GRADED IDENTITIES OF SEVERAL TENSOR PRODUCTS OF THE GRASSMANN ALGEBRA

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Abstract. Let F be an infinite field of characteristic different from two and E be the unitary Grassmann algebra of an infinite dimensional F-vector space L. Denote by E_{gr} an arbitrary $\mathbb{Z}_2$-grading on E such that the subspace L is homogeneous. We consider $E_{gr} \otimes E^{\otimes n}$ as a $(\mathbb{Z}_2 \times \mathbb{Z}_2^n)$-graded algebra, where the grading on E is supposed to be the canonical one, and we find its graded ideal of identities.

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1. Introduction

The way approaching the study in PI theory changed radically in 1972 after the paper by Regev [19] about the existence of identities in the tensor product of two PI-algebras. Two decades later Kemer in his famous work [16] proved that the ideal of identities of a given associative PI-algebra over a field of characteristic 0 is the same as the ideal of identities (is PI-equivalent to $\sim$) of the Grassmann envelope of a finite dimensional superalgebra. We recall if $A = A^0 \oplus A^1$ is a superalgebra, then its Grassmann envelope is $G(A) = (A^0 \otimes E^0) \oplus (A^1 \otimes E^1)$, where $E = E^0 \oplus E^1$ is the infinite dimensional Grassmann algebra endowed with its canonical $\mathbb{Z}_2$-grading. In particular as a consequence of Kemer's theory (see [15]) we have the following set of PI-equivalences also called Tensor Product Theorem (TPT) over a field of characteristic 0:

$$E \otimes E \sim_{PI} M_{1,1}(E),$$
$$M_{a,b}(E) \otimes E \sim_{PI} M_{a+b}(E),$$
$$M_{a,b}(E) \otimes M_{c,d}(E) \sim_{PI} M_{ac+bd,ad+bc}(E),$$

where the algebra $M_{a,b}(E)$ is a block-subalgebra of $M_{a+b}(E)$ in which the upper left and lower right blocks are respectively of size $a \times a$ and $b \times b$ and fulfilled with elements of E^0 and all the other entries are from E^1. We want to point out that the algebras $M_n(F)$, $M_n(E)$ and $M_{a,b}(E)$ are the building blocks of Kemer's Theory because they generate the only T-prime varieties over a field of characteristic 0. The behavior of the T-ideals of T-prime algebras in positive characteristic has been studied in [1], [3] and [4]. It has also been proved that TPT is still valid over infinite fields of characteristic $p > 2$ as long as one considers multilinear polynomials only. Moreover, another proof of TPT theorem can be found in the paper [18] by Regev.

Due to the importance of TPT type problems and the key role of the infinite dimensional Grassmann algebra in PI-theory, in [12] Di Vincenzo and Nardozza compared the G-graded identities of a G-graded algebra $A = \bigoplus_{g \in G} A^g$(over a field of characteristic 0) with the $(G \times \mathbb{Z}_2)$-graded identities of $A \otimes E$, where the $(G \times \mathbb{Z}_2)$-components are $A^g \otimes E^i$ and E is endowed with its canonical grading. Using the ideas of Kemer, they construct a map ζ_J (where J is any subset of $\mathbb{N}$) such that if S is a generator set of the G-graded polynomial identities of A, then $\{\zeta_J(S)\}_{J \subseteq \mathbb{N}}$ is a set of generators for the $(G \times \mathbb{Z}_2)$-graded identities of $A \otimes E$.

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At the light of the arguments above and the work by Di Vincenzo and Nardozza, if A is a G-graded algebra over an infinite field, it seems interesting to investigate how do the $(G \times \mathbb{Z}_2)$-graded identities of $A \otimes E$ behave with respect to the G-graded identities of A. In [10] the authors studied the case of $UT_2(E)$, the algebra of upper triangular matrices with entries from E of size 2, endowed with the $\mathbb{Z}_2$-grading inherited by the canonical grading of E. It turns out that the ideal of $\mathbb{Z}_2$-graded identities of $UT_2(E)$ is generated by the same finite set of multilinear polynomials over any infinite field of characteristic different from 2 (the same result can be obtained as a consequence of [8] too). Notice that in [7] the author proved that the ideal of $\mathbb{Z}_2$-graded identities of E endowed with the canonical grading is also generated by the same finite set of multilinear polynomials provided the field is infinite. In [9] the authors deal with the case of $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-graded identities of $E_{k^*} \otimes E$, where E_{k^*} denotes one of the three up to isomorphism kinds (E_{k^*}, E_∞, E_k) of homogeneous $\mathbb{Z}_2$-gradings of E which will be denoted by E_{gr} without distinguish them. In the latter paper they show the graded ideal of $E_{k^*} \otimes E$ is not generated by multilinear identities, then it cannot be covered by Di Vincenzo and Nardozza's results. We want to point out that by TPT the algebra $E^{\otimes n}$ is PI-equivalent to $M_{2^{\frac{n-2}{2}},2^{\frac{n-2}{2}}}(E)$ if n is even whereas is PI-equivalent to $M_{2^{\frac{n-1}{2}}}(E)$ otherwise and in both of the cases the graded identities are of interest on its own. For this purpose we want to cite the paper by Popov [17] in which he found a basis of polynomial identities of $E \otimes E$ in characteristic 0, the paper by Diniz da Silva [11] in which he found a basis for the $\mathbb{Z}_2$-graded identities of $E_{gr} \otimes E_{gr}$ for some particular cases over an infinite field.

Following this line of research, in this paper we generalize the results in [9] finding an explicit set of generators and a basis for the graded relatively free algebra for $E_{gr} \otimes E^{\otimes n}$ as a $(\mathbb{Z}_2 \times \mathbb{Z}_2^n)$-graded algebra, where E is supposed endowed with its canonical $\mathbb{Z}_2$-grading. As a consequence we get the relatively free graded algebras of E_{gr} do grow at the same rate under tensorization by the Grassmann algebra endowed with its canonical $\mathbb{Z}_2$-grading. The last result is a particular case of the conjecture formulated by the authors in [9].

2. Preliminaries

All algebras we refer to are associative with unit, all fields are to be considered infinite with characteristic different from 2, and all groups we refer to are finite and abelian unless explicitly written.

Let $G = \{g_1, \ldots, g_s\}$ be any group of finite order s and let F be a field. If A is an F-algebra, we say that A is a *G-graded algebra* if there are subspaces A^g for each $g \in G$ such that

$$A = \bigoplus_{g \in G} A^g \text{ and } A^g A^h \subseteq A^{gh}.$$

If $0 \neq a \in A^g$ we say that a is *homogeneous of G-degree g* or *G-graded homogeneous of G-degree g*, and we write $\deg(a) = g$.

Let $\{X^g \mid g \in G\}$ be a family of disjoint countable sets. Set $X = \bigcup_{g \in G} X^g$ and denote by $F\langle X|G\rangle$ the free associative algebra freely generated by the set X. An indeterminate (or variable) $x \in X$ is said to be of *homogeneous G-degree* g, written $\deg(x) = g$, if $x \in X^g$. We always write x^g if $x \in X^g$. The homogeneous G-degree of a monomial $m = x_{i_1} x_{i_2} \cdots x_{i_k}$ is defined to be $\deg(m) = \deg(x_{i_1}) \cdot \deg(x_{i_2}) \cdots \deg(x_{i_k})$. For every $g \in G$ we denote by $F\langle X|G\rangle^g$ the subspace of $F\langle X|G\rangle$ spanned by all monomials having homogeneous G-degree g. Notice that $F\langle X|G\rangle^g F\langle X|\mathbb{Z}_2\rangle^{g'} \subseteq F\langle X|G\rangle^{gg'}$ for all $g, g' \in G$. Thus

$$F\langle X|G\rangle = \bigoplus_{g \in G} F\langle X|G\rangle^g$$

and $F\langle X|G\rangle$ is a G-graded algebra. We refer to the elements of $F\langle X|G\rangle$ as *G-graded polynomials* or just *graded polynomials*. An ideal I of $F\langle X|G\rangle$ is said to be a *T_G-ideal* (or *graded T-ideal*) if it is invariant under all G-graded endomorphisms $\varphi : F\langle X|G\rangle \to F\langle X|G\rangle$ such that $\varphi(F\langle X|G\rangle^g) \subseteq F\langle X|G\rangle^g$ for all $g \in G$. If A is a G-graded algebra, a G-graded polynomial $f(x_1, \ldots, x_n)$ is said to be a *graded polynomial identity* of A if $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in \bigcup_{g \in G} A^g$ such that $a_k \in A^{\deg(x_k)}$, $k = 1, \ldots, n$. We denote by $T_G(A)$ the ideal of all graded polynomial identities of A. It is a T_G-ideal of $F\langle X|G\rangle$ in the sense that it is invariant under all graded homomorphism of $F\langle X|G\rangle$. We shall call *substitution* with elements of A any graded homomorphism $F\langle X|G\rangle \to A$ and we sometimes use the notation $\overline{x} = a \in A$ in order to denote explicitly such evaluation of the variable x.

Given a subset $S \subseteq F\langle X|\mathbb{Z}_2\rangle$ one can talk about the least T_G-ideal of $F\langle X|G\rangle$ containing the set S. Such T_G-ideal will be denoted by $\langle S\rangle_{T_G}$ and will be called the *T_G-ideal generated by S*. We say that elements of $\langle S\rangle_{T_G}$ are *consequences* of elements of S, or simply that they follow from S. If $T_G(A) = \langle S\rangle_{T_G}$, we say that S is a *basis* for the graded polynomial identities of A.

Definition 1. We consider the vector space

$$P_n^G = \operatorname{span}_F\{x_{\sigma(1)}^{g_1} x_{\sigma(2)}^{g_2} \cdots x_{\sigma(n)}^{g_n} \mid g_i \in G, \sigma \in S_n\}.$$

We shall refer to the elements of P_n^G as *multilinear polynomials* of degree n.

Since the ground field F is infinite, a standard *Vandermonde-argument* yields that a polynomial f is a G-graded polynomial identity for A if and only if its (multi)homogeneous components (with respect to the ordinary $(\mathbb{Z}^\infty)\mathbb{Z}$-grading), are identities as well. Moreover, if the characteristic of the ground field is 0, the well known multilinearization process shows that the T_G-ideal of a G-graded algebra A is determined by its multilinear polynomials.

Now, for $n_{g_1}, \ldots, n_{g_s} \in \mathbb{N}$ we consider the subspace of P_n^G ($n = n_{g_1} + \cdots + n_{g_s}$) spanned by the multilinear monomials in the following set of variables:

$$\{x_1^{g_1}, \ldots, x_{n_{g_1}}^{g_1}, x_{n_{g_1}+1}^{g_2}, \ldots, x_{n_{g_1}+n_{g_2}}^{g_2}, \ldots, x_{n_{g_1}+\cdots+n_{g_s}}^{g_s}\}.$$

We denote this vector space by $P_{n_{g_1},\ldots,n_{g_s}}^G$ and its quotient space by the graded identities by

$$P_{n_{g_1},\ldots,n_{g_s}}^G(A) := P_{n_{g_1},\ldots,n_{g_s}}^G / (P_{n_{g_1},\ldots,n_{g_s}}^G \cap T_G(A)).$$

It is well known that studying the graded identities of A in P_n^G is equivalent to studying the graded identities of A in $P_{n_{g_1},\ldots,n_{g_s}}^G$ for all $(n_{g_1}, \ldots, n_{g_s})$ such that $n_{g_1} + \cdots + n_{g_s} = n$.

As we said in the introduction, we shall study the behavior of graded T-ideals of graded algebras tensorized by the Grassmann algebra. When the ground field has characteristic 0, a complete description is given in [12]. For this purpose, we recall one of the main results obtained in [12] by Di Vincenzo and Nardozza.

The relatively free algebra $F\langle X|G \times \mathbb{Z}_2\rangle$ is both a G-graded algebra and a $\mathbb{Z}_2$-graded algebra, then let us consider $F\langle X|G \times \mathbb{Z}_2\rangle$ as a $\mathbb{Z}_2$-graded algebra. Let m be a multilinear monomial in $F\langle X|G \times \mathbb{Z}_2\rangle$ and let $i_1 < \cdots < i_k$ be the indices of the variables with odd $\mathbb{Z}_2$-degree occurring in m. Then we may write $m = m_0 z_{\sigma(i_1)} m_1 z_{\sigma(i_2)} \cdots m_{k-1} z_{\sigma(i_k)} m_k$, where $m_0, \ldots, m_k$ are multilinear monomials in even variables and z_{i_j} are odd variables, for some σ in the symmetric group $S_k(\{i_1, \ldots, i_k\})$. We define $\zeta(m) := (-1)^\sigma m$ as in the work by Kemer [16] and note that $\zeta(\zeta(m)) = m$. Similarly, we define a map from the free G-graded algebra to the free $(G \times \mathbb{Z}_2)$-graded algebra as below.

Definition 2. Given $J \subseteq \mathbb{N}$, we denote for every $r \in \mathbb{N}$

$$\theta_{r,J} := \begin{cases} 0 & \text{if } r \notin J \\ 1 & \text{if } r \in J. \end{cases}$$

Let $\varphi_J : F\langle X|G\rangle \to F\langle X|G \times \mathbb{Z}_2\rangle$ be the unique G-homomorphism given by

$$\varphi_J(x_r^g) = x_r^{(g,\theta_{r,J})}, \qquad \text{for any } r \in \mathbb{N}.$$

Moreover if $m \in P_n^G$ is multilinear, define $\zeta_J(m) := \zeta(\varphi_J(m))$. We may extend the map ζ_J by linearity to the space of all G-graded multilinear polynomials P_n^G. If $f \in P_n^G$, then $\zeta_J(f)$ is a multilinear element of $F\langle X|G \times \mathbb{Z}_2\rangle$.

Remark 3. In particular, for the Lie commutator $[x_1^{g_1}, x_2^{g_2}] := x_1^{g_1} x_2^{g_2} - x_2^{g_2} x_1^{g_1}$, if $J \subseteq \mathbb{N}$, then

$$\zeta_J([x_1^{g_1}, x_2^{g_2}]) = \zeta([x_1^{(g_1,\theta_{1,J})}, x_2^{(g_2,\theta_{2,J})}] = x_1^{(g_1,\theta_{1,J})} x_2^{(g_2,\theta_{2,J})} - (-1)^{\theta_{1,J}\theta_{2,J}} x_2^{(g_2,\theta_{2,J})} x_1^{(g_1,\theta_{1,J})}.$$

Remark 4. Given a v-tuple $V = (r_1, \ldots, r_v)$ of pairwise distinct positive integers and the v-tuples $S = (s_{r_1}, \ldots, s_{r_v}) \in \mathbb{Z}_2^v$ and $(g_{r_1}, \ldots, g_{r_v}) \in G^v$, then $\zeta(x_{r_1}^{(g_{r_1},s_{r_1})} \cdots x_{r_v}^{(g_{r_v},s_{r_v})})$ depends on the v-tuples V e S only. Let us denote by $c_{V,S}$ the coefficient of $x_{r_1}^{(g_{r_1},s_{r_1})} \cdots x_{r_v}^{(g_{r_v},s_{r_v})}$ in $\zeta(x_{r_1}^{(g_{r_1},s_{r_1})} \cdots x_{r_v}^{(g_{r_v},s_{r_v})})$, that is, $c_{V,S}$ is defined by the equality $\zeta(x_{r_1}^{(g_{r_1},s_{r_1})} \cdots x_{r_v}^{(g_{r_v},s_{r_v})}) = c_{V,S}\, x_{r_1}^{(g_{r_1},s_{r_1})} \cdots x_{r_v}^{(g_{r_v},s_{r_v})}$.

Let $J \subseteq \mathbb{N}$ and set

$$\theta(V,J) := (\theta_{r_1,J}, \dots, \theta_{r_v,J}),$$

then

$$\zeta_J(x_{r_1}^{g_{r_1}} \cdots x_{r_v}^{g_{r_v}}) = \zeta(x_{r_1}^{(g_{r_1},\theta_{r_1,J})} \cdots x_{r_v}^{(g_{r_v},\theta_{r_v,J})}) = c_{V,\theta(V,J)}\, x_{r_1}^{(g_{r_1},\theta_{r_1,J})} \cdots x_{r_v}^{(g_{r_v},\theta_{r_v,J})}.$$

Let us denote by E the unitary infinite dimensional Grassmann algebra with generators $\epsilon_1, \epsilon_2, \dots$ subject to the condition $\epsilon_i\epsilon_j = -\epsilon_j\epsilon_i$ for all i,j. Let A be a G-graded algebra and let us consider E endowed with its canonical $\mathbb{Z}_2$-grading $E = E^0 \oplus E^1$, where E^0 (resp. E^1) is the span of the monomials in the $\epsilon_i's$ of even length (resp. odd length). We consider $A \otimes E$ endowed with the $(G \times \mathbb{Z}_2)$-grading such that its $(G\times\mathbb{Z}_2)$-components are $(A\otimes E)^{(g,i)} := A^g \otimes E^i$.

Proposition 5 (Lemma 4 of [12])**.** *Let f be a multilinear G-graded polynomial identity of A of degree n, and let $J \subseteq \{1,\dots,n\}$. Then $\zeta_J(f) \in T_{G\times\mathbb{Z}_2}(A\otimes E)$.*

Theorem 6 (Theorem 11 of [12])**.** *Let S be a system of multilinear generators for $T_G(A)$. If the characteristic of the ground field F is zero, then*

$$\{\zeta_J(f) \in F\langle X|G\times\mathbb{Z}_2\rangle \mid f \in S, J \subseteq \mathbb{N}\}$$

is a set of multilinear generators for $T_{G\times\mathbb{Z}_2}(A\otimes E)$.

3. $(G\times\mathbb{Z}_2^n)$-graded identities of $A\otimes E^{\otimes n}$

In this paper we deal with $(G\times\mathbb{Z}_2^n)$-graded identities of $A\otimes E^{\otimes n} := A \otimes \underbrace{E\otimes\cdots\otimes E}_{n \text{ times}}$ over an infinite field of characteristic different from 2, where A is a G-graded algebra, $n \geq 1$ and E is supposed to be $\mathbb{Z}_2$-graded with its canonical grading. We consider $A\otimes E^{\otimes n}$ endowed with the $(G\times\mathbb{Z}_2^n)$-grading such that its $(G\times\mathbb{Z}_2^n)$-components are $(A\otimes E^{\otimes n})^{(g,\mathbf{g})} := A^g \otimes (E^{\otimes n})^{\mathbf{g}}$, where $(E^{\otimes n})^{\mathbf{g}} = E^{g_1}\otimes\cdots\otimes E^{g_n}$ when $\mathbf{g} = (g_1,\dots,g_n) \in \mathbb{Z}_2^n$. Motivated by Remark 3, we start off by the following.

Definition 7. Let B be a $\mathbb{Z}_2^n$-graded algebra. We define the $\mathbb{Z}_2^n$-*graded commutator*

$$[b,b']_{\mathbb{Z}_2^n} := bb' - (-1)^{\sum_{r=1}^n i_r i'_r} b'b,$$

for every homogeneous elements $b, b' \in B$, where $\deg(b) = (i_1,\dots,i_n)$ and $\deg(b') = (i'_1,\dots,i'_n)$.

Notice that $A\otimes E^{\otimes n}$ as well as the free algebra $F\langle X \mid G\times\mathbb{Z}_2^n\rangle$ are both G-graded algebras and $\mathbb{Z}_2^n$-graded algebras. Therefore, referring to the $\mathbb{Z}_2^n$-grading, one considers in $A\otimes E^{\otimes n}$ and also in $F\langle X \mid G\times\mathbb{Z}_2^n\rangle$, the $\mathbb{Z}_2^n$-graded commutator as defined above.

Remark 8. Let $J \subseteq \mathbb{N}$. We can see that the notation introduced above helps us to rewrite big expressions in a compact way. In particular,

$$\zeta_J([x_1^{g_1}, x_2^{g_2}]) = x_1^{(g_1,\theta_{1,J})} x_2^{(g_2,\theta_{2,J})} - (-1)^{\theta_{1,J}\theta_{2,J}} x_2^{(g_2,\theta_{2,J})} x_1^{(g_1,\theta_{1,J})} = \left[x_1^{(g_1,\theta_{1,J})}, x_2^{(g_2,\theta_{2,J})}\right]_{\mathbb{Z}_2},$$

for all $g_1, g_2 \in G$.

Furthermore, if $V = (r_1,\dots,r_v,l_1,\dots,l_t)$ is a $(v+t)$-tuple of pairwise distinct positive integers (with t even) and $(g_{r_1},\dots,g_{r_v},g_{l_1},\dots,g_{l_t}) \in G^{v+t}$, then

$$\zeta_J(x_{r_1}^{g_{r_1}}\cdots x_{r_v}^{g_{r_v}}[x_{l_1}^{g_{l_1}}, x_{l_2}^{g_{l_2}}]\cdots[x_{l_{t-1}}^{g_{l_{t-1}}}, x_{l_t}^{g_{l_t}}]) =$$

$$= c_{V,\theta(V,J)}\; x_{r_1}^{(g_{r_1},\theta_{r_1,J})}\cdots x_{r_v}^{(g_{r_v},\theta_{r_v,J})}\left[x_{l_1}^{(g_{l_1},\theta_{l_1,J})}, x_{l_2}^{(g_{l_2},\theta_{l_2,J})}\right]_{\mathbb{Z}_2}\cdots\left[x_{l_{t-1}}^{(g_{l_{t-1}},\theta_{l_{t-1},J})}, x_{l_t}^{(g_{l_t},\theta_{l_t,J})}\right]_{\mathbb{Z}_2}.$$

We introduce the generalized analog of the ζ_J function. Let $l \in \mathbb{N}$, $J_l \subseteq \mathbb{N}$ and consider the map

$$\varphi_{J_l} : F\langle X|G\times\mathbb{Z}_2^{l-1}\rangle \to F\langle X|G\times\mathbb{Z}_2^{l}\rangle$$

such that

$$x_r^{(g,i_1,\ldots,i_{l-1})} \mapsto x_r^{(g,i_1,\ldots,i_{l-1},\theta_{r,J_l})}.$$

We observe that $F\langle X|G\times\mathbb{Z}_2^l\rangle$ is a $\mathbb{Z}_2$-graded algebra via its natural $(1\times 1^{l-1}\times\mathbb{Z}_2)$-grading. Hence we can define

$$\zeta_{J_l}(m) := \zeta(\varphi_{J_l}(m))$$

for every monomial m of $F\langle X|G\times\mathbb{Z}_2^l\rangle$ as in the definition of Di Vincenzo and Nardozza.

The following generalization of Remark 8 holds.

Remark 9. Consider the subsets $J_1,\ldots,J_n\subseteq\mathbb{N}$. Then for any $g_1,g_2\in G$,

$$\begin{aligned}(\zeta_{J_n}\circ\cdots\circ\zeta_{J_1})([x_1^{g_1},x_2^{g_2}]) &= x_1^{(g_1,\theta_{1,J_1},\ldots,\theta_{1,J_n})}x_2^{(g_2,\theta_{2,J_1},\ldots,\theta_{2,J_n})} - (-1)^{\sum_{i=1}^n\theta_{1,J_i}\theta_{2,J_i}}x_2^{(g_2,\theta_{2,J_1},\ldots,\theta_{2,J_n})}x_1^{(g_1,\theta_{1,J_1},\ldots,\theta_{1,J_n})}\\ &= [x_1^{(g_1,\mathbf{g}_1)},x_2^{(g_2,\mathbf{g}_2)}]_{\mathbb{Z}_2^n},\end{aligned}$$

where $\mathbf{g}_1=(\theta_{1,J_1},\ldots,\theta_{1,J_n})$ and $\mathbf{g}_2=(\theta_{2,J_1},\ldots,\theta_{2,J_n})$.

Moreover, given an even integer t and the $(v+t)$-tuples $V=(r_1,\ldots,r_v,l_1,\ldots,l_t)$ of pairwise distinct positive integers and $(g_{r_1},\ldots,g_{r_v},g_{l_1},\ldots,g_{l_t})\in G^{v+t}$, one obtains

$$\begin{aligned}&(\zeta_{J_n}\circ\cdots\circ\zeta_{J_1})(x_{r_1}^{g_{r_1}}\cdots x_{r_v}^{g_{r_v}}[x_{l_1}^{g_{l_1}},x_{l_2}^{g_{l_2}}]\cdots[x_{l_{t-1}}^{g_{l_{t-1}}},x_{l_t}^{g_{l_t}}]) =\\ &= \left(\prod_{s=1}^n c_{V,\theta(V,J_s)}\right)\; x_{r_1}^{(g_{r_1},\mathbf{g}_{r_1})}\cdots x_{r_v}^{(g_{r_v},\mathbf{g}_{r_v})}\left[x_{l_1}^{(g_{l_1},\mathbf{g}_{l_1})},x_{l_2}^{(g_{l_2},\mathbf{g}_{l_2})}\right]_{\mathbb{Z}_2^n}\cdots\left[x_{l_{t-1}}^{(g_{l_{t-1}},\mathbf{g}_{l_{t-1}})},x_{l_t}^{(g_{l_t},\mathbf{g}_{l_t})}\right]_{\mathbb{Z}_2^n},\end{aligned}$$

where $\mathbf{g}_i=(\theta_{i,J_1},\ldots,\theta_{i,J_n})$, for any $i\in\mathbb{N}$.

In what follows we say that a subset $S=\{l_1,\ldots,l_s\}$ of $\{1,\ldots,m\}$ is *ordered* if $l_1<\cdots<l_s$.

Definition 10. Given $m\geq 1$ and an even integer $t\geq 0$, let $T=\{l_1,\ldots,l_t\}$ be an ordered subset of $\{1,\ldots,m\}$. Consider $\{r_1,\ldots,r_v\}$ the ordered complementary set of T in $\{1,\ldots,m\}$, (that is, $r_1<\cdots<r_v$, $v+t=m$, $\{r_1,\ldots,r_v,l_1,\ldots,l_t\}=\{1,\ldots,m\}$). Given the m-tuple $(g_1,\ldots,g_m)\in G^m$ and the n-tuples $\mathbf{g}_i:=(h_{i,1},\ldots,h_{i,n})\in\mathbb{Z}_2^n$, with $i=1,\ldots,m$, let us set

- $f_T(x_1^{g_1},\ldots,x_m^{g_m}) := x_{r_1}^{g_{r_1}}\cdots x_{r_v}^{g_{r_v}}[x_{l_1}^{g_{l_1}},x_{l_2}^{g_{l_2}}]\cdots[x_{l_{t-1}}^{g_{l_{t-1}}},x_{l_t}^{g_{l_t}}]$,
- $f_{T,n}(x_1^{(g_1,\mathbf{g}_1)},\ldots,x_m^{(g_m,\mathbf{g}_m)}) := x_{r_1}^{(g_{r_1},\mathbf{g}_{r_1})}\cdots x_{r_v}^{(g_{r_v},\mathbf{g}_{r_v})}\left[x_{l_1}^{(g_{l_1},\mathbf{g}_{l_1})},x_{l_2}^{(g_{l_2},\mathbf{g}_{l_2})}\right]_{\mathbb{Z}_2^n}\cdots\left[x_{l_{t-1}}^{(g_{l_{t-1}},\mathbf{g}_{l_{t-1}})},x_{l_t}^{(g_{l_t},\mathbf{g}_{l_t})}\right]_{\mathbb{Z}_2^n}$,
- $\mathcal{G}:=(\mathbf{g}_1,\ldots,\mathbf{g}_m)$,
- $\mathbf{C}_{T,\mathcal{G}}:=\prod_{s=1}^n c_{V,(h_{r_1,s},\ldots,h_{r_v,s},h_{l_1,s},\ldots,h_{l_t,s})}$,

where $V=(r_1,\ldots,r_v,l_1,\ldots,l_t)$.

Remark 11. Consider $m\geq 1$ and an even integer $t\geq 0$, let $T=\{l_1,\ldots,l_t\}$ be an ordered subset of $\{1,\ldots,m\}$. Given the m-tuple $(g_1,\ldots,g_m)\in G^m$ and the subsets $J_1,\ldots,J_n\subseteq\mathbb{N}$, we obtain

$$(\zeta_{J_n}\circ\cdots\circ\zeta_{J_1})(f_T(x_1^{g_1},\ldots,x_m^{g_m})) = \mathbf{C}_{T,\mathcal{G}}f_{T,n}(x_1^{(g_1,\mathbf{g}_1)},\ldots,x_m^{(g_m,\mathbf{g}_m)}),$$

where $\mathbf{g}_i=(\theta_{i,J_1},\ldots,\theta_{i,J_n})$, for each $i=1,\ldots,m$.

In Lemma 6 of [9] the authors showed up some properties of the $(G\times\mathbb{Z}_2)$-graded identities of $A\otimes E$, where A is a G-graded algebra, relating them with the G-graded identities of A. Down here we generalize such result for the $(G\times\mathbb{Z}_2^n)$-graded identities of $A\otimes E^{\otimes n}$.

Proposition 12. *Let A be a G-graded algebra. Then the following properties hold.*

(a) *If* $[x_1^{g_1},x_2^{g_2}]\in T_G(A)$ *then* $\left[x_1^{(g_1,\mathbf{g}_1)},x_2^{(g_2,\mathbf{g}_2)}\right]_{\mathbb{Z}_2^n}\in T_{G\times\mathbb{Z}_2^n}(A\otimes E^{\otimes n})$;

(b) *if* $[x_1^{g_1},x_2^{g_2},x_3^{g_3}]\in T_G(A)$ *then* $\left[x_1^{(g_1,\mathbf{g}_1)},x_2^{(g_2,\mathbf{g}_2)},x_3^{(g_3,\mathbf{g}_3)}\right]_{\mathbb{Z}_2^n}\in T_{G\times\mathbb{Z}_2^n}(A\otimes E^{\otimes n})$;

(c) *if* $[x_1^{g_1}, x_2^{g_2}][x_3^{g_3}, x_4^{g_4}] + [x_1^{g_1}, x_4^{g_4}][x_3^{g_3}, x_2^{g_2}] \in T_G(A)$ *then*

$$\left[x_1^{(g_1,\mathbf{g}_1)}, x_2^{(g_2,\mathbf{g}_2)}\right]_{\mathbb{Z}_2^n}\left[x_3^{(g_3,\mathbf{g}_3)}, x_4^{(g_4,\mathbf{g}_4)}\right]_{\mathbb{Z}_2^n} + c\left[x_1^{(g_1,\mathbf{g}_1)}, x_4^{(g_4,\mathbf{g}_4)}\right]_{\mathbb{Z}_2^n}\left[x_3^{(g_3,\mathbf{g}_3)}, x_2^{(g_2,\mathbf{g}_2)}\right]_{\mathbb{Z}_2^n} \in T_{G\times\mathbb{Z}_2^n}(A\otimes E^{\otimes n}),$$

where $c = (-1)^{\sum_{s=1}^{n} h_{2,s}h_{3,s}+h_{2,s}h_{4,s}+h_{3,s}h_{4,s}}$;

(d) *if* $x_1^{g_1}x_2^{g_2}\cdots x_m^{g_m} \in T_G(A)$ *then* $x_1^{(g_1,\mathbf{g}_1)}x_2^{(g_2,\mathbf{g}_2)}\cdots x_m^{(g_m,\mathbf{g}_m)} \in T_{G\times\mathbb{Z}_2^n}(A\otimes E^{\otimes n})$;

(e) *if* $T_i \subseteq \{1,\ldots,m\}$ *is ordered,* $\alpha_i \in F$ $(i = 1,\ldots,s)$*, and* $\sum_{i=1}^{s}\alpha_i f_{T_i}(x_1^{g_1},\ldots,x_m^{g_m}) \in T_G(A)$*, then* $\sum_{i=1}^{s}\alpha_i\,\mathbf{C}_{T_i,\mathcal{G}}\,f_{T_i,n}(x_1^{(g_1,\mathbf{g}_1)},\ldots,x_m^{(g_m,\mathbf{g}_m)}) \in T_{G\times\mathbb{Z}_2^n}(A\otimes E^{\otimes n})$,

for every $\mathbf{g}_i = (h_{i,1},\ldots,h_{i,n}) \in G^n$*, with* $i \in \mathbb{N}$.

Proof. It follows directly from Proposition 5 and Remarks 9 and 11. □

Now, if the elements of the subset $\{\epsilon_1,\epsilon_2,\ldots\} \subseteq E$ are the generators of E as an algebra, then a linear basis of E is given by $B := \{1, \epsilon_{i_1}\cdots\epsilon_{i_j} \mid i_1 < \cdots < i_j; j \geq 1\}$. Therefore, since E is endowed with its canonical grading, then $B^{\otimes n} := \{b_1\otimes\cdots\otimes b_n \mid b_1,\ldots,b_n \in B\}$ is an homogeneous basis of $E^{\otimes n}$.

The next is an easy, but very useful, result.

Lemma 13. *Let A be a G-graded algebra, then for every homogeneous elements $a_1, a_2 \in A$ and $b_1, b_2 \in B^{\otimes n}$, we have*

$$[a_1\otimes b_1, a_2\otimes b_2]_{\mathbb{Z}_2^n} = [a_1,a_2]\otimes b_1b_2.$$

As a consequence the following holds.

Remark 14. Consider the polynomials f_T and $f_{T,n}$ given in Definition 10. If $\overline{x}_i^{(g_i,\mathbf{g}_i)} = a_i\otimes b_i$, where $a_i \in A^{g_i}$, $b_i \in (B^{\otimes n})^{\mathbf{g}_i}$, then

$$\begin{aligned}
&f_{T,n}(\overline{x}_1^{(g_1,\mathbf{g}_1)},\ldots,\overline{x}_m^{(g_m,\mathbf{g}_m)})\\
&\quad= (a_{r_1}\otimes b_{r_1})\cdots(a_{r_v}\otimes b_{r_v})\,[a_{l_1}\otimes b_{l_1}, a_{l_2}\otimes b_{l_2}]_{\mathbb{Z}_2^n}\cdots\left[a_{l_{t-1}}\otimes b_{l_{t-1}}, a_{l_t}\otimes b_{l_t}\right]_{\mathbb{Z}_2^n}\\
&\quad= (a_{r_1}\cdots a_{r_v})\otimes(b_{r_1}\cdots b_{r_v})([a_{l_1},a_{l_2}]\otimes b_{l_1}b_{l_2})\cdots([a_{l_{t-1}},a_{l_t}]\otimes b_{l_{t-1}}b_{l_t})\\
&\quad= a_{r_1}\cdots a_{r_v}[a_{l_1},a_{l_2}]\cdots[a_{l_{t-1}},a_{l_t}]\otimes(b_{r_1}\cdots b_{r_v}b_{l_1}\cdots b_{l_t})\\
&\quad= \pm f_T(a_1,\ldots,a_m)\otimes(b_1\cdots b_m).
\end{aligned}$$

4. The homogeneous $\mathbb{Z}_2$-gradings of the Grassmann algebra

In the following sections we shall focus on the description of the $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded identities of $E_{gr}\otimes E^{\otimes n}$, where E_{gr} is the unitary infinite dimensional Grassmann algebra generated by $\{e_1,e_2,\ldots,\}$ and endowed with an arbitrary $\mathbb{Z}_2$-grading such that the subspace $L := \mathrm{span}_F\{e_1,e_2,\ldots\}$ is homogeneous. We start by recalling the known results about the $\mathbb{Z}_2$-graded identities of E_{gr}.

We denote by $B_{gr} := \{1, e_{i_1}\cdots e_{i_j} \mid i_1 < \cdots < i_j; j \geq 1\}$ the homogeneous basis of E_{gr} and by $\ell(a)$ the length of a basis element $a \in B_{gr}$.

Given $k \geq 0$, let us consider the following maps defined on the generators of the Grassmann algebra:

$$|e_i|_\infty := \begin{cases}0, & i \text{ even}\\ 1, & i \text{ odd},\end{cases} \qquad |e_i|_k := \begin{cases}0, & i = 1,\ldots,k\\ 1, & \text{otherwise}\end{cases} \qquad \text{and} \qquad |e_i|_{k^*} := \begin{cases}1, & i = 1,\ldots,k\\ 0, & \text{otherwise.}\end{cases}$$

We denote by E_∞, E_k and E_{k^*} the $\mathbb{Z}_2$-gradings of the Grassmann algebra endowed with the $\mathbb{Z}_2$-grading induced by the maps $|\cdot|_\infty$, $|\cdot|_k$ and $|\cdot|_{k^*}$, respectively. It is well known that we may assume any E_{gr} being one among E_∞, E_k or E_{k^*} (see for instance [2]). Therefore, in order to describe the $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded identities of $E_{gr}\otimes E^{\otimes n}$ for any E_{gr}, it is enough to describe the $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded identities of $E_\infty\otimes E^{\otimes n}$, $E_k\otimes E^{\otimes n}$ and $E_{k^*}\otimes E^{\otimes n}$.

We recall that since $[x_1, x_2, x_3]$ is an ordinary identity of E then $[x_1^{g_1}, x_2^{g_2}, x_3^{g_3}]$ is a $\mathbb{Z}_2$-graded identity of E_{gr}, for all $g_1, g_2, g_3 \in \mathbb{Z}_2$. Remembering that $[x_1, x_2][x_3, x_4] + [x_1, x_4][x_3, x_2] \in \langle [x_1, x_2, x_3] \rangle_T$ we obtain

$$[x_1^{g_1}, x_2^{g_2}][x_3^{g_3}, x_4^{g_4}] + [x_1^{g_1}, x_4^{g_4}][x_3^{g_3}, x_2^{g_2}] \in \left\langle [x_1^{h_1}, x_2^{h_2}, x_3^{h_3}];\ h_1, h_2, h_3 \in \mathbb{Z}_2 \right\rangle_{T_{\mathbb{Z}_2}},$$

for all $g_1, g_2, g_3, g_4 \in \mathbb{Z}_2$.

We define the following set of polynomials involving the polynomials f_T given in Definition 10.

Definition 15. Given $m \geq 1$ and the m-tuple $(g_1, \dots, g_m) \in \mathbb{Z}_2^m$, put

$$\gamma_1(x_1^{g_1}) := x_1^{g_1}$$

and, for $m \geq 2$, set

$$\gamma_m(x_1^{g_1}, \dots, x_m^{g_m}) := \sum_{\substack{T \subseteq \{1, \dots, m\}; \\ |T| \text{ even}}} (-2)^{-\frac{|T|}{2}} f_T(x_1^{g_1}, \dots, x_m^{g_m}).$$

The generators of the graded identities of E_{gr} were described by Di Vincenzo and da Silva in [13] when the ground field has characteristic 0.

Theorem 16 (Theorems 10 and 38 of [13])**.** *Given $k \geq 0$ and a field F of characteristic zero, let $T_{\mathbb{Z}_2}(E_d)$ be the $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities for the F-superalgebra E_d, with $d = \infty, k^*, k$. Then*

(a) *$T_{\mathbb{Z}_2}(E_\infty)$ is generated by the triple commutators $[x_1^{g_1}, x_2^{g_2}, x_3^{g_3}]$, with $g_1, g_2, g_3 \in \mathbb{Z}_2$,*

(b) *$T_{\mathbb{Z}_2}(E_{k^*})$ is generated by the set of the following polynomials:*

- $[x_1^{g_1}, x_2^{g_2}, x_3^{g_3}]$, *with* $g_1, g_2, g_3 \in \mathbb{Z}_2$,
- $x_1^1 x_2^1 \cdots x_{k+1}^1$,

(c) *$T_{\mathbb{Z}_2}(E_k)$ is generated by the set of the following polynomials:*

- $[x_1^{g_1}, x_2^{g_2}, x_3^{g_3}]$, *with* $g_1, g_2, g_3 \in \mathbb{Z}_2$,
- $[x_1^0, x_2^0] \cdots [x_{k-1}^0, x_k^0][x_{k+1}^0, x_{k+2}^{g_{k+2}}]$, *with* $g_{k+2} \in \mathbb{Z}_2$, (*if k is even*)
- $[x_1^0, x_2^0] \cdots [x_{k-2}^0, x_{k-1}^0][x_k^0, x_{k+1}^0]$ (*if k is odd*)
- $\gamma_{k-l+2}(x_1^1, \dots, x_{k-l+2}^1)[x_{k-l+3}^0, x_{k-l+4}^0] \cdots [x_{k+1}^0, x_{k+2}^0]$ ($\forall\, l \leqslant k$, *l even*)
- $[\gamma_{k-l+2}(x_1^1, \dots, x_{k-l+2}^1), x_{k-l+3}^0][x_{k-l+4}^0, x_{k-l+5}^0] \cdots [x_{k+1}^0, x_{k+2}^0]$ ($\forall\, l \leqslant k$, *l odd*)
- $\gamma_{k-l+2}(x_1^1, \dots, x_{k-l+2}^1)[x_{k-l+3}^1, x_{k-l+4}^0][x_{k-l+5}^0, x_{k-l+6}^0] \cdots [x_{k+2}^0, x_{k+3}^0]$ ($\forall\, l \leqslant k$, *l odd*).

As a consequence of the results of [13] we also obtain the following

Remark 17. Given $n_0, n_1 \geq 0$ with $n_0 + n_1 \geq 1$, consider, for each ordered subset T of $\{1, \dots, n_0 + n_1\}$, the polynomial $f_T := f_T(x_1^0, \dots, x_{n_0}^0, x_{n_0+1}^1, \dots, x_{n_0+n_1}^1) \in P_{n_0,n_1}^{\mathbb{Z}_2}$ (given in Definition 10) and let

$$r_T := |\{1 \leq i \leq n_0 \mid i \in T\}| \qquad \text{and} \qquad s_T := |\{n_0 + 1 \leq i \leq n_0 + n_1 \mid i \notin T\}|.$$

Then a basis for the quotient space $P_{n_0,n_1}^{\mathbb{Z}_2}(E_k)$ is given by the polynomials f_T satisfying the following conditions:

- $r_T + s_T \leq k + 1$
- if $r_T + s_T = k + 1$ then $s_T \neq 0$ and $n_0 + 1 \notin T$.

When the characteristic of the ground field is $p > 2$, in [7] Centrone showed that a list of generators of the $T_{\mathbb{Z}_2}$-ideal of E_d (with $d = \infty, k^*, k$) is given by the polynomials described in Theorem 16 plus the monomial $(x_1^1)^p$.

5. The $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded identities of $E_{gr}\otimes E^{\otimes n}$ in characteristic zero

In this section we will give an explicit set of generators for the $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded identities of $E_{gr}\otimes E^{\otimes n}$ over a field of characteristic zero. Notice that the next results are a direct consequence of a combination of Theorem 6 (Theorem 11 of [12]), Theorem 16 and Proposition 12.

Theorem 18. *Let F be a field of characteristic 0, then the $T_{\mathbb{Z}_2\times\mathbb{Z}_2^n}$-ideal of $E_\infty\otimes E^{\otimes n}$ is generated by $\left[x_1^{(g_1,\mathbf{g}_1)},x_2^{(g_2,\mathbf{g}_2)},x_3^{(g_3,\mathbf{g}_3)}\right]_{\mathbb{Z}_2^n}$, with $g_1,g_2,g_3\in\mathbb{Z}_2$, $\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3\in\mathbb{Z}_2^n$.*

Theorem 19. *Let F be a field of characteristic 0 and consider $k\geq 0$. Then the $T_{\mathbb{Z}_2\times\mathbb{Z}_2^n}$-ideal of $E_{k^*}\otimes E^{\otimes n}$ is generated by*

- $\left[x_1^{(g_1,\mathbf{g}_1)},x_2^{(g_2,\mathbf{g}_2)},x_3^{(g_3,\mathbf{g}_3)}\right]_{\mathbb{Z}_2^n}$, *with $g_1,g_2,g_3\in\mathbb{Z}_2$,*
- $x_1^{(1,\mathbf{g}_1)}\cdots x_{k+1}^{(1,\mathbf{g}_{k+1})}$,

with $\mathbf{g}_i\in\mathbb{Z}_2^n$, for all i.

By using the notation given in Definition 10, let us introduce the generalization of the polynomials given in Definition 15. The latter will have a crucial role in the description of the graded T-ideals of $E_k\otimes E^{\otimes n}$.

Definition 20. Given $m\geq 1$, the m-tuple $(g_1,\dots,g_m)\in\mathbb{Z}_2^m$ and the n-tuples $\mathbf{g}_i\in\mathbb{Z}_2^n$, with $i=1,\dots,m$, put

$$\gamma_{1,n}(x_1^{(g_1,\mathbf{g}_1)}):=x_1^{(g_1,\mathbf{g}_1)}$$

and, for $m\geq 2$, set

$$\gamma_{m,n}(x_1^{(g_1,\mathbf{g}_1)},\dots,x_m^{(g_m,\mathbf{g}_m)}):=\sum_{\substack{T\subseteq\{1,\dots,m\};\\ |T|\text{ even}}}(-2)^{-\frac{|T|}{2}}\,\mathbf{C}_{T,\mathcal{G}}f_{T,n}(x_1^{(g_1,\mathbf{g}_1)},\dots,x_m^{(g_m,\mathbf{g}_m)}).$$

As above, the next result is a direct consequence of a combination of Theorem 6 (Theorem 11 of [12]), Theorem 16 and Proposition 12.

Theorem 21. *Let F be a field of characteristic 0 and consider $k\geq 0$. Then the $T_{\mathbb{Z}_2\times\mathbb{Z}_2^n}$-graded ideal of identities of $E_k\otimes E^{\otimes n}$ is generated by the following set of polynomials:*

- $\left[x_1^{(g_1,\mathbf{g}_1)},x_2^{(g_2,\mathbf{g}_2)},x_3^{(g_3,\mathbf{g}_3)}\right]_{\mathbb{Z}_2^n}$, *with $g_1,g_2,g_3\in\mathbb{Z}_2$,*
- $\left[x_1^{(0,\mathbf{g}_1)},x_2^{(0,\mathbf{g}_2)}\right]_{\mathbb{Z}_2^n}\cdots\left[x_{k-1}^{(0,\mathbf{g}_{k-1})},x_k^{(0,\mathbf{g}_k)}\right]_{\mathbb{Z}_2^n}\left[x_{k+1}^{(0,\mathbf{g}_{k+1})},x_{k+2}^{(g_{k+2},\mathbf{g}_{k+2})}\right]_{\mathbb{Z}_2^n}$, *with $g_{k+2}\in\mathbb{Z}_2$ (if k is even)*
- $\left[x_1^{(0,\mathbf{g}_1)},x_2^{(0,\mathbf{g}_2)}\right]_{\mathbb{Z}_2^n}\cdots\left[x_k^{(0,\mathbf{g}_k)},x_{k+1}^{(0,\mathbf{g}_{k+1})}\right]_{\mathbb{Z}_2^n}$ *(if k is odd)*
- $\gamma_{k-l+2,n}(x_1^{(1,\mathbf{g}_1)},\dots,x_{k-l+2}^{(1,\mathbf{g}_{k-l+2})})\left[x_{k-l+3}^{(0,\mathbf{g}_{k-l+3})},x_{k-l+4}^{(0,\mathbf{g}_{k-l+4})}\right]_{\mathbb{Z}_2^n}\cdots\left[x_{k+1}^{(0,\mathbf{g}_{k+1})},x_{k+2}^{(0,\mathbf{g}_{k+2})}\right]_{\mathbb{Z}_2^n}$, *($\forall\, l\leqslant k$, l even)*
- $\left[\gamma_{k-l+2,n}(x_1^{(1,\mathbf{g}_1)},\dots,x_{k-l+2}^{(1,\mathbf{g}_{k-l+2})}),x_{k-l+3}^{(0,\mathbf{g}_{k-l+3})}\right]_{\mathbb{Z}_2^n}\left[x_{k-l+4}^{(0,\mathbf{g}_{k-l+4})},x_{k-l+5}^{(0,\mathbf{g}_{k-l+5})}\right]_{\mathbb{Z}_2^n}\cdots\left[x_{k+1}^{(0,\mathbf{g}_{k+1})},x_{k+2}^{(0,\mathbf{g}_{k+2})}\right]_{\mathbb{Z}_2^n}$, *($\forall\, l\leqslant k$, l odd)*
- $\gamma_{k-l+2,n}(x_1^{(1,\mathbf{g}_1)},\dots,x_{k-l+2}^{(1,\mathbf{g}_{k-l+2})})\left[x_{k-l+3}^{(1,\mathbf{g}_{k-l+3})},x_{k-l+4}^{(0,\mathbf{g}_{k-l+4})}\right]_{\mathbb{Z}_2^n}\cdots\left[x_{k+2}^{(0,\mathbf{g}_{k+2})},x_{k+3}^{(0,\mathbf{g}_{k+3})}\right]_{\mathbb{Z}_2^n}$, *($\forall\, l\leqslant k$, l odd),*

with $\mathbf{g}_i\in\mathbb{Z}_2^n$, for all i.

6. The $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded identities of $E_{gr}\otimes E^{\otimes n}$ in odd characteristic

Now we are going to exploit the case of ground field having odd characteristic.

Before listing some crucial $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded identities of $E_{gr}\otimes E^{\otimes n}$ in positive characteristic, we introduce the following "test elements" of $E_{gr}\otimes E^{\otimes n}$ jointly with a relation involving only the elements of the $\mathbb{Z}_2^n$ part. We recall that $B^{\otimes n}=\{b_1\otimes\cdots\otimes b_n\,|\,b_1,\ldots,b_n\in B\}$ is an homogeneous basis of $E^{\otimes n}$, whereas $B_{gr}=\{1,e_{i_1}\cdots e_{i_j}\,|\,i_1<\cdots<i_j; j\geq 1\}$ is the homogeneous basis of E_{gr} and $\ell(a)$ is the length of a basis element $a\in B_{gr}$.

Definition 22. Consider $g\in\mathbb{Z}_2$, $\mathbf{g}=(g_1,\ldots,g_n)\in\mathbb{Z}_2^n$, and non-negative integers $h\leq m$.

- An $S_{m,h}^{(g,\mathbf{g})}$-element is an element of the form
$$a_1\otimes b_1+\cdots+a_h\otimes b_h+a_{h+1}\otimes b_{h+1}+\cdots+a_m\otimes b_m,$$
where $b_1,\ldots,b_m\in(B^{\otimes n}-\{1\})\cap(E^{\otimes n})^{\mathbf{g}}$, $a_1,\ldots,a_m\in(B_{gr}-\{1\})\cap E_{gr}^g$, and $\ell(a_i)$ is even for $1\leq i\leq h$, whereas $\ell(a_i)$ is odd for $h+1\leq i\leq m$.
- If $g_1+\cdots+g_n\equiv g \pmod 2$, then we say that $\mathbf{g}\sim_2 g$.

The next results are a generalization of Lemmas 12, 13 and 16 of [9] and their proofs follow word by word the ones given in that paper.

Lemma 23. *Let $u=1\otimes 1+a_1\otimes b_1$ and $v=c\otimes d$, where $a_1\otimes b_1$ is an $S_{1,0}^{(0,0)}$-element, $c\in B_{gr}$ and $d\in B^{\otimes n}$ are homogeneous elements. Then, for all $m\geq 1$,*

I) $u^m=1\otimes 1+m\,a_1\otimes b_1$

II) $$u^{m-1}[u,v]_{\mathbb{Z}_2^n}=\begin{cases}2(a_1c)\otimes(b_1d), & \textit{if } \ell(c) \textit{ is odd}\\ 0, & \textit{otherwise.}\end{cases}$$

Lemma 24. *Let $u=a_1\otimes b_1+\cdots+a_h\otimes b_h+a_{h+1}\otimes b_{h+1}+\cdots+a_m\otimes b_m$ be an $S_{m,h}^{(g,\mathbf{g})}$-element, where $\mathbf{g}\sim_2 0$. Let c,d be homogeneous elements such that $c\in B_{gr}$ and $d\in B^{\otimes n}$. Then by setting $v=c\otimes d$ we obtain*

I) $$u^m=\begin{cases}m!\,(a_1\cdots a_m)\otimes(b_1\cdots b_m), & \textit{if } h=m-1,m\\ 0, & \textit{otherwise}\end{cases}$$

II) $$u^{m-1}[u,v]_{\mathbb{Z}_2^n}=\begin{cases}2(m-1)!\,w, & \textit{if } h=m-1 \textit{ and } \ell(c) \textit{ is odd}\\ 0, & \textit{otherwise}\end{cases}$$

where $w=(a_1\cdots a_mc)\otimes(b_1\cdots b_md)$.

Lemma 25. *Let $u=a_1\otimes b_1+\cdots+a_h\otimes b_h+a_{h+1}\otimes b_{h+1}+\cdots+a_m\otimes b_m$ be an $S_{m,h}^{(g,\mathbf{g})}$-element, where $\mathbf{g}\sim_2 1$. Let c,d be homogeneous elements of E_{gr} such that $c\in B_{gr}$ and $d\in B^{\otimes n}$. Then by setting $v=c\otimes d$ we obtain*

I) $$u^m=\begin{cases}m!\,(a_1\cdots a_m)\otimes(b_1\cdots b_m), & \textit{if } h=0\\ (m-1)!\,(a_1\cdots a_m)\otimes(b_1\cdots b_m), & \textit{if } h=1 \textit{ and } m \textit{ is odd}\\ 0, & \textit{otherwise}\end{cases}$$

II) $$u^{m-1}vu=\begin{cases}m!(-1)^{\ell(c)}(-1)^{\tilde{\ell}(d)}\,w, & \textit{if } h=0\\ (m-1)!(-1)^{\tilde{\ell}(d)}((-1)^{\ell(c)}-1)\,w, & \textit{if } h=1 \textit{ and } m \textit{ is even}\\ (m-1)!(-1)^{\tilde{\ell}(d)}\,w, & \textit{if } h=1 \textit{ and } m \textit{ is odd}\\ 0, & \textit{otherwise}\end{cases}$$

III) $$u^{m-1}[u,v]_{\mathbb{Z}_2^n}=\begin{cases}2m!\,w, & \textit{if } h=0 \textit{ and } \ell(c) \textit{ is odd}\\ 2(m-1)!\,w, & \textit{if } h=1,\ m \textit{ is even and } \ell(c) \textit{ is odd}\\ 0, & \textit{otherwise.}\end{cases}$$

where $w=(a_1\cdots a_mc)\otimes(b_1\cdots b_md)$, and $\tilde{\ell}(d)$ is such that $\deg(d)\sim_2\tilde{\ell}(d)$.

Because of Lemmas 24 and 25 we get the next result.

Proposition 26. *Let F be an infinite field of characteristic $p > 2$, then*

- $(x^{(g,\mathbf{g})})^p$, *with* $\mathbf{g} \sim_2 0$ *and* $(g,\mathbf{g}) \neq (0,0)$;
- $(x^{(g,\mathbf{g})})^p \, x^{(h,\mathbf{h})} \, x^{(g,\mathbf{g})}$, *with* $\mathbf{g} \sim_2 1$;
- $(x^{(g,\mathbf{g})})^{p-1} \left[x^{(g,\mathbf{g})}, x^{(h,\mathbf{h})}\right]_{\mathbb{Z}_2^n}$, *with* $\mathbf{g} \sim_2 1$;

are $(\mathbb{Z}_2 \times \mathbb{Z}_2^n)$-graded identities for $E_{gr} \otimes E^{\otimes n}$.

We observe that if $(g,\mathbf{g}) \in \mathbb{Z}_2 \times \mathbb{Z}_2^n$, then

$$\left[x^{(g,\mathbf{g})}, x^{(g,\mathbf{g})}\right]_{\mathbb{Z}_2^n} = \begin{cases} 0, & \text{if } \mathbf{g} \sim_2 0 \\ 2(x^{(g,\mathbf{g})})^2, & \text{if } \mathbf{g} \sim_2 1. \end{cases}$$

Then we may rewrite Lemma 25 in the following way.

Lemma 27. *Let $u = a_1 \otimes b_1 + \cdots + a_h \otimes b_h + a_{h+1} \otimes b_{h+1} + \cdots + a_m \otimes b_m$ be an $S_{m,h}^{(g,\mathbf{g})}$-element, where $\mathbf{g} \sim_2 1$. Let c, d be homogeneous elements of E_{gr} such that $c \in B_{gr}$ and $d \in B^{\otimes n}$, and set $v = c \otimes d$.*

If m is even, then

$$\text{I)}\quad ([u,u]_{\mathbb{Z}_2^n})^{\frac{m}{2}} = \begin{cases} 2^{\frac{m}{2}} m!\,(a_1 \cdots a_m) \otimes (b_1 \cdots b_m), & \textit{if } h = 0 \\ 0, & \textit{otherwise} \end{cases}$$

$$\text{II)}\quad u([u,u]_{\mathbb{Z}_2^n})^{\frac{m-2}{2}} \, [u,v]_{\mathbb{Z}_2^n} = \begin{cases} 2^{\frac{m}{2}} m!\, w, & \textit{if } h = 0 \textit{ and } \ell(c) \textit{ is odd} \\ 2^{\frac{m}{2}} (m-1)!\, w, & \textit{if } h = 1 \textit{ and } \ell(c) \textit{ is odd} \\ 0, & \textit{otherwise.} \end{cases}$$

If m is odd, then

$$\text{I')}\quad ([u,u]_{\mathbb{Z}_2^n})^{\frac{m-1}{2}} \, [u,v]_{\mathbb{Z}_2^n} = \begin{cases} 2^{\frac{m+1}{2}} m!\, w, & \textit{if } h = 0 \textit{ and } \ell(c) \textit{ is odd} \\ 0, & \textit{otherwise} \end{cases}$$

$$\text{II')}\quad u([u,u]_{\mathbb{Z}_2^n})^{\frac{m-1}{2}} = \begin{cases} 2^{\frac{m-1}{2}} m!(a_1 \cdots a_m) \otimes (b_1 \cdots b_m), & \textit{if } h = 0 \\ 2^{\frac{m-1}{2}} (m-1)!(a_1 \cdots a_m) \otimes (b_1 \cdots b_m), & \textit{if } h = 1 \\ 0, & \textit{otherwise,} \end{cases}$$

where $w = (a_1 \cdots a_m c) \otimes (b_1 \cdots b_m d)$.

Let us recall the notation used by the authors in [9] when working with the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-graded identities of $E_{k^*} \otimes E$. Given a multiset

$$\mathcal{B} = \{x_{i_1}^{(g_{i_1},\mathbf{g}_{i_1})}, \ldots, x_{i_l}^{(g_{i_l},\mathbf{g}_{i_l})}\},$$

we denote by $P_{\mathcal{B}}$ the vector space generated by all monomials obtained by permuting the order of the variables in

$$x_{i_1}^{(g_{i_1},\mathbf{g}_{i_1})} \cdots x_{i_l}^{(g_{i_l},\mathbf{g}_{i_l})}.$$

We denote by $m_{\mathcal{B}}(u)$ the multiplicity of the element u in the multiset $\mathcal{B}$. We say that a multiset $\mathcal{A}$ is a submultiset of $\mathcal{B}$ if, up to multiplicity, all the members of $\mathcal{A}$ are also members of $\mathcal{B}$, and $m_{\mathcal{A}}(u) \leq m_{\mathcal{B}}(u)$, for all u. Given a submultiset $\mathcal{A}$ of $\mathcal{B}$, we denote by $\mathcal{B} - \mathcal{A}$ the complementary multiset, that is, $\mathcal{B} - \mathcal{A}$ is the submultiset of $\mathcal{B}$ such that the multiplicity of each member u is given by $m_{\mathcal{B}}(u) - m_{\mathcal{A}}(u)$.

In what follows we shall consider the natural order on $\mathbb{Z}_2 \times \mathbb{Z}_2^n$. Moreover, let us assume, for simplicity, that the elements of the multiset $\mathcal{B}$ are the variables $x_1^{(g,\mathbf{g})}, \ldots, x_{n_{(g,\mathbf{g})}}^{(g,\mathbf{g})}$, where $(g,\mathbf{g}) \in \mathbb{Z}_2 \times \mathbb{Z}_2^n$ and $x_i^{(g,\mathbf{g})}$ has multiplicity $m_{\mathcal{B}}(x_i^{(g,\mathbf{g})})$. We denote

$$|\mathcal{B}| := \sum_{(g,\mathbf{g}) \in \mathbb{Z}_2 \times \mathbb{Z}_2^n} \sum_{i=1}^{n_{(g,\mathbf{g})}} m_{\mathcal{B}}(x_i^{(g,\mathbf{g})}).$$

At the light light of Proposition 26 we reformulate the notion of an adequate multiset given in Definition 19 of [9].

Definition 28. Let F be a field of characteristic $p > 2$. We call a multiset $\mathcal{B}$ *adequate* if $0 \leq m_{\mathcal{B}}(x_i^{(g,\mathbf{g})}) \leq p$, for every $i = 1, \ldots, n_{(g,\mathbf{g})}$ with $(g, \mathbf{g}) \in \mathbb{Z}_2 \times \mathbb{Z}_2^n$, $(g, \mathbf{g}) \neq (0, 0)$.

We are in position to introduce a generalization of the polynomials $f_{T,n}$ given in Definition 10.

Definition 29. Given an adequate multiset $\mathcal{B}$, let $\mathcal{T} = \{u_1, \ldots, u_t\}$ be a submultiset of $\mathcal{B}$ and $\mathcal{B} - \mathcal{T} = \{v_1, \ldots, v_{|\mathcal{B}|-t}\}$ its complementary set such that:

(*i*) t is even

(*ii*) $0 \leq m_{\mathcal{T}}(x_i^{(g,\mathbf{g})}) \leq 1$ if $\mathbf{g} \sim_2 0$

(*iii*) $0 \leq m_{\mathcal{B}-\mathcal{T}}(x_i^{(g,\mathbf{g})}) \leq 1$ if $\mathbf{g} \sim_2 1$

(*iv*) $u_1 \leq \cdots \leq u_t$

(*v*) $v_1 \leq \cdots \leq v_{|\mathcal{B}|-t}$.

The following "objects" will play a special role in the sequel and we will use them intensively in our next proofs. Let us denote

- $f_{\mathcal{T}}^{\mathcal{B}} := v_1 \cdots v_{|\mathcal{B}|-t} \, [u_1, u_2]_{\mathbb{Z}_2^n} \cdots [u_{t-1}, u_t]_{\mathbb{Z}_2^n}$,
- $r_{\mathcal{T}} := \sum_{\mathbf{g} \in \mathbb{Z}_2^n} \sum_{i=1}^{n_{(0,\mathbf{g})}} m_{\mathcal{T}}(x_i^{(0,\mathbf{g})})$,
- $s_{\mathcal{T}} := \sum_{\mathbf{g} \in \mathbb{Z}_2^n} \sum_{i=1}^{n_{(1,\mathbf{g})}} m_{\mathcal{B}-\mathcal{T}}(x_i^{(1,\mathbf{g})})$.

Furthermore we shall use the following substitution and we shall call it *canonical substitution.* For each multiset $\mathcal{T}$ we consider the evaluation $\overline{\mathcal{T}}$ given by

$$\overline{x}_i^{(0,0)} = \begin{cases} 1 \otimes 1 & \text{if } x_i^{(0,0)} \notin \mathcal{T} \\ 1 \otimes 1 + \tau_i^{(0,0)} & \text{if } x_i^{(0,0)} \in \mathcal{T}, \end{cases}$$

where $\tau_i^{(0,0)}$ is an $S_{1,0}^{(0,0)}$-element. Still, for every $(g, \mathbf{g}) \neq (0, 0)$ and $i = 1, \ldots, n_{(g,\mathbf{g})}$, we put $\overline{x}_i^{(g,\mathbf{g})} = \tau_i^{(g,\mathbf{g})}$, where $\tau_i^{(g,\mathbf{g})}$ is an $S_{m,h}^{(g,\mathbf{g})}$-element with $m = m_{\mathcal{B}}(x_i^{(g,\mathbf{g})})$ and

$$h = \begin{cases} m & \text{if } \mathbf{g} \sim_2 0 \text{ and } x_i^{(g,\mathbf{g})} \notin \mathcal{T} \\ m-1 & \text{if } \mathbf{g} \sim_2 0 \text{ and } x_i^{(g,\mathbf{g})} \in \mathcal{T} \\ 0 & \text{if } \mathbf{g} \sim_2 1 \text{ and } x_i^{(g,\mathbf{g})} \notin \mathcal{B} - \mathcal{T} \\ 1 & \text{if } \mathbf{g} \sim_2 1 \text{ and } x_i^{(g,\mathbf{g})} \in \mathcal{B} - \mathcal{T}. \end{cases}$$

Moreover, all of the elements and summands have disjoint supports.

We are ready to state our main results.

6.1. The case of $E_\infty \otimes E^{\otimes n}$.

The generators of the $(\mathbb{Z}_2 \times \mathbb{Z}_2^n)$-graded identities of $E_\infty \otimes E^{\otimes n}$ are given by the set of generators of the $(\mathbb{Z}_2 \times \mathbb{Z}_2^n)$-graded identities of $E_\infty \otimes E^{\otimes n}$ in characteristic 0 as in Theorem 18 and the polynomials described in Proposition 26. Actually, we have the following.

Theorem 30. *Let F be an infinite field of characteristic $p > 2$. The generators of the $(\mathbb{Z}_2 \times \mathbb{Z}_2^n)$-graded identities of $E_\infty \otimes E^{\otimes n}$ are given by the following polynomials:*

(*i*) $\left[x_1^{(g_1,\mathbf{g}_1)}, x_2^{(g_2,\mathbf{g}_2)}, x_3^{(g_3,\mathbf{g}_3)}\right]_{\mathbb{Z}_2^n}$, *with* $g_1, g_2, g_3 \in \mathbb{Z}_2$, $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \in \mathbb{Z}_2^n$.

(*ii*) $(x^{(g,\mathbf{g})})^p$, *with* $\mathbf{g} \sim_2 0$ *and* $(g, \mathbf{g}) \neq (0, 0)$*;*

(*iii*) $(x^{(g,\mathbf{g})})^p \, x^{(h,\mathbf{h})} \, x^{(g,\mathbf{g})}$, *with* $\mathbf{g} \sim_2 1$*;*

(*iv*) $(x^{(g,\mathbf{g})})^{p-1} \left[x^{(g,\mathbf{g})}, x^{(h,\mathbf{h})}\right]_{\mathbb{Z}_2^n}$, *with* $\mathbf{g} \sim_2 1$.

Moreover, if $\mathcal{B}$ is an adequate multiset, then a basis for $P_{\mathcal{B}}(E_\infty \otimes E^{\otimes n})$ is given by the polynomials $f_{\mathcal{T}}^{\mathcal{B}}$ such that:

($*$) *for each* $(g,\mathbf{g})\neq(0,0)$ *and* $1\leq i\leq n_{(g,\mathbf{g})}$ *with* $m_{\mathcal{B}}(x_i^{(g,\mathbf{g})})=p$*, we have* $x_i^{(g,\mathbf{g})}\in\mathcal{T}$ *if* $\mathbf{g}\sim_2 0$ *whereas* $x_i^{(g,\mathbf{g})}\in\mathcal{B}-\mathcal{T}$ *if* $\mathbf{g}\sim_2 1$.

Proof. Let $\tilde{I}_{p,\infty}$ be the $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded ideal generated by the polynomials (i) to (iv). Clearly $\tilde{I}_{p,\infty}\subseteq T_{\mathbb{Z}_2\times\mathbb{Z}_2^n}(E_\infty\otimes E^{\otimes n})$ and, for any adequate multiset $\mathcal{B}$, each polynomial of $P_{\mathcal{B}}$ may be written, modulo $\tilde{I}_{p,\infty}$, as a linear combination of polynomials $f_{\mathcal{T}}^{\mathcal{B}}\in P_{\mathcal{B}}$ satisfying $(*)$.

Let f be a $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded identity of $E_\infty\otimes E^{\otimes n}$ such that

$$f=\sum_{\mathcal{T}}\gamma_{\mathcal{T}}f_{\mathcal{T}}^{\mathcal{B}}\qquad(\gamma_{\mathcal{T}}\in F),$$

where the sum runs over all submultisets $\mathcal{T}$ of $\mathcal{B}$ satisfying $(*)$. We want to prove $\gamma_{\mathcal{T}}=0$ for all such $\mathcal{T}$.

We shall denote by $\overline{f_{\mathcal{T}}^{\mathcal{B}}}$ the evaluation of $f_{\mathcal{T}}^{\mathcal{B}}$ by a substitution. For each multiset $\mathcal{T}$ we consider the canonical evaluation. Let us denote by $\mathcal{O}(\mathcal{T})$ the cardinality of the union of the sets

$$\left(\bigcup_{\mathbf{g}\sim_2 0}\bigcup_{r\in\{0,1\}}\{x_i^{(r,\mathbf{g})}\,|\,x_i^{(r,\mathbf{g})}\notin\mathcal{T}\}\right)\bigcup\left(\bigcup_{\mathbf{g}\sim_2 1}\bigcup_{r\in\{0,1\}}\{x_i^{(r,\mathbf{g})}\,|\,x_i^{(r,\mathbf{g})}\in\mathcal{B}-\mathcal{T}\}\right).$$

Let $\mathcal{T}'$ be a multiset such that $\mathcal{O}(\mathcal{T}')$ is maximal and let us consider the evaluation $\overline{\mathcal{T}'}$. Then by Lemmas 23, 24 and 27 we obtain $\overline{f_{\mathcal{T}}^{\mathcal{B}}}\neq 0$ if, and only if, $\mathcal{T}=\mathcal{T}'$. Hence $\gamma_{\mathcal{T}'}=0$, then by proceeding inductively we conclude the linear independence.

Recalling that in the case of an infinite field all the identities are consequences of the homogeneous ones, we conclude our statements. ☐

6.2. **The case of $E_{k^*}\otimes E^{\otimes n}$.** Let $\tilde{I}_{p,k^*}$ be the $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded ideal generated by the polynomials in Theorem 19 and the ones in Proposition 26. Following verbatim the proof of Theorem 30, we obtain the description of the graded relatively free algebra of $E_{k^*}\otimes E^{\otimes n}$ as well we state $T_{\mathbb{Z}_2\times\mathbb{Z}_2^n}(E_{k^*}\otimes E^{\otimes n})=\tilde{I}_{p,k^*}$. The reader should pay attention to the fact that the canonical substitution fits in our computations in this case, too. Nevertheless, we have some restrictions, actually there exists only k distinct elements $e_i\in E_{gr}^1$.

Theorem 31. *Let F be an infinite field of characteristic $p>2$. The generators of the $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded identities of $E_{k^*}\otimes E^{\otimes n}$ are given by the following polynomials:*

(i) $\left[x_1^{(g_1,\mathbf{g}_1)},x_2^{(g_2,\mathbf{g}_2)},x_3^{(g_3,\mathbf{g}_3)}\right]_{\mathbb{Z}_2^n}$*, with* $g_1,g_2,g_3\in\mathbb{Z}_2$*,* $\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3\in\mathbb{Z}_2^n$.

(ii) $x_1^{(1,\mathbf{g}_1)}\cdots x_{k+1}^{(1,\mathbf{g}_{k+1})}$*, with* $\mathbf{g}_1,\ldots,\mathbf{g}_k\in\mathbb{Z}_2^n$*;*

(iii) $(x^{(g,\mathbf{g})})^p$*, with* $\mathbf{g}\sim_2 0$ *and* $(g,\mathbf{g})\neq(0,0)$*;*

(iv) $(x^{(g,\mathbf{g})})^p\,x^{(h,\mathbf{h})}\,x^{(g,\mathbf{g})}$*, with* $\mathbf{g}\sim_2 1$*;*

(v) $(x^{(g,\mathbf{g})})^{p-1}\left[x^{(g,\mathbf{g})},x^{(h,\mathbf{h})}\right]_{\mathbb{Z}_2^n}$*, with* $\mathbf{g}\sim_2 1$.

Moreover, if $\mathcal{B}$ is an adequate multiset, then a basis for $P_{\mathcal{B}}(E_{k^*}\otimes E^{\otimes n})$ is given by the $f_{\mathcal{T}}^{\mathcal{B}}$ such that:

- *for each* $(g,\mathbf{g})\neq(0,0)$ *and* $1\leq i\leq n_{(g,\mathbf{g})}$ *such that* $m_{\mathcal{B}}(x_i^{(g,\mathbf{g})})=p$*, we have* $x_i^{(g,\mathbf{g})}\in\mathcal{T}$ *if* $\mathbf{g}\sim_2 0$ *whereas* $x_i^{(g,\mathbf{g})}\in\mathcal{B}-\mathcal{T}$ *if* $\mathbf{g}\sim_2 1$*;*
- $\sum_{\mathbf{g}\in\mathbb{Z}_2^n}\sum_{i=1}^{n_{(1,\mathbf{g})}}m_{\mathcal{B}}(x_i^{(1,\mathbf{g})})\leq k$.

6.3. **The case of $E_k\otimes E^{\otimes n}$.** Given an adequate multiset $\mathcal{B}$, let

$$\mathcal{M}_p:=\{x_i^{(g,\mathbf{g})}\mid(g,\mathbf{g})\neq(0,0)\text{ and }m_{\mathcal{B}}(x_i^{(g,\mathbf{g})})=p\}.$$

Consider a submultiset $\mathcal{T}\subseteq\mathcal{B}$ satisfying conditions (i) to (v) of Definition 29 and such that for each $(g,\mathbf{g})\neq(0,0)$ and $1\leq i\leq n_{(g,\mathbf{g})}$ with $x_i^{(g,\mathbf{g})}\in\mathcal{M}_p$, we have $x_i^{(g,\mathbf{g})}\in\mathcal{T}$ if $\mathbf{g}\sim_2 0$ whereas $x_i^{(g,\mathbf{g})}\in\mathcal{B}-\mathcal{T}$ if $\mathbf{g}\sim_2 1$. By Lemmas 23, 24 and 27 we get that, in order to obtain a non-zero evaluation of $f_{\mathcal{T}}^{\mathcal{B}}$, the amount of distinct elements $e_j\in E_{gr}^0$ required, for each variable $x_i^{(g,\mathbf{g})}\in\mathcal{B}$, is at least

- $m_{\mathcal{T}}(x_i^{(g,\mathbf{g})})$, if $g=0$;
- $m_{\mathcal{B}-\mathcal{T}}(x_i^{(g,\mathbf{g})})$, if $g=1$ and one of the following conditions is satisfied:
  - ⋄ $\mathbf{g}\sim_2 0$ and $x_i^{(g,\mathbf{g})}\in\mathcal{T}$ or
  - ⋄ $\mathbf{g}\sim_2 1$, $x_i^{(1,\mathbf{g})}\in\mathcal{M}_p$ and $x_i^{(1,\mathbf{g})}\in\mathcal{B}-\mathcal{T}$ or
  - ⋄ $\mathbf{g}\sim_2 1$, $x_i^{(1,\mathbf{g})}\notin\mathcal{M}_p$ and $x_i^{(1,\mathbf{g})}\notin\mathcal{B}-\mathcal{T}$;
- $m_{\mathcal{B}-\mathcal{T}}(x_i^{(g,\mathbf{g})})-1$, if $g=1$ and one of the following conditions is satisfied:
  - ⋄ $\mathbf{g}\sim_2 0$, $x_i^{(g,\mathbf{g})}\notin\mathcal{M}_p$ and $x_i^{(g,\mathbf{g})}\notin\mathcal{T}$ or
  - ⋄ $\mathbf{g}\sim_2 1$, $x_i^{(g,\mathbf{g})}\notin\mathcal{M}_p$ and $x_i^{(g,\mathbf{g})}\in\mathcal{B}-\mathcal{T}$.

Therefore, since there exist only k distinct elements $e_j\in E^0_{gr}$, we have the next.

Proposition 32. *If $r_{\mathcal{T}}+s_{\mathcal{T}}=k+1$ and each variable $x_i^{(1,\mathbf{g})}\in\mathcal{B}$ satisfies one of the following conditions:*

- $\mathbf{g}\sim_2 0$ *and* $x_i^{(1,\mathbf{g})}\in\mathcal{T}$ *or*
- $\mathbf{g}\sim_2 1$, $x_i^{(1,\mathbf{g})}\in\mathcal{M}_p$ *and* $x_i^{(1,\mathbf{g})}\in\mathcal{B}-\mathcal{T}$ *or*
- $\mathbf{g}\sim_2 1$, $x_i^{(1,\mathbf{g})}\notin\mathcal{M}_p$ *and* $x_i^{(1,\mathbf{g})}\notin\mathcal{B}-\mathcal{T}$,

then $f^{\mathcal{B}}_{\mathcal{T}}$ is a graded identity for $E_k\otimes E^{\otimes n}$.

Finally, let $\tilde{I}_{p,k}$ be the $(\mathbb{Z}_2\times\mathbb{Z}_2^n)$-graded ideal generated by the polynomials in Theorem 19 and the ones in Propositions 26 and 32.

Theorem 33. *Let F be an infinite field of characteristic $p>2$. Then $T_{\mathbb{Z}_2\times\mathbb{Z}_2^n}(E_k\otimes E^{\otimes n})=\tilde{I}_{p,k}$.*

Moreover, if $\mathcal{B}$ is an adequate multiset, then a basis for $P_{\mathcal{B}}(E_k\otimes E^{\otimes n})$ is given by the polynomials $f^{\mathcal{B}}_{\mathcal{T}}$ such that:

- *for each $(g,\mathbf{g})\neq(0,0)$ and $1\leq i\leq n_{(g,\mathbf{g})}$ such that $m_{\mathcal{B}}(x_i^{(g,\mathbf{g})})=p$, we have $x_i^{(g,\mathbf{g})}\in\mathcal{T}$ if $\mathbf{g}\sim_2 0$ whereas $x_i^{(g,\mathbf{g})}\in\mathcal{B}-\mathcal{T}$ if $\mathbf{g}\sim_2 1$;*
- *$r_{\mathcal{T}}+s_{\mathcal{T}}\leq k+1$;*
- *if $r_{\mathcal{T}}+s_{\mathcal{T}}=k+1$, then $s_{\mathcal{T}}\neq 0$. Let $(\mathbf{g}_0,i_0)$ be the smallest element in $\mathbb{Z}_2^n\times\mathbb{N}$ such that $x_{i_0}^{(1,\mathbf{g}_0)}\in\mathcal{B}$ and $x_{i_0}^{(1,\mathbf{g}_0)}\notin\mathcal{M}_p$. Then $x_{i_0}^{(1,\mathbf{g}_0)}\notin\mathcal{T}$ if $\mathbf{g}\sim_2 0$ whereas $x_{i_0}^{(1,\mathbf{g}_0)}\in\mathcal{B}-\mathcal{T}$ if $\mathbf{g}\sim_2 1$.*

Proof. The fact that the polynomials above form a system of generators of $P_{\mathcal{B}}(E_k\otimes E^{\otimes n})$ follows from Theorem 21, Propositions 26 and 32 and the techniques used in [13].

For the linear independence we shall argue as in the proof of Theorems 30 and 31 using the canonical substitution for all variables, except for the variable $x_{i_0}^{(1,\mathbf{g}_0)}$ whenever $r_{\mathcal{T}}+s_{\mathcal{T}}=k+1$. In this case we take $\overline{x}_{i_0}^{(1,\mathbf{g}_0)}=\tau_{i_0}^{(1,\mathbf{g}_0)}$, where $\tau_{i_0}^{(1,\mathbf{g}_0)}$ is an $S_{m,h}^{(1,\mathbf{g}_0)}$-element with $m=m_{\mathcal{B}}(x_{i_0}^{(1,\mathbf{g}_0)})$ and

$$h=\begin{cases} m-1 & \text{if } \mathbf{g}\sim_2 0\\ 0 & \text{if } \mathbf{g}\sim_2 1.\end{cases}$$

Note that this substitution fits in our computation because we need $m_{\mathcal{T}}(x_i^{(0,\mathbf{g})})$ distinct elements $e_j\in E^0_{gr}$ for a substitution of the type $\overline{x}_i^{(0,\mathbf{g})}$, $m_{\mathcal{B}-\mathcal{T}}(x_i^{(1,\mathbf{g})})$ distinct elements $e_j\in E^0_{gr}$ for a substitution of the type $\overline{x}_i^{(1,\mathbf{g})}$ if either $r_{\mathcal{T}}+s_{\mathcal{T}}\leq k$ or $r_{\mathcal{T}}+s_{\mathcal{T}}=k+1$ and $x_i^{(1,\mathbf{g})}\neq x_{i_0}^{(1,\mathbf{g}_0)}$, whereas $m_{\mathcal{B}-\mathcal{T}}(x_{i_0}^{(1,\mathbf{g}_0)})-1$ distinct elements $e_j\in E^0_{gr}$ for the remaining case. Finally, we need a total of $r_{\mathcal{T}}+s_{\mathcal{T}}$ distinct elements $e_j\in E^0_{gr}$ if $r_{\mathcal{T}}+s_{\mathcal{T}}\leq k$ and $r_{\mathcal{T}}+s_{\mathcal{T}}-1$ distinct elements $e_j\in E^0_{gr}$ if $r_{\mathcal{T}}+s_{\mathcal{T}}=k+1$. □

We want to spend some words towards the growth of the relatively free graded algebras of $E_{gr}\otimes E^{\otimes n}$. In [6] we can find a survey on graded Gelfand-Kirillov dimension of graded relatively free algebras. In [9] the authors made the following conjecture.

Conjecture 34. *Let A be a G-graded PI-algebra over an infinite field and let m be a non-negative integer. Then*

$$\mathrm{GKdim}_m^G(A)=\mathrm{GKdim}_m^{G\times\mathbb{Z}_2}(A\otimes E).$$

Comparing Theorems 30, 31 and 33 with the results in [5] we have another generalized result in that direction. In particular, we get the next.

Corollary 35. *Let $k \geq 0$ $n, m \geq 1$ be integer and F be an infinite field. Then the following equalities hold.*

$$\mathrm{GKdim}_m^{\mathbb{Z}_2\times\mathbb{Z}_2^n}(E_\infty \otimes E^{\otimes n}) = \mathrm{GKdim}_m^{\mathbb{Z}_2}(E_\infty) = 2m,$$
$$\mathrm{GKdim}_m^{\mathbb{Z}_2\times\mathbb{Z}_2^n}(E_{k^*} \otimes E^{\otimes n}) = \mathrm{GKdim}_m^{\mathbb{Z}_2}(E_{k^*}) = m,$$
$$\mathrm{GKdim}_m^{\mathbb{Z}_2\times\mathbb{Z}_2^n}(E_k \otimes E^{\otimes n}) = \mathrm{GKdim}_m^{\mathbb{Z}_2}(E_k) = m.$$

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