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QUANTIZATION OF THE POISSON TYPE CENTRAL LIMIT THEOREM (1)

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Abstract. A sequence of binomial random variables, both classical and algebraic, is modelized in terms of the creation–annihilation operators in a natural way and each of these random variables is a sum of four terms. By taking a proper interacting Fock structure, these random variables verify a certain pre–given (classical, Boolean, free, monotone, anti–monotone, etc) independence and the sum of finite independent binomial random variables formulates the corresponding Bernoulli sequence. With the help of such a structure, the Poisson type central limit theorem is quantized by considering individually the contribution of those four terms to the limit. Moreover, its *off–diagonal part* gives a quantization of the Laplace–de Moivre type central limit theorem.

1. Introduction

The main goal of the present paper and [10], [11], [12] is to set a **quantization** of both the classical and the algebraic (Boolean, free, monotone, anti–monotone, etc) Poisson central limit theorem (CLT in short).

In this paper, we give
- a general discussion of the problem;
- an outline of the main results;
- a full treatment of a quantization of the classical Poisson CLT.

Other cases will be discussed elsewhere (see [10], [11] and [12]).

1.1. Main problem, both the classical and the algebraic cases. Recall that the classical Poisson CLT can be formulated as follows:

If $\{p_n\}_{n=1}^{\infty} \subset [0,1]$ *verifies* $np_n \to \lambda$, *then in the weak convergence,*

$$\lim_{n\to\infty} \left((1-p_n)\,\delta_0 + p_n\delta_1 \right)^{*n} = \mu_\lambda \tag{1.1}$$

hereinafter,
- δ_x *is the Dirac measure centred on* x *for any* $x \in \mathbb{R}$;
- $*$ *is the classical convolution*;

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• μ_λ *is the Poisson distribution with the parameter* λ, *i.e.,*

$$\mu_\lambda := \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda k}}{k!} \, \delta_k$$

in particular, $\mu_0 := \delta_0$.

It is well known (see [16], [18], [19]) that, replacing the classical convolution by either Boolean, or free, or monotone (anti–monotone) convolution, *there exists such a probability measure* μ_λ *that (1.1) holds whenever* $np_n \to \lambda$. These results are called the corresponding Poisson CLT, μ_λ is named as the corresponding Poisson distribution with the parameter λ and in particular, $\mu_0 := \delta_0$.

One can also reformulate all the Poisson type CLT (classical or algebraic) in terms of *algebraic* random variables and the independences as follows:

Let $(\mathcal{X}, \psi)$ *be an algebraic probability space and let* $\{\xi_{n,k} : n \in \mathbb{N}^*,\ k \le n\}$ *be a* $\psi-$*binomial family of* **algebraic** *random variables, that is*

• $\psi\big(\xi_{n,k}^m\big) = \psi\left(\xi_{n,k}\right)$ and $p_n := \psi\left(\xi_{n,k}\right) \in [0,1]$ for any $n, m \in \mathbb{N}^*$ and $k \le n$ (one says naturally that any $\xi_{n,k}$ has the binomial distribution $b\left(1, p_n\right)$ with respect to ψ);

• *for any* $n \in \mathbb{N}^*$, $\{\xi_{n,k} : k \le n\}$ *is an independent (either classical or Boolean, or free, or monotone, or anti–monotone, etc) family with respect to* ψ.

Then, the $\psi-$*distribution of* $\sum_{k=1}^n \xi_{n,k}$ *goes to the corresponding Poisson distribution with the parameter* λ *whenever* $np_n \to \lambda$.

The main problem to be treated in this paper and [10], [11], [12] is a **quantization** of the above Poisson type CLT.

In order to understand the meaning and results of this quantization, one notices first of all that the above reformulation of the Poisson type CLT in terms of random variables does **not** depend specific construction of the algebraic probability space and random variables. In other words, if we take a **specific** $(\mathcal{A}, \phi)$ and $\{X_{n,k} : n \in \mathbb{N}^*,\ k \le n\}$, versus the general $(\mathcal{X}, \psi)$ and $\{\xi_{n,k} : n \in \mathbb{N}^*,\ k \le n\}$, the Poisson type CLT does **not** change whenever

• for any $n \in \mathbb{N}^*$ and $k \le n$, the $\phi-$distribution of $X_{n,k}$ is the same as the $\psi-$distribution of $\xi_{n,k}$;

• for any $n \in \mathbb{N}^*$, $\phi-$independence of family $\{X_{n,k} : k \le n\}$ is the same as the $\psi-$independence of the family $\{\xi_{n,k} : k \le n\}$.

Thanks to this observation, one can take a specific–concrete $(\mathcal{A}, \phi)$ and a $\phi-$binomial family $\{X_{n,k} : n \in \mathbb{N},\ k \le n\}$ such that the Poisson type CLT can be realized and one gets, in addition, some new information. In fact, we will take the following *Fock random variables*

$$X_{n,k} := \sqrt{p_n\left(1 - p_n\right)}\left(a_k + a_k^+\right) + p_n a_k a_k^+ + \left(1 - p_n\right) a_k^+ a_k \tag{1.2}$$

for any $n \in \mathbb{N}^*$ and $k \le n$, where

• $\{p_n\}_{n=1}^\infty \subset [0,1]$;

• for any $k \in \mathbb{N}^*$, a_k (respectively, a_k^+) is the annihilation (respectively, creation) operator $a(e_k)$ (respectively, $a^+(e_k)$) defined on a certain interacting Fock space (IFS in short and see, e.g. [3], [8] and [9] for the detail) over a given (pre–)Hilbert space $\mathcal{H}$ with orthogonal normal family $\{e_k\}_{k=1}^\infty$.

• ϕ :=the vacuum state.

The structure of the IFS said before will be taken in such a proper way that

(H1) $\phi-$distribution of $X_{n,k}$ is the binomial distribution $b(1,p_n)$ ($X_{n,k} \overset{\phi}{\sim} b(1,p_n)$ in short) for any $n\in\mathbb{N}^*$ and $k\le n$;

(H2) with respect to ϕ, $\{\mathcal{A}_k\}_{k=1}^{\infty}$ is a family verifying a pre–given (classical, Boolean, free, monotone, anti–monotone, etc, according requirement) independence, hereinafter, $\mathcal{A}_k$ is the algebra generated by a_k and a_k^+, i.e., $\mathcal{A}_k$ is the totality of polynomials in a_k and a_k^+ with the degree greater than or equals to 1.

By taking $\mathcal{A}$ the algebra generated by the family $\{\mathcal{A}_k\}_{k=1}^{\infty}$, one has an algebraic probability space $(\mathcal{A},\phi)$. Since $\sum_{k=1}^n X_{n,k}$ is a sum of n independent (classical, Boolean, free, monotone, anti–monotone, etc) random variables of the distribution $b(1,p_n)$, it is natural to say that it has the (classical, Boolean, free monotone, anti–monotone, etc) **binomial distribution** with the parameter (n,p_n), i.e., $\sum_{k=1}^n X_{n,k} \overset{\phi}{\sim} b(n,p_n)$.

<u>Remark</u> 1) As a consequence of the above (H2), for any $n\in\mathbb{N}^*$ and $\varepsilon\in\{-1,0,1,2\}^n$, the family $\left\{a_k^{(\varepsilon(k))} : k\le n\right\}$ is independent (with respect to a pre–given independence), and so $\{X_{n,k}:k\le n\}$ is independent, hereinafter

$$a_k^{(\varepsilon)} := \begin{cases} a_k, & \text{if } \varepsilon=-1\\ a_k^+, & \text{if } \varepsilon=1\\ a_ka_k^+, & \text{if } \varepsilon=0\\ a_k^+a_k, & \text{if } \varepsilon=2\end{cases} \quad ,\quad \forall k\in\mathbb{N}^* \tag{1.3}$$

2) Different independence requires different interacting Fock structure and the concrete constructions will be given case by case.

By using notations introduced in (1.3), one writes (1.2) to

$$X_{n,k} = \sqrt{p_n(1-p_n)}\left(a_k^{(-1)}+a_k^{(+1)}\right) + p_n a_k^{(0)} + (1-p_n)\,a_k^{(2)} \tag{1.4}$$

Since $\sqrt{p_n(1-p_n)}a_k^{(-1)}$ and $\sqrt{p_n(1-p_n)}a_k^{(+1)}$ are conjugate each other, we call them the **off–diagonal components** of $X_{n,k}$; similarly, since $(1-p_n)\,a_k^{(2)}$ and $p_na_k^{(0)}$ are self–adjoint, we call them the **diagonal components** of $X_{n,k}$.

As the above said, the Poisson type CLT can be realized in terms of the concrete Fock random variables $\{X_{n,k}:n\in\mathbb{N}^*,\, k\le n\}$ with $X_{n,k}$ given in (1.4). Furthermore, one can read from (1.4) more information: **each $X_{n,k}$ is a sum of four terms and each term has a specific coefficient.** This information is absent if one considers general algebraic random variables.

By denoting

$$B_n^{(\pm1)} := \sqrt{p_n(1-p_n)}\sum_{k=1}^n a_k^{(\pm1)};\quad B_n^{(0)} := p_n\sum_{k=1}^n a_k^{(0)};\quad B_n^{(2)} := (1-p_n)\sum_{k=1}^n a_k^{(2)}$$

$$B_n := B_n^{(-1)}+B_n^{(+1)}+B_n^{(0)}+B_n^{(2)} = \sum_{k=1}^n X_{n,k} \tag{1.5}$$

then the usual Poisson type CLT is to consider the weak limit of the $\phi-$distribution of $\{B_n\}_{n=1}^{\infty}$, while the *quantization* of the Poisson type CLT is:

1) to examine, for any $m \in \mathbb{N}$ and $\varepsilon \in \{-1,0,1,2\}^m$, the limit

$$\lim_{n\to\infty} \phi\left(B_n^{(\varepsilon(1))} \dots B_n^{(\varepsilon(m))}\right) \tag{1.6}$$

i.e., to see *individually* the limits of $B_n^{(\varepsilon)}$ for any $\varepsilon \in \{-1,0,1,2\}$, not only their sum;

2) to study, for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$, the weak limit of the $\phi-$distribution of the sequence $\left\{c_1\left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)} + c_2 B_n^{(2)}\right\}_{n=1}^{\infty}$ instead of $\{B_n\}_{n=1}^{\infty}$;

3) to give a suitable representation to the above limits.

Clearly, by taking $c_0 = c_1 = c_2 = 1$ in the above 2), one has the usual Poisson type CLT. Moreover, in the case of $np_n \to \lambda$ (recall that this is a standard assumption for performing the Poisson type CLT), one has

$$B_n^{(\pm 1)} := \sqrt{p_n\left(1-p_n\right)} \sum_{k=1}^n a_k^{(\pm 1)} \approx \sqrt{\lambda} \frac{1}{\sqrt{n}} \sum_{k=1}^n a_k^{(\pm 1)}$$

and so by taking $c_0 = c_2 = 0$ and $c_1 = 1$, one gets the usual Laplace–de Moivre type CLT. Therefore, **the usual Laplace–de Moivre type CLT is in fact the off–diagonal part the quantization of the Poisson type CLT**. In this way, the quantization of the Poisson type CLT gives a view to understand the relationship between the (classical, Boolean, free monotone, anti–monotone, etc) Poisson type CLT and the corresponding Laplace–de Moivre type CLT: roughly speaking, *the Laplace–de Moivre type CLT is one part of the Poisson type CLT*. Furthermore, by using the *representation* mentioned in the above 3), one can also understand the relationship between the (classical, Boolean, free monotone, anti–monotone, etc) Poisson distribution and the corresponding Gaussian distribution. Hereinafter,

• the classical Gaussian distribution:=the Normal distribution;

• the Boolean Gaussian distribution:= the 2 points symmetric distribution with equi—probability;

• the free Gaussian distribution:=the Wigner distribution;

• the monotone (anti–monotone) Gaussian distribution:=the arc–sine distribution.

Moreover, the determinative *standard* could be added before the above four distributions, it presumes zero–mean and uni–variance.

1.2. Re–scaling in (1.5). The coefficients of operators in (1.5), as well as in (1.4), are in fact some kinds of **re–scaling**. These re–scaling are not put by hand, in fact they have a natural motivation and now we see it in classical case.

The starting point is the following common accepted **convention** (see, e.g. [1], [2], [7], [14], [17]): by denoting $\mathcal{M}_2$:=the set of all 2×2 complex matrices, *the matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *as an algebraic random variable on the algebraic probability* *space* $(\mathcal{M}_2, \phi_1)$ *with* $\phi_1(\cdot) := \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$, *is a natural quantization of classical random variable with the distribution* $\frac{1}{2}(\delta_{-1} + \delta_1)$.

A step forward from the above staring point, one asks naturally, on the algebraic probability space $(\mathcal{M}_2, \phi_1)$, *what* $A := \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix}$ *(where* $\{a, c\} \subset \mathbb{R}$ *and* $b \in \mathbb{C}$*) does give a quantization of a random variable distributed* $(1-p)\delta_{-1} + p\delta_1$, *instead of* $\frac{1}{2}(\delta_{-1} + \delta_1)$, *with a given* $p \in [0,1]$?

It is easy to find an answer as follows.

On the one hand, the definitions of A and ϕ_1 say that

$$\phi_1(A) = a;\ \phi_1(A^2) = a^2 + |b|^2;\ \ \phi_1(A^3) = a\left(a^2 + |b|^2\right) + |b|^2(a+c) \qquad (1.7)$$

On the other hand, the fact $A \overset{\phi_1}{\sim} (1-p)\delta_{-1} + p\delta_1$ gives

$$\phi_1(A) = 2p - 1;\ \ \phi_1(A^2) = 1;\ \ \phi_1(A^3) = 2p - 1 \qquad (1.8)$$

So one gets, by combining (1.7) with (1.8),

$$a = 2p-1;\ b = 2e^{i\theta}\sqrt{p(1-p)};\ c = 1 - 2p$$

with $\theta \in \mathbb{R}$, i.e.,

$$A = A_{\theta,p} = \begin{pmatrix} 2p-1 & 2e^{i\theta}\sqrt{p(1-p)} \\ 2e^{-i\theta}\sqrt{p(1-p)} & 1-2p \end{pmatrix} \qquad (1.9)$$

Moreover, in the classical case, a random variable ξ distributes $(1-p)\delta_{-1} + p\delta_1$, if and only if $\frac{1}{2}(1+\xi)$ distributes $(1-p)\delta_0 + p\delta_1$. That is, a natural quantization of a classical random variable with the distribution $(1-p)\delta_0 + p\delta_1$ is $s_{\theta,p} := \frac{1}{2}(\mathbf{1} + A_{\theta,p})$, i.e.

$$s_{\theta,p} = \begin{pmatrix} p & e^{i\theta}\sqrt{p(1-p)} \\ e^{-i\theta}\sqrt{p(1-p)} & 1-p \end{pmatrix}$$

Notice that $a := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (respectively $a^+ := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$) is nothing else the *baby–annihilator* (respectively, the *baby–creator*), $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = aa^+$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = a^+a$. So (without loss of generality, one takes $\theta = 0$), a natural quantization of a classical random variable with the distribution $(1-p)\delta_0 + p\delta_1$ (i.e. the binomial distribution $b(1,p)$) is

$$\sqrt{p(1-p)}\left(a + a^+\right) + paa^+ + (1-p)a^+a \qquad (1.10)$$

1.3. Outline of main results. The main results of the quantization of the Poisson type CLT could be formulated as follows:

1) In all cases (by assuming anyone of the classical, Boolean, free, monotone, anti–monotone, etc, independence), the limit (1.6) exists for any $m \in \mathbb{N}^*$ and $\varepsilon \in \{-1, 0, 1, 2\}^m$. Moreover the limit (1.6) can be represented to

$$\left\langle \Phi, b^{(\varepsilon(1))} \ldots b^{(\varepsilon(m))}\Phi \right\rangle \lambda^{\sum_{k=1}^{m}(1-|\varepsilon(k)|/2)} \qquad (1.11)$$

where, Φ and $b^{(\varepsilon)}$'s will be given in the below 3). In other words, $B_n^{(\varepsilon)} \longrightarrow b^{(\varepsilon)}\lambda^{1-|\varepsilon|/2}$ in *the convergence of mixed—moments* for any $\varepsilon \in \{-1, 0, 1, 2\}$.

2) The weak limit of the vacuum distribution of the sequence $\Big\{c_1\big(B_n^{(-1)} + B_n^{(+1)}\big) + c_0 B_n^{(0)} + c_2 B_n^{(2)}\Big\}_{n=1}^{\infty}$ goes to the vacuum distribution of the operator $\sqrt{\lambda}c_1\left(b^{(-1)} + b^{(+1)}\right) + \lambda c_0 b^{(0)} + c_2 b^{(2)}$.

3) $b^{(\varepsilon)}$'s are operators defined on a certain Fock space Γ over a (pre–)Hilbert space $\mathcal{H}$ and Φ is the vacuum vector of Γ. The precise construction of the Fock space Γ depends on the pre–given independence. In addition, in the monotone (anti–monotone) case, $b^{(\pm 1)}$ depends also on a certain *test function* (i.e., an element of $\mathcal{H}$). In more detail

• if one assumes the classical independence, the above Fock space Γ is the one–mode interacting Fock space (1M–IFS in short and see [3] for the detail) $\Gamma\left(\mathbb{C}, \{\omega_n\}_n\right)$ with $\omega_n = n$ for any $n \in \mathbb{N}$; $b^{(-1)}$ and $b^{(+1)}$ are the annihilation–creation operators (recall that in this case, the vacuum distribution of $b^{(-1)} + b^{(+1)}$ is the *standard* normal distribution $N(0,1)$); $b^{(0)} = \mathbf{1}$:=the identity of Γ and $b^{(2)} = \omega_\Lambda = \Lambda = b^{(+1)}b^{(-1)}$, hereinafter, in all cases, if without any alternative mention, Λ :=the number operator on the Fock space Γ; $\omega_\Lambda := \omega \circ \Lambda$ and ω is the application from $\mathbb{N}$ to $[0, +\infty)$, defined as $\omega(n) := \omega_n$ for any $n \in \mathbb{N}$;

• if one assumes the Boolean independence, the above Fock space Γ is the 1M–IFS $\Gamma\left(\mathbb{C}, \{\omega_n\}_n\right)$ with $\omega_1 = 1$ and $\omega_n = 0$ for any $n \geq 2$; $b^{(-1)}$ and $b^{(+1)}$ are the annihilation–creation operators (recall that in this case, the vacuum distribution of $b^{(-1)} + b^{(+1)}$ is $\frac{1}{2}(\delta_{-1} + \delta_1)$, i.e. the *standard* Boolean–Gaussian distribution); $b^{(0)} = P_\Phi = b^{(-1)}b^{(+1)}$ and $b^{(2)} = \omega_\Lambda = \mathbf{1} - P_\Phi = b^{(+1)}b^{(-1)}$, hereinafter, P_Φ :=the projector to the vacuum if without any alternative mention;

• if one assumes the free independence, the above Fock space Γ is the 1M–IFS $\Gamma\left(\mathbb{C}, \{\omega_n\}_n\right)$ with $\omega_n = 1$ for any $n \in \mathbb{N}^*$; $b^{(-1)}$ and $b^{(+1)}$ are the annihilation–creation operators (recall that in this case, the vacuum distribution of $b^{(-1)} + b^{(+1)}$ is the *standard* free–Gaussian distribution); $b^{(0)} = \mathbf{1} = b^{(-1)}b^{(+1)}$ and $b^{(2)} = \omega_\Lambda = \mathbf{1} - P_\Phi = b^{(+1)}b^{(-1)}$;

• if one assumes the monotone (anti–monotone) independence, the above Fock space Γ is the monotone (anti–monotone) Fock space over $\mathbf{L}^2([0,1])$ (no more a 1M–IFS), $b^{(-1)}$ and $b^{(+1)}$ are the annihilation–creation operators *with the test function* $\chi_{[0,1]} \in \mathbf{L}^2([0,1])$ (recall that in this case, the vacuum distribution of $b^{(-1)} + b^{(+1)}$ is the *standard* monotone–Gaussian distribution); $b^{(0)} = b^{(-1)}b^{(+1)}$ and $b^{(2)} = \mathbf{1} - P_\Phi$.

Remark In the monotone (anti–monotone) case, it is impossible like in other cases to represent (1.11) on a 1M–IFS such that $b^{(-1)}$ and $b^{(+1)}$ are the corresponding annihilation–creation operators. For the detail, see [12].

2. Quantization of the Classical Poisson CLT in Terms of the Moments

In the classical case, we use the *baby–Fock space* as model space.

Recall that on the baby–Fock space, the annihilation–creation operators are defined as follows:

$$a := a^{(-1)} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^{+} := a^{(+1)} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Let's introduce

$$a^{(0)} := aa^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a^{(2)} := a^{+}a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$\begin{aligned} & a^{(-1)}a^{(+1)} = a^{(0)}; \quad a^{(+1)}a^{(-1)} = a^{(2)}; \quad a^{(0)}a^{(+1)} = 0 \\ & a^{(+1)}a^{(0)} = a^{(+1)}; \quad a^{(2)}a^{(+1)} = a^{(+1)}; \quad a^{(+1)}a^{(2)} = 0 \\ & a^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - a^{(2)}; \quad \left(a^{(\pm 1)}\right)^{*} = a^{(\mp 1)}; \quad \left(a^{(\varepsilon)}\right)^{*} = a^{(\varepsilon)}, \quad \forall \varepsilon \in \{0, 2\} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} s_p := \text{the expression (1.10) } &= \sqrt{p\,(1-p)}\left(a^{(-1)} + a^{(+1)}\right) + pa^{(0)} + (1-p)\,a^{(2)} \\ &= \begin{pmatrix} p & \sqrt{p\,(1-p)} \\ \sqrt{p\,(1-p)} & 1-p \end{pmatrix} \end{aligned}$$

Moreover, s_p is a projector, $\phi_1(s_p) = p$ (recall that $\phi_1(\cdot) := \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$) and so the ϕ_1-distribution of s_p is $b(1,p)$.

Let's introduce $\Phi_n := \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes n}$ and $\phi_n(\cdot) = \langle \Phi_n, \cdot \Phi_n \rangle$ for any n, then ϕ_n is a state on $\mathcal{M}_2^{\otimes n}$ and ϕ :=the inductive limit of $\{\phi_n\}_n$ is a state on $\bigotimes_{k=1}^{\infty} \mathcal{M}_2$. Let's introduce also the elements of $\bigotimes_{k=1}^{\infty} \mathcal{M}_2$

$$a_k^{(\varepsilon)} := \mathbf{1}^{\otimes (k-1)} \otimes a^{(\varepsilon)} \otimes \mathbf{1}, \quad \forall k \text{ and } \forall \varepsilon \in \{-1, 1\} \tag{2.2}$$

then

$$\mathbf{1}^{\otimes (k-1)} \otimes s_p \otimes \mathbf{1} = \sum_{\varepsilon \in \{-1,0,1,2\}} \mathbf{1}^{\otimes (k-1)} \otimes a^{(\varepsilon)} \otimes \mathbf{1} = \sum_{\varepsilon \in \{-1,0,1,2\}} a_k^{(\varepsilon)}$$

is clearly a projector and has the $\phi-$distribution $b(1,p)$. So, with respect to the state ϕ,

- $X_{n,k}$ introduced in (1.4) is a projector and has the $\phi-$distribution $b(1, p_n)$;
- $\{X_{n,k} : k \le n\}$ is a classical independent family.

Moreover, it follows easily from (2.1) and (2.2) that

$$\begin{aligned} & \left[a_k^{(-1)}, a_h^{(+1)}\right] = \delta_{k,h}\left(a_k^{(0)} - a_k^{(2)}\right), \quad \left[a_k^{(0)}, a_h^{(+1)}\right] = -\delta_{k,h} a_k^{(+1)}, \\ & \left[a_k^{(2)}, a_h^{(+1)}\right] = \delta_{k,h} a_k^{(+1)}, \quad a_k^{(0)} = \mathbf{1} - a_k^{(2)}, \\ & \left(a_k^{(\pm 1)}\right)^{*} = a_k^{(\mp 1)}, \quad \left(a_k^{(\varepsilon)}\right)^{*} = a_k^{(\varepsilon)}, \quad \forall \varepsilon \in \{0, 2\} \text{ and } \forall k, h \end{aligned} \tag{2.3}$$

By using these notations, the well–known classical Poisson CLT can be formulated *algebraically*. In fact, in the following Proposition 2.1,

• the second affirmation is nothing else than the classical Poisson CLT in terms of the annihilation–creation operators on the baby–Fock space;
• the first affirmation is its moment version.

Proposition 2.1. *If* $\lim_{N\to\infty} Np_N = \lambda$, *one has,*
1) for any $m \in \mathbb{N}$,

$$\lim_{N\to\infty} \phi\Big(\Big(\sum_{k=1}^{N} X_{N,k}\Big)^m\Big) = \lim_{N\to\infty} \Big\langle \Phi_N, \Big(\sum_{k=1}^{N} X_{N,k}\Big)^m \Phi_N\Big\rangle$$

$$= m\text{–th moment of the Poisson distribution with the parameter } \lambda \tag{2.4}$$

2) for any $t \in \mathbb{R}$,

$$\lim_{N\to\infty} \phi\Big(\exp\Big(it\sum_{k=1}^{N} X_{N,k}\Big)\Big) = \lim_{N\to\infty} \Big\langle \Phi_N, \exp\Big(it\sum_{k=1}^{N} X_{N,k}\Big)\Phi_N\Big\rangle$$

$$= \exp\left(\lambda\left(e^{it}-1\right)\right) \tag{2.5}$$

Proof. Since each $X_{N,k}$ is a projector (so, $X_{N,k}^j = X_{N,k}$ for any $j \in \mathbb{N}^*$) and since

$$\sum_{\substack{1\le j_1,\dots,j_r\le m\\ j_1+\dots+j_r=m}} 1 = r^m, \quad \forall m \in \mathbb{N}^*,\ 1 \le r \le m$$

one gets, for any m,

$$\Big(\sum_{k=1}^{N} X_{N,k}\Big)^m = \sum_{r=1}^{m} \sum_{1\le k_1<\dots<k_r\le N} \sum_{\substack{1\le j_1,\dots,j_r\le m\\ j_1+\dots+j_r=m}} X_{N,k_1}^{j_1}\dots X_{N,k_r}^{j_r}$$

$$= \sum_{r=1}^{m} \sum_{1\le k_1<\dots<k_r\le N} \sum_{\substack{1\le j_1,\dots,j_r\le m\\ j_1+\dots+j_r=m}} X_{N,k_1}\dots X_{N,k_r}$$

$$= \sum_{r=1}^{m} \sum_{1\le k_1<\dots<k_r\le N} r^m X_{N,k_1}\dots X_{N,k_r}$$

Moreover, the ϕ–independence of the family $\{X_{N,k} : 1 \le k \le N\}$ gives

$$\phi\big(X_{N,k_1}\dots X_{N,k_r}\big) = \langle \Phi_N, X_{N,k_1}\dots X_{N,k_r}\Phi_N\rangle$$

$$= \prod_{h=1}^{r} \langle \Phi_N, X_{N,k_h}\Phi_N\rangle = \prod_{h=1}^{r} \phi\big(X_{N,k_h}\big)$$

and the fact $X_{N,k} \overset{\phi}{\sim} b(1,p_N)$ guarantees that $\phi\big(X_{N,k}\big) = p_N$. So, one finds

$$\phi\Big(\Big(\sum_{k=1}^{N} X_{N,k}\Big)^m\Big) = \sum_{r=1}^{m} \sum_{1\le k_1<\dots<k_r\le N} r^m\, \phi\big(X_{N,k_1}\dots X_{N,k_r}\big)$$

$$= \sum_{r=1}^{m} \sum_{1\le k_1<\dots<k_r\le N} r^m p_N^r = \sum_{r=1}^{m} \binom{N}{r} r^m p_N^r \tag{2.6}$$

where, the last equality is obtained thanks to the fact

$$\sum_{1\le k_1<\ldots<k_r\le N} 1 = \binom{N}{r}, \quad \forall N\in\mathbb{N}^*,\ 1\le r\le N$$

Let, for any $N\in\mathbb{N}$, $\{\xi_{N,1},\ldots,\xi_{N,N}\}$ be independent random variables defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ and $\xi_{N,k}\sim b(1,p_N)$ for all $k\le N$, then $\sum_{k=1}^N \xi_{N,k}\sim b(N,p_N)$ and so for any m,

$$\begin{aligned}\mathbf{E}\Big(\Big(\sum_{k=1}^N \xi_{N,k}\Big)^m\Big) &= \sum_{\substack{0\le j_1,\ldots,j_N\le m\\ j_1+\ldots+j_N=m}} \frac{m!}{j_1!\ldots j_N!}\mathbf{E}\left(\xi_{N,1}^{j_1}\ldots\xi_{N,N}^{j_N}\right)\\ &= \sum_{\substack{0\le j_1,\ldots,j_N\le m\\ j_1+\ldots+j_N=m}} \frac{m!}{j_1!\ldots j_N!}\mathbf{E}\left(\xi_{N,1}^{j_1}\right)\ldots\mathbf{E}\left(\xi_{N,N}^{j_N}\right)\end{aligned}$$

Because $\xi_{N,k}^j = \begin{cases}1, & \text{if } j=0\\ \xi_{N,k}, & \text{if } j\ge 1\end{cases}$ and $\xi_{N,k}\sim b(1,p_N)$, one gets $\mathbf{E}\left(\xi_{N,k}^j\right) = \begin{cases}1, & \text{if } j=0\\ p_N, & \text{if } j\ge 1\end{cases}$. Moreover, by noticing that the set

$$\big\{(j_1,\ldots,j_N): 0\le j_1,\ldots,j_N\le m,\ j_1+\ldots+j_N=m\big\}$$

equals to

$$\bigcup_{r=1}^m \bigcup_{1\le k_1<\ldots<k_r\le N} \big\{(i_1,\ldots,i_N): i_{k_1},\ldots,i_{k_r}\ge 1,\ i_1+\ldots+i_N=m\big\}$$

one has

$$\begin{aligned}\mathbf{E}\Big(\Big(\sum_{k=1}^N \xi_{N,k}\Big)^m\Big) &= \sum_{r=1}^m \sum_{1\le k_1<\ldots<k_r\le N} \sum_{\substack{1\le j_1,\ldots,j_r\le m\\ j_1+\ldots+j_r=m}} \mathbf{E}\left(\xi_{N,k_1}^{j_1}\ldots\xi_{N,k_r}^{j_r}\right)\\ &= \sum_{r=1}^m \sum_{1\le k_1<\ldots<k_r\le N} \sum_{\substack{1\le j_1,\ldots,j_r\le m\\ j_1+\ldots+j_r=m}} \mathbf{E}\left(\xi_{N,k_1}^{j_1}\right)\ldots\mathbf{E}\left(\xi_{N,k_r}^{j_r}\right)\\ &= \sum_{r=1}^m \sum_{1\le k_1<\ldots<k_r\le N} \sum_{\substack{1\le j_1,\ldots,j_r\le m\\ j_1+\ldots+j_r=m}} \mathbf{E}\left(\xi_{N,k_1}\right)\ldots\mathbf{E}\left(\xi_{N,k_r}\right)\\ &= \sum_{r=1}^m \sum_{1\le k_1<\ldots<k_r\le N} \sum_{\substack{1\le j_1,\ldots,j_r\le m\\ j_1+\ldots+j_r=m}} p_N^r\\ &= \sum_{r=1}^m \binom{N}{r} r^m p_N^r \qquad (2.7)\end{aligned}$$

and so

$$\Big\langle\Phi_N, \Big(\sum_{k=1}^N X_{N,k}\Big)^m\Phi_N\Big\rangle = \mathbf{E}\Big(\Big(\sum_{k=1}^N \xi_{N,k}\Big)^m\Big) \qquad (2.8)$$

On the other hand, in virtue of the fact $\sum_{k=1}^{N}\xi_{N,k}\sim b(N,p_N)$ and the assumption $Np_N\longrightarrow\lambda$, one gets

$$\mathbf{E}\Big(\Big(\sum_{k=1}^{N}\xi_{N,k}\Big)^m\Big)=\sum_{k=0}^{N}k^m\binom{N}{k}p_N^k\left(1-p_N\right)^{n-k}$$
$$=\sum_{k=0}^{N}\frac{k^m}{k!}N\left(N-1\right)\dots\left(N-k+1\right)p_N^k\left(1-p_N\right)^{N-k}\longrightarrow\sum_{k=0}^{\infty}\frac{k^m}{k!}\lambda^k e^{-\lambda}\tag{2.9}$$

and the affirmation 1) is proved by combining together (2.8) and (2.9).

For each N, $\{X_{N,k}:k\le N\}$ is a commutative family and so

$$\exp\Big(it\sum_{k=1}^{N}X_{N,k}\Big)=\prod_{k=1}^{N}\exp\left(itX_{N,k}\right)$$

Since $\begin{pmatrix} p_N & \sqrt{p_N\left(1-p_N\right)}\\ \sqrt{p_N\left(1-p_N\right)} & 1-p_N\end{pmatrix}$ is a projector, one finds

$$\exp\left(itX_{N,k}\right)=\exp\Big(it\mathbf{1}^{\otimes(k-1)}\otimes\begin{pmatrix} p_N & \sqrt{p_N\left(1-p_N\right)}\\ \sqrt{p_N\left(1-p_N\right)} & 1-p_N\end{pmatrix}\otimes\mathbf{1}\Big)$$
$$=\mathbf{1}^{\otimes(k-1)}\otimes\Big(\mathbf{1}+\begin{pmatrix} p_N & \sqrt{p_N\left(1-p_N\right)}\\ \sqrt{p_N\left(1-p_N\right)} & 1-p_N\end{pmatrix}\sum_{m=1}^{\infty}\frac{(it)^m}{m!}\Big)\otimes\mathbf{1}$$
$$=\mathbf{1}^{\otimes(k-1)}\otimes\Big(\mathbf{1}+\begin{pmatrix} p_N & \sqrt{p_N\left(1-p_N\right)}\\ \sqrt{p_N\left(1-p_N\right)} & 1-p_N\end{pmatrix}\left(e^{it}-1\right)\Big)\otimes\mathbf{1}$$

and so

$$\Big\langle\begin{pmatrix}1\\0\end{pmatrix}^{\otimes N},\prod_{k=1}^{N}\exp\left(itX_{N,k}\right)\begin{pmatrix}1\\0\end{pmatrix}^{\otimes N}\Big\rangle$$
$$=\prod_{k=1}^{N}\Big\langle\begin{pmatrix}1\\0\end{pmatrix},\Big(\mathbf{1}+\begin{pmatrix} p_N & \sqrt{p_N\left(1-p_N\right)}\\ \sqrt{p_N\left(1-p_N\right)} & 1-p_N\end{pmatrix}\left(e^{it}-1\right)\Big)\begin{pmatrix}1\\0\end{pmatrix}\Big\rangle$$

Consequently, by combining these with the fact

$$\Big\langle\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix} p_N & \sqrt{p_N\left(1-p_N\right)}\\ \sqrt{p_N\left(1-p_N\right)} & 1-p_N\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\Big\rangle=p_N$$

one gets, thanks to the assumption $Np_N\longrightarrow\lambda$,

$$\Big\langle\Phi_N,\exp\Big(it\sum_{k=1}^{N}X_{N,k}\Big)\Phi_N\Big\rangle=\prod_{k=1}^{N}\left\langle\Phi_N,\exp\left(itX_{N,k}\right)\Phi_N\right\rangle$$
$$=\left(1+p_N\left(e^{it}-1\right)\right)^N\longrightarrow e^{\lambda\left(e^{it}-1\right)}$$

□

The following result plays crucial role in the proof of Theorem 2.3 and moreover it has its own interest as well.

Lemma 2.2. *1) On the baby–Fock space, by denoting* $S_N^{(\varepsilon)} := \frac{1}{N^{1-|\varepsilon|/2}} \sum_{k=1}^N a_k^{(\varepsilon)}$ *for any* $\varepsilon \in \{-1, 0, 1, 2\}$*, one has*

$$S_N^{(0)} \Phi_N = \Phi_N; \quad S_N^{(\varepsilon)} \Phi_N = 0, \ \forall \varepsilon \in \{-1, 2\}; \quad \left\langle \Phi_N, S_N^{(+1)} \Phi_N \right\rangle = 0 \tag{2.10}$$

and

$$\begin{aligned} \left[S_N^{(2)}, S_N^{(+1)}\right] &= S_N^{(+1)}; \quad \left[S_N^{(0)}, S_N^{(+1)}\right] = -\frac{1}{N} S_N^{(+1)} \\ \left[S_N^{(-1)}, S_N^{(+1)}\right] &= S_N^{(0)} - \frac{1}{N} S_N^{(2)}; \quad S_N^{(0)} = \mathbf{1} - \frac{1}{N} S_N^{(2)} \end{aligned} \tag{2.11}$$

2) On the 1M-IFS $\Gamma(\mathbb{C}, \{\omega_n\}_n)$ *with the* $\omega_n = n$ *for all* $n \in \mathbb{N}$*, by denoting* b *(*b^+ *and* Λ *respectively) the annihilation operator (the creation operator and the number operator respectively) and*

$$b^{(\varepsilon)} := \begin{cases} b, & \text{if } \varepsilon = -1 \\ b^+, & \text{if } \varepsilon = 1 \\ \mathbf{1}, & \text{if } \varepsilon = 0 \\ \Lambda, & \text{if } \varepsilon = 2 \end{cases} \tag{2.12}$$

for any $\varepsilon \in \{-1, 0, 1, 2\}$ *, one has*

$$\left[b^{(-1)}, b^{(+1)}\right] = b^{(0)}; \quad \left[b^{(2)}, b^{(+1)}\right] = b^{(+1)}; \quad \left[b^{(0)}, b^{(+1)}\right] = 0 \tag{2.13}$$

Proof. 1) One gets (2.10) just by the definition; the last equality of (2.11) is trivial in virtue of the 4^{th} equality in (2.3), i.e. $a_k^{(0)} = \mathbf{1} - a_k^{(2)}$ for all k.

By applying the first three equalities in (2.3) and by summing k and h, one finds

$$\begin{aligned} \left[S_N^{(-1)}, S_N^{(+1)}\right] &= \frac{1}{N} \sum_{1 \le k,h \le N} \left[a_k^{(-1)}, a_h^{(+1)}\right] \\ &= \frac{1}{N} \sum_{1 \le k,h \le N} \delta_{k,h} \left(a_k^{(0)} - a_k^{(2)}\right) = S_N^{(0)} - \frac{1}{N} S_N^{(2)} \\ \left[S_N^{(0)}, S_N^{(+1)}\right] &= \frac{1}{N^{3/2}} \sum_{1 \le k,h \le N} \left[a_k^{(0)}, a_h^{(+1)}\right] \\ &= -\frac{1}{N^{3/2}} \sum_{1 \le k,h \le N} \delta_{k,h} a_k^{(+1)} = -\frac{1}{N} S_N^{(+1)} \\ \left[S_N^{(2)}, S_N^{(+1)}\right] &= \frac{1}{\sqrt{N}} \sum_{1 \le k,h \le N} \left[a_k^{(2)}, a_h^{(+1)}\right] \\ &= \frac{1}{\sqrt{N}} \sum_{1 \le k,h \le N} \delta_{k,h} a_k^{(+1)} = S_N^{(+1)} \end{aligned} \tag{2.14}$$

2) The last equality of (2.13) is trivial because of $b^{(0)} = \mathbf{1}$. Recall that on any 1M–IFS $\Gamma(\mathbb{C}, \{\omega_n\}_n)$, one has always

$$bb^+ = \omega_{\Lambda+1}, \quad b^+ b = \omega_\Lambda, \quad b^+ \omega_{\Lambda+1} = \omega_\Lambda b^+$$

In particular, in the case of $\omega_n = n$ for any $n \in \mathbb{N}$,

$$bb^+ = \Lambda + 1, \quad b^+ b = \Lambda, \quad b^+ (\Lambda + 1) = \Lambda b^+$$

So, the first and the second equalities of (2.13) are obtained. □

Remark As noted in Subsection 1.3, the vacuum distribution of the sum of creation and annihilation operators (i.e., $b + b^+$) on the 1M-IFS $\Gamma(\mathbb{C}, \{\omega_n\}_n)$ with the $\omega_n = n$ for all $n \in \mathbb{N}$ is the *standard* normal distribution $N(0,1)$.

The following result gives practically the limit (1.6), i.e., $B_n^{(\varepsilon)} \to b^{(\varepsilon)} \lambda^{1-|\varepsilon|/2}$ in the convergence of mixed—moments for any $\varepsilon \in \{-1, 0, 1, 2\}$ as said in Subsection 1.3.

Theorem 2.3. *If* $\lim_{N\to\infty} Np_N = \lambda$*, then for any* $n \in \mathbb{N}$ *and* $\varepsilon \in \{-1, 0, 1, 2\}^n$*,*

$$\begin{aligned}&\lim_{N\to\infty} \left\langle \Phi_N, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n))} \Phi_N \right\rangle \\ =&\left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))} \Phi \right\rangle \lambda^{\sum_{k=1}^n (1-|\varepsilon(k)|/2)}\end{aligned} \tag{2.15}$$

and

$$\lim_{N\to\infty} \left\langle \Phi_N, S_N^{(\varepsilon(1))} \dots S_N^{(\varepsilon(n))} \Phi_N \right\rangle = \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))} \Phi \right\rangle \tag{2.16}$$

hereinafter,

• Φ *is the vacuum vector of the 1M-IFS* $\Gamma(\mathbb{C}, \{\omega_n\}_n)$ *with the* $\omega_n = n$ *for all* $n \in \mathbb{N}$*;*

• b^ε*'s (where,* $\varepsilon \in \{-1, 0, 1, 2\}$*) are operators on the above 1M-IFS, defined in* (2.12)*.*

Proof. The definitions of $B_N^{(\varepsilon)}$ and $S_N^{(\varepsilon)}$ say that

$$B_N^{(\pm 1)} = \sqrt{Np_N (1-p_N)} S_N^{(\pm 1)}; \quad B_N^{(0)} = Np_N S_N^{(0)}; \quad B_N^{(2)} = (1-p_N) S_N^{(2)}$$

and so

$$B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n))} = (Np_N)^{\sum_{k=1}^N (1-|\varepsilon(k)|/2)} (1-p_N)^{\sum_{k=1}^N |\varepsilon(k)|/2} S_N^{(\varepsilon(1))} \dots S_N^{(\varepsilon(n))}$$

The assumption $Np_N \longrightarrow \lambda$ gives clearly

$$\lim_{N\to\infty} (Np_N)^{\sum_{k=1}^N (1-|\varepsilon(k)|/2)} (1-p_N)^{\sum_{k=1}^N |\varepsilon(k)|/2} = \lambda^{\sum_{k=1}^N (1-|\varepsilon(k)|/2)}$$

So, (2.15) and (2.16) are equivalent, we are going to prove (2.16).

The formula (2.10) and the definition of $b^{(\varepsilon)}$'s (i.e., (2.12)) tell us that

$$\left\langle \Phi_N, S_N^{(\varepsilon)} \Phi_N \right\rangle = \begin{cases} 0, & \text{if } \varepsilon \in \{-1, 1, 2\} \\ 1, & \text{if } \varepsilon = 0 \end{cases} = \left\langle \Phi, b^{(\varepsilon)} \Phi \right\rangle$$

i.e., (2.16) holds for $n = 1$.

Suppose that the thesis is proved up to n and let's see it for $n + 1$, i.e., we go to show that, for any $\varepsilon \in \{-1, 0, 1, 2\}^{n+1}$

$$\lim_{N\to\infty} \left\langle \Phi_N, S_N^{(\varepsilon(1))} \dots S_N^{(\varepsilon(n+1))} \Phi_N \right\rangle = \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n+1))} \Phi \right\rangle \tag{2.17}$$

If $\varepsilon(n+1) \in \{-1, 2\}$, by using the second equality of (2.10) and its analogy $b^{(\varepsilon)}\Phi = 0$ for any $\varepsilon \in \{-1, 2\}$, the two scalar products in (2.17) are zero.

If $\varepsilon(n+1)=0$, by using the first equality of (2.10) and its analogy $b^{(0)}\Phi=\Phi$ (in fact $b^{(0)}=\mathbf{1}$), the assumption of the induction gives

$$\begin{aligned}&\left\langle \Phi_N, S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(n+1))}\Phi_N\right\rangle = \left\langle \Phi_N, S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(n))}\Phi_N\right\rangle\\ \longrightarrow &\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}\Phi\right\rangle = \left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}b^{(0)}\Phi\right\rangle\\ =&\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(n+1))}\Phi\right\rangle\end{aligned}$$

The last case to be examined is $\varepsilon(n+1)=1$. We have first of all

$$\begin{aligned}S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(n+1))} &= S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(n))}S_N^{(+1)} = S_N^{(+1)}S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(n))}\\ &+\sum_{k=1}^{n} S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(k-1))}\left[S_N^{(\varepsilon(k))}, S_N^{(+1)}\right]S_N^{(\varepsilon(k+1))}\dots S_N^{(\varepsilon(n))}\end{aligned}$$

and analogously,

$$\begin{aligned}&b^{(\varepsilon(1))}\dots b^{(\varepsilon(n+1))} = b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}b^{(+1)}\\ =&b^{(+1)}b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))} + \sum_{k=1}^{n} b^{(\varepsilon(1))}\dots b^{(\varepsilon(k-1))}\left[b^{(\varepsilon(k))}, b^{(+1)}\right]b^{(\varepsilon(k+1))}\dots b^{(\varepsilon(n))}\end{aligned}$$

Since

$$\left\langle \Phi_N, S_N^{(+1)}S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(n))}\Phi_N\right\rangle = \left\langle S_N^{(-1)}\Phi_N, S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(n))}\Phi_N\right\rangle = 0$$

and since

$$\left\langle \Phi, b^{(+1)}b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}\Phi\right\rangle = \left\langle b\Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(n))}\Phi\right\rangle = 0$$

the thesis will be obtained if one is able to show that for any $k\in\{1,\dots,n\}$,

$$\begin{aligned}&\lim_{n\to\infty}\left\langle \Phi_N, S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(k-1))}\left[S_N^{(\varepsilon(k))}, S_N^{(+1)}\right]S_N^{(\varepsilon(k+1))}\dots S_N^{(\varepsilon(n))}\Phi_N\right\rangle\\ =&\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(k-1))}\left[b^{(\varepsilon(k))}, b^{(+1)}\right]b^{(\varepsilon(k+1))}\dots b^{(\varepsilon(n))}\Phi\right\rangle \qquad (2.18)\end{aligned}$$

and now we turn to prove (2.18).

(2.18) is trivial if $\varepsilon(k)=1$: in this case, both of commutators in (2.18) are zero.

If $\varepsilon(k)=-1$, the third formula in (2.11) tells us that the scalar product in the left hand side of (2.18) equals to

$$\begin{aligned}&\left\langle \Phi_N, S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(k-1))}S_N^{(0)}S_N^{(\varepsilon(k+1))}\dots S_N^{(\varepsilon(n))}\Phi_N\right\rangle\\ &-\frac{1}{N}\left\langle \Phi_N, S_N^{(\varepsilon(1))}\dots S_N^{(\varepsilon(k-1))}S_N^{(2)}S_N^{(\varepsilon(k+1))}\dots S_N^{(\varepsilon(n))}\Phi_N\right\rangle\end{aligned}$$

which goes, thanks to the assumption of induction, to

$$\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(k-1))}b^{(0)}b^{(\varepsilon(k+1))}\dots b^{(\varepsilon(n))}\Phi\right\rangle$$

i.e., because of the first formula in (2.13)

$$\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(k-1))}\left[b^{(-1)}, b^{(+1)}\right]b^{(\varepsilon(k+1))}\dots b^{(\varepsilon(n))}\Phi\right\rangle$$

If $\varepsilon(k) = 2$, the first formula in (2.11) tells us that the scalar product in the left hand side of (2.18) equals to

$$\left\langle \Phi_N, S_N^{(\varepsilon(1))} \dots S_N^{(\varepsilon(k-1))} S_N^{(+1)} S_N^{(\varepsilon(k+1))} \dots S_N^{(\varepsilon(n))} \Phi_N \right\rangle$$

which goes, thanks to the assumption of induction, to

$$\left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(k-1))} b^{(+1)} b^{(\varepsilon(k+1))} \dots b^{(\varepsilon(n))} \Phi \right\rangle$$

i.e., in virtue of the second formula in (2.13).

$$\left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(k-1))} \left[b^{(2)}, b^{(+1)} \right] b^{(\varepsilon(k+1))} \dots b^{(\varepsilon(n))} \Phi \right\rangle$$

If $\varepsilon(k) = 0$, the second formula in (2.11) tells us that the scalar product in the left hand side of (2.18) equals to

$$-\frac{1}{N} \left\langle \Phi_N, S_N^{(\varepsilon(1))} \dots S_N^{(\varepsilon(k-1))} S_N^{(+1)} S_N^{(\varepsilon(k+1))} \dots S_N^{(\varepsilon(n))} \Phi_N \right\rangle$$

which goes to zero, in virtue of the assumption of induction. On the other hand, the third formula in (2.13) makes sure that

$$\left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(k-1))} \left[b^{(0)}, b^{(+1)} \right] b^{(\varepsilon(k+1))} \dots b^{(\varepsilon(n))} \Phi \right\rangle = 0$$

Summing up, the induction principle gives the thesis. $\square$

Corollary If $\lim_{N\to\infty} Np_N = \lambda$, then for any $n \in \mathbb{N}$ and $\{c_{-1}, c_0, c_1, c_2\} \subset \mathbb{C}$

$$\lim_{N\to\infty} \left\langle \Phi_N, \Big(\sum_{\varepsilon\in\{-1,0,1,2\}} c_\varepsilon B_N^{(\varepsilon)} \Big)^n \Phi_N \right\rangle$$
$$= \left\langle \Phi, \left(\sqrt{\lambda} \left(c_{-1} b + c_1 b^+ \right) + c_0 \lambda + c_2 \Lambda \right)^n \Phi \right\rangle \tag{2.19}$$

In particular,

• by taking $c_{-1} = c_1 = 1$ and $c_0 = c_2 = 0$

$$\lim_{N\to\infty} \left\langle \Phi_N, \left(B_N^{(-1)} + B_N^{(+1)} \right)^n \Phi_N \right\rangle = \lambda^{n/2} \left\langle \Phi, \left(b + b^+ \right)^n \Phi \right\rangle \tag{2.20}$$

• the $\langle \Phi, \cdot \Phi \rangle$ −distribution of $\sqrt{\lambda}\,(b + b^+) + \lambda \mathbf{1} + \Lambda$ is the Poisson distribution with the parameter λ (recall that the Poisson distribution with the parameter 0 is understood as the Dirac measure δ_0).

Proof. The thesis is obtained just by combining Theorem 2.3 with the fact

$$\Big(\sum_{\varepsilon\in\{-1,0,1,2\}} c_\varepsilon B_N^{(\varepsilon)} \Big)^n = \sum_{\varepsilon\in\{-1,0,1,2\}^n} B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n))} \prod_{k=1}^n c_{\varepsilon(k)} \qquad \square$$

3. Quantization of the Classical Poisson CLT in Terms of the Generalized Characteristic Function

Now let's examine the limit of an arbitrary real affine combination of $B_N^{(\varepsilon)}$'s in the weak convergence.

Theorem 3.1. *If $\lim_{N\to\infty} Np_N = \lambda$, then for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$,*

$$\begin{aligned}
&\lim_{N\to\infty} \left\langle \Phi_N, \exp\left(it\left(c_1\big(B_N^{(-1)} + B_N^{(+1)}\big) + c_0 B_N^{(0)} + c_2 B_N^{(2)}\right)\right)\Phi_N\right\rangle \\
&= \left\langle \Phi, \exp\left(it\left(c_1\left(b^{(-1)} + b^{(+1)}\right) + c_0 b^{(0)} + c_2 b^{(2)}\right)\right)\Phi\right\rangle \\
&= \begin{cases} e^{it\lambda c_o}, & \text{if } c_1 = 0 \\ \exp\left(-\frac{t^2\lambda c_1^2}{2} + it\lambda c_0\right), & \text{if } c_1 \neq 0 \text{ and } c_2 = 0 \\ \exp\left(\frac{\lambda c_1^2}{c_2^2}\left(e^{itc_2} - 1\right) + \frac{it\lambda\left(c_0c_2 - c_1^2\right)}{c_2}\right), & \text{if } c_1 \neq 0 \text{ and } c_2 \neq 0 \end{cases}
\end{aligned} \tag{3.1}$$

i.e., the weak limit of the vacuum distribution of the sequence $\left\{c_1\big(B_N^{(-1)} + B_N^{(+1)}\big) + c_0 B_N^{(0)} + c_2 B_N^{(2)}\right\}_{N=1}^{\infty}$ *is*

$$\begin{cases} \delta_{\lambda c_0}, & \text{if } c_1 = 0 \\ N\left(\lambda c_0, \lambda c_1^2\right), & \text{if } c_1 \neq 0 \text{ and } c_2 = 0 \\ P_{c_0,c_1,c_2}(\lambda), & \text{if } c_1 \neq 0 \text{ and } c_2 \neq 0 \end{cases} \tag{3.2}$$

where

- *both $N\left(\lambda c_0, \lambda c_1^2\right)\Big|_{\lambda=0}$ and $P_{c_0,c_1,c_2}(\lambda)\Big|_{\lambda=0}$ are understood as δ_0;*
- *for any $a \in \mathbb{R}$ and $\sigma > 0, N\left(a, \sigma^2\right)$ is the normal distribution with the mean a and the variance σ^2;*
- *for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$ and $\lambda > 0$, $P_{c_0,c_1,c_2}(\lambda)$ is a* **linear transformation of the Poisson distribution,** *more precisely, $P_{c_0,c_1,c_2}(\lambda)$ is the distribution of $c_2\Big(\xi + \frac{\lambda\left(c_0c_2 - c_1^2\right)}{c_2^2}\Big)$ with ξ having the Poisson distribution of parameter $\frac{\lambda c_1^2}{c_2^2}$, in other words,*

$$P_{c_0,c_1,c_2}(\lambda) = \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda c_1^2}{c_2^2}\right)^k e^{-\frac{\lambda c_1^2}{c_2^2}}}{k!}\delta_{c_2\left(k + \frac{\lambda\left(c_0c_2 - c_1^2\right)}{c_2^2}\right)} \tag{3.3}$$

__Remark__ For understanding well the distribution $P_{c_0,c_1,c_2}(\lambda)$, it is better to notice that $P_{1,1,1}(\lambda)$ is the Poisson distribution with the parameter λ, i.e., $\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}\delta_k$.

In order to prove Theorem 3.1, we begin with the ϕ_1−distribution of a generic element of $\mathbb{M}_{2,sys}(\mathbb{R})$:=the set of all symmetric 2×2 real matrices.

Lemma 3.2. *Let* $A := \begin{pmatrix} a & c \\ c & b \end{pmatrix} \in \mathbb{M}_{2,sys}(\mathbb{R})$.

1) If $c = 0$, one has

$$\exp(itA) = \begin{pmatrix} e^{ita} & 0 \\ 0 & e^{itb} \end{pmatrix}; \qquad \phi_1(\exp(itA)) = e^{ita}$$

or equivalently, the ϕ_1-distribution of A is δ_a.

2) If $c \neq 0$, the matrix A has two different eigenvalues $\frac{1}{2}\Big(b+a\pm\sqrt{(b-a)^2+4c^2}\Big)$ and the corresponding eigenvectors $\begin{pmatrix} 1 \\ v_\pm \end{pmatrix}$ with

$$v_\pm := \frac{1}{2c}\Big(b-a\pm\sqrt{(b-a)^2+4c^2}\Big) \tag{3.4}$$

Moreover

$$v_+v_- = -1, \quad \frac{1}{1+v_+^2}+\frac{1}{1+v_-^2} = 1, \quad \frac{v_+}{1+v_+^2}+\frac{v_-}{1+v_-^2} = 0 \tag{3.5}$$

and

$$\begin{aligned} &\phi_1(\exp(itA)) \\ =&\frac{1}{1+v_+^2}e^{\frac{it}{2}\left(b+a+\sqrt{(b-a)^2+4c^2}\right)}+\frac{1}{1+v_-^2}e^{\frac{it}{2}\left(b+a-\sqrt{(b-a)^2+4c^2}\right)} \end{aligned} \tag{3.6}$$

i.e. the ϕ_1-distribution of A is the two points distribution

$$\frac{1}{1+v_+^2}\delta_{\frac{1}{2}\left(b+a+\sqrt{(b-a)^2+4c^2}\right)}+\frac{1}{1+v_-^2}\delta_{\frac{1}{2}\left(b+a-\sqrt{(b-a)^2+4c^2}\right)} \tag{3.7}$$

Proof. The affirmation 1) is obvious and we prove the affirmation 2). Notice that $c \neq 0$ in this case.

The fact

$$\det(\alpha\mathbf{1}-A) = (\alpha-a)(\alpha-b)-c^2 = \alpha^2-(b+a)\alpha-c^2+ab$$

tells us that A has eigenvalues $\alpha_\pm = \frac{1}{2}\Big(b+a\pm\sqrt{(b-a)^2+4c^2}\Big)$. Moreover

- $\alpha_+ \neq \alpha_-$ (in fact $\alpha_- < \alpha_+$) since $(b-a)^2+4c^2 \geq 4c^2 > 0$ in virtue of the fact $c \neq 0$;
- $\alpha_+\alpha_- = 0$ (i.e. A has an eigenvalue zero) if and only if $ab = c^2$ (so a and b have the same sign), in more detail

$$0 = \alpha_- < \alpha_+ \text{ if and only if } ab = c^2 \text{ and } a+b > 0$$
$$\alpha_- < \alpha_+ = 0 \text{ if and only if } ab = c^2 \text{ and } a+b < 0$$

- $v_\pm$ introduced in (3.4) is $\frac{\alpha_\pm - a}{c}$ (so $a + cv_\pm = \alpha_\pm$).

Furthermore, by rewriting the equation $\alpha_\pm^2-(b+a)\alpha_\pm-c^2+ab = 0$ to

$$c^2+b(\alpha_\pm-a) = \alpha_\pm(\alpha_\pm-a)$$

one gets

$$c+b\frac{\alpha_\pm-a}{c} = \alpha_\pm\frac{\alpha_\pm-a}{c}$$

i.e.,

$$c+bv_\pm = \alpha_\pm v_\pm$$

So

$$A\begin{pmatrix}1\\ v_\pm\end{pmatrix}=\begin{pmatrix}a+cv_\pm\\ c+bv_\pm\end{pmatrix}=\begin{pmatrix}\alpha_\pm\\ \alpha_\pm v_\pm\end{pmatrix}=\alpha_\pm\begin{pmatrix}1\\ v_\pm\end{pmatrix} \tag{3.8}$$

In other words, $\begin{pmatrix}1\\ v_\pm\end{pmatrix}$ is the eigenvector of A with the corresponding eigenvalue $\alpha_\pm$. Moreover, one gets the first formula of (3.5):

$$v_+v_-=\frac{1}{(2c)^2}\Big(b-a+\sqrt{(b-a)^2+4c^2}\Big)\Big(b-a-\sqrt{(b-a)^2+4c^2}\Big)=-1 \tag{3.9}$$

and in particular, $v_\pm\neq 0$. Consequently,

$$v_+^2v_-^2=1;\qquad \left\langle\begin{pmatrix}1\\ v_-\end{pmatrix},\begin{pmatrix}1\\ v_+\end{pmatrix}\right\rangle=0 \tag{3.10}$$

$$(1+v_+^2)(1+v_-^2)=1+v_+^2v_-^2+v_+^2+v_-^2=2+v_+^2+v_-^2 \tag{3.11}$$

$$v_+(1+v_-^2)+v_-(1+v_+^2)=v_++v_+v_-^2+v_-+v_-v_+^2=0 \tag{3.12}$$

and so one finds the second and the third formulae of (3.5):

$$\frac{1}{1+v_+^2}+\frac{1}{1+v_-^2}=\frac{1+v_+^2+1+v_-^2}{(1+v_+^2)(1+v_-^2)}\overset{(3.11)}{=}1$$

$$\frac{v_+}{1+v_+^2}+\frac{v_-}{1+v_-^2}=\frac{v_+(1+v_-^2)+v_-(1+v_+^2)}{(1+v_+^2)(1+v_-^2)}\overset{(3.12)}{=}0$$

Let's introduce

$$T_\pm:=\frac{1}{\sqrt{1+v_\pm^2}}\begin{pmatrix}1\\ v_\pm\end{pmatrix};\quad T:=(T_+,T_-)=\begin{pmatrix}\frac{1}{\sqrt{1+v_+^2}} & \frac{1}{\sqrt{1+v_-^2}}\\ \frac{v_+}{\sqrt{1+v_+^2}} & \frac{v_-}{\sqrt{1+v_-^2}}\end{pmatrix}$$

then

$$T^*T=\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad\text{and}\qquad T^*AT=\begin{pmatrix}\alpha_+&0\\0&\alpha_-\end{pmatrix} \tag{3.13}$$

Moreover, as a consequence of the first formula of (3.13), one knows (see e.g., [4]) $TT^*=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

By using above results, one finds that

$$\phi_1\big(\exp(itA)\big)=\left\langle T^*\begin{pmatrix}1\\0\end{pmatrix},T^*\exp(itA)TT^*\begin{pmatrix}1\\0\end{pmatrix}\right\rangle$$

$$=\left\langle\begin{pmatrix}\frac{1}{\sqrt{1+v_+^2}}\\ \frac{1}{\sqrt{1+v_-^2}}\end{pmatrix},\begin{pmatrix}e^{it\alpha_+}&0\\0&e^{it\alpha_-}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{1+v_+^2}}\\ \frac{1}{\sqrt{1+v_-^2}}\end{pmatrix}\right\rangle$$

$$=\frac{1}{1+v_+^2}e^{it\alpha_+}+\frac{1}{1+v_-^2}e^{it\alpha_-}$$

which is nothing else than the characteristic function of the distribution (3.7). □

A particular important case is that $a=c_0p$, $b=c_2(1-p)$ and $c=c_1\sqrt{p(1-p)}$ with $p\in[0,1]$ and $\{c_0,c_1,c_2\}\subset\mathbb{R}$. In this case, the distribution (3.7) will be called

generalized binomial distribution with the parameters (c_0, c_1, c_2, p), which is $\delta_{c_0 p}$ if either $c_1 = 0$ or $p \in \{0, 1\}$.

Proposition 3.3. *Let b, b^+ and Λ be the annihilation–creation operators and the number operator on the 1M–IFS $\Gamma(\mathbb{C}, \{\omega_n\}_n)$ with a sequence of the canonical Jacobi coefficients $\{\omega_n\}_n$ (hereinafter, one says that a sequence of the Jacobi coefficients $\{\omega_n\}_n$ is canonical if $\omega_{m+k} = 0$ for all $k \in \mathbb{N}$ whenever $\omega_m = 0$). One has, by denoting $\Phi :=$ the vacuum vector of $\Gamma(\mathbb{C}, \{\omega_n\}_n)$ and*

$$\mathbb{F}_+ := \{f : \mathbb{Z} \mapsto \mathbb{C} : f(k) = 0 \ \forall k < 0\} \tag{3.14}$$

1) for any $n \in \mathbb{N}$ and $\varepsilon \in \{-1, 1\}^n$, for any $f_1, \ldots, f_n, f \in \mathbb{F}_+$

$$\begin{aligned} & f_1(\Lambda) b^{(\varepsilon(1))} f_2(\Lambda) b^{(\varepsilon(2))} \ldots f_n(\Lambda) b^{(\varepsilon(n))} f(\Lambda) \Phi \\ =& f(0) \prod_{m=1}^{n} f_m\Big(\sum_{k=m}^{n} \varepsilon(k)\Big) b^{(\varepsilon(1))} b^{(\varepsilon(2))} \ldots b^{(\varepsilon(n))} \Phi \end{aligned} \tag{3.15}$$

2) for any n being odd, for any $\varepsilon \in \{-1, 1\}^n$ and $f_1, \ldots, f_n, f \in \mathbb{F}_+$

$$\Big\langle \Phi, f_1(\Lambda) b^{(\varepsilon(1))} f_2(\Lambda) b^{(\varepsilon(2))} \ldots f_n(\Lambda) b^{(\varepsilon(n))} f(\Lambda) \Phi \Big\rangle = 0$$

3) for any $n \in \mathbb{N}$ and $c \in \mathbb{C}$, for any $f \in \mathbb{F}_+$,

$$\begin{aligned} & \left\langle \Phi, \left(c\left(b + b^+\right) + f(\Lambda)\right)^n \Phi \right\rangle = \left\langle \Phi, \left(-c\left(b + b^+\right) + f(\Lambda)\right)^n \Phi \right\rangle \\ =& \sum_{\substack{0 \le m \le n \\ m \text{ is even}}} c^m \sum_{\substack{k, k_1, \ldots, k_m \ge 0 \\ k + k_1 + \ldots + k_m = n - m}} \sum_{\varepsilon \in \{-1,1\}^m} \\ & \Big\langle \Phi, (f(\Lambda))^{k_1} b^{(\varepsilon(1))} (f(\Lambda))^{k_2} b^{(\varepsilon(2))} \ldots (f(\Lambda))^{k_m} b^{(\varepsilon(m))} (f(\Lambda))^k \Phi \Big\rangle \end{aligned} \tag{3.16}$$

In particular, if $c \in \mathbb{R}$

- *with respect to the vacuum state, $c(b + b^+) + f(\Lambda)$ and $-c(b + b^+) + f(\Lambda)$ have the same moments;*
- *in case of the vacuum distribution of $c(b + b^+) + f(\Lambda)$ being determined by its moments, the distribution is invariant with respect to the sign of c.*

Proof. Because of $f(\Lambda)\Phi = f(0)\Phi$, one needs to see the affirmation 1) only in the case of f being the constant 1.

(3.15) holds trivially for $n = 1$. Suppose that it holds for n, one gets

$$\begin{aligned} & f_1(\Lambda) b^{(\varepsilon(1))} f_2(\Lambda) b^{(\varepsilon(2))} \ldots f_{n+1}(\Lambda) b^{(\varepsilon(n+1))} \Phi \\ =& f_1(\Lambda) b^{(\varepsilon(1))} \prod_{m=2}^{n+1} f_m\Big(\sum_{k=m}^{n+1} \varepsilon(k)\Big) b^{(\varepsilon(2))} \ldots b^{(\varepsilon(n+1))} \Phi \\ =& \prod_{m=1}^{n+1} f_m\Big(\sum_{k=m}^{n+1} \varepsilon(k)\Big) b^{(\varepsilon(1))} b^{(\varepsilon(2))} \ldots b^{(\varepsilon(n+1))} \Phi \end{aligned} \tag{3.17}$$

where,

- the first equality follows from the induction's assumption;

• the second equality dues to the fact

$$f_1(\Lambda)\, b^{(\varepsilon(1))} b^{(\varepsilon(2))} \dots b^{(\varepsilon(n+1))}\Phi = f_1\Big(\sum_{k=1}^{n+1}\varepsilon(k)\Big) b^{(\varepsilon(1))} b^{(\varepsilon(2))} \dots b^{(\varepsilon(n+1))}\Phi$$

The affirmation 2) is a trivially consequence of the affirmation 1) and the fact $\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))}\Phi\rangle = 0$ for n being odd.

Now we turn to prove the affirmation 3). Just by expanding the power n terms in (3.16), one has

$$\begin{aligned}
&\left(\pm c\left(b+b^{+}\right)+f(\Lambda)\right)^{n}=\sum_{m=0}^{n}(\pm c)^{m} \sum_{\substack{k,k_1,\dots,k_m\geq 0\\ k+k_1+\dots+k_m=n-m}} \\
&(f(\Lambda))^{k_1}\left(b+b^{+}\right)(f(\Lambda))^{k_2}\left(b+b^{+}\right)\dots(f(\Lambda))^{k_m}\left(b+b^{+}\right)(f(\Lambda))^{k} \\
=&\sum_{m=0}^{n}(\pm c)^{m} \sum_{\substack{k,k_1,\dots,k_m\geq 0\\ k+k_1+\dots+k_m=n-m}} \sum_{\varepsilon\in\{-1,1\}^m} \\
&(f(\Lambda))^{k_1} b^{(\varepsilon(1))}(f(\Lambda))^{k_2} b^{(\varepsilon(2))}\dots(f(\Lambda))^{k_m} b^{(\varepsilon(m))}(f(\Lambda))^{k}
\end{aligned}$$

By combining this formula with the affirmation 2), one obtains

$$\begin{aligned}
&\left\langle\Phi,\left(\pm c\left(b+b^{+}\right)+f(\Lambda)\right)^{n}\Phi\right\rangle=\sum_{m=0}^{n}(\pm c)^{m} \sum_{\substack{k,k_1,\dots,k_m\geq 0\\ k+k_1+\dots+k_m=n-m}} \sum_{\varepsilon\in\{-1,1\}^m} \\
&\left\langle\Phi,(f(\Lambda))^{k_1} b^{(\varepsilon(1))}(f(\Lambda))^{k_2} b^{(\varepsilon(2))}\dots(f(\Lambda))^{k_m} b^{(\varepsilon(m))}(f(\Lambda))^{k}\Phi\right\rangle \\
=&\sum_{\substack{0\leq m\leq n\\ m \text{ is even}}} c^{m} \sum_{\substack{k,k_1,\dots,k_m\geq 0\\ k+k_1+\dots+k_m=n-m}} \sum_{\varepsilon\in\{-1,1\}^m} \\
&\left\langle\Phi,(f(\Lambda))^{k_1} b^{(\varepsilon(1))}(f(\Lambda))^{k_2} b^{(\varepsilon(2))}\dots(f(\Lambda))^{k_m} b^{(\varepsilon(m))}(f(\Lambda))^{k}\Phi\right\rangle \qquad \square
\end{aligned}$$

The proof of Theorem 3.1 Without loss of generality, we assume that $p_N = \frac{\lambda}{N}$.

By the definition, for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$

$$\begin{aligned}
&c_1\left(B_N^{(-1)}+B_N^{(+1)}\right)+c_0 B_N^{(0)}+c_2 B_N^{(2)} \\
=&\sum_{k=1}^{N}\mathbf{1}^{\otimes(k-1)}\otimes\begin{pmatrix} c_0 p_N & c_1\sqrt{p_N(1-p_N)} \\ c_1\sqrt{p_N(1-p_N)} & c_2(1-p_N)\end{pmatrix}\otimes\mathbf{1}
\end{aligned}$$

So by taking, in Lemma 3.2, the element of $\mathbb{M}_{2,sys}(\mathbb{R})$ as the matrix

$$\begin{pmatrix} c_0 p_N & c_1\sqrt{p_N(1-p_N)} \\ c_1\sqrt{p_N(1-p_N)} & c_2(1-p_N)\end{pmatrix}$$

one finds

$$\begin{aligned}
&\left\langle \Phi_N, \exp\left(it\left(c_1\left(B_N^{(-1)}+B_N^{(+1)}\right)+c_0B_N^{(0)}+c_2B_N^{(2)}\right)\right)\Phi_N\right\rangle\\
=&\prod_{k=1}^N \phi_1\Big(\exp\Big(it\begin{pmatrix} c_0p_N & c_1\sqrt{p_N(1-p_N)}\\ c_1\sqrt{p_N(1-p_N)} & c_2(1-p_N)\end{pmatrix}\Big)\Big)\\
=&\Big(\frac{1}{1+v_+^2}e^{\frac{it}{2}\left(b+a+\sqrt{(b-a)^2+4c^2}\right)}+\frac{1}{1+v_-^2}e^{\frac{it}{2}\left(b+a-\sqrt{(b-a)^2+4c^2}\right)}\Big)^N
\end{aligned} \tag{3.18}$$

and where

$$\begin{aligned}
\sqrt{(b-a)^2+4c^2}&=\sqrt{(c_2(1-p_N)-c_0p_N)^2+4c_1^2p_N(1-p_N)}\\
&=\sqrt{c_2^2+2\left(2c_1^2-c_2(c_0+c_2)\right)p_N+\left((c_0+c_2)^2-4c_1^2\right)p_N^2}
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
&b+a\pm\sqrt{(b-a)^2+4c^2}\\
=&c_2+(c_0-c_2)p_N\pm\sqrt{c_2^2+2\left(2c_1^2-c_2(c_0+c_2)\right)p_N+\left((c_0+c_2)^2-4c_1^2\right)p_N^2}
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
v_\pm&=\frac{c_2(1-p_N)-c_0p_N\pm\sqrt{(c_2(1-p_N)-c_0p_N)^2+4c_1^2p_N(1-p_N)}}{2c_1\sqrt{p_N(1-p_N)}}\\
&=\frac{c_2-(c_0+c_2)p_N\pm\sqrt{c_2^2+2\left(2c_1^2-c_2(c_0+c_2)\right)p_N+\left((c_0+c_2)^2-4c_1^2\right)p_N^2}}{2c_1\sqrt{p_N(1-p_N)}}
\end{aligned} \tag{3.21}$$

The 1^{st} case: $c_1=0$. In this case, Lemma 3.2 says that the expression in (3.18) equals to $\prod_{k=1}^N e^{itc_op_N}$, i.e., $e^{itc_oNp_N}$, which goes to $e^{it\lambda c_o}$ as $N\to\infty$. So the weak limit of the $\phi-$distribution of $c_0B_N^{(0)}+c_2B_N^{(2)}$ is $\delta_{\lambda c_0}$. On the other hand, the vacuum distribution of $c_0b^{(0)}+c_2b^{(2)}$ (i.e., $\lambda c_0\mathbf{1}+c_2\Lambda$) is $\delta_{\lambda c_0}$ as well, since

$$\left\langle \Phi, \left(c_0b^{(0)}+c_2b^{(2)}\right)^n\Phi\right\rangle=\langle\Phi,(\lambda c_0\mathbf{1}+c_2\Lambda)^n\Phi\rangle=(\lambda c_0)^n,\quad\forall n$$

The 2^{nd} case: $c_1\neq 0$ and $c_2=0$. In this case, one has

$$b+a\pm\sqrt{(b-a)^2+4c^2}=a\pm\sqrt{a^2+4c^2}$$

and the expression in (3.18) equals, in virtue of Lemma 3.2, to

$$\Big(\frac{1}{1+v_+^2}e^{\frac{it}{2}\left(a+\sqrt{a^2+4c^2}\right)}+\frac{1}{1+v_-^2}e^{\frac{it}{2}\left(a-\sqrt{a^2+4c^2}\right)}\Big)^N \tag{3.22}$$

i.e.,

$$e^{\frac{ita}{2}N}\Big(\frac{1}{1+v_+^2}e^{\frac{it}{2}\sqrt{a^2+4c^2}}+\frac{1}{1+v_-^2}e^{-\frac{it}{2}\sqrt{a^2+4c^2}}\Big)^N$$

where, as particular cases of the formulae (3.19), (3.20) and (3.21), one has (recall the $p_N = \frac{\lambda}{N}$)

$$a = \frac{\lambda c_0}{N}$$

$$\sqrt{a^2 + 4c^2} = \sqrt{\left(\frac{\lambda c_0}{N}\right)^2 + 4\lambda c_1^2 \frac{(N-\lambda)}{N^2}} = \frac{1}{\sqrt{N}} \sqrt{\frac{\lambda^2 \left(c_0^2 - 4c_1^2\right)}{N} + 4\lambda c_1^2}$$

$$v_{\pm} = \frac{1}{2c}\left(-a \pm \sqrt{a^2 + 4c^2}\right) = \frac{1}{2c_1\sqrt{\lambda}}\left(- \frac{\lambda c_0}{\sqrt{N-\lambda}} \pm \sqrt{\frac{\lambda^2 c_0^2}{N-\lambda} + 4c_1^2 \lambda}\right)$$

Since for any $A > 0$

$$\left[1 + \frac{1}{A}\left(\sqrt{A + x^2} \mp x\right)^2\right]^{-1} = \frac{1}{2} \pm \frac{x}{2\sqrt{A}} + o\left(x^2\right) \tag{3.23}$$

one gets, by taking $A := 4c_1^2\lambda$ and $x := \frac{\lambda c_0}{\sqrt{N-\lambda}}$,

$$\begin{aligned}\frac{1}{1 + v_{\pm}^2} &= \frac{1}{1 + \frac{1}{4c_1^2\lambda}\left(- \frac{\lambda c_0}{\sqrt{N-\lambda}} \pm \sqrt{\left(\frac{\lambda c_0}{\sqrt{N-\lambda}}\right)^2 + 4c_1^2 \lambda}\right)^2} \\ &= \frac{1}{2} \pm \frac{\lambda c_0}{4\sqrt{c_1^2 \lambda}\sqrt{N-\lambda}} + o\left(\frac{1}{N}\right)\end{aligned} \tag{3.24}$$

Because for any $A \in \mathbb{R}$, $B > 0$ and $C \in \mathbb{C}$

$$e^{Ax\sqrt{B + Cx^2}} = 1 + A\sqrt{B}x + \frac{A^2 B}{2}x^2 + o\left(x^2\right) \tag{3.25}$$

one gets, by taking $A := \frac{\pm it}{2}$, $B := 4\lambda c_1^2$, $C := \lambda^2\left(c_0^2 - 4c_1^2\right)$ and $x := \frac{1}{\sqrt{N}}$

$$e^{\frac{\pm it}{2}\sqrt{a^2+4c^2}} = e^{\frac{\pm it}{2}\frac{1}{\sqrt{N}}\sqrt{4\lambda c_1^2 + \frac{\lambda^2\left(c_0^2 - 4c_1^2\right)}{N}}} = 1 \pm \frac{it\sqrt{\lambda c_1^2}}{\sqrt{N}} - \frac{t^2 \lambda c_1^2}{2N} + o\left(\frac{1}{N}\right) \tag{3.26}$$

Consequently

$$\begin{aligned}&\frac{1}{1 + v_+^2} e^{\frac{it}{2}\sqrt{a^2+4c^2}} + \frac{1}{1 + v_-^2} e^{-\frac{it}{2}\sqrt{a^2+4c^2}} \\ =&\left(\frac{1}{2} + \frac{\lambda c_0}{4\sqrt{c_1^2\lambda}\sqrt{N-\lambda}} + o\left(\frac{1}{N}\right)\right)\left(1 + \frac{it\sqrt{\lambda c_1^2}}{\sqrt{N}} - \frac{t^2\lambda c_1^2}{2N} + o\left(\frac{1}{N}\right)\right) \\ &+ \left(\frac{1}{2} - \frac{\lambda c_0}{4\sqrt{c_1^2\lambda}\sqrt{N-\lambda}} + o\left(\frac{1}{N}\right)\right)\left(1 - \frac{it\sqrt{\lambda c_1^2}}{\sqrt{N}} - \frac{t^2\lambda c_1^2}{2N} + o\left(\frac{1}{N}\right)\right) \\ =&1 + \frac{it\lambda c_0}{2\sqrt{N(N-\lambda)}} - \frac{t^2\lambda c_1^2}{2N} + o\left(\frac{1}{N}\right) \\ =&1 + \frac{1}{N}\left(\frac{it\lambda c_0}{2} - \frac{t^2\lambda c_1^2}{2}\right) + o\left(\frac{1}{N}\right)\end{aligned} \tag{3.27}$$

and so,

$$\begin{aligned}
&\lim_{N\to\infty}\phi\left(\exp\left(it\left(c_1\left(B_N^{(-1)}+B_N^{(+1)}\right)+c_0B_N^{(0)}\right)\right)\right)\\
=&\lim_{N\to\infty}e^{\frac{it\lambda c_0}{2}}\left(1+\frac{1}{N}\left(\frac{it\lambda c_0-t^2\lambda c_1^2}{2}\right)+o\left(\frac{1}{N}\right)\right)^N\\
=&\exp\Big(-\frac{t^2\lambda c_1^2}{2}+it\lambda c_0\Big)
\end{aligned}\tag{3.28}$$

In other words, the weak limit of the $\phi-$distribution of $\Big\{c_1\left(B_N^{(-1)}+B_N^{(+1)}\right)$ $+c_0B_N^{(0)}\Big\}_{N=1}^{\infty}$ is the normal distribution $N\left(\lambda c_0,\lambda c_1^2\right)$. On the other hand, it is obvious that the vacuum distribution of $\sqrt{\lambda}c_1\left(b+b^+\right)+\lambda c_0b^{(0)}$ (i.e. $\sqrt{\lambda}c_1\left(b+b^+\right)+\lambda c_0\mathbf{1}$) is $N\left(\lambda c_0,\lambda c_1^2\right)$ as well.

The 3^{rd} case: $c_1\neq 0$ and $c_2\neq 0$. In this case one introduces

$$s(x):=sign(x),\quad \forall x\in\mathbb{R}$$

then

$$\begin{aligned}
s(b)&=s(c_2);\quad x=s(x)|x|,\quad \forall x\in\mathbb{R};\\
b+a\pm\sqrt{(b-a)^2+4c^2}&=s(b)|b|+a\pm\sqrt{(b-a)^2+4c^2}\\
v_{s(b)}=v_{s(c_2)}&=\frac{b-a+s(b)\sqrt{(b-a)^2+4c^2}}{2c_1\sqrt{p_N(1-p_N)}}
\end{aligned}\tag{3.29}$$

So,

$$\begin{aligned}
&\frac{1}{1+v_+^2}e^{\frac{it}{2}\left(b+a+\sqrt{(b-a)^2+4c^2}\right)}+\frac{1}{1+v_-^2}e^{\frac{it}{2}\left(b+a-\sqrt{(b-a)^2+4c^2}\right)}\\
=&\frac{e^{\frac{it}{2}\left(b+a-s(b)\sqrt{(b-a)^2+4c^2}\right)}}{1+v_{-s(b)}^2}\Big(\frac{1+v_{-s(b)}^2}{1+v_+^2}e^{\frac{it}{2}(s(b)+1)\sqrt{(b-a)^2+4c^2}}\\
&\qquad+\frac{1+v_{-s(b)}^2}{1+v_-^2}e^{\frac{it}{2}(s(b)-1)\sqrt{(b-a)^2+4c^2}}\Big)\\
=&\frac{e^{\frac{it}{2}\left(b+a-s(b)\sqrt{(b-a)^2+4c^2}\right)}}{1+v_{-s(b)}^2}\sum_{\varepsilon\in\{-1,1\}}\frac{1+v_{-s(b)}^2}{1+v_\varepsilon^2}e^{\frac{it}{2}(s(b)+\varepsilon)\sqrt{(b-a)^2+4c^2}}
\end{aligned}$$

Thus one is able to rewrites the expression in (3.18) as follows

$$\begin{aligned}
&\left(\frac{1}{1+v_+^2}e^{\frac{it}{2}\left(b+a+\sqrt{(b-a)^2+4c^2}\right)}+\frac{1}{1+v_-^2}e^{\frac{it}{2}\left(b+a-\sqrt{(b-a)^2+4c^2}\right)}\right)^N\\
=&\frac{e^{\frac{it}{2}N\left(b+a-s(b)\sqrt{(b-a)^2+4c^2}\right)}}{\left(1+v_{-s(b)}^2\right)^N}\Big(\sum_{\varepsilon\in\{-1,1\}}\frac{1+v_{-s(b)}^2}{1+v_\varepsilon^2}e^{\frac{it}{2}(s(b)+\varepsilon)\sqrt{(b-a)^2+4c^2}}\Big)^N
\end{aligned}\tag{3.30}$$

Now we see its limit as $N\to\infty$.

First of all, by taking

$$A := c_2^2, \quad A_1 := \frac{itc_2}{2}, \ A_2 := \frac{it\left(c_0 - c_2\right)}{2}, \ A_3 := -\frac{it}{2} s\left(c_2\right)$$
$$A_4 := 2\left(2c_1^2 - c_2\left(c_0 + c_2\right)\right), \ A_5 := \left(c_0 + c_2\right)^2 - 4c_1^2$$

one obtains, thanks to the fact $p_N = \frac{\lambda}{N}$,

$$\begin{aligned} &\frac{it}{2}\Big(b + a - s\left(b\right)\sqrt{\left(b-a\right)^2 + 4c^2}\Big) \\ =&A_1 + A_2\frac{\lambda}{N} + A_3\sqrt{A + A_4\frac{\lambda}{N} + A_5\left(\frac{\lambda}{N}\right)^2} \end{aligned} \tag{3.31}$$

Since

$$A_1 + A_3\sqrt{A} = \frac{it}{2}\left(c_2 - s\left(c_2\right)\left|c_2\right|\right) = 0 \tag{3.32}$$

$$A_2 + \frac{A_3A_4}{2\sqrt{A}} = \frac{it}{2}\Big(c_0 - c_2 - \frac{s\left(c_2\right)\left(2c_1^2 - c_2\left(c_0 + c_2\right)\right)}{\left|c_2\right|}\Big) = \frac{it\left(c_0c_2 - c_1^2\right)}{c_2} \tag{3.33}$$

and since for any $\{A_1, A_2, A_3\} \subset \mathbb{C}$, $\{A_4, A_5\} \subset \mathbb{R}$ and $A > 0$,

$$\begin{aligned} &e^{A_1 + A_2x + A_3\sqrt{A + A_4x + A_5x^2}} = e^{A_1 + A_3\sqrt{A}} + \Big(A_2 + \frac{A_3A_4}{2\sqrt{A}}\Big)e^{A_1 + A_3\sqrt{A}}x + o\left(x\right) \\ =&e^{A_1 + A_3\sqrt{A}}\Big(1 + \Big(A_2 + \frac{A_3A_4}{2\sqrt{A}}\Big)x\Big) + o\left(x\right) \\ \overset{(3.32)}{=}&1 + \Big(A_2 + \frac{A_3A_4}{2\sqrt{A}}\Big)x + o\left(x\right) \end{aligned} \tag{3.34}$$

one finds, by taking $x := \frac{\lambda}{N}$

$$\begin{aligned} &e^{\frac{it}{2}N\left(b + a - s(b)\sqrt{(b-a)^2 + 4c^2}\right)} = \Big(e^{\frac{it}{2}\left(b + a - s(b)\sqrt{(b-a)^2 + 4c^2}\right)}\Big)^N \\ \overset{(3.31)}{=}&\Big(e^{A_1 + A_2\frac{\lambda}{N} + A_3\sqrt{A + A_4\frac{\lambda}{N} + A_5\left(\frac{\lambda}{N}\right)^2}}\Big)^N \overset{(3.34)}{=} \Big(1 + \Big(A_2 + \frac{A_3A_4}{2\sqrt{A}}\Big)x + o\left(x\right)\Big)^N \\ \overset{(3.33)}{=}&\Big(1 + \frac{it\left(c_0c_2 - c_1^2\right)}{c_2}\frac{\lambda}{N} + o\left(\frac{1}{N}\right)\Big)^N \longrightarrow \exp\Big(\frac{it\left(c_0c_2 - c_1^2\right)\lambda}{c_2}\Big) \end{aligned} \tag{3.35}$$

Second of all, (3.5) permits us to rewrite $1 + v_\mp^2$ in different forms:

$$1 + v_\mp^2 = 1 + \frac{1}{v_\pm^2} = 1 + v_\pm^{-2} = \frac{1 + v_\pm^2}{v_\pm^2} \tag{3.36}$$

By introducing

$$A_1 := \left|c_2\right| = s\left(c_2\right)c_2 = s\left(b\right)c_2, \ A_2 := -s\left(c_2\right)\left(c_0 + c_2\right) = -s\left(b\right)\left(c_0 + c_2\right)$$
$$A_3 := 2\left(2c_1^2 - c_2\left(c_0 + c_2\right)\right), \quad A_4 := \left(c_0 + c_2\right)^2 - 4c_1^2, \quad A := c_2^2, \ B := 4c_1^2$$

and by applying (3.20), one has

$$v_{s(b)}^{-2} = \Big(\frac{2c_1\sqrt{p_N(1-p_N)}}{b-a+s(b)\sqrt{(b-a)^2+4c^2}}\Big)^2 = \Big(\frac{2c_1\sqrt{p_N(1-p_N)}}{s(b)(b-a)+\sqrt{(b-a)^2+4c^2}}\Big)^2$$
$$= \frac{B\frac{\lambda}{N}\left(1-\frac{\lambda}{N}\right)}{\Big(A_1+A_2\frac{\lambda}{N}+\sqrt{A+A_3\frac{\lambda}{N}+A_4\left(\frac{\lambda}{N}\right)^2}\Big)^2}$$

Since for any such a $\{B, A, A_1, A_2, A_3, A_4\} \subset \mathbb{R}$ that $A > 0$ and $A_1 \neq -\sqrt{A}$ (in our case, $A_1 = \sqrt{A} = |c_2| > 0$)

$$\frac{x(1-x)}{\left(A_1+A_2x+\sqrt{A+A_3x+A_4x^2}\right)^2} = \frac{x}{\left(A_1+\sqrt{A}\right)^2} + o(x)$$

and

$$\frac{1}{1+\frac{Bx(1-x)}{\left(A_1+A_2x+\sqrt{A+A_3x+A_4x^2}\right)^2}} = 1 - \frac{Bx}{\left(A_1+\sqrt{A}\right)^2} + o(x)$$

one obtains, by taking $x := \frac{\lambda}{N}$ and noting that $\frac{B}{\left(A_1+\sqrt{A}\right)^2} = \frac{c_1^2}{c_2^2}$

$$v_{s(b)}^{-2} = v_{s(c_2)}^{-2} = \frac{c_1^2}{c_2^2}\frac{\lambda}{N} + o\left(\frac{1}{N}\right); \quad \frac{1}{1+v_{s(b)}^{-2}} = 1 - \frac{c_1^2}{c_2^2}\frac{\lambda}{N} + o(x) \tag{3.37}$$

This gives surely

$$\Big(\frac{1}{1+v_{-s(b)}^2}\Big)^N \overset{(3.36)}{=} \Big(\frac{1}{1+v_{s(b)}^{-2}}\Big)^N = \Big(1 - \frac{c_1^2}{c_2^2}\frac{\lambda}{N} + o\left(\frac{1}{N}\right)\Big)^N \longrightarrow e^{-\frac{\lambda c_1^2}{c_2^2}} \tag{3.38}$$

Finally, since

$$\Big[\frac{1+v_{-s(b)}^2}{1+v_\varepsilon^2} e^{\frac{it}{2}(s(b)+\varepsilon)\sqrt{(b-a)^2+4c^2}}\Big]_{\varepsilon=-s(b)} = 1$$

one gets, thanks to (3.37), that

$$\sum_{\varepsilon\in\{-1,1\}} \frac{1+v_{-s(b)}^2}{1+v_\varepsilon^2} e^{\frac{it}{2}(s(b)+\varepsilon)\sqrt{(b-a)^2+4c^2}} = 1 + \frac{1}{v_{s(b)}^2} e^{its(b)\sqrt{(b-a)^2+4c^2}}$$
$$= 1 + \Big(\frac{c_1^2}{c_2^2}\frac{\lambda}{N} + o\left(\frac{1}{N}\right)\Big) e^{its(b)\sqrt{(b-a)^2+4c^2}} \tag{3.39}$$

By taking

$$A := c_2^2, \quad A_1 := 0, \ A_2 := 0, \ A_3 := its(c_2)$$
$$A_4 := 2\left(2c_1^2 - c_2(c_0+c_2)\right), \ A_5 := \left((c_0+c_2)^2 - 4c_1^2\right)$$

one finds

$$A_1 + A_3\sqrt{A} = itc_2; \quad A_2 + \frac{A_3A_4}{2\sqrt{A}} = it\left(2c_1 - (c_0+c_2)\right)$$

So, (3.19) and (3.34) give us

$$e^{its(b)\sqrt{(b-a)^2+4c^2}} = e^{itc_2}\Big(1 + \frac{it\left(2c_1 - (c_0+c_2)\right)\lambda}{N} + o\left(\frac{1}{N}\right)\Big) \tag{3.40}$$

By applying this formula, (3.39) becomes to

$$\begin{aligned}&\sum_{\varepsilon\in\{-1,1\}} \frac{1+v_{-s(b)}^2}{1+v_\varepsilon^2} e^{\frac{it}{2}(s(b)+\varepsilon)\sqrt{(b-a)^2+4c^2}}\\ =&1+\Big(\frac{c_1^2}{c_2^2}\frac{\lambda}{N}+o\left(\frac{1}{N}\right)\Big)e^{itc_2}\Big(1+\frac{it\left(2c_1-(c_0+c_2)\right)\lambda}{N}+o\left(\frac{1}{N}\right)\Big)\\ =&1+\frac{c_1^2}{c_2^2}e^{itc_2}\frac{\lambda}{N}+o\left(\frac{1}{N}\right)\end{aligned}$$

and consequently

$$\begin{aligned}&\Big(\frac{1+v_{-s(b)}^2}{1+v_+^2}e^{\frac{it}{2}(s(b)+1)\sqrt{(b-a)^2+4c^2}}+\frac{1+v_{-s(b)}^2}{1+v_-^2}e^{\frac{it}{2}(s(b)-1)\sqrt{(b-a)^2+4c^2}}\Big)^N\\ =&\Big(1+\frac{c_1^2}{c_2^2}e^{itc_2}\frac{\lambda}{N}+o\left(\frac{1}{N}\right)\Big)^N \longrightarrow \exp\Big(\frac{\lambda c_1^2}{c_2^2}e^{itc_2}\Big)\end{aligned} \tag{3.41}$$

Summing up, we have proved

$$\begin{aligned}&\lim_{N\to\infty}\Big(\frac{1}{1+v_+^2}e^{\frac{it}{2}\left(b+a+\sqrt{(b-a)^2+4c^2}\right)}+\frac{1}{1+v_-^2}e^{\frac{it}{2}\left(b+a-\sqrt{(b-a)^2+4c^2}\right)}\Big)^N\\ =&\exp\Big(\frac{\lambda c_1^2}{c_2^2}\left(e^{itc_2}-1\right)+\frac{it\lambda\left(c_0c_2-c_1^2\right)}{c_2}\Big)\end{aligned}$$

and so the weak limit of ϕ−distribution of $c_1\left(B_N^{(-1)}+B_N^{(+1)}\right)+c_0B_N^{(0)}+c_2B_N^{(2)}$ is $P_{c_0,c_1,c_2}(\lambda)$ given in (3.3).

On the other hand, since

$$\begin{aligned}&c_1\left(b^{(-1)}+b^{(+1)}\right)+c_0b^{(0)}+c_2b^{(2)}\\ =&c_1\sqrt{\lambda}\left(b+b^+\right)+c_0\lambda+c_2\Lambda=c_2\Big(\frac{c_1\sqrt{\lambda}}{c_2}\left(b+b^+\right)+\frac{c_1^2\lambda}{c_2^2}+\Lambda\Big)+\frac{\left(c_0c_2-c_1^2\right)\lambda}{c_2}\end{aligned}$$

and since the vacuum distribution of $\frac{c_1\sqrt{\lambda}}{c_2}\left(b+b^+\right)+\frac{c_1^2\lambda}{c_2^2}+\Lambda$ is the Poisson distribution with the parameter $\frac{c_1^2\lambda}{c_2^2}$, one knows that the vacuum distribution of $c_1\left(b^{(-1)}+b^{(+1)}\right)+c_0b^{(0)}+c_2b^{(2)}$ is $P_{c_0,c_1,c_2}(\lambda)$. □

Remark The proof of the above affirmation 2) becomes much easier if in additional $c_0=0$ (i.e. the case of $c_1\neq 0$ and $c_2=c_0=0$). In fact, in this case

$$\begin{pmatrix} c_0p_N & c_1\sqrt{p_N(1-p_N)}\\ c_1\sqrt{p_N(1-p_N)} & c_2(1-p_N)\end{pmatrix} = c_1\sqrt{p_N(1-p_N)}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

and so

$$\phi_1\Big(\exp\Big(it\begin{pmatrix} c_0p_N & c_1\sqrt{p_N(1-p_N)}\\ c_1\sqrt{p_N(1-p_N)} & c_2(1-p_N)\end{pmatrix}\Big)\Big)=\cos\Big(c_1t\sqrt{p_N(1-p_N)}\Big)$$

Since $\cos(x) = 1 - \frac{x^2}{2} + o(x^2)$, one gets

$$\lim_{N\to\infty} \prod_{k=1}^{N} \phi_1\Big(\exp\Big(it \begin{pmatrix} c_0 p_N & c_1\sqrt{p_N(1-p_N)} \\ c_1\sqrt{p_N(1-p_N)} & c_2(1-p_N) \end{pmatrix}\Big)\Big) = e^{-\frac{\lambda c_1^2 t^2}{2}} \tag{3.42}$$

So the weak limit of the ϕ−distribution of $c_1\left(B_N^{(-1)} + B_N^{(+1)}\right)$ is the normal distribution $N\left(0, \lambda c_1^2\right)$. On the other hand, the vacuum distribution of $c_1\left(b^{(-1)} + b^{(+1)}\right)$ is $N\left(0, \lambda c_1^2\right)$ as well.

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