This is the pre-print of the paper
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NONLINEAR ANALYSIS 119 (2015) 443-456
https://dx.doi.org/10.1016/j.na.2014.10.037

CRITICAL BEHAVIOUR FOR THE POLYHARMONIC OPERATOR WITH HARDY POTENTIAL

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Abstract. Let us consider the Dirichlet problem

$$\begin{cases} \mathcal{L}_\mu[u] := (-\Delta)^m u - \mu \dfrac{u}{|x|^{2m}} = u^{2^*-1} + \lambda u, \qquad u > 0 \text{ in } B \\ D^\beta u|_{\partial B} = 0 \qquad \text{for } |\beta| \leq m-1 \end{cases}$$

where B is the unit ball in $\mathbb{R}^n$, $n > 2m$, $2^* = 2n/(n-2m)$. We find that, whatever n may be, this problem is critical (in the sense of Pucci–Serrin and Grunau) depending on the value of $\mu \in [0, \overline{\mu})$, $\overline{\mu}$ being the best constant in Rellich inequality. The present work extends to the perturbed operator $(-\Delta)^m - \mu|x|^{-2m} I$ a well–known result by Grunau regarding the polyharmonic operator (see [6]).

1. Introduction

The present paper deals with non–existence results for weak solutions to the problem

$$\begin{cases} \mathcal{L}_\mu[u] := (-\Delta)^m u - \mu \dfrac{u}{|x|^{2m}} = u^{2^*-1} + \lambda u, \qquad u > 0 \text{ in } B \\ u \in H^m_{0,r}(B) \end{cases} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^n$, $n \geq 2m+1$ and $H^m_{0,r}(B)$ is the space of the functions $v \in H^m_0(B)$ with spherical symmetry.

Throughout this paper we shall assume that $0 \leq \mu < \overline{\mu}$, where $\overline{\mu}$ is the best constant for the Rellich inequality (see the Notations below)

$$\int_{\mathbb{R}^n} |D^m u|^2 \, dx \geq \overline{\mu} \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2m}} \, dx \qquad \forall\, u \in \mathcal{D}^{m,2}(\mathbb{R}^n) \tag{1.2}$$

which is not achieved by any $u \in \mathcal{D}^{m,2}(\mathbb{R}^n)$ (see [3], [12]). Being $\mu < \overline{\mu}$, $\mathcal{L}_\mu$ is positive defined.

Let us set

$$P_\mu(z) = (-1)^m \prod_{i=1}^{m} (z + n - 2i)(z + 2 - 2i) - \mu\,; \tag{1.3}$$

this polynomial will play a crucial role in all our discussion, as it is a sort of "symbol" for $\mathcal{L}_\mu$. We know from [3], [12] that

$$\overline{\mu} = P_0(m - n/2) = (-4)^m (1 - m/2 - n/4)^{\overline{m}} (n/4 - m/2)^{\overline{m}} \tag{1.4}$$

2010 *Mathematics Subject Classification.* 35J60, 31B30, 35B33.

Key words and phrases. Polyharmonic problems, Hardy perturbation, Critical behaviour, Pohozaev identity.

where $a^{\overline{h}} := \prod_{j=0}^{h-1}(a+j)$ (see the Notations below).

The behavior of problem (1.1) is deeply influenced by the amount of μ, and we shall obtain non–existence results depending on μ and λ.

More precisely, let us define

$$\mu_1 := \begin{cases} P_0(-n/2) = (-4)^m(1-n/4)^{\overline{m}}(n/4)^{\overline{m}} & n \geq 4m+1; \\ 0 & 2m+1 \leq n \leq 4m. \end{cases} \tag{1.5}$$

An elementary investigation about $P_0(x)$ for $x \in [2m-n, 0]$ shows that $0 \leq \mu_1 < \overline{\mu}$.

Definition 1. *We say that μ is* critical *for $\mathcal{L}_\mu$ if $\mu_1 < \mu < \overline{\mu}$ when $n \geq 4m$, or $\mu_1 \leq \mu < \overline{\mu}$ when $2m+1 \leq n \leq 4m-1$.*

In other words, any $\mu \in [0, \overline{\mu})$ is critical when $2m+1 \leq n \leq 4m-1$; any $\mu \in (\mu_1, \overline{\mu})$ is critical for $n \geq 4m$.

Now we may state our theorem.

Theorem 1. *If μ is critical for $\mathcal{L}_\mu$, then there exists $\lambda_* = \lambda_*(\mu, n) > 0$ such that for $\lambda < \lambda_*$ problem* (1.1) *admits no nontrivial positive radial weak solutions in $H_0^m(B)$.*

A few words of comment. Theorem 1 generalizes to the case of problem (1.1) the well-known result by Grunau (see [6]) regarding the case $\mu = 0$, i.e. when the linear operator is the polyharmonic operator $(-\Delta)^m$, and indeed, when possible, we have tried to transpose to our case Grunau's reasonment, which in turn originates from Theorem 1.2" of [1].

In [6] Grunau shows that, when $n = 2m+1 \ldots 4m-1$, $(-\Delta)^m$ has a critical behavior, which means that there exists $\lambda_* > 0$ such that the critical problem for $(-\Delta)^m$ has not *positive* radial solution for $\lambda < \lambda_*$; this was a considerable step forward in proving the well-known conjecture by Pucci-Serrin (see [13]), which states the same claim, but without the restriction of the positivity of u.

Now, if we consider the fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^n$, i.e. $|x|^{2m-n}$, we may remark that $|x|^{2m-n}$ belongs to L^2_{loc} iff $n = 2m+1 \ldots 4m-1$. In the light of the results of Pucci-Serrin and Grunau, this is not a coincidence: in [7] it is shown for some classes of problems, each class depending on a continuous parameter, that

critical behavior occours when the (generalized) fundamental solution (depending on the parameter) belongs to L^2_{loc}.

For more detailed motivation of this principle we refer to [7]; what is relevant here is that this principle applies in the present work. To see this, let us remark (see Section 2) that $|x|^\sigma$ solves $\mathcal{L}_\mu[|x|^\sigma] = 0$ in $\mathbb{R}^n \setminus \{0\}$ iff $P_\mu(\sigma) = 0$; now, if we denote by $\beta_1 = \beta_1(\mu)$ the continuous branch among the roots of P_μ which starts from $2m-n$ when $\mu = 0$, we may reasonably call $|x|^{\beta_1}$ the (generalized) fundamental solution of $\mathcal{L}_\mu$. Then it is easy to see that μ is critical in the sense of our Definition 1 iff $\beta_1 > -n/2$, which means that the (generalized) fundamental solution is in L^2_{loc}.

When $m = 1, 2$, the nonlinear critical problem for $\mathcal{L}_\mu$ has been extensively studied in [7], [2] respectively, where the analogous of Theorem 1 has been proved in a stronger version: namely it is proved that, when μ is critical, there exist no nontrivial radial solutions u for $\lambda > 0$ sufficiently small, without any assumption about the positivity of u (indeed the theorem in [7] is enounced for positive solutions, but from the proof it is evident that the theorem holds for any radial solution). This is achieved by means of sharp radial Pohozaev identities and, when $m = 2$, suitable Hardy inequalities; this technique does not seem to apply to $\mathcal{L}_\mu$ for general m.

Another remark: many results about critical behavior of nonlinear critical problems state nonexistence theorems of *classical* solutions. But in our case, when $\mu > 0$, we must face singular (hence weak) solutions, which in general have a pole at the origin. This, among other technicalities, leads to state a Pohozaev identity for weak solutions (see Section 4) in a ball.

This paper is organized as follows: in Section 2 we give an explicit representation formula for the solution to the linear problem

$$\begin{cases} \mathcal{L}_\mu[u] = f & \text{in } B\,, \\ u \in H^m_{0,r}(B) \end{cases} \tag{1.6}$$

in terms of the roots of the polynomial $P_\mu(z)$. Section 3 is devoted to the study of the auxiliary function w_μ, which solves the problem

$$\begin{cases} \mathcal{L}_\mu[w_\mu] = 0 & \text{in } B \\ w_\mu \in H^m_r(B) \\ w_\mu(1) = \cdots = w_\mu^{(m-2)}(1) = 0, w_\mu^{(m-1)}(1) = (-1)^{m-1}\,, \end{cases} \tag{1.7}$$

where $H^m_r(B)$ is the closed subspace of the functions of $H^m(B)$ with spherical symmetry; by coupling problems (1.6) and (1.7) we shall get useful estimates about u when f is the right hand side of problem (1.1).

In Section 4 we derive a Pohozaev identity for weak solutions to (1.1), and finally in Section 5 we collect together all the informations, so proving Theorem 1.

Notations

u^* — The Schwarz symmetrization of u (see, for instance, [10]).

2^* — $\dfrac{2n}{n-2m}$, the limit exponent for the Sobolev embedding $H^m(\Omega) \subset L^p(\Omega)$;

$a^{\overline{h}}$ — *Rising factorial power* (see [5]). For $a \in \mathbb{R}$ and h non negative integer it is defined as

$$a^{\overline{h}} = \begin{cases} \displaystyle\prod_{j=0}^{h-1}(a+j) & m \geq 1\,; \\ 1 & h = 0\,. \end{cases}$$

$a^{\underline{h}}$ — *Falling factorial power* (see [5]). For $a \in \mathbb{R}$ and h non negative integer it is defined as

$$a^{\underline{h}} = \begin{cases} \displaystyle\prod_{j=0}^{h-1}(a-j) & m \geq 1\,; \\ 1 & h = 0\,. \end{cases}$$

ρ^z — For any $\rho > 0$ and $z = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ we set

$$\rho^z = \rho^\alpha \big(\cos(\beta \log \rho) + i \sin(\beta \log \rho)\big)\,.$$

$H(x)$ — *Heaviside function.* $H(x) = 1$ for $x \geq 0$, $H(x) = 0$ for $x < 0$.

$\delta_\rho(s)$ — *One dimensional Dirac delta at point* ρ. The distribution $\delta_\rho(s)$ is defined as $\langle \delta_\rho(s), \varphi(s) \rangle = \varphi(\rho)$ for any test function $\varphi \in C_c(\mathbb{R})$.

$D^m u$ — $\Delta^{m/2}u$ if m is even; $\nabla\Delta^{(m-1)/2}u$ if m is odd.

$\|u\|_{m,2}$ — $\|D^m u\|_{L^2(\Omega)}$

$\mathcal{D}^{m,2}(\mathbb{R}^n)$ — The completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|_{m,2}$.

$H^m(\Omega)$ — Hilbertian Sobolev space of the m–times weakly differentiable functions in Ω with L^2 derivatives.

$H_0^m(\Omega)$ — In bounded domains Ω, the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|u\|_{m,2}$.

B — $B = \{x \in \mathbb{R}^n : |x| < 1\}$

X_r — If X is any function space on $\mathbb{R}^n$ or B, X_r is the subspace of the functions in X with spherical symmetry.

$\Gamma(z)$ — Euler's Gamma function; $\Gamma(z) = \displaystyle\int_0^\infty e^{-t} t^{z-1}\, dt \qquad (z \in \mathbb{C},\ \mathrm{Re}\, z > 0)$.

ω_n — The $n-1$ dimensional measure of the euclidean sphere $\mathbb{S}^n$; $\omega_n = \dfrac{n\pi^{n/2}}{\Gamma(1+\frac{n}{2})}$.

2. The linear radial problem

The main goal of this section is to give an explicit representation formula for the solution to the equation

$$\begin{cases} (-\Delta)^m u - \mu \dfrac{u}{|x|^{2m}} = f \text{ in } B \\ u \in H^m_{0,r}(B) \end{cases} \tag{2.1}$$

where $f \in L_r^{2^{*\prime}}(B)$. We shall achieve this kind of results by means of elementary ODE techniques.

Lemma 2.1. *Let $z_1, z_2 \dots z_k$ be k distinct complex numbers. Then:*

$$\sum_{i=1}^{k} \frac{z_i^h}{\prod\limits_{j\neq i}(z_i - z_j)} = \begin{cases} 0 & 0 \leq h \leq k-2 \\ 1 & h = k-1 \\ z_1 + \cdots + z_m & h = k\,; \end{cases} \tag{2.2}$$

$$\sum_{i=1}^{k} \frac{z_i^{\underline{h}}}{\prod\limits_{j\neq i}(z_i - z_j)} = \begin{cases} 0 & 0 \leq h \leq k-2 \\ 1 & h = k-1 \\ z_1 + \cdots + z_m - \dfrac{k(k-1)}{2} & h = k\,. \end{cases} \tag{2.3}$$

Proof. As for (2.2) see [11], Section 1.2.3, Exercise 33. Equation (2.3) is a trivial consequence of (2.2) by means of linear combinations. □

Definition 2. *Let*

$$\Gamma = \frac{\mathrm{d}^k}{\mathrm{d}\rho^k} + \sum_{i=1}^{k} \frac{a_i}{\rho^i} \frac{\mathrm{d}^{k-i}}{\mathrm{d}\rho^{k-i}} \tag{2.4}$$

be an homogeneous linear differential operator defined for $\rho \in (0, \infty)$ with coefficients $a_i \in \mathbb{R}$. We define the symbol *of Γ as the polynomial*

$$P(z) = z^{\underline{k}} + \sum_{i=1}^{k} a_i z^{\underline{k-i}} \tag{2.5}$$

and we call characteristic roots *its (real or conjugate complex) roots.*

Remark 1. We obviously have that $\Gamma[\rho^z] = 0$ if and only if $P(z) = 0$. Moreover, if z is a root with multiplicity p, then $\Gamma[\rho^z \log \rho] = \cdots = \Gamma[\rho^z (\log \rho)^{p-1}] = 0$.

Proposition 2.2. *Let Γ as in Definition 2, and let us suppose that its symbol P has distinct roots $z_1, z_2 \dots z_k$. Finally, let*

$$\psi(\rho, s) = -\left(\sum_{i=1}^{k} \frac{\rho^{z_i} s^{k-1-z_i}}{\prod\limits_{j\neq i}(z_i - z_j)} \right) H(s - \rho)\,, \qquad s > 0\,. \tag{2.6}$$

Then $\Gamma(\psi(\rho, s)) = \delta_\rho(s)$; moreover, if $k \geq 2$, then $\psi(\rho, s)$ is of C^{k-2} class with respect to ρ for any fixed $s > 0$.

Proof. Let us define

$$g(\rho,s)=-\sum_{i=1}^{k}\frac{\rho^{z_i}s^{k-1-z_i}}{\prod\limits_{j\neq i}(z_i-z_j)}\,; \tag{2.7}$$

then, by (2.3), we get

$$\frac{\partial^h g}{\partial\rho^h}(\rho,\rho)=\begin{cases}0 & 0\le h\le k-2\\ -1 & h=k-1\end{cases} \tag{2.8}$$

and therefore

$$\frac{\partial^h\psi}{\partial\rho^h}(\rho,s)=\begin{cases}\dfrac{\partial^h g}{\partial\rho^h}(\rho,s)H(s-\rho) & 0\le h\le k-1\,;\\[2ex] \dfrac{\partial^k g}{\partial\rho^k}(\rho,s)H(s-\rho)+\delta_\rho(s) & h=k\,.\end{cases} \tag{2.9}$$

Hence

$$\Gamma(\psi(\rho,s))=-\left(\sum_{i=1}^{k}\frac{\Gamma(\rho^{z_i})s^{k-1-z_i}}{\prod\limits_{j\neq i}(z_i-z_j)}\right)H(s-\rho)+\delta_\rho(s)=\delta_\rho(s)\,.$$

As for the regularity of $\psi(\rho,s)$ with respect to ρ, it is an immediate consequence of (2.8). ☐

Remark 2. Obviously the same conclusion of Proposition 2.2 holds for any function $\psi_1(\rho,s)=\psi(\rho,s)+\sum_{i=1}^{k}f_i(s)\rho^{z_i}$ with arbitrary $f_i(s)$.

Remark 3. When double roots occour, (2.6) must be modified accordingly. If, say, $z_2=z_1$, then (2.6) becomes

$$\begin{aligned}\psi(\rho,s)=\Bigg(&-\frac{\rho^{z_1}s^{k-1-z_1}}{\prod\limits_{j\ge3}(z_1-z_j)}\Big(\sum_{j\ge3}\frac{1}{z_1-z_j}-\log\frac{\rho}{s}\Big)\\ &-\sum_{i=3}^{k}\frac{\rho^{z_i}s^{k-1-z_i}}{(z_1-z_i)^2\prod\limits_{j\ge3}(z_i-z_j)}\Bigg)H(s-\rho)\,,\qquad s>0\,.\end{aligned} \tag{2.10}$$

Of course (2.10) is nothing but the limit of (2.6) for $z_2\to z_1$.

From now on we shall denote by $\mathcal{L}_\mu$ *both* the partial differential operator in cartesian coordinates $(-\Delta)^m-\frac{\mu}{|x|^{2m}}I$ and the ordinary differential operator

$$\Big(-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}-\frac{(n-1)}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^m-\frac{\mu}{\rho^{2m}}I\,.$$

From the context it will be always clear if we consider $\mathcal{L}_\mu$ as a PDE or a ODE operator.

We specialize Proposition 2.2 (together with Remark 2) to the case $\Gamma = \mathcal{L}_\mu$. To this aim, we must discuss about the characteristic roots of $\mathcal{L}_\mu$. The following assertions, contained in Remarks 4-8, are quite easily verified:

Remark 4. Let

$$P_\mu(z) = (-1)^m \prod_{i=1}^{m} (x+n-2i)(x+2-2i) - \mu\,; \tag{2.11}$$

then P_μ is the symbol of $\mathcal{L}_\mu$.

Remark 5. The line $x = m - n/2$ plays a relevant role, as $\overline{\mu} = P_0(m - n/2)$ and $P_\mu(z)$ has the following symmetry property:

$$P_\mu(z) = P_\mu(2m - n - z)\,. \tag{2.12}$$

Hence $P_\mu(z)$ has m roots in the half plane $\mathrm{Re}(z) \geq m - n/2$ and m roots in the half plane $\mathrm{Re}(z) \leq m - n/2$.

Remark 6. The roots of $P_\mu(z)$ are all real for $\mu > 0$ sufficiently small. When μ increases, up to $m-1$ pairs of complex conjugate roots may appear, their number depending on n, m, μ. Anyway, for any $\mu \in [0, \overline{\mu})$ there is one and only one real root of $P_\mu(z)$ in the interval $(m - n/2, 0]$. Hence, when $\mu \in [0, \overline{\mu})$ we set

$$\begin{array}{ll} \alpha_1 & \text{the unique real root in } (m-n/2, 0]; \\ \alpha_i\,,\ i = 2 \dots m & \text{the remaining } m-1 \text{ roots} \\ & \text{in the half plane } \mathrm{Re}(z) \geq m - n/2; \\ \beta_1 = n - 2m - \alpha_1 & \text{the unique real root in } [2m-n, m-n/2); \\ \beta_i = n - 2m - \alpha_i\,,\ i = 2 \dots m & \text{the remaining } m-1 \text{ roots} \\ & \text{in the half plane } \mathrm{Re}(z) \leq m - n/2. \end{array} \tag{2.13}$$

Remark 7. $P_0'(z)$ has $2m-1$ distinct real roots. Being $P_\mu'(z) = P_0'(z)$, $P_\mu(z)$ has all distinct roots but for a finite number of values of μ. Coincident roots may only be real ones, and they may have only multiplicity equal to 2. When $\mu \in (0, \overline{\mu})$, this may happen only if $m \geq 3$ and $\mu = P_0(\overline{x}_k)$, where $\overline{x}_k$ is the unique root of $P_0'(x)$ in the interval $[4k-2, 4k]$, $1 \leq k \leq (m-1)/2$. In particular this implies that, when $\mu \in [0, \overline{\mu})$,

$$\begin{aligned} &\alpha_1, \mathrm{Re}(\alpha_2) \dots \mathrm{Re}(\alpha_m) > m - n/2\,; \\ &\beta_1, \mathrm{Re}(\beta_2) \dots \mathrm{Re}(\beta_m) < m - n/2\,. \end{aligned} \tag{2.14}$$

Remark 8. It holds $\mu_1 = \max\{0, P_0(-n/2)\}$. Hence μ is critical for $\mathcal{L}_\mu$ iff $\beta_1 > -n/2$.

Figure 1 shows the graph of $P_0(x)$ compared with different values of μ, which provides a visual representation of some of the preceeding remarks.

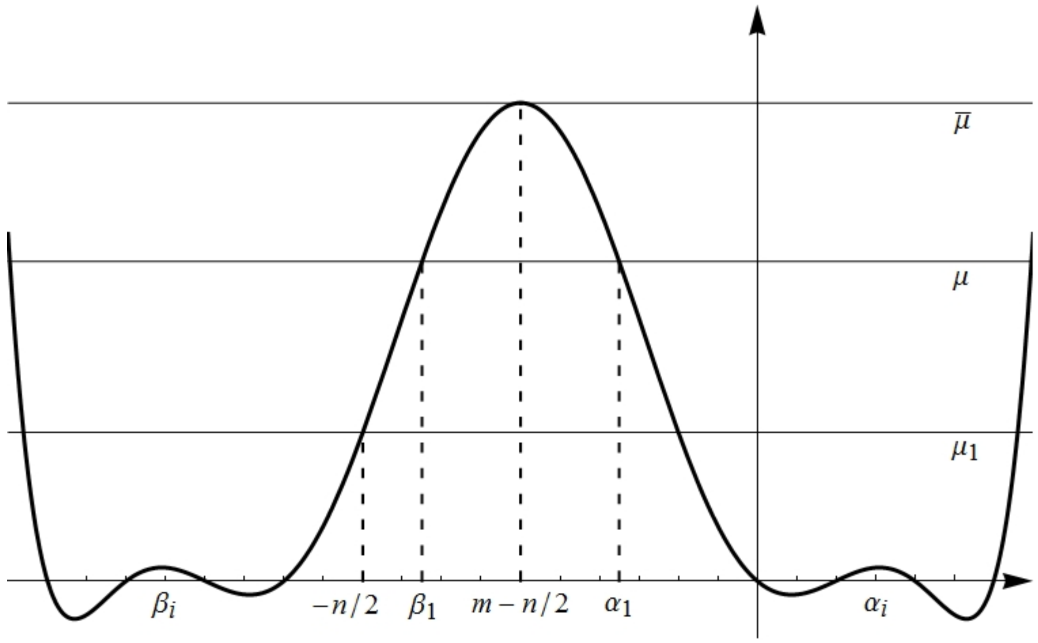

FIGURE 1. $P_0(x) = \mu$

Now we want to prove some more subtle properties regarding the location of the roots of $P_\mu(z)$. To this aim we need the following

Proposition 2.3. *Let $f \in L_r^{2^{*\prime}}(\mathbb{R}^n)$. Then the problem*

$$(-\Delta)^m u - \frac{\mu}{|x|^{2m}} u = f\,; \qquad u \in \mathcal{D}_r^{m,2} \tag{2.15}$$

has one and only one solution. Moreover, if f is positive and decreasing, then u is positive and decreasing too.

Proof. The solution u to (2.15) is the unique minimum for the strictly convex functional $J : \mathcal{D}_r^{m,2} \to \mathbb{R}$ defined by

$$J[w] = \frac{1}{2}\int_{\mathbb{R}^n} |D^m w|^2 - \frac{\mu}{2}\int_{\mathbb{R}^n} \frac{u^2}{|x|^{2m}} - \int_{\mathbb{R}^n} f u\,. \tag{2.16}$$

Now let f be positive and decreasing, so that $f = f^*$, where $(\quad)^*$ denotes the Schwarz symmetrization. Let $\{u_n\} \subset \mathcal{D}_r$ be a sequence of smooth functions such that $u_n \to u$ in $\mathcal{D}_r^{m,2}$, and let v_n in $\mathcal{D}_r^{m,2}$ such that

$$(-\Delta)^k v_n = ((-\Delta)^k u_n)^*\,, \quad k = \begin{cases} m/2 & m \text{ even} \\ (m-1)/2 & m \text{ odd} \end{cases} \tag{2.17}$$

Arguing as in [9], by means of Talenti comparison theorem (see [15]) we know that $v_n \geq u_n^*$, hence

$$
\begin{aligned}
&\|D^m v_n\|_2^2 = \|D^m u_n\|_2^2 \quad (m \text{ even}); \qquad \|D^m v_n\|_2^2 \leq \|D^m u_n\|_2^2 \quad (m \text{ odd});\\
&\int_{\mathbb{R}^n} \frac{v_n^2}{|x|^{2m}} \geq \int_{\mathbb{R}^n} \frac{u_n^2}{|x|^{2m}}\,; \qquad \int_{\mathbb{R}^n} f v_n \geq \int_{\mathbb{R}^n} f u_n^* = \int_{\mathbb{R}^n} f^* u_n^* \geq \int_{\mathbb{R}^n} f u_n\,;
\end{aligned}
\tag{2.18}
$$

therefore $J[v_n] \leq J[u_n]$. Being Schwarz symmetrization non expansive, v_n is a Cauchy sequence in $\mathcal{D}_r^{m,2}$; if $v = \lim_n v_n$, we have that $v = u$, and therefore u is positive and decreasing.

□

Now we give a representation for the solution u to (2.15) in terms of the characteristic roots of $P_\mu(z)$:

Proposition 2.4. *Let $\mu \in [0, \overline{\mu})$ be such that $P_\mu(z)$ has distinct roots, and let*

$$
\begin{aligned}
g_1(\rho, s) &= (-1)^m \sum_{i=1}^m \frac{\rho^{\beta_i} s^{2m-1-\beta_i}}{\prod_{j\neq i}(\beta_i - \beta_j) \prod_j (\beta_i - \alpha_j)}\\
g_2(\rho, s) &= (-1)^{m-1} \sum_{i=1}^m \frac{\rho^{\alpha_i} s^{2m-1-\alpha_i}}{\prod_{j\neq i}(\alpha_i - \alpha_j) \prod_j (\alpha_i - \beta_j)}
\end{aligned}
\tag{2.19}
$$

where α_i, β_i are defined in (2.13)*. Then the solution u to* (2.15) *is given by*

$$
u(\rho) = \int_0^\rho g_1(\rho, s) f(s)\, ds + \int_\rho^\infty g_2(\rho, s) f(s)\, ds\,, \qquad \rho > 0\,. \tag{2.20}
$$

Proof. Let us suppose at first that $f \in \mathcal{D}_r$. By means of Proposition 2.2 we easily get that

$$
\mathcal{L}_\mu[(g_2(\rho, s) - g_1(\rho, s)) H(s - \rho)] = \delta_\rho(s) \tag{2.21}
$$

and therefore (see also Remark 2)

$$
\mathcal{L}_\mu[g_1(\rho, s) + (g_2(\rho, s) - g_1(\rho, s)) H(s - \rho)] = \delta_\rho(s)\,; \tag{2.22}
$$

hence the function $u(\rho)$ defined by (2.20) is a solution in the interval $(0, \infty)$ to the ODE $\mathcal{L}_\mu[u] = f$, and all the other solutions to this ODE are given by $u + \sum_i c_i' \rho^{\beta_i} + c_i'' \rho^{\alpha_i}$ for arbitrary constants c_i', c_i''. We want to show that the solution to (2.15) is the one with $c_i' = c_i'' = 0$ for $i = 1 \ldots m$, i.e. it is $u(\rho)$ as defined in (2.20). To this aim, let us estimate u when $\rho \to 0$ and when $\rho \to \infty$.

Remembering that f is bounded and that $\mathrm{Re}(\beta_i) < 0$, from

$$
\left| \int_0^\rho \rho^{\beta_i} s^{2m-1-\beta_i} f(s)\, ds \right| \leq \int_0^\rho \left(\frac{s}{\rho}\right)^{-\mathrm{Re}(\beta_i)} s^{2m-1} |f(s)|\, ds \tag{2.23}
$$

we get

$$
\int_0^\rho g_1(\rho, s) f(s)\, ds = O(\rho^{2m}) \qquad (\rho \to 0)\,. \tag{2.24}
$$

As for g_2, let us distinguish between $\mathrm{Re}(\alpha_i) \geq 0$ and $\mathrm{Re}(\alpha_i) < 0$. In the first case we have

$$\left|\int_\rho^\infty \rho^{\alpha_i} s^{2m-1-\alpha_i} f(s)\,ds\right| \leq \int_\rho^\infty \left(\frac{\rho}{s}\right)^{\mathrm{Re}(\alpha_i)} s^{2m-1}|f(s)|\,ds \leq \int_0^\infty s^{2m-1}|f(s)|\,ds = C \tag{2.25}$$

while, when $\mathrm{Re}(\alpha_i) < 0$, we can argue as follows:

$$\begin{aligned} &\left|\int_\rho^\infty \rho^{\alpha_i} s^{2m-1-\alpha_i} f(s)\,ds\right| = \rho^{\alpha_i}\left|\int_\rho^\infty s^{2m-1-\alpha_i} f(s)\,ds\right| \\ &\leq \rho^{\alpha_i}\int_0^\infty s^{2m-1-\mathrm{Re}(\alpha_i)}|f(s)|\,ds = C\rho^{\alpha_i}\,. \end{aligned} \tag{2.26}$$

Setting $\gamma_1 = \min\{\alpha_1, \mathrm{Re}(\alpha_2)\ldots\mathrm{Re}(\alpha_m)\}$, we can conclude from the preceeding relations that

$$u(\rho) = O(\rho^{\gamma_1}) \qquad (\rho \to 0)\,. \tag{2.27}$$

Now let us estimate $u(\rho)$ for $\rho \to \infty$. Having f compact support we get

$$\int_\rho^\infty g_2(\rho, s) f(s)\,ds = 0 \qquad \text{for } \rho \text{ sufficiently large;} \tag{2.28}$$

on the other hand

$$\begin{aligned} &\left|\int_0^\rho \rho^{\beta_i} s^{2m-1-\beta_i} f(s)\,ds\right| = \rho^{\beta_i}\left|\int_0^\rho s^{2m-1-\beta_i} f(s)\,ds\right| \\ &\leq \rho^{\beta_i}\int_0^\infty s^{2m-1-\mathrm{Re}(\beta_i)}|f(s)|\,ds = C\rho^{\beta_i}\,. \end{aligned} \tag{2.29}$$

Setting $\gamma_2 = \max\{\beta_1, \mathrm{Re}(\beta_2)\ldots\mathrm{Re}(\beta_m)\}$, we can conclude

$$u(\rho) = O(\rho^{\gamma_2}) \qquad (\rho \to \infty)\,. \tag{2.30}$$

Summing up, $u(\rho)$ is a continuous function on $(0, \infty)$ which verifies (2.27) and (2.30); being $\gamma_2 < m - n/2 < \gamma_1$, we get that $u(x) \in L^{2^*}(\mathbb{R}^n)$, and indeed the solution to equation (2.15) has this summability, belonging to $\mathcal{D}^{m,2}$. Hence the conclusion of the proof in the case $f \in \mathcal{D}_r$ follows by observing that $u + \sum_i c_i' \rho^{\beta_i} + c_i'' \rho^{\alpha_i}$ belongs to $L^{2^*}(\mathbb{R}^n)$ if and only if $c_i' = c_i'' = 0$ for $i = 1 \ldots m$.

Now we want to pass to the general case $f \in L_r^{2^{*\prime}}(\mathbb{R}^n)$. Let $\{f_n\} \subset \mathcal{D}_r$ such that $f_n \to f$ in $L_r^{2^{*\prime}}(\mathbb{R}^n)$, and let us denote by u_n, u the solutions to (2.15) with right hand side equal to f_n, f respectively; then, in particular, $u_n(\rho) \to u(\rho)$. Hence, if we show that for any $\rho > 0$

$$\begin{aligned} &\int_0^\rho g_1(\rho, s) f_n(s)\,ds \to \int_0^\rho g_1(\rho, s) f(s)\,ds \\ &\int_\rho^\infty g_2(\rho, s) f_n(s)\,ds \to \int_\rho^\infty g_2(\rho, s) f(s)\,ds \end{aligned} \tag{2.31}$$

we are done. But (2.31) holds true; indeed, by (2.14) we know that, for any $\rho > 0$,

$$|x|^{2m-n-\beta_i} \in L^{2^*}(\{x : |x| < \rho\})\,; \quad |x|^{2m-n-\alpha_i} \in L^{2^*}(\{x : |x| > \rho\})\,; \tag{2.32}$$

therefore

$$\begin{aligned}&\left|\int_0^\rho s^{2m-1-\beta_i}(f(s)-f_n(s))\,ds\right| \leq \\ &\int_0^\rho s^{2m-n-\mathrm{Re}(\beta_i)}|f(s)-f_n(s)|s^{n-1}\,ds \leq C\|f-f_n\|_{2^{*\prime}}\,; \\ &\left|\int_\rho^\infty s^{2m-1-\alpha_i}(f(s)-f_n(s))\,ds\right| \leq \\ &\int_\rho^\infty s^{2m-n-\mathrm{Re}(\alpha_i)}|f(s)-f_n(s)|s^{n-1}\,ds \leq C\|f-f_n\|_{2^{*\prime}}\,.\end{aligned} \tag{2.33}$$

□

Remark 9. When $\mu = \tilde{\mu}$ is one of those (finite number of) values for which double real roots occour, the definition of g_1, g_2 in (2.19) must be modified; the expression of g_1, g_2 may be easily computated passing to the limit for $\mu \to \tilde{\mu}$ (see Remark 3). However this is not relevant at all in what follows.

As a consequence of Proposition 2.4 we get the following result of location of the characteristic roots:

Corollary 2.5. *The following estimate holds true:*

$$\alpha_1 \leq \mathrm{Re}(\alpha_i)\,, \quad \beta_1 \geq \mathrm{Re}(\beta_i)\,, \qquad i = 2\ldots m\,. \tag{2.34}$$

Proof. Let $f \in \mathcal{D}_r$ be positive and decreasing, and let u the solution to (2.15) for such f. From Proposition 2.3 we know that u is positive decreasing, while from Proposition 2.4 and its proof we know that

$$u(\rho) = \sum_{\mathrm{Re}(\alpha_i)<0} k_i \rho^{\alpha_i} \int_0^\infty s^{2m-1-\alpha_i} f(s)\,ds + O(1) \qquad (\rho \to 0) \tag{2.35}$$

where k_i are universal constants from (2.19). If there exists $j \geq 2$ such that $\mathrm{Re}(\alpha_j) < \alpha_1$, by choosing $f \in \mathcal{D}_r$ positive decreasing and such that $\int_0^\infty s^{2m-1-\alpha_i} f(s)\,ds \neq 0$, we get by (2.35) that u oscillates around the ρ-axis near $\rho = 0$, which is absurd. Quite analogous reasonment if two or more characteristic roots verify $\mathrm{Re}(\alpha_j) < \alpha_1$. Obviously the result about the β_i follows by symmetry. □

Now let us come to the representation formula for the solution to problem (2.1). All we have to do is to modify (2.20) (we may think $f \equiv 0$ for $\rho > 1$) in such a way that $u(1) = \cdots = u^{(m-1)}(1) = 0$.

To get this we must add to $g_1(\rho, s), g_2(\rho, s)$ a suitable linear combination of the functions $\rho^{\alpha_i}, \rho^{\beta_i}$ in such a way that the boundary conditions at $\rho = 1$ are fulfilled. So let us denote by $g_3(\rho, s)$ the term which will be added to $g_1(\rho, s), g_2(\rho, s)$; we have

$$g_3(\rho, s) = \sum_{i=1}^m c_i(s)\rho^{\alpha_i} + \sum_{i=1}^m d_i(s)\rho^{\beta_i} \tag{2.36}$$

and we want to determine $c_i(s), d_i(s)$ in such a way that

$$u(\rho) = \int_0^\rho g_1(\rho,s)f(s)\,ds + \int_\rho^1 g_2(\rho,s)f(s)\,ds + \int_0^1 g_3(\rho,s)f(s)\,ds\,, \qquad \rho\in(0,1]\,. \tag{2.37}$$

solves problem (2.1).

But u must belong to $H^m_{0,r}(B)$, and this implies that $d_1(s)=\cdots=d_m(s)=0$; now the boundary conditions $u(1)=\cdots=u^{(m-1)}(1)=0$ become m relations in the m unknown functions $c_i(s)$; namely

$$\sum_{i=1}^m \alpha_i^h c_i(s) + \left(\frac{\partial g_1}{\partial\rho}\right)^h (1,s) = 0\,, \qquad h=0\dots m-1\,. \tag{2.38}$$

If P_μ has all distinct roots, the coefficient matrix in system is invertible. After some calculations, we find

$$c_i(s) = (-1)^{m-1}\sum_j \frac{s^{2m-1-\beta_j}}{(\beta_j-\alpha_i)\prod_{h\neq i}(\alpha_i-\alpha_h)\prod_{h\neq j}(\beta_j-\beta_h)}\,. \tag{2.39}$$

Hence we get the following

Proposition 2.6. *Let $\mu\in[0,\overline{\mu})$ be such that $P_\mu(z)$ has distinct roots; let g_1, g_2 defined by* (2.19) *and*

$$g_3(\rho,s) = (-1)^{m-1}\sum_{i,j} \frac{s^{2m-1-\beta_j}\rho^{\alpha_i}}{(\beta_j-\alpha_i)\prod_{h\neq i}(\alpha_i-\alpha_h)\prod_{h\neq j}(\beta_j-\beta_h)} \tag{2.40}$$

where α_i, β_i are defined in (2.13). *Finally, let*

$$h(\rho,s) = g_1(\rho,s) + (g_2(\rho,s)-g_1(\rho,s))H(s-\rho) + g_3(\rho,s)\,. \tag{2.41}$$

Then the solution u to (2.1) *is given by*

$$u(\rho) = \int_0^1 h(\rho,s)\,f(s)\,ds\,, \qquad \rho\in(0,1]\,. \tag{2.42}$$

Remark 10. Again, as in Remark 9, when double real characteristic roots occour, formula (2.42) must be changed accordingly. This, however, has no relevance in what follows.

Proposition 2.7. *The solution $u(\rho)$ to* (2.1) *belongs to $C^{2m-1}\big((0,1]\big)$, and*

$$\left(\frac{\partial}{\partial\rho}\right)^j u(\rho) = \int_0^1 \left(\frac{\partial}{\partial\rho}\right)^j h(\rho,s)\,f(s)\,ds\,, \qquad \rho\in(0,1]\,. \tag{2.43}$$

Proof. The proof is similar to the proof of Proposition 2.2. Indeed, in our case (2.8) becomes

$$\left(\frac{\partial}{\partial\rho}\right)^j (g_2-g_1)\big|_{s=\rho} = 0\,, \qquad j=1\dots 2m-2\,; \tag{2.44}$$

therefore, in the distributional sense,

$$\begin{aligned}\Big(\frac{\partial}{\partial\rho}\Big)^j h(\rho,s) &= \Big(\frac{\partial}{\partial\rho}\Big)^j g_1(\rho,s) + \Big(\Big(\frac{\partial}{\partial\rho}\Big)^j g_2(\rho,s) - \Big(\frac{\partial}{\partial\rho}\Big)^j g_1(\rho,s)\Big) H(s-\rho) \\ &\quad + \Big(\frac{\partial}{\partial\rho}\Big)^j g_3(\rho,s)\,, \qquad 0 \le j \le 2m-1\,;\end{aligned} \tag{2.45}$$

now our claim easily follows. □

As a consequence we may state the following two propositions:

Proposition 2.8. *Let $\mu \in [0,\overline{\mu})$, and let u be a solution to (2.1), with $f \in L_r^{2^{*\prime}}(B)$ such that $\rho^{n/2+m} f(\rho)$ is bounded. Then, for any $\delta > 0$ $\exists C_\delta$ such that*

$$|u^{(h)}(\rho)| \le C_\delta \rho^{-n/2+m-h-\delta} \qquad 0 \le h \le 2m-1, \quad \rho \in (0,1]\,. \tag{2.46}$$

Proof. Let us fix the order of derivation h. We shall use (2.43) and the structure of $h(\rho,s)$. The right hand side of (2.43) is a linear combination of terms like

$$\rho^{\beta_i - h}\int_0^\rho s^{2m-1-\beta_i} f(s)\,ds\,, \quad \rho^{\alpha_i-h}\int_\rho^1 s^{2m-1-\alpha_i} f(s)\,ds\,, \quad \rho^{\alpha_i-h}\int_\rho^1 s^{2m-1-\beta_j} f(s)\,ds \tag{2.47}$$

which we shall estimate. Taking (2.14) into account, let $\delta > 0$ so small that $\beta_1 + \delta < -n/2+m$. We have

$$\begin{aligned}&\left|\rho^{\beta_i-h}\int_0^\rho s^{2m-1-\beta_i} f(s)\,ds\right| = \\ &\left|\rho^{-n/2+m-h-\delta}\int_0^\rho \left(\frac{s}{\rho}\right)^{-n/2+m-\beta_i-\delta} s^{n/2+m} f(s) s^{-1+\delta}\,ds\right| \le C_\delta\,\rho^{-n/2+m-h-\delta}\end{aligned} \tag{2.48}$$

where we used the boundedness of $\rho^{n/2+m} f(\rho)$ and the inequality $-n/2+m-\mathrm{Re}(\beta_i)-\delta > 0$. Analogously, remembering that $n/2-m+\mathrm{Re}(\alpha_i)+\delta > 0$, we get

$$\begin{aligned}&\left|\rho^{\alpha_i-h}\int_\rho^1 s^{2m-1-\alpha_i} f(s)\,ds\right| = \\ &\left|\rho^{-n/2+m-h-\delta}\int_\rho^1 \left(\frac{\rho}{s}\right)^{n/2-m+\alpha_i+\delta} s^{n/2+m} f(s) s^{-1+\delta}\,ds\right| \le C_\delta\,\rho^{-n/2+m-h-\delta}\end{aligned} \tag{2.49}$$

and finally

$$\begin{aligned}&\left|\rho^{\alpha_i-h}\int_0^1 s^{2m-1-\beta_j} f(s)\,ds\right| = \\ &\left|\rho^{\alpha_i-h}\int_0^1 s^{-n/2+m-\beta_j-\delta} s^{n/2+m} f(s) s^{-1+\delta}\,ds\right| \le C_\delta\,\rho^{\mathrm{Re}(\alpha_i)-h} \le C_\delta \rho^{-n/2+m-h-\delta}\,.\end{aligned} \tag{2.50}$$

□

Proposition 2.9. *Let $\mu \in [0,\overline{\mu})$, and let u be a solution to (2.1); moreover, let us suppose that $\rho^{\alpha_1} f(\rho) = \rho^{\alpha_1}\mathcal{L}_\mu[u] \in L^1_r(B)$. Then*

$$|u(\rho)| \le C\|\rho^{\alpha_1}\mathcal{L}_\mu[u]\|_1 \rho^{\beta_1}\,; \tag{2.51}$$

moreover, if μ is critical for $\mathcal{L}_\mu$, then

$$\|u\|_2 \le C\|\rho^{\alpha_1}\mathcal{L}_\mu[u]\|_1\,. \tag{2.52}$$

Proof. The proof is similar to the proof of Proposition 2.8. Remembering that $2m-1 = n-1+\alpha_1+\beta_1$, we have

$$\left|\rho^{\beta_i}\int_0^\rho s^{2m-1-\beta_i}f(s)\,ds\right| = \left|\rho^{\beta_1}\int_0^\rho \left(\frac{s}{\rho}\right)^{\beta_1-\beta_i} s^{n-1}s^{\alpha_1}f(s)\,ds\right| \le C\|g\|_1\rho^{\beta_1}\,; \tag{2.53}$$

$$\left|\rho^{\alpha_i}\int_\rho^1 s^{2m-1-\alpha_i}f(s)\,ds\right| = \left|\rho^{\beta_1}\int_\rho^1 \left(\frac{\rho}{s}\right)^{\alpha_i-\beta_1} s^{n-1}s^{\alpha_1}f(s)\,ds\right| \le C\|g\|_1\rho^{\beta_1}\,; \tag{2.54}$$

$$\left|\rho^{\alpha_i}\int_0^1 s^{2m-1-\beta_j}f(s)\,ds\right| = \left|\rho^{\alpha_i}\int_0^1 s^{\beta_1-\beta_j}s^{n-1}s^{\alpha_1}f(s)\,ds\right| \le C\|g\|_1\rho^{\alpha_i} \le C\|g\|_1\rho^{\beta_1}\,. \tag{2.55}$$

From the above relations (2.51) immediately follows. Taking into account that μ is critical for $\mathcal{L}_\mu$ iff $\beta_1 > -n/2$, we see that (2.52) is a consequence of (2.51).

□

3. An auxiliary function

Now we want to study the properties of an auxiliary function, which throughout the rest of the paper we shall denote by $w_\mu(\rho)$; this function will be useful in what follows.

We define $w_\mu(\rho)$ to be the solution to the problem

$$\begin{cases} \mathcal{L}_\mu[w_\mu] = 0 & \text{in } B\,, \\ w_\mu \in H^m_r(B)\,, \\ w_\mu(1) = \cdots = w_\mu^{(m-2)}(1) = 0, w_\mu^{(m-1)}(1) = (-1)^{m-1}\,. \end{cases} \tag{3.1}$$

Proposition 3.1. *Let $\mu \in [0,\overline{\mu})$. Then $w_\mu(\rho)$ is positive and decreasing.*

Proof. Let $\mu : P_\mu(z)$ has all distinct roots. By (2.3) and (2.14) we easily get that

$$w_\mu(\rho) = \sum_{i=1}^m \frac{\rho^{\alpha_i}}{\prod\limits_{j\neq i}(\alpha_j-\alpha_i)}\,, \tag{3.2}$$

while, if for a certain μ' double roots occour, then, as usual, $w_{\mu'}(\rho) = \lim\limits_{\mu\to\mu'} w_\mu(\rho)$. Moreover, an elementary verification (see [6]) shows that

$$w_0(\rho) = \frac{(1-\rho^2)^{m-1}}{2^{m-1}(m-1)!}\,. \tag{3.3}$$

We claim that

$$\begin{aligned}&\text{if } \mu \text{ is such that } w_\mu \ge 0 \text{ in } B, \text{then } w_\mu(\rho) \text{ is a decreasing function,}\\ &\text{and } w_\mu(\rho) \ge w_0(\rho)\,.\end{aligned} \tag{3.4}$$

Indeed, if we set $w_\mu = w_0 + v_\mu$, then v_μ is the solution to the problem

$$\begin{cases} (-\Delta)^m v_\mu = \dfrac{\mu}{|x|^{2m}} w_\mu \\ v_\mu \in H^m_{0,r}(B)\,. \end{cases} \tag{3.5}$$

From [14] (see also [4]) we know that, if $(-\Delta)^m v \geq 0$ in B and $v = 0$ on ∂B, then $v \geq 0$ in B and v is radially decreasing. Hence, for any μ such that $w_\mu \geq 0$, we have that $w_\mu = w_0 + v_\mu$ is the sum of two positive decreasing functions, so that (3.4) holds true. Therefore, to end the proof we must show that $w_\mu \geq 0$ for any $\mu \in [0, \overline{\mu})$.

Now let $I = \{\mu \in [0, \overline{\mu}) : w_\sigma(\rho) \geq 0 \ \forall \sigma \in [0, \mu^*]\}$. By its very definition I is an interval, and $I \neq \emptyset$, because $0 \in I$. We must show that $I = [0, \overline{\mu})$, and we argue by contradiction. So, let $\tilde{\mu} = \sup I$ and let us suppose that $\tilde{\mu} < \overline{\mu}$. By continuity it is obvious that $\tilde{\mu} \in I$. To fix ideas, let us suppose that $P_{\tilde{\mu}}(z)$ has distinct roots.

Due to boundary conditions and continuous dependence on the parameter μ, there exists $\delta_1 > 0$ such that $w_\mu(\rho) \geq 0$ for $\mu \in [\tilde{\mu}, \tilde{\mu} + \delta_1]$ and $\rho \in (1 - \delta_1, 1]$. Now let us estimate $w_\mu(\rho)$ in the remaining part of the interval, namely $(0, 1 - \delta_1]$. By (3.2), making explicit the dependence on μ, we have $w_\mu(\rho) = \rho^{\alpha_1(\mu)} \psi_\mu(\rho)$, where

$$\psi_\mu(\rho) = c_1(\mu) + c_2(\mu)\rho^{\alpha_2(\mu) - \alpha_1(\mu)} + \cdots + c_m(\mu)\rho^{\alpha_m(\mu) - \alpha_1(\mu)} \tag{3.6}$$

for suitable $c_i(\mu)$.

Taking into account (3.4), we know that $\psi_{\tilde{\mu}}(\rho) > 0$ in $(0, 1 - \delta_1]$; indeed, being $\mathrm{Re}(\alpha_i(\mu) - \alpha_1(\mu)) \geq 0$ for any $\mu \in [0, \overline{\mu})$ (see Corollary 2.5), a moment's thought shows that $\exists \eta > 0 :$ $\psi_{\tilde{\mu}}(\rho) \geq \eta$ in $(0, 1 - \delta_1]$.

By continuity, which we can invoke as $\mathrm{Re}(\alpha_i(\mu) - \alpha_1(\mu)) \geq 0$, there exists a $\delta_2 > 0$ such that $\psi_\mu(\rho) \geq \eta/2 > 0$ in $(0, 1 - \delta_1]$ for any $\mu \in [\tilde{\mu}, \tilde{\mu} + \delta_2]$.

Summing up, $w_\mu(\rho)$ keeps non-negative for $\mu \in [\tilde{\mu}, \tilde{\mu} + \delta]$, where $\delta = \delta_1 \wedge \delta_2$, an absurd. Hence $\tilde{\mu} = \overline{\mu}$. The reasonment is perfectly analogous if double roots occour for $\mu = \tilde{\mu}$.

□

Remark 11. From the proof of Proposition 3.1 it is clear that there exists $\gamma > 0$ such that

$$\gamma\, w_\mu(\rho) \geq \rho^{\alpha_1}\,, \qquad \rho \in (0, 1/2)\,. \tag{3.7}$$

Proposition 3.2. *Let $\mu \in [0, \overline{\mu})$, and let $f(\rho)$ be a measurable, positive and nonincreasing function on $(0,1)$. Then there exists $C = C(n, \mu) > 0$ such that*

$$\int_0^1 \rho^{n-1+\alpha_1} f(\rho)\, d\rho \leq C \int_0^1 \rho^{n-1} w_\mu(\rho) f(\rho)\, d\rho\,. \tag{3.8}$$

Proof. By means of (3.7) we get

$$
\begin{aligned}
&\int_0^1 \rho^{n-1+\alpha_1} f(\rho)\,d\rho = \int_0^{1/2} \rho^{n-1+\alpha_1} f(\rho)\,d\rho + \int_{1/2}^1 \rho^{n-1+\alpha_1} f(\rho)\,d\rho \le \\
&\int_0^{1/2} \rho^{n-1+\alpha_1} f(\rho)\,d\rho + 2^{-\alpha_1} f(1/2) \int_{1/2}^1 \rho^{n-1}\,d\rho = \\
&\int_0^{1/2} \rho^{n-1+\alpha_1} f(\rho)\,d\rho + (2^n-1)2^{-\alpha_1} f(1/2) \int_0^{1/2} \rho^{n-1}\,d\rho \le \\
2^n &\int_0^{1/2} \rho^{n-1+\alpha_1} f(\rho)\,d\rho \le 2^n\gamma \int_0^{1/2} \rho^{n-1} w_\mu(\rho) f(\rho)\,d\rho \le 2^n\gamma \int_0^1 \rho^{n-1} w_\mu(\rho) f(\rho)\,d\rho\,.
\end{aligned} \tag{3.9}
$$

□

Proposition 3.3. *Let $\mu \in [0,\overline{\mu})$, and let $u \in H^m_{0,r}(B) \cap C^{2m}(\overline{B}\setminus\{0\})$ be a solution to (2.1), with $f \in L^{2^{*\prime}}_r(B)$ such that $\rho^{n/2+m} f(\rho)$ is bounded. Then*

$$
\left(\int_B w_\mu \mathcal{L}_\mu[u]\right)^2 = \omega_n^2 (u^{(m)}(1))^2\,. \tag{3.10}
$$

Proof. Let Ω be a bounded smooth region in $\mathbb{R}^n$ with outward normal ν, and let $\varphi,\psi \in C^{2m}(\overline{\Omega})$ such that

$$
D^\alpha\varphi|_{\partial\Omega} = 0\,, \quad D^\beta\psi|_{\partial\Omega} = 0\,, \qquad |\alpha| \le m-1, \quad |\beta| \le m-2\,; \tag{3.11}
$$

then, by Gauss–Green formula we get

$$
\begin{aligned}
&\int_\Omega \psi(-\Delta)^m\varphi - \int_\Omega \varphi(-\Delta)^m\psi = \\
&\begin{cases} \displaystyle\int_{\partial\Omega} \big(\nabla((-\Delta)^{m/2-1}\psi)\cdot\nu\big)\big((-\Delta)^{m/2}\varphi\big) & m \text{ even;} \\ \displaystyle\int_{\partial\Omega} \big((-\Delta)^{(m-1)/2}\psi\big)\big(\nabla((-\Delta)^{(m-1)/2}\varphi)\cdot\nu\big) & m \text{ odd.} \end{cases}
\end{aligned} \tag{3.12}
$$

For any $\varepsilon \in (0,1)$ let $B_\varepsilon = \{x : \varepsilon < |x| < 1\}$. Taking into account (3.1) and (3.12) with $\Omega = B_\varepsilon$, $\varphi = u$ and $\psi = w_\mu$ we get

$$
\int_{B_\varepsilon} w_\mu \mathcal{L}_\mu[u] = (-1)^m \omega_n u^{(m)}(1) + \varepsilon^{n-1} \sum_{h=0}^{2m-1} c_h u^{(h)}(\varepsilon) w_\mu^{(2m-1-h)}(\varepsilon) \qquad \forall\, \varepsilon \in (0,1)\,. \tag{3.13}
$$

for suitable fixed constants c_h.

By (2.14) we know that $\alpha_1 > -n/2+m$, so let us choose $\delta > 0$ such that $\alpha_1 + n/2 - m - \delta > 0$. By Proposition (2.8) we have

$$
|u^{(h)}(\varepsilon)| \le C\varepsilon^{-n/2+m-h-\delta} \tag{3.14}
$$

while

$$
|w_\mu^{(2m-1-h)}(\varepsilon)| \le C\varepsilon^{\alpha_1-2m+1+h}\,. \tag{3.15}
$$

Therefore

$$|\varepsilon^{n-1}\sum_{h=0}^{2m-1} c_h u^{(h)}(\varepsilon)w_\mu^{(2m-1-h)}(\varepsilon)| \le C\varepsilon^{\alpha_1+n/2-m-\delta} \to 0 \qquad (\varepsilon\to 0)\,. \tag{3.16}$$

On the other hand, $w_\mu\mathcal{L}_\mu[u]\in L^1(B)$, and therefore

$$\int_{B_\varepsilon} w_\mu\mathcal{L}_\mu[u] \to \int_B w_\mu\mathcal{L}_\mu[u] \qquad (\varepsilon\to 0)\,, \tag{3.17}$$

so that (3.10) follows from (3.13), (3.16) and (3.17). □

4. A Pohozaev identity

The keystone of most of non existence theorems in nonlinear critical problems is the *Pohozaev identity*, which is obtained classically by means of multiplication of the equation by suitable testing expressions and integration by parts. In the present work we shall establish the following Pohozaev–type result:

Proposition 4.1. *Let $u\in H^m_{0,r}(B)$ be a weak solution to*

$$\begin{cases}\mathcal{L}_\mu[u]=|u|^{2^*-2}u+\lambda u & \text{in } B\\ D^\beta u|_{\partial B}=0 & \text{for } |\beta|\le m-1\end{cases} \tag{4.1}$$

where $\lambda\in\mathbb{R}$. Then, the following identity holds:

$$2m\lambda\int_B u^2\,dx=\omega_n(u^{(m)}(1))^2 \tag{4.2}$$

Remark 12. The identity (4.2) is well-known in the non-singular case, i.e. when $\mu=0$, under the hypothesis that the solution u is *classical*, i.e. $u\in C^{2m}(\overline{B})$ (see [6] and Theorem 7.27 in [4], where it is proved in general domains Ω). In our case the solution u is no more of C^{2m} class; to move around this problem, we shall resort to a method which requires weaker assumptions: namely, we shall compare two admissible variations of the involved functional.

To prove Proposition 4.1 we need the following two lemmas:

Lemma 4.2. *Let $u\in H^m_{0,r}(B)$. For any $\gamma\in(0,1)$ let $\widetilde{\varphi}_\gamma(\rho):[0,1]\to[0,1]$ be a smooth non-increasing function such that $\widetilde{\varphi}_\gamma(\rho)\equiv 1$ in $[0,1-\gamma]$, $\widetilde{\varphi}_\gamma(\rho)\equiv 0$ in $[1-\gamma/2,1]$, and let $\varphi_\gamma(\rho)=\rho\,\widetilde{\varphi}_\gamma(\rho)$. Finally, let us set $\sigma_{\gamma,\varepsilon}(\rho)=\rho+\varepsilon\varphi_\gamma(\rho)$ and $u_{\gamma,\varepsilon}(\rho)=u(\sigma_{\gamma,\varepsilon}(\rho))$. Then*

$$\begin{aligned}&\lim_{\gamma\to 0}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\frac{\omega_n}{2}\int_0^1\rho^{n-1}(u^{(m)}_{\gamma,\varepsilon}(\rho))^2\,d\rho\right]_{\varepsilon=0}=\\ &-\frac{n-2m}{2}\,\omega_n\int_0^1\rho^{n-1}(u^{(m)}(\rho))^2\,d\rho-\frac{\omega_n}{2}(u^{(m)}(1))^2\,.\end{aligned} \tag{4.3}$$

Proof. See [8], Lemma 4.2. □

Lemma 4.3. *Let $u\in H^m_{0,r}(B)$. Then*

$$\lim_{\rho\to 0}u(\rho)\rho^{n/2-m}=0\,. \tag{4.4}$$

Proof. By classical Hardy–Rellich inequalities (see for instance [3], [12]), the functions $u(\rho)\rho^{n/2-1/2-m}$, $u'(\rho)\rho^{n/2+1/2-m}$ belong to $L^2(0,1)$. Hence, if we set

$$\varphi(\rho) = u^2(\rho)\rho^{n-2m-1}\,, \quad \psi(\rho) = \rho\,\varphi(\rho) = u^2(\rho)\rho^{n-2m} \tag{4.5}$$

we get that $\varphi(\rho), \psi'(\rho) \in L^1(0,1)$. As $\varphi(\rho) \in L^1(0,1)$ we have

$$\liminf_{\rho\to 0} \psi(\rho) = \liminf_{\rho\to 0} \rho\,\varphi(\rho) = 0\,, \tag{4.6}$$

whch implies $\lim_{\rho\to 0} \psi(\rho) = 0$, because $\psi(\rho)$ is an absolutely continuous function. □

Proof of Proposition 4.1.

Let u as in the statement of the proposition; then u is a critical point of the functional $J : H^m_{0,r}(B) \to \mathbb{R}$ defined by

$$\begin{aligned} J[v] &= \frac{1}{2}\int_B \big|D^m v(x)\big|^2\,dx - \frac{\mu}{2}\int_B \frac{v^2(x)}{|x|^{2m}}\,dx - \frac{\lambda}{2}\int_B v^2(x)\,dx - \frac{1}{2^*}\int_B |v(x)|^{2^*}\,dx \\ &= \frac{\omega_n}{2}\int_0^1 \rho^{n-1}(v^{(m)}(\rho))^2\,d\rho - \frac{\mu}{2}\omega_n\int_0^1 \rho^{n-2m-1}v^2(\rho)\,d\rho \\ &\qquad - \frac{\lambda}{2}\omega_n\int_0^1 \rho^{n-1}v^2(\rho)\,d\rho - \frac{1}{2^*}\int_0^1 \rho^{n-1}|v(\rho)|^{2^*}\,d\rho\,. \end{aligned} \tag{4.7}$$

Let $u_{\gamma,\varepsilon} \in H^m_{0,r}(B)$ as in Lemma 4.2. Then

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_0^1 \rho^{n-2m-1}|u_{\gamma,\varepsilon}(\rho)|^2\,d\rho\right]_{\varepsilon=0} = 2\int_0^1 \rho^{n-2m-1}u(\rho)u'(\rho)\varphi_\gamma(\rho)\,d\rho\,; \\ &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_0^1 \rho^{n-1}u^2_{\gamma,\varepsilon}(\rho)\,d\rho\right]_{\varepsilon=0} = 2\int_0^1 \rho^{n-1}u(\rho)u'(\rho)\varphi_\gamma(\rho)\,d\rho\,; \\ &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_0^1 \rho^{n-1}|u_{\gamma,\varepsilon}(\rho)|^{2^*}\,d\rho\right]_{\varepsilon=0} = 2^*\int_0^1 \rho^{n-1}|u(\rho)|^{2^*-2}u(\rho)u'(\rho)\varphi_\gamma(\rho)\,d\rho\,. \end{aligned} \tag{4.8}$$

Taking into account (4.4), we see that we may integrate by parts in (4.8), getting

$$\begin{aligned} &\lim_{\gamma\to 0}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[-\frac{\mu}{2}\omega_n\int_0^1 \rho^{n-2m-1}u^2_{\gamma,\varepsilon}(\rho)\,d\rho\right]_{\varepsilon=0} = \frac{n-2m}{2}\mu\,\omega_n\int_0^1 \rho^{n-1}u^2(\rho)\,d\rho\,; \\ &\lim_{\gamma\to 0}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[-\frac{\lambda}{2}\omega_n\int_0^1 \rho^{n-1}u^2_{\gamma,\varepsilon}(\rho)\,d\rho\right]_{\varepsilon=0} = \frac{n}{2}\lambda\,\omega_n\int_0^1 \rho^{n-1}u^2(\rho)\,d\rho\,; \\ &\lim_{\gamma\to 0}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[-\frac{\omega_n}{2^*}\int_0^1 \rho^{n-1}|u_{\gamma,\varepsilon}(\rho)|^{2^*}\,d\rho\right]_{\varepsilon=0} = \frac{n-2m}{2}\,\omega_n\int_0^1 \rho^{n-1}|u(\rho)|^{2^*}\,d\rho\,. \end{aligned} \tag{4.9}$$

Now let us compare two admissible variations for J. Being u a critical point for J, we know that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big[J[u_{\gamma,\varepsilon}]\big]_{\varepsilon=0} = 0 \Longrightarrow \lim_{\gamma\to 0}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big[J[u_{\gamma,\varepsilon}]\big]_{\varepsilon=0} = 0\,; \tag{4.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big[J[(1+\varepsilon)u]\big]_{\varepsilon=0} = 0\,. \tag{4.11}$$

Taking into account (4.7), from (4.3), (4.9) and (4.10) we get

$$\begin{aligned}-\frac{n-2m}{2}\,\omega_n\int_0^1 \rho^{n-1}(u^{(m)}(\rho))^2\,d\rho-\frac{\omega_n}{2}(u^{(m)}(1))^2+\\ \frac{n}{2}\lambda\,\omega_n\int_0^1 \rho^{n-1}u^2(\rho)\,d\rho+\frac{n-2m}{2}\omega_n\int_0^1 \rho^{n-1}|u(\rho)|^{2^*}\,d\rho=0\end{aligned} \tag{4.12}$$

while from (4.11) we obtain

$$\omega_n\int_0^1 \rho^{n-1}(u^{(m)}(\rho))^2\,d\rho-\lambda\omega_n\int_0^1 \rho^{n-1}u^2(\rho)\,d\rho-\omega_n\int_0^1 \rho^{n-1}|u(\rho)|^{2^*}\,d\rho=0\,. \tag{4.13}$$

Now the thesis follows by adding to (4.12) the equation (4.13) multiplied by $(n/2-m)$.

□

5. The proof of Theorem 1

If $\lambda<0$, Theorem 1 follows immediately from (4.2), while, if $\lambda=0$, the claim follows by comparing (3.10) and (4.2) (and indeed problem (1.1) admits no nontrivial solutions for any $\lambda\le 0$, whatever μ may be in $[0,\overline{\mu})$); hence, for the rest of the section, let $\lambda>0$.

The proof is based upon the following chain of equalities–inequalities, which holds true, if μ is critical for $\mathcal{L}_\mu$, for any $u\in H^m_{0,r}(B)$ weak positive radial solution to (1.1):

$$\lambda\|u\|_2^2 \underset{(I)}{=} \frac{1}{2m\omega_n}\|w\mathcal{L}_\mu[u]\|_1^2 \underset{(II)}{\ge} C_1\|\rho^{\alpha_1}\mathcal{L}_\mu[u]\|_1^2 \underset{(III)}{\ge} C_2\|u\|_2^2 \tag{5.1}$$

where C_1, C_2 are strictly positive constants.

Proof of (5.1)-(I).

By standard regularity arguments, $u\in H^m_{0,r}(B)\cap C^{2m}(\overline{B}\setminus\{0\})$; moreover, by Lemma 4.3 we get that, setting $f=u^{2^*-1}+\lambda u$, the function $\rho^{n/2+m}f(\rho)$ is bounded. Hence we may use Proposition 3.3; now (5.1)-(I) follows by comparing (3.10) and (4.2).

Proof of (5.1)-(II).

Being u a radial positive solution to (1.1), it solves

$$(-\Delta)^m u = g := \mu\frac{u}{|x|^{2m}}+u^{2^*-1}+\lambda u \qquad u\in H^m_{0,R}(B)\,; \tag{5.2}$$

the right hand side g in (5.2) is positive, therefore from [14], [4] we know that u is radially decreasing, and so the same is true for $u^{2^*-1}+\lambda u$ and for $\mathcal{L}_\mu[u]$. Now (5.1)-(II) follows from Proposition (3.2)

Proof of (5.1)-(III).

The inequality (5.1)-(III) is nothing but (2.52), which holds true as μ is critical for $\mathcal{L}_\mu$.

□

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