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Number of steady states of quantum evolutions

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We prove sharp universal upper bounds on the number of linearly independent steady and asymptotic states of discrete- and continuous-time Markovian evolutions of open quantum systems. We show that the bounds depend only on the dimension of the system and not on the details of the dynamics. A comparison with similar bounds deriving from a recent spectral conjecture for Markovian evolutions is also provided.

Spectral theory is still a hot topic in quantum mechanics. Indeed, quantum theory was developed at the beginning of the last century in order to explain the energy spectra of atoms¹.

In particular, the dynamics of a closed quantum system, namely isolated from its surroundings, is encoded in the eigenvalues (energy levels) of its Hamiltonian². Similarly, for an open quantum system under the Markovian approximation³, studying the spectrum of the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) generator (the open-system analogue of the Hamiltonian) allows us to obtain information about the dynamics of the system⁴.

In spite of this general interplay between spectrum and dynamics, a complete understanding of open-quantum-system evolutions still remains a formidable task. However, a more detailed analysis may be performed if we restrict our attention to the large-time dynamics of open systems. This amounts to study the steady and, more generally, asymptotic states towards which the evolution converges in the asymptotic limit.

This topic was already investigated at the dawn of the theory of open quantum systems in various works^{5–7} (see also the review⁸), focusing on the existence and the uniqueness of a steady state for Markovian evolutions. Moreover, the structure of steady and asymptotic manifolds were taken into account in several later articles^{4,9–17}.

Besides their theoretical importance, stationary states also play a central role in reservoir engineering^{18–20}, consisting of properly choosing the system-environment coupling for preparing a target quantum state, or in phase-locking and synchronization of quantum systems²¹. Moreover, GKLS generators with multiple steady states²² may be used in order to drive a dissipative system into (degenerate) subspaces protected from noise²³ or decoherence²⁴, in which only a unitary evolution, related to purely imaginary eigenvalues of the generator¹¹, may be exploited for the realization of quantum gates^{25–27}. For this reason, the analysis of stationary states and, more generally, the study of the relaxation of an open quantum system towards the equilibrium is needed for applications in quantum information storage and processing^{28–31}.

The asymptotic properties of open quantum systems have also been deeply studied in quantum statistical mechanics. In particular, dissipative quantum phase transitions^{32,33}, as well as driven-dissipative systems^{34,35}, require the study of the large-time dynamical behaviour of the system. More generally, determining the steady states of an open system sheds light on the transport properties of the system itself. In particular, the existence of discontinuities of the dimension of the steady-state manifold should correspond to a jump for the transport features of the system^{36,37}.

Finally, open-quantum-system asymptotics naturally emerges in quantum implementations of Hopfield-type attractor neural networks³⁸. Indeed, the stored memories of such type of network may be identified with the stationary states of its (non-unitary) evolution^{39,40}.

Despite the much effort devoted to the asymptotic dynamics of open quantum systems, general constraints for the number of steady and asymptotic states of quantum evolutions are still to be found, as far as we know. Besides the theoretical relevance of this problem, they may allow us to elucidate the potential of some of the above mentioned applications.

In this Article, we find sharp *universal* upper bounds on the number of linearly independent steady and asymptotic states of discrete-time and Markovian continuous-time quantum evolutions. Importantly, these bounds are only related to the dimension of the system and not on the properties of the dynamics. These findings quantify the effects of noise and decoherence in the implementation of unitary gates in a concrete experimental setup. More precisely, these bounds allow us to understand the minimum number of physical qubits needed to design decoherence-free subspaces with a desired dimension, i.e. a targeted number of logical qubits, for quantum information and computation tasks.

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The Article is organized as follows. After introducing some preliminary notions in Sect. “Preliminaries”, we will discuss our main results in Sect. “Bounds on the dimensions of the asymptotic manifolds, then we will provide explicit examples proving the sharpness of the bounds in Sect. “Sharpness of the bounds”. Subsequently, before proving the theorems in Sect. “Proofs of Theorems 1–4”, our results will be compared with analogous bounds derived from a recent universal spectral conjecture proposed in⁴¹ in Sect. “Relation with the Chruściński–Kimura–Kossakowski–Shishido bound”. Finally, we will draw the conclusions of the work in Sect. “Conclusions and outlooks”.

Preliminaries

In the present Section we will recall some basic notions about evolutions of finite-dimensional open quantum systems, see also Sect. “Proofs of Theorems 1–4” for more details.

The state of an arbitrary d -level open quantum system is given by a density operator ρ , namely a positive semidefinite operator on a Hilbert space \mathcal{H} ($d = \dim \mathcal{H}$) with $\text{Tr}(\rho) = 1$, whereas its dynamics in a given time interval $[0, \tau]$ with $\tau > 0$ is described by a quantum channel Φ , namely a completely positive trace-preserving map (a superoperator) on $\mathcal{B}(\mathcal{H})$, the space of linear operators on \mathcal{H} ⁴².

If the system state at time $t = 0$ is ρ , its *discrete-time evolution* at time $t = n\tau$, with $n \in \mathbb{N}$, will be given by the action of the n -fold composition Φ^n of the map Φ , namely,

$$\rho(n\tau) = \Phi^n(\rho), \quad n = 0, 1, \dots \tag{1}$$

As the Hilbert space \mathcal{H} is finite-dimensional, $\mathcal{B}(\mathcal{H})$ is isomorphic to the space of complex matrices of order d . We will indicate the space of $d \times d'$ matrices with complex entries by $\mathcal{M}_{d,d'}(\mathbb{C})$ and, for the sake of simplicity, $\mathcal{M}_d(\mathbb{C}) \equiv \mathcal{M}_{d,d}(\mathbb{C})$.

Let $\mu_\alpha, \alpha = 0, \dots, n - 1$, with $n \leq d^2$ be the distinct eigenvalues of Φ , namely

$$\Phi(A_\alpha) = \mu_\alpha A_\alpha, \tag{2}$$

with A_α being an eigenoperator corresponding to μ_α . The spectrum $\text{spect}(\Phi)$ is the set of eigenvalues of Φ . Let ℓ_α be the algebraic multiplicity⁴³ of the eigenvalue μ_α , so that $\sum_{\alpha=0}^{n-1} \ell_\alpha = d^2$. It is well known¹² that:

- (i) The spectrum is contained in the unit disk,

$$\text{spect}(\Phi) \subseteq \mathbb{D}, \quad \mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}; \tag{3}$$

- (ii) 1 is always an eigenvalue, namely,

$$\mu_0 = 1 \in \text{spect}(\Phi); \tag{4}$$

- (iii) The spectrum is symmetric with respect to the real axis, i.e.,

$$\mu_\alpha \in \text{spect}(\Phi) \Rightarrow \mu_\alpha^* \in \text{spect}(\Phi), \quad \text{and} \quad \Phi(A_\alpha^\dagger) = \mu_\alpha^* A_\alpha^\dagger, \tag{5}$$

- (iv) The unimodular or *peripheral* eigenvalues $\mu_\alpha \in \partial\mathbb{D}$, the boundary of \mathbb{D} , are semisimple, i.e. their algebraic multiplicity ℓ_α coincides with their geometric multiplicity.

The eigenspace $\text{Fix}(\Phi)$ corresponding to $\mu_0 = 1$, called the *fixed-point space* of Φ , is spanned by a set of ℓ_0 density operators, which are the *steady* (or *stationary*) states of the channel Φ .

Also, the space $\text{Attr}(\Phi)$ corresponding to the peripheral eigenvalues $\mu_\alpha \in \partial\mathbb{D}$ is known as the asymptotic⁴⁴ or the *attractor subspace*^{45,46} of the channel Φ , since the evolution $\Phi^n(\rho)$ of any initial state ρ asymptotically moves towards this space for large times, i.e. as $n \rightarrow \infty$, see Sect. “Proofs of Theorems 1–4” for more details. These limiting states may be called *oscillating* or *asymptotic states*, and it is always possible to construct a basis of such states for the subspace $\text{Attr}(\Phi)$, analogously to $\text{Fix}(\Phi)$.

Note that closed-system evolutions are described by a *unitary channel*

$$\Phi(X) = UXU^\dagger, \quad \text{for all } X \in \mathcal{B}(\mathcal{H}), \tag{6}$$

and some unitary U . Importantly, a quantum channel is unitary if and only if $\text{spect}(\Phi) \subseteq \partial\mathbb{D}$, i.e. all its eigenvalues belong to the unit circle¹².

The *Markovian continuous-time evolution* of an open quantum system is described by a quantum dynamical semigroup⁸

$$\rho(t) = \Phi_t(\rho) = e^{t\mathcal{L}}\rho, \quad t \geq 0, \tag{7}$$

where the generator \mathcal{L} takes the well-known *GKLS form*^{47,48}

$$\mathcal{L}(X) = -i[H, X] + \sum_{k=1}^{d^2-1} \left(A_k X A_k^\dagger - \frac{1}{2} \{A_k^\dagger A_k, X\} \right) = \mathcal{L}_H(X) + \mathcal{L}_D(X), \quad X \in \mathcal{B}(\mathcal{H}), \tag{8}$$

where the square (curly) brackets represent the (anti)commutator, $H = H^\dagger$ is the system Hamiltonian, the noise operators A_k are arbitrary, and the first and second terms \mathcal{L}_H and \mathcal{L}_D in Eq. 8 are called the Hamiltonian and dissipative parts of the generator, respectively. Notice that the GKLS form 8 is not unique and, in particular, so

is the decomposition of \mathcal{L} into Hamiltonian and dissipative contributions. \mathcal{L} is called a *Hamiltonian generator* if $\mathcal{L}_D = 0$ for one (and hence all) GKLS representation [8](#).

If $\lambda_\alpha, \alpha = 0, \dots, m-1$, with $m \leq d^2$, denote the distinct eigenvalues of \mathcal{L} , from the GKLS form one obtains that $\lambda_0 = 0$ and, given an eigenoperator $X_0 \geq 0$ corresponding to this eigenvalue, then $X_0/\text{Tr}(X_0)$ is a steady state of $\Phi_t = e^{t\mathcal{L}}$. The kernel of \mathcal{L} , i.e. the eigenspace corresponding to the zero eigenvalue, will be denoted by $\text{Ker}(\mathcal{L})$. Moreover,

$$\lambda_\alpha \in \text{spect}(\mathcal{L}) \Rightarrow \lambda_\alpha^* \in \text{spect}(\mathcal{L}), \quad \text{and} \quad \text{Re}(\lambda_\alpha) = -\Gamma_\alpha \leq 0 \quad (9)$$

with Γ_α being the *relaxation rates* of \mathcal{L} . These parameters, describing the relaxation properties of an open system⁵⁰, may be experimentally measured. A condition for the relaxation rates of a quantum dynamical semigroup, recently conjectured in⁴¹ and which we will call Chruściński–Kimura–Kossakowski–Shishido (CKKS) bound, is recalled in Sect. “[Relation with the Chruściński–Kimura–Kossakowski–Shishido bound](#)” in order to investigate its relation with the main results of this work, stated in Sect. “[Bounds on the dimensions of the asymptotic manifolds](#)”.

Finally, note that the purely imaginary (peripheral) eigenvalues of \mathcal{L} are semisimple and are related to the large-time dynamics of $\Phi_t = e^{t\mathcal{L}}$, as the space corresponding to such eigenvalues is the asymptotic manifold $\text{Attr}(\mathcal{L})$ of the Markovian evolution, see Sect. “[Proofs of Theorems 1–4](#)” for details. Importantly, as for unitary channels, the generator \mathcal{L} is Hamiltonian if and only if $\Gamma_\alpha = 0$ for all $\alpha = 0, \dots, m-1$, i.e. all its eigenvalues are peripheral.

Bounds on the dimensions of the asymptotic manifolds

In this Section we will present the main results of this work, whose proofs are postponed to Sect. “[Proofs of Theorems 1–4](#)”. First, let us introduce the quantities involved in our findings. Remember that we denoted with μ_α the α -th distinct eigenvalue of Φ and with ℓ_α its algebraic multiplicity with $\alpha = 0, \dots, n-1$. In particular, ℓ_0 is the algebraic multiplicity of $\mu_0 = 1$, and coincides with the dimension of its eigenspace, the steady-state manifold, i.e.

$$\ell_0 = \dim \text{Fix}(\Phi). \quad (10)$$

We define the *peripheral multiplicity* ℓ_P of Φ as the sum of the multiplicities of all peripheral eigenvalues, which coincides with the dimension of the attractor subspace $\text{Attr}(\Phi)$, made up of asymptotic states. Namely,

$$\ell_P = \sum_{\mu_\alpha \in \partial\mathbb{D}} \ell_\alpha = \dim \text{Attr}(\Phi). \quad (11)$$

Physically, ℓ_0 and ℓ_P are respectively the number of independent steady and asymptotic states of the evolution described by Φ .

Analogously, denote with m_α ($\alpha = 0, \dots, m-1$) the algebraic multiplicity of the α -th *distinct* eigenvalue λ_α of the generator \mathcal{L} of the continuous-time semigroup [7](#). In particular m_0 denotes the multiplicity of the zero eigenvalue $\lambda_0 = 0$, so that

$$m_0 = \dim \text{Ker}(\mathcal{L}). \quad (12)$$

Moreover, the *peripheral multiplicity* m_P of \mathcal{L} is the sum of the multiplicities of its purely imaginary eigenvalues and measures the dimension of its attractor manifold:

$$m_P = \sum_{\lambda_\alpha \in i\mathbb{R}} m_\alpha = \dim \text{Attr}(\mathcal{L}). \quad (13)$$

The integers m_0 and m_P represent respectively the number of independent steady and asymptotic states of the Markovian evolution $\Phi_t = e^{t\mathcal{L}}$ generated by \mathcal{L} .

Now we will provide sharp upper bounds on such multiplicities. Let us call a quantum channel *non-trivial* if it is different from the identity channel, $\Phi(\rho) = \rho$.

Theorem 1 (Unitary discrete-time evolution) *Let Φ be a non-trivial unitary quantum channel on a d -dimensional system. Then the multiplicity ℓ_0 of the eigenvalue 1 and the peripheral multiplicity ℓ_P of Φ satisfy*

$$\ell_0 \leq d^2 - 2d + 2, \quad \ell_P = d^2. \quad (14)$$

Theorem 2 (Non-unitary discrete-time evolution) *Let Φ be a non-unitary quantum channel. Then the multiplicity ℓ_0 of the eigenvalue 1 and the peripheral multiplicity ℓ_P of Φ obey*

$$\ell_0 \leq \ell_P \leq d^2 - 2d + 2. \quad (15)$$

The content of the latter result is schematically illustrated in Fig. [1](#). Now, it is possible to construct quantum channels with ℓ_0 and ℓ_P attaining the equalities in Eqs. [14](#) and [15](#), namely all the upper bounds are *sharp*, see Sect. “[Sharpness of the bounds](#)” for explicit examples. Obviously, for a trivial quantum channel $\ell_0 = \ell_P = d^2$, therefore the bounds [14](#) and [15](#) are not valid.

The above results, valid for discrete-time evolutions [1](#) are perfectly mirrored by the following results on Markovian continuous-time evolutions [7](#), with GKLS generators [8](#).

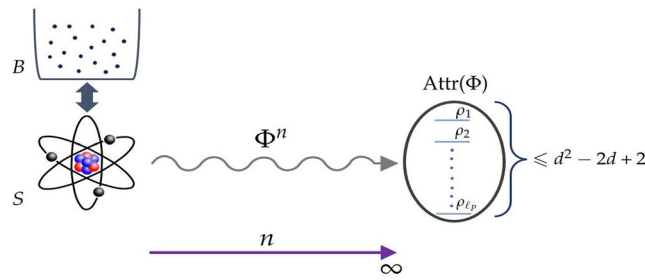


Figure 1. Schematic representation of the content of Theorem 2. A system S coupled to a bath B evolves according to the non-unitary discrete-time evolution Φ^n with $n \geq 1$. The asymptotic states $\rho_1, \dots, \rho_{\ell_P}$ of S , spanning the attractor subspace $\text{Attr}(\Phi)$, are at most $d^2 - 2d + 2$, where d is the dimension of the system.

Theorem 3 (Hamiltonian generator) *Let \mathcal{L} be a non-zero Hamiltonian GKLS generator. Then the multiplicity m_0 of the zero eigenvalue and the peripheral multiplicity m_P of \mathcal{L} fulfill*

$$m_0 \leq d^2 - 2d + 2, \quad m_P = d^2. \tag{16}$$

Theorem 4 (Non-Hamiltonian GKLS generator) *Let \mathcal{L} be a non-Hamiltonian GKLS generator. Then the multiplicity m_0 of the zero eigenvalue and the peripheral multiplicity m_P of \mathcal{L} satisfy*

$$m_0 \leq m_P \leq d^2 - 2d + 2. \tag{17}$$

The bounds 16 and 17 are also sharp as the previous ones, see Sect. “Sharpness of the bounds”. Clearly, the two latter theorems do not apply to the zero operator because in such case $m_0 = m_P = d^2$.

Theorem 1 provides a tight universal upper bound on the number of linearly independent steady states of a (non-trivial) unitary quantum channel Φ , depending only on the dimension d of the system. Similarly, Theorem 2 shows that the number of linearly independent steady and asymptotic states of a non-unitary channel Φ is bounded from above by the same d -dependent quantity. Theorems 3 and 4 provide analogous constraints for non-zero Hamiltonian and non-Hamiltonian generators respectively and, indeed, Theorem 4 easily follows from Theorem 2, as shown in Sect. “Proofs of Theorems 1–4”.

Interestingly, Theorem 4 implies that when we add to a Hamiltonian generator a dissipative part, no matter how small, the peripheral multiplicity m_P undergoes a jump not smaller than the gap

$$\Delta = 2(d - 1), \tag{18}$$

varying linearly with d . Consequently, we have

$$N_f = 2(d - 1) \tag{19}$$

forbidden values for m_P . The same minimum jump 18 for the peripheral multiplicity ℓ_P occurs when we pass from unitary channels to non-unitary ones according to the bound 15 and, consequently, the same number of non-allowed values 19 for ℓ_P . Therefore, the bounds in Theorems 1–4 quantify how the implementation of a desired decoherence-free logical gate is affected by the non-trivial interaction of the system with the environment.

For example, in the case $d = 4$, viz. a two-qubit system, the upper bound for the dimensions ℓ_0 and ℓ_P (m_0 and m_P) in the non-unitary (non-Hamiltonian) case is $d^2 - 2d + 2 = 10$, thus the $N_f = 6$ forbidden values for the dimensions are 11, 12, 13, 14, 15, 16. Therefore, on a two-qubit system one can exploit up to 10 (linearly independent) asymptotic states in order to implement unitary gates within decoherence-free subspaces. Equivalently, at least three physical qubits are needed to implement a decoherence-free subspace with dimension higher than 10, i.e. spanned by more than 10 asymptotic states.

As a final remark, it is worth stressing that the obtained bounds are a consequence of complete positivity of the dynamics. Indeed, positivity alone is not sufficient for implying the bounds in Theorems 1 and 2. For instance, consider the qubit transposition map

$$T(X) = X^T, \quad X \in \mathcal{B}(\mathbb{C}^2), \tag{20}$$

with respect to the computational basis of \mathbb{C}^2 , which is known to be positive but not completely positive⁴². It is easy to check that the peripheral multiplicity of T is $\ell_P = 4 = d^2$, while its fixed-point multiplicity is $\ell_0 = 3 > d^2 - 2d + 2 = 2$, i.e. it violates the bound 14. Similarly, if we consider the qubit positive map

$$\Phi(X) = pT(X) + (1 - p)X, \quad X \in \mathcal{B}(\mathbb{C}^2), \quad p \in (0, 1), \tag{21}$$

then its fixed-point and peripheral multiplicities are $\ell_0 = \ell_P = 3$, and do not satisfy the upper bound in 15.

Interestingly, one can show that Theorems 1 and 2 remain valid for Schwarz maps¹², an important class of positive maps which are not necessarily completely positive. In fact, the Schwarz property, which lies between positivity and complete positivity, has proven to be sufficient for the analysis of the asymptotic dynamics of open quantum systems^{15,16}.

Sharpness of the bounds

In this Section we will prove the sharpness of the bounds stated in Theorems 1-4. Let us start with the proof of the sharpness of the bound 16 for non-zero Hamiltonian GKLS generators. If we take

$$\mathcal{L}(X) = -i[H, X], \quad H = h_1|e_1\rangle\langle e_1| + h_2 \sum_{i=2}^d |e_i\rangle\langle e_i|, \quad h_1 \neq h_2 \in \mathbb{R}, \quad X \in \mathcal{B}(\mathcal{H}), \quad (22)$$

for some basis $\{|e_i\rangle\}_{i=1}^d$ of \mathcal{H} , then it is immediate to check that

$$\text{Ker}(\mathcal{L}) = \text{span}\{|e_1\rangle\langle e_1|, |e_j\rangle\langle e_k| : j, k = 2, \dots, d\}, \quad (23)$$

whence $m_0 = (d - 1)^2 + 1 = d^2 - 2d + 2$. Furthermore, if we require that

$$h_1 - h_2 \neq 2k\pi, \quad \forall k \in \mathbb{Z}, \quad (24)$$

the multiplicity ℓ_0 of the corresponding unitary channel $\Phi = e^{\mathcal{L}}$ attains the inequality in Eq. 14. Note that condition 24 guarantees that Φ is not trivial.

Let us now turn our attention to the sharpness of the bounds 17 for GKLS generators. Recall that the commutant S' of a system of operators $S = \{A_k\}_{k=1}^M \subset \mathcal{B}(\mathcal{H})$ is defined as

$$S' = \{B \in \mathcal{B}(\mathcal{H}) : A_k B = B A_k, \quad k = 1, \dots, M\}. \quad (25)$$

Now consider the system $S = \{A_k\}_{k=1}^N$ of diagonal operators with respect to the basis $\{|e_i\rangle\}_{i=1}^d$ of \mathcal{H} with

$$A_k = \lambda_1^{(k)} P_1 + \lambda_2^{(k)} P_2, \quad k = 1, \dots, N. \quad (26)$$

Here, $\lambda_1^{(k)}, \lambda_2^{(k)} \in \mathbb{C}$, with $\lambda_1^{(k)} \neq \lambda_2^{(k)}$, are the eigenvalues of A_k , and

$$P_1 = |e_1\rangle\langle e_1|, \quad P_2 = \mathbb{I} - P_1, \quad (27)$$

are the corresponding spectral projections, with \mathbb{I} being the identity operator on \mathcal{H} . Note that, by construction, the eigenvalues $\lambda_1^{(k)}, \lambda_2^{(k)}$ have respectively multiplicities $m_1 = 1, m_2 = d - 1$ for all $k = 1, \dots, N$. Let us now take into account the generator

$$\mathcal{L}(X) = \sum_{k=1}^N \left(A_k X A_k^\dagger - \frac{1}{2} \{A_k^\dagger A_k, X\} \right), \quad (28)$$

for which $\mathcal{L}(\mathbb{I}) = 0$. We have

$$m_0 = \dim \text{Ker}(\mathcal{L}) = \dim S' = \dim \{P_1\}' = d^2 - 2d + 2. \quad (29)$$

Here, the second and fourth equalities follow respectively from Proposition 8 and Corollary 5.1 in Sect. "Proofs of Theorems 1-4", whereas the third one is a consequence of Eq. 26. Moreover, as \mathcal{L} is non-Hamiltonian by construction, we necessarily have

$$m_0 = m_p = d^2 - 2d + 2, \quad (30)$$

by Theorem 4. A quantum channel saturating the equalities in Eq. 15 is simply $\Phi = e^{\mathcal{L}}$, with \mathcal{L} given by Eq. 28.

In particular, we can construct a more physically transparent example of GKLS generator saturating the bounds 17 by taking $S = \{P_1, P_2\}$. The associated Markovian channel Φ acts as follows with respect to the basis $\{|e_i\rangle\}_{i=1}^d$

$$\mathcal{B}(\mathcal{H}) \ni X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \mapsto \Phi(X) = \begin{pmatrix} X_{11} & X_{12}e^{-1} \\ X_{21}e^{-1} & X_{22} \end{pmatrix}, \quad (31)$$

with $X_{11} \in \mathbb{C}, X_{22} \in \mathcal{M}_{d-1}(\mathbb{C}), X_{12} \in \mathcal{M}_{1,d-1}(\mathbb{C}),$ and $X_{21} \in \mathcal{M}_{d-1,1}(\mathbb{C})$.

Therefore we realize that Φ is a phase-damping channel causing an exponential suppression of the coherences $x_{12}, \dots, x_{1d} \in X_{12}$, and we immediately see that it attains the equalities in the bounds 15, in line with the discussion above.

Relation with the Chruściński–Kimura–Kossakowski–Shishido bound

In this Section we will make a comparison between the bounds given in Theorems 1-4 and similar bounds arising from a recent spectral conjecture discussed in⁴¹. As already noted in Sect. "Introduction", the real parts of the eigenvalues λ_α of a quantum dynamical semigroup $\Phi_t = e^{t\mathcal{L}}$ with GKLS generator \mathcal{L} are non-positive. However, it was recently conjectured in⁴¹ that the relaxation rates $\Gamma_\alpha = -\text{Re}(\lambda_\alpha)$ are not arbitrary non-negative numbers, but they must obey the CKKS bound

$$\Gamma_\alpha \leq \frac{1}{d} \sum_{\beta=1}^{m-1} m_\beta \Gamma_\beta, \quad \alpha = 1, \dots, m - 1, \quad (32)$$

where m_β is the algebraic multiplicity of λ_β . This upper bound was not proved yet in general, but it holds for qubit systems, while for $d \geq 3$ it is valid for generators of unital semigroups, i.e. with $\mathcal{L}(\mathbb{I}) = 0$, and for a class of generators obtained in the weak coupling limit⁴¹ (see also⁵¹ for further results). Also, it was experimentally demonstrated for two-level systems^{52,53}.

The CKKS bound 32 implies the following inequalities for non-Hamiltonian generators

$$m_0 \leq m_P \leq d^2 - d. \tag{33}$$

Indeed, summing Eq. 32 over the bulk, i.e. non-peripheral, eigenvalues of \mathcal{L} yields

$$\sum_{\Gamma_\alpha < 0} m_\alpha \Gamma_\alpha \leq \frac{m_B}{d} \sum_{\Gamma_\beta < 0} m_\beta \Gamma_\beta, \tag{34}$$

where $m_B = \sum_{\Gamma_\alpha < 0} m_\alpha$ is the number of the repeated eigenvalues in the bulk. If \mathcal{L} is not Hamiltonian, viz. $m_B \neq 0$ as noted in Sect. ‘‘Preliminaries’’, this implies that

$$m_B \geq d \Rightarrow m_P \leq d^2 - d, \tag{35}$$

namely the assertion.

Interestingly, the CKKS bound 32 implies also the following bound on the real parts x_α of the eigenvalues μ_α of an arbitrary quantum channel Φ ⁴¹

$$\sum_{\beta=0}^{n-1} \ell_\beta x_\beta \leq d(d-1) + dx_\alpha, \quad \alpha = 1, \dots, n-1, \tag{36}$$

where ℓ_β is the algebraic multiplicity of μ_β .

Although Eq. 36 does not yield an upper bound similar to Eq. 33 for the peripheral multiplicity ℓ_P of Φ , the multiplicity ℓ_0 of the eigenvalue $\mu_0 = 1$, i.e. the number of steady states of Φ , satisfies

$$\ell_0 \leq d^2 - d, \tag{37}$$

if Φ is not trivial. The proof goes as follows: when $\ell_0 = d^2$ we have the identity channel, so suppose $\ell_0 = d^2 - d + N$ with $0 \leq N \leq d - 1$. Then from Eq. 36 one gets

$$x_\alpha \geq \frac{1}{d} \sum_{\beta=1}^{n-1} \ell_\beta x_\beta + \frac{N}{d}, \quad \alpha = 1, \dots, n-1. \tag{38}$$

Now, the right-hand side of Eq. 38 is the arithmetic mean of the set

$$S = \{\underbrace{x_1, \dots, x_1}_{\ell_1}, \dots, \underbrace{x_{n-1}, \dots, x_{n-1}}_{\ell_{n-1}}, \underbrace{1, \dots, 1}_N\}, \tag{39}$$

therefore condition 38 is equivalent to require that all the elements of S exceed their arithmetic mean, which is true if and only if $N = 0$ and

$$x_1 = \dots = x_{n-1} = x \in [-1, 1), \tag{40}$$

which concludes the proof of Eq. 37.

Furthermore, from Eq. 33 it follows that

$$\ell_0 \leq \ell_P \leq d^2 - d, \tag{41}$$

for non-unitary Markovian channels, viz. of the form $\Phi = e^{\mathcal{L}}$ with \mathcal{L} non-Hamiltonian generator.

Now let us compare the bounds 33, 37, and 41 arising from the CKKS conjecture 32 discussed in the present Section with the ones stated in Sect. ‘‘Bounds on the dimensions of the asymptotic manifolds’’. First, the upper bound 15 for ℓ_P is also valid for non-Markovian channels, differently from the bound 41 and, in the Markovian case, it is stricter than Eq. 41 when $d \neq 2$. Analogously, the bound in Eq. 14 and the one for ℓ_0 in Eq. 15 hold down to Eq. 37 in the two-dimensional case, but they are stricter otherwise.

Similarly, the bounds 33 for m_0 and m_P are not tight for all $d \geq 3$, whereas they are equivalent to condition 17 in the case $d = 2$. Consequently, the jump for m_P is also predicted by the bound 33 but $\Delta = d$, which is loose for all $d \neq 2$. In conclusion, the bounds given in Theorems 1-4 imply the bounds 33, 37, and 41 deriving from the CKKS conjecture 32, in favor of the validity of the conjecture itself.

Proofs of Theorems 1–4

In this Section we will prove Theorems 1-4 stated in Sect. ‘‘Bounds on the dimensions of the asymptotic manifolds’’. To this purpose, let us recall several preliminary concepts, besides the ones introduced in Sect. ‘‘Preliminaries’’.

First, given a quantum channel Φ , it always admits a Kraus representation⁴²,

$$\Phi(X) = \sum_{k=1}^N B_k X B_k^\dagger, \quad \sum_{k=1}^N B_k^\dagger B_k = \mathbb{I}, \quad \text{with } X \in \mathcal{B}(\mathcal{H}), \tag{42}$$

in terms of some operators $\{B_k\}_{k=1}^N \subset \mathcal{B}(\mathcal{H})$. Note that the second equation in 42 expresses the trace-preservation condition.

Let $\mu_\alpha, \alpha = 0, \dots, n - 1$, with $\mu_0 = 1$ ($\lambda_\alpha, \alpha = 0, \dots, m - 1$, with $\lambda_0 = 0$) denote the n (m) distinct eigenvalues of Φ (\mathcal{L}). Let \mathcal{L}_α (\mathcal{M}_α) be the algebraic eigenspace of Φ (\mathcal{L}) corresponding to the eigenvalue μ_α (λ_α), whose dimension is the algebraic multiplicity ℓ_α (m_α) of the eigenvalue. The attractor subspaces of Φ and \mathcal{L} read

$$\text{Attr}(\Phi) = \bigoplus_{\mu_\alpha \in \partial\mathbb{D}} \mathcal{L}_\alpha, \tag{43}$$

$$\text{Attr}(\mathcal{L}) = \bigoplus_{\lambda_\alpha \in i\mathbb{R}} \mathcal{M}_\alpha, \tag{44}$$

whose dimensions are the peripheral multiplicities ℓ_P and m_P of Φ and \mathcal{L} defined in Eqs. 11 and 13, respectively. Let $\text{Fix}(\Phi)$ stand for the fixed-point space of Φ , i.e.

$$\text{Fix}(\Phi) = \{A \in \mathcal{B}(\mathcal{H}) : \Phi(A) = A\}, \tag{45}$$

and $\text{Fix}(\Phi^*)$ indicate the fixed-point space of the dual Φ^* of Φ , defined via

$$\text{Tr}(A\Phi(B)) = \text{Tr}(\Phi^*(A)B), \quad A, B \in \mathcal{B}(\mathcal{H}). \tag{46}$$

Note that Φ^* has the same eigenvalues with the same algebraic multiplicities of Φ ⁵⁴. In addition, the spectral projections \mathcal{P} and \mathcal{P}_P onto $\text{Fix}(\Phi)$ and $\text{Attr}(\Phi)$ are quantum channels themselves¹². Finally, let $\mathcal{M}_0 \equiv \text{Ker}(\mathcal{L})$ be the kernel of \mathcal{L} , given by

$$\text{Ker}(\mathcal{L}) = \{A \in \mathcal{B}(\mathcal{H}) : \mathcal{L}(A) = 0\}. \tag{47}$$

Before discussing the proofs of Theorems 1-4, we need a few preparatory results.

Consider $A \in \mathcal{B}(\mathcal{H})$ with spectrum $\text{spect}(A) = \{\lambda_k\}_{k=1}^N$. If m_k, n_k are the algebraic and geometric multiplicities of the eigenvalue λ_k , let $d_{j,k}$ with $j = 1, \dots, n_k$ and $k = 1, \dots, N$ indicate the dimension of the j -th Jordan block corresponding to the eigenvalue λ_k of the Jordan normal form J of A ⁴³.

Proposition 5 ⁽⁵⁵⁾ *Let $A \in \mathcal{B}(\mathcal{H})$ with Jordan normal form $J \in \mathcal{M}_d(\mathbb{C})$. Then*

$$c_A = \dim\{A\}' = \sum_{k=1}^N \sum_{i=1}^{m_k} s_{i,k}^2, \tag{48}$$

where

$$s_{i,k} = |\{j = 1, \dots, n_k : d_{j,k} \geq i\}|, \tag{49}$$

with $i = 1, \dots, m_k, k = 1, \dots, N$ and $|I|$ being the cardinality of the set I .

Corollary 5.1 *Let $A \in \mathcal{B}(\mathcal{H})$ with $A \neq c\mathbb{I}, c \in \mathbb{C}$. Then*

$$c_A \leq d^2 - 2d + 2. \tag{50}$$

In particular, the equality holds if and only if A is diagonalizable with spectrum $\text{spect}(A) = \{\lambda_1, \lambda_2\}$ having algebraic multiplicities $m_1 = 1$ and $m_2 = d - 1$.

Proof By definition $\{s_{i,k}\}_{i=1}^{m_k}$ is a partition of m_k , so

$$c_A \leq \sum_{k=1}^N m_k^2, \tag{51}$$

where the equality holds if and only if A is diagonalizable, viz. $m_k = n_k$ for all $k = 1, \dots, N$. Now, by the fundamental theorem of algebra⁴³, $\{m_k\}_{k=1}^d$ is a partition of d , consequently

$$c_A \leq d^2, \tag{52}$$

where the equality holds if and only if $A = c\mathbb{I}, c \in \mathbb{C}$. If A is a non-scalar matrix, then the maximum value is attained when A is diagonalizable and has spectrum $\text{spect}(A) = \{\lambda_1, \lambda_2\}$ with multiplicities $m_1 = 1$ and $m_2 = d - 1$, and it reads

$$c_A^{(\max)} = m_1^2 + m_2^2 = d^2 - 2d + 2, \tag{53}$$

which concludes the proof. □

Let us now recall several known facts about open-system asymptotics. Let us start with the following definition.

Definition 6 ⁽⁴⁴⁾ A quantum channel Φ is said to be faithful if it admits an invertible steady state, i.e. $\Phi(\rho) = \rho > 0$ invertible state.

The structure of the fixed-point space of the dual of a quantum channel is related to its Kraus operators in the following way.

Proposition 7 ⁽¹⁰⁾ Let Φ be a quantum channel with Kraus operators $\mathcal{B} = \{B_k, B_k^\dagger\}_{k=1}^N$. Then

$$\mathcal{B}' \subseteq \text{Fix}(\Phi^*). \quad (54)$$

Furthermore, if Φ is faithful, then the equality holds in Eq. 54.

Let us now state the analogue of the latter result for GKLS generators, exploiting the representation 8.

Proposition 8 ⁽¹²⁾ Let \mathcal{L} be a GKLS generator of the form 8 with $\mathcal{A} = \{H, A_k, A_k^\dagger\}_{k=1}^{d^2-1}$. Then

$$\mathcal{A}' \subseteq \text{Ker}(\mathcal{L}^*), \quad (55)$$

and the equality is satisfied if there exists an invertible state $0 < \rho \in \text{Ker}(\mathcal{L})$.

Finally, the following proposition shows that we can reduce to faithful channels for the analysis of the fixed-point space.

Proposition 9 ⁽¹²⁾ Given a quantum channel Φ , define the map φ_{00} as

$$\Phi(X) = \varphi_{00}(X_0) \oplus 0, \quad X = X_0 \oplus 0 \in \mathcal{B}(\mathcal{H}_0) \oplus 0, \quad (56)$$

where $\mathcal{H}_0 := \text{supp}(\mathcal{P}(\mathbb{I}))$, i.e. the support space² of $\mathcal{P}(\mathbb{I})$. Then φ_{00} is a faithful quantum channel and

$$\text{Fix}(\Phi) = \text{Fix}(\varphi_{00}) \oplus 0, \quad (57)$$

with $\text{Fix}(\varphi_{00})$ indicating the fixed-point space of φ_{00} .

Now we are ready to prove Theorems 1–4.

Proof Theorem 1: Let $\Phi \neq 1$ be a unitary quantum channel with unitary U . Then it is easy to see from the spectral decomposition of U that

$$\mu_{k\ell} = \lambda_k \lambda_\ell^*, \quad k, \ell = 1, \dots, d, \quad (58)$$

where λ_k , with $k = 1, \dots, d$ are the (repeated) unimodular eigenvalues of U . Thus the maximum value of the algebraic multiplicity ℓ_0 of the eigenvalue 1 of Φ is $d^2 - 2d + 2$, achieved when $\lambda_1 = \dots = \lambda_{d-1} \neq \lambda_d$.

The equality $\ell_P = d^2$ is trivial because all the eigenvalues of a unitary channel are peripheral, as it is clear from Eq. 58. \square

Proof Theorem 2: First, let us prove that $\ell_0 \leq d^2 - 2d + 2$ for any non-unitary channel. Let Φ be a non-unitary channel with Kraus operators $\mathcal{B} = \{B_k, B_k^\dagger\}_{k=1}^N$. In the faithful case, see Definition 6, by applying Corollary 5.1 and Proposition 7, we obtain

$$\ell_0 = \dim \text{Fix}(\Phi^*) = \dim \mathcal{B}' \leq d^2 - 2d + 2. \quad (59)$$

If Φ is not faithful, then we can define the faithful channel φ_{00} as in Eq. 56, therefore we have as a consequence of Proposition 9

$$\ell_0 = \dim \text{Fix}(\varphi_{00}^*) = \dim \mathcal{B}'_0 \leq d_0^2 < d^2 - 2d + 2. \quad (60)$$

where $d_0 = \dim \mathcal{H}_0 \leq d - 1$ and $\mathcal{B}'_0 = \{B_{0,k}, B_{0,k}^\dagger\}_{k=1}^{N_0}$ is the system of Kraus operators of φ_{00} .

Let us now prove the analogous bound on the peripheral multiplicity ℓ_P of Φ . Observe that the spectral projection \mathcal{P}_P of Φ onto $\text{Attr}(\Phi)$ satisfies

$$\mathcal{P}_P \neq 1, \quad (61)$$

because not all the eigenvalues of the non-unitary channel Φ are peripheral. Indeed, \mathcal{P}_P is a non-unitary channel as \mathcal{P}_P is non-invertible. Therefore, since the fixed-point space of \mathcal{P}_P is $\text{Attr}(\Phi)$, it is sufficient to apply the bound 60 to \mathcal{P}_P . \square

Proof Theorem 3: Let \mathcal{L} be a non-zero Hamiltonian GKLS generator. Then it is straightforward to show that²⁷

$$\lambda_{k\ell} = -i(h_k - h_\ell), \quad k, \ell = 1, \dots, d, \quad (62)$$

where h_k , with $k = 1, \dots, d$, are the (repeated) real eigenvalues of the Hamiltonian H . Therefore this implies that the maximum value of the algebraic multiplicity m_0 of the zero eigenvalue of \mathcal{L} is $d^2 - 2d + 2$, obtained by setting $h_1 = h_2 = \dots = h_{d-1} \neq h_d$. The equality $m_P = d^2$ follows immediately from Eq. 62. \square

Proof Theorem 4: Let \mathcal{L} be a non-Hamiltonian GKLS generator. The first inequality is trivial. Since

$$m_P = \dim \text{Attr}(\Phi) = \dim \text{Attr}(\mathcal{L}), \quad (63)$$

where $\text{Attr}(\Phi)$ is the attractor subspace of the non-unitary channel $\Phi = e^{\mathcal{L}}$, the second inequality follows from Theorem 2. \square

Notice that the universal bounds given in Theorems 1–4 may also be proved by using the structure theorems on the asymptotic evolution of quantum channels^{12,14}.

Conclusions and outlooks

We found dimension-dependent sharp upper bounds on the number of independent steady states of non-trivial unitary quantum channels and an analogous bound on the number of independent asymptotic states of non-unitary channels. Moreover, similar sharp upper bounds on the number of independent steady and asymptotic states of GKLS generators were also obtained. We further made a comparison of our bounds with similar ones obtained from the CKKS conjecture 32 and 36.

Interestingly, the upper bound on the peripheral multiplicity of GKLS generators reveals that adding a dissipative perturbation to an initially Hamiltonian generator causes a jump for the peripheral multiplicity across a gap linearly depending on the dimension, and an analogous remark may be made for the peripheral multiplicity ℓ_P of quantum channels on the basis of condition 15. These bounds provide the number of physical qubits needed to implement quantum information gates within decoherence-free subspaces of a desired dimension.

These findings may be framed in a series of works, addressing the general spectral properties of open quantum systems^{12,41,51,56–59}, in particular Markovian ones, and can motivate further study of the spectral properties of channels and generators, far from being completely understood. In particular, the bounds found in this Article may be the consequence of a generalization of the CKKS bound 32 involving also the imaginary parts of the eigenvalues of a GKLS generator. Moreover, structure theorems on the asymptotic dynamics^{12,14} may be employed in order to find further constraints for the quantities studied in this work.

Data availability

All data generated or analysed during this study are included in this article.

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Author contributions

D.A. contributed to the conceptualization, investigation, methodology, and the writing of the manuscript. P.F. contributed to the conceptualization, investigation, methodology, and the writing of the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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