;\text{ if }\; |\mathbf{u}| > \rho^*. \tag{5.11}$$

Summing up, from (5.10) and (5.11), we have that

$$G(x,\mathbf{u}) \le \overline{\lambda}\sum_i |u_i|^{p_i} + \sigma^*\sum_i |u_i|^{\overline{q}_i} \quad \text{for a.e. } x\in\Omega\text{ , all } \mathbf{u}\in\mathbb{R}^m. \tag{5.12}$$

Then, taking $\mathbf{u} \in X$, from (1.4), assumption (h_7) and inequality (5.12) we have that

$$\mathcal{J}(\mathbf{u}) \ge \sum_i \left(\mu_3\int_\Omega |\nabla u_i|^{p_i}dx - \overline{\lambda}\int_\Omega |u_i|^{p_i}dx - \sigma^*\int_\Omega |u_i|^{\overline{q}_i}dx\right),$$

or better, (3.4), (4.16), (5.1), the definition of λ_1^* and also (3.11) imply that

$$\begin{aligned} \mathcal{J}(\mathbf{u}) \;&\ge\; \sum_i \left(\left(\mu_3 - \frac{\overline{\lambda}}{\lambda_{i,1}}\right)\|u_i\|_{W_i}^{p_i} - \bar{\sigma}\|u_i\|_{W_i}^{\overline{q}_i}\right) \\ &\ge \sum_i \left(\|u_i\|_{W_i}^{p_i}\left(\mu_3 - \frac{\overline{\lambda}}{\lambda_1^*} - \bar{\sigma}\|\mathbf{u}\|_W^{\overline{q}_i-p_i}\right)\right) \end{aligned} \tag{5.13}$$

for a suitable $\bar{\sigma} > 0$. We note that from (4.16) and (5.9) some constants R_0, ρ_1 exist such that

$$0 < R_0 < 2m \qquad \text{and} \qquad \mu_3 - \frac{\overline{\lambda}}{\lambda_1^*} - \bar{\sigma} R_0^{\overline{q}_i - p_i} \geq \rho_1 > 0 \tag{5.14}$$

for all $i \in \{1, \dots, m\}$. Since (3.11) and $\|\mathbf{u}\|_W = R_0$ imply that $\|u_j\|_{W_j} \geq \frac{R_0}{2m}$ for some $j \in \{1, \dots, m\}$, from (5.13) and (5.14) we have that

$$\mathcal{J}(\mathbf{u}) \geq \rho_1 \sum_i \|u_i\|_{W_i}^{p_i} \geq \rho_1 \|u_j\|_{W_j}^{p_j} \geq \rho_1 \left(\frac{R_0}{2m}\right)^{\bar{p}}$$

with $\bar{p} = \max\{p_i : \ 1 \leq i \leq m\}$. So, a constant $\rho_0 > 0$ can be found so that

$$\mathbf{u} \in X, \quad \|\mathbf{u}\|_W = R_0 \qquad \Longrightarrow \qquad \mathcal{J}(\mathbf{u}) \geq \rho_0. \tag{5.15}$$

Finally, in order to prove that also the geometric condition (ii) in Theorem 2.2 is satisfied, fixing any $i \in \{1, \dots, m\}$, we take $\varphi_{i,1} \in X_i$ as in (5.2). Then, from (1.4), (4.10), (5.6), (5.8) and direct computations, for any $s > 0$ we obtain that

$$\begin{aligned}
\mathcal{J}(s\varphi_{i,1}\mathbf{e}_i) \ &= \int_\Omega A_i(x, s\varphi_{i,1}, s\nabla\varphi_{i,1}) dx + \sum_{j \neq i} \int_\Omega A_j(x, 0, \mathbf{0}_N) dx \\
&- \int_\Omega G(x, s\varphi_{i,1}\mathbf{e}_i) dx \\
\leq&\ b_1^* \mathrm{meas}(\Omega) + b_1^* s^{\frac{1}{\theta_i}(1 - \frac{\mu_2}{\eta_1})} \int_\Omega |\varphi_{i,1}|^{\frac{1}{\theta_i}(1 - \frac{\mu_2}{\eta_1})} dx + b_2^* s^{p_i} \int_\Omega |\nabla\varphi_{i,1}|^{p_i} dx \\
&+ b_2^* s^{\frac{1}{\theta_i}(1 - \frac{\mu_2}{\eta_1})} \int_\Omega |\varphi_{i,1}|^{\frac{1}{\theta_i}(1 - \frac{\mu_2}{\eta_1}) - p_i} |\nabla\varphi_{i,1}|^{p_i} dx \\
&- s^{\frac{1}{\theta_i}} \int_\Omega h_i(x) |\varphi_{i,1}|^{\frac{1}{\theta_i}} dx + \sigma_i \mathrm{meas}(\Omega),
\end{aligned}$$

where (5.2) and Remark 4.3 imply that all the integrals are finite and

$$\int_\Omega h_i(x) |\varphi_{i,1}|^{\frac{1}{\theta_i}} dx > 0.$$

Thus, since $\eta_1, \mu_2 > 0$, from (4.9) it follows that

$$\mathcal{J}(s\varphi_{i,1}\mathbf{e}_i) \to -\infty \quad \text{ as } s \to +\infty.$$

Hence, $v \in X_i$ exists such that

$$\|v\mathbf{e}_i\|_W > R_0 \quad \text{ and } \quad \mathcal{J}(v\mathbf{e}_i) < \varrho_0, \tag{5.16}$$

R_0 and ϱ_0 as in (5.15).
So, since (1.4), (5.6) and (5.7) imply that $\mathcal{J}(\mathbf{0}) = 0$, by means of Theorem 2.2 we have that Propositions 3.5 and 4.8 together with (5.15) and (5.16) imply the existence of a critical point $\mathbf{u}^* \in X$ such that $\mathcal{J}(\mathbf{u}^*) \geq \varrho_0 > \mathcal{J}(\mathbf{0})$. □ □

of Theorem 5.2. Since (h_8) and (g_5) imply that $\mathcal{J}$ is an even functional on X, in order to apply Corollary 2.4 we have to prove that assumption $(\mathcal{H}_\varrho)$ holds for infinitely many ϱ.
For simplicity, here and in the following, b_l will denote any strictly positive constant independent of the index $i \in \{1,\dots,m\}$.
Firstly, we prove that, taking any finite dimensional subspace V of X, a radius $R_V > 0$ exists such that

$$\mathcal{J}(\mathbf{u}) \leq 0 \qquad \text{for all } \mathbf{u} \in V,\ \|\mathbf{u}\|_X \geq R_V. \tag{5.17}$$

To this aim, for any $i \in \{1,\dots,m\}$ and $u_i \in X_i$, from (5.8), and, then, (3.13), direct computations imply that

$$\begin{aligned} &\int_\Omega A_i(x,u_i,\nabla u_i)dx \leq b_1^* \mathrm{meas}(\Omega)\left(1 + |u_i|_\infty^{\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)}\right) \\ &+ b_2^*\left(1 + |u_i|_\infty^{\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)-p_i}\right)\|u_i\|_{W_i}^{p_i} \leq b_1 + b_2\|u_i\|_{X_i}^{\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)} + b_2^*\|u_i\|_{X_i}^{p_i}. \end{aligned} \tag{5.18}$$

On the other hand, from hypothesis (g_4) it follows that $\bar{\bar{\lambda}}$ exists such that

$$\liminf_{|\mathbf{u}|\to+\infty} \frac{G(x,\mathbf{u})}{\sum_i |u_i|^{\frac{1}{\theta_i}}} > \bar{\bar{\lambda}} > 0 \quad \text{uniformly a.e. in } \Omega,$$

so $R_1 > 0$ exists so that

$$G(x,\mathbf{u}) \geq \bar{\bar{\lambda}} \sum_i |u_i|^{\frac{1}{\theta_i}} \quad \text{for a.e. } x \in \Omega, \text{ if } |\mathbf{u}| \geq R_1,$$

while from (3.9) we obtain

$$\Big|G(x,\mathbf{u}) - \bar{\bar{\lambda}} \sum_i |u_i|^{\frac{1}{\theta_i}}\Big| \leq \bar{\bar{\sigma}} \quad \text{for a.e. } x \in \Omega, \text{ if } |\mathbf{u}| \leq R_1.$$

Hence, we have that

$$G(x,\mathbf{u}) \geq \bar{\bar{\lambda}} \sum_i |u_i|^{\frac{1}{\theta_i}} - \bar{\bar{\sigma}} \quad \text{for a.e. } x \in \Omega, \text{ for all } \mathbf{u} \in \mathbb{R}^m. \tag{5.19}$$

Thus, from (1.4), (5.18) and (5.19) we obtain that

$$\mathcal{J}(\mathbf{u}) \leq \sum_i \left(b_3 + b_2\|u_i\|_{X_i}^{\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)} + b_2^*\|u_i\|_{X_i}^{p_i} - \bar{\bar{\lambda}}\int_\Omega |u_i|^{\frac{1}{\theta_i}}dx\right) \tag{5.20}$$

for all $\mathbf{u} \in X$. Now, if V is a finite dimensional subspace of X, $V = V_1 \times \cdots \times V_m$, for any $i \in \{1,\dots,m\}$ also the projection V_i onto X_i is a finite dimensional subspace, so all the norms are equivalent and in particular $\nu_i > 0$ exists such that

$$\left(\int_\Omega |v|^{\frac{1}{\theta_i}}dx\right)^{\theta_i} \geq \nu_i\|v\|_{X_i} \qquad \text{for all } v \in V_i.$$

So, from (5.20) it follows that

$$\mathcal{J}(\mathbf{u}) \leq \sum_i \left(b_3 + b_2 \|u_i\|_{X_i}^{\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)} + b_2^* \|u_i\|_{X_i}^{p_i} - b_4 \|u_i\|_{X_i}^{\frac{1}{\theta_i}} \right) \tag{5.21}$$

for all $\mathbf{u} \in V$ or better, taking $i \in \{1, \dots, m\}$ and

$$\zeta_i : s \in [0, +\infty[\mapsto \zeta_i(s) = b_3 + b_2 s^{\frac{1}{\theta_i}\left(1-\frac{\mu_2}{\eta_1}\right)} + b_2^* s^{p_i} - b_4 s^{\frac{1}{\theta_i}} \in \mathbb{R},$$

estimate (5.21) reduces to

$$\mathcal{J}(\mathbf{u}) \leq \sum_i \zeta_i(\|u_i\|_{X_i}) \qquad \text{for all } \mathbf{u} \in V. \tag{5.22}$$

We note that for each $i \in \{1, \dots, m\}$ the continuous function ζ_i is such that $\zeta_i(0) = b_3 > 0$ and $\zeta_i(s) \to -\infty$ as $s \to +\infty$. Hence, a constant $b_5 > 0$ and a radius $\bar{R} > 0$ exist such that

$$\max_{s \geq 0} \zeta_i(s) \leq b_5 \quad \text{and} \quad \zeta_i(s) < -m b_5 \ \text{ if } \ s \geq \bar{R} \quad \text{for all } i \in \{1, \dots, m\}. \tag{5.23}$$

Thus, if $(s_1, \dots, s_m) \in (\mathbb{R}_+)^m$ is such that $\sum_i s_i > m\bar{R}$, we have that $s_j \geq \bar{R}$ for some index $j \in \{1, \dots, m\}$, then (5.23) and direct computations allow one to prove that

$$\sum_i \zeta_i(s_i) < 0 \quad \text{if} \quad \sum_i s_i > m\bar{R}.$$

Whence, (5.17) holds from (5.22) with $R_V \geq m\bar{R}$, $s_i = \|u_i\|_{X_i}$.
Now, we want to prove that for any fixed $\varrho > 0$ there exists $k = k(\varrho) \geq 1$ and $R_k > 1$ such that

$$\mathbf{u} \in Y_k^X, \quad \|\mathbf{u}\|_W = R_k \qquad \Longrightarrow \qquad \mathcal{J}(\mathbf{u}) \geq \varrho \tag{5.24}$$

with Y_k^X as in (5.5).
To this aim, we note that if $\mathbf{u} \in X$ estimate (4.6) gives

$$\int_\Omega A_i(x, u_i, \nabla u_i) dx \geq \lambda(\theta_i \mu_1 + \mu_2) \int_\Omega |\nabla u_i|^{p_i} dx - \eta_3 \text{meas}(\Omega),$$

for each $i \in \{1, \dots, m\}$. Then, (1.4) and (4.15) imply that

$$\begin{aligned} \mathcal{J}(\mathbf{u}) \ \geq \ \sum_i \Bigg(& \lambda(\theta_i \mu_1 + \mu_2) \int_\Omega |\nabla u_i|^{p_i} dx - \eta_3 \text{meas}(\Omega) \\ & -\sigma_0 \int_\Omega |u_i|^{\bar{q}_i} dx - \sigma_0 \text{meas}(\Omega) \Bigg). \end{aligned} \tag{5.25}$$

We note that, for every $i \in \{1, \dots, m\}$, inequality (4.16) allows us to take $0 < r_i < p_i$ such that

$$\frac{r_i}{p_i} + \frac{\bar{q}_i - r_i}{p_i^*} = 1, \quad \text{i.e.} \quad r_i = p_i \frac{p_i^* - \bar{q}_i}{p_i^* - p_i},$$

so that from (3.4), standard interpolation arguments and, fixing any $k \in \mathbb{N}$, condition (5.4), it follows that

$$\int_\Omega |u_i|^{\bar{q}_i} dx \le \tau_{i,p_i^*}^{\bar{q}_i - r_i} \left(\frac{1}{\lambda_{i,k+1}}\right)^{\frac{r_i}{p_i}} \|u_i\|_{W_i}^{\bar{q}_i} \quad \text{for all } u_i \in Y_k^{X_i}. \tag{5.26}$$

Thus, from (5.25) and (5.26), taking b_6, b_7, $b_8 > 0$ such that

$$b_6 = \min_{1 \le i \le m} \lambda(\theta_i \mu_1 + \mu_2), \quad b_7 = \max_{1 \le i \le m} \tau_{i,p_i^*}^{\bar{q}_i - r_i} \sigma_0, \quad b_8 = m(\eta_3 + \sigma_0)\text{meas}(\Omega),$$

we obtain that

$$\mathcal{J}(\mathbf{u}) \ge \sum_i \left(\|u_i\|_{W_i}^{p_i} \left(b_6 - \frac{b_7}{\bar{\lambda}_k} \|u_i\|_{W_i}^{\bar{q}_i - p_i} \right)\right) - b_8 \quad \text{for all } \mathbf{u} \in Y_k^X, \tag{5.27}$$

with

$$\bar{\lambda}_k = \min_{1 \le i \le m} (\lambda_{i,k+1})^{\frac{r_i}{p_i}}.$$

We note that (5.3) implies

$$\bar{\lambda}_k \to +\infty \quad \text{as } k \to +\infty, \tag{5.28}$$

then, if we assume

$$R_k = \left(\frac{b_6}{2b_7} \bar{\lambda}_k\right)^{\frac{1}{\bar{q} - p}},$$

with

$$p = \min_{1 \le i \le m} p_i, \qquad \bar{q} = \max_{1 \le i \le m} \bar{q}_i,$$

$\bar{q} > p > 1$ from (4.16), limit (5.28) gives

$$R_k \to +\infty \quad \text{as } k \to +\infty. \tag{5.29}$$

Thus, an integer $k_1 \in \mathbb{N}$ exists such that for all $k \ge k_1$ it is $R_k \ge 1$ and

$$R_k^{\bar{q}_i - p_i} \le R_k^{\bar{q} - p} \quad \text{for all } i \in \{1, \dots, m\}.$$

Or better, if $k_2 \ge k_1$ is such that for all $k \ge k_2$ we have $R_k \ge 2m$, taking $k \ge k_2$ and $\mathbf{u} \in Y_k^X$ such that $\|\mathbf{u}\|_W = R_k$, direct computations imply not only that

$$b_6 - \frac{b_7}{\bar{\lambda}_k} \|u_i\|_{W_i}^{\bar{q}_i - p_i} \ge b_6 - \frac{b_7}{\bar{\lambda}_k} R_k^{\bar{q}_i - p_i} \ge \frac{b_6}{2} \quad \text{for all } i \in \{1, \dots, m\},$$

but also

$$\sum_i \|u_i\|_{W_i}^{p_i} \ge \left(\frac{R_k}{2m}\right)^p.$$

Hence, (5.27) gives

$$\mathcal{J}(\mathbf{u}) \ge \frac{b_6}{2} \left(\frac{R_k}{2m}\right)^p - b_8 \quad \text{if } \mathbf{u} \in Y_k^X \text{ is such that } \|\mathbf{u}\|_W = R_k.$$

Thus, taking any $\varrho > 0$, (5.24) follows from (5.29) if $k = k(\varrho) \in \mathbb{N}$ is large enough.
Moreover, taking $\bar{k} > k$, the $\bar{k}$–dimensional space $V_{\bar{k}}$, defined as in (5.5), is such that not only $\mathrm{codim} Y_k < \dim V_{\bar{k}}$ but also (5.17) holds. Whence, assumption $(\mathcal{H}_\varrho)$ is verified with $\mathcal{M}_\varrho = \{\mathbf{u} \in X : \|\mathbf{u}\|_W = R_k\}$.
Finally, since (5.6) and (5.7) give $\mathcal{J}(\mathbf{0}) = 0$, from Propositions 3.5, 4.8 and the arbitrariness of ϱ for condition $(\mathcal{H}_\varrho)$, we have that Corollary 2.4 applies to $\mathcal{J}$ in X and a sequence of diverging critical levels exists. □ □

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