Letter

Central charge in quantum optics

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The product of two unitaries can normally be expressed as a single exponential through the famous Baker-Campbell-Hausdorff formula. We present here a counterexample in quantum optics, by showing that an expression in terms of a single exponential is possible only at the expense of the introduction of a new element (a central extension of the algebra), implying that there will be unitaries, generated by a sequence of gates, that cannot be generated by any time-independent quadratic Hamiltonian. A quantum-optical experiment is proposed that brings to light this phenomenon.

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Introduction. Given an ensemble of (time-independent) Hamiltonians, consider the evolution obtained by applying them in a given temporal sequence. There are many examples of this sort in quantum information and applications, where one applies quantum gates or performs quantum operations [1], in evolutions and approximations involving the Trotter product formula [2,3], and in topics related to the quantum Zeno effect [4,5] and quantum control [6].

It is often useful (and desirable) to find one (time-independent) Hamiltonian that generates the same evolution. Such a Hamiltonian normally belongs to the algebra of the initial set of operators (technically called the dynamical algebra of the system, obtained by taking linear combinations of commutators [6]). As we will discuss in this Letter, this is not true for optical gates generated by quadratic operators, describing the evolution of Gaussian states. In such a case, there will be unitaries, obtained by applying a sequence of gates, that cannot be generated by any time-independent quadratic Hamiltonian.

We will consider an explicit example that involves squeezing and phase shift. By using these two optical elements, we will obtain a transformation that cannot be obtained by applying only one optical element. Interestingly, we will see that this reflects a divergence of the Baker-Campbell-Hausdorff (BCH) series [7,8] and is technically related to the appearance of a central extension of the operator algebra [9], a phenomenon that we will thoroughly analyze in the following. We observe that such a phenomenon is familiar in quantum field theory [10] and condensed-matter physics, but not in the context of quantum optics and quantum information.

Central extensions and central charges are physically interesting for several reasons. They arise in the study of symmetry groups and their representations and can modify the algebra of the generators of symmetries and conservation laws, providing a deeper understanding of the underlying structure of physical theories. In high-energy physics, they play a significant role in the study of conformal symmetries and conservation laws [11,12], as well as anomalies in quantum field theories [13], that arise when classical symmetries are not preserved at the quantum level; in this context, they are crucial for maintaining the consistency of the theory. They are very relevant in grand unified theories [14] and string theory [15]. In condensed-matter physics, central extensions are very important in the context of the topological phases of matter [16], in the study of topological insulators [17], and in the quantum Hall effect [18]. They are also closely related to the concept of fractionalization and the emergence of anyonic excitations [19,20] and play a prominent role in the formulation of the fractional quantum Hall effect, where the Laughlin wave function can be understood in terms of a Chern-Simons theory with a central extension [21,22]. We propose here a practical example in terms of a simple quantum-optical setting and discuss a possible experiment that allows a direct observation of this phenomenon.

Applicability of the BCH formula. The product of two unitaries is unitary and thus it is expected to be expressed as a single exponential

$$e^A e^B = e^C, \tag{1}$$

where A, B, and C are anti-Hermitian operators. In the usual naive formulation of the BCH formula, one writes C as a series expansion of nested commutators,

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \frac{1}{12}[[A, B], B] + \cdots . \tag{2}$$

An important observation, in the present context, is that, if the commutators close, C can be constructed in terms of the elements of the (closed) algebra. This is at the basis of the naive expectation that by taking commutators, as in the above BCH expansion, one does not "leave" the algebra. This is what will make the appearance of a central charge counterintuitive in the following.

The BCH formula is derived under the assumption of small exponents. The convergence of the series (2) has been a subject of discussion (see, e.g., Ref. [23]) and there is no general treatment valid for large exponents, except for some simple solvable cases (a classical example being the case of $[A, B]$ commuting with both A and B). It would therefore be desirable to have a nontrivial case where the exponent C can be obtained analytically. We stress that the group is connected: $e^A e^B$ is continuously connected to the identity, e.g., by the continuous path $U(s) = e^{sA}e^{sB}$, with $s \in [0, 1]$, $U(0) = 1$, and $U(1) = e^A e^B$.

In this Letter we derive an analytical form of C in a particular yet interesting physical case and endeavor to shed light on the above issue. Our purpose is threefold. First, we provide an illustrative example in quantum optics where the exponent C requires an extension of the symplectic algebra $\mathrm{sp}(2, \mathbb{R})$. The parameters that appear in our analysis characterize the size of operators A and B, which need not be small. Second, we examine the behavior of C, especially at large values of the parameters, and show that C exhibits a bifurcation at a certain critical value of a parameter, which results in the introduction of a new element that is not included in the original algebra. A naive extension (or a wrong choice of the branch) would yield a diverging C at a horizon beyond which no single exponent exists without the extension of the algebra. We will see that such a bifurcation usually only appears when the algebra is not compact. Finally, we discuss these results on the basis of an interferometric quantum-optical example.

Case study. Consider the quadratic quantum-optical Hamiltonians, representing phase shift h_H and squeezings $h_\pm$,

$$h_H = \frac{1}{4}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger), \quad h_+ = \frac{1}{4}(\hat{a}^{\dagger 2} + \hat{a}^2),$$
$$h_- = \frac{1}{4i}(\hat{a}^{\dagger 2} - \hat{a}^2), \tag{3}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, satisfying $[\hat{a}, \hat{a}^\dagger] = 1$. They form a closed algebra under the commutation relations

$$[h_+, h_H] = -ih_-, \quad [h_H, h_-] = -ih_+, \quad [h_-, h_+] = ih_H. \tag{4}$$

Notice that the operators h_H and $h_\pm$ are Hermitian. The one-parameter (t) actions of these Hamiltonians on $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = \frac{1}{\sqrt{2}i}(\hat{a} - \hat{a}^\dagger)$ are represented by the elements of the symplectic group $\mathrm{Sp}(2, \mathbb{R})$ (see, e.g., Ref. [24]),

$$e^{ith_-}\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}e^{-ith_-} = e^{-t\frac{\sigma_3}{2}}\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}, \tag{5}$$

$$e^{ith_H}\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}e^{-ith_H} = e^{it\frac{\sigma_2}{2}}\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}. \tag{6}$$

The generators $\{-\frac{\sigma_1}{2}, \frac{i\sigma_2}{2}, -\frac{\sigma_3}{2}\}$, with σ_i ($i = 1, 2, 3$) the Pauli matrices, corresponding to $\{-ih_+, -ih_H, -ih_-\}$, are the elements of the algebra $\mathrm{sp}(2, \mathbb{R})$. Clearly, these generators satisfy the same commutation relations as (4), but without i on the right-hand sides.

The Lie algebra generated by the quantum-optical operators (3) is an infinite-dimensional representation of the symplectic algebra $\mathrm{sp}(2, \mathbb{R})$, called metaplectic representation in the mathematical literature [25,26]. The unitaries generated by those Hamiltonians form in fact a covering group of the symplectic group [with a mechanism analogous to SU(2) and SO(3)]. For a physically characterized discussion of this covering group, see, e.g., Ref. [27].

Observe that the product of a particular 2π phase shift and a squeezing operator, which is surely an element of $\mathrm{Sp}(2, \mathbb{R})$, cannot be expressed as a single exponential of $\mathrm{sp}(2, \mathbb{R})$, because

$$e^{-t\sigma_3/2}e^{\pi i\sigma_2} = \begin{pmatrix}-e^{-t/2} & 0\\ 0 & -e^{t/2}\end{pmatrix} = e^{-t\sigma_3/2 + \pi i\hat{1}}. \tag{7}$$

The extra identity operator $\hat{1}$ multiplied by πi, which is not an element of $\mathrm{sp}(2, \mathbb{R})$ and commutes with any of its elements, is necessary to express the product: this clearly shows a breakdown of the BCH formula. Technically, this signifies the appearance of a central extension of the operator algebra [9], a phenomenon that we will analyze in the following. We observe that such a phenomenon is familiar in quantum field theory [10] and condensed-matter physics, but not in the present quantum-optical context.

In order to gain additional insight into this issue, we look at a specific example and endeavor to find an analytic expression of the exponent $C = ih$ for the product (1) of two unitaries with $A = i\omega h_H$ and $B = i\eta h_-$,

$$e^{i\omega h_H}e^{i\eta h_-} = e^{ih}, \tag{8}$$

where ω and η are real parameters and h is a Hermitian operator. By parametrizing the exponent as

$$h = a\cosh\xi\,\sin\frac{\omega}{2}\,h_+ + a\cosh\xi\,\cos\frac{\omega}{2}\,h_- + a\sinh\xi\,h_H$$
$$= \alpha(\omega, \eta)\,h_+ + \beta(\omega, \eta)\,h_- + \gamma(\omega, \eta)\,h_H, \tag{9}$$

we obtain the parameters a and ξ as the solutions of the equations

$$\sinh\frac{a}{2}\cosh\xi = \sinh\frac{\eta}{2}, \tag{10}$$

$$\sinh\frac{a}{2}\sinh\xi = \cosh\frac{\eta}{2}\sin\frac{\omega}{2}, \tag{11}$$

$$\cosh\frac{a}{2} = \cosh\frac{\eta}{2}\cos\frac{\omega}{2}. \tag{12}$$

Notice that the parameters a and ξ are determined through (10)–(12) as functions of ω and η and are in general complex value. Clearly, one must make sure that the solution, obtained by inverting the trigonometric functions in (10)–(12), yields a unitary operator in (8).

Summarizing, given the parameters ω and η, which determine the two unitaries on the left-hand side of (8) and characterize the size of their exponents, their successive action can be reproduced by a unitary evolution generated by the Hamiltonian h, provided we can determine the (finite) coefficients in (9). A schematic picture of the problem is shown in Fig. 1.

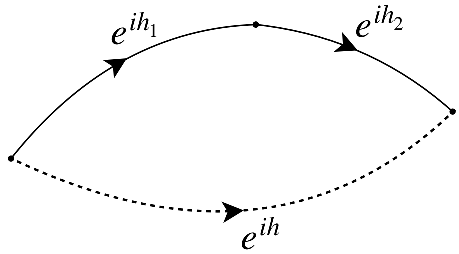

FIG. 1. Can the product of two unitaries $e^{ih_1}e^{ih_2}$ always be reproduced by a single unitary e^{ih} in linear optics?

Solution and validity range: The central charge. The full solution of the problem is involved and is detailed in the Supplemental Material [28]. We summarize here its most salient features.

It is straightforward to explicitly obtain h for small values of ω and η. On the other hand, when the parameters ω and η become larger, the problem becomes much more involved. One observes the presence of critical curves in the (ω, η) plane, which are solutions of the equation

$$\cosh\frac{\eta}{2}\cos\frac{\omega}{2} = -1. \tag{13}$$

These curves can be viewed as horizons, in the sense that the coefficients appearing in (9) diverge at these curves. This is shown in Fig. 2: Beyond the horizons it becomes impossible to express h as a linear combination of the elements of the algebra (4) and a central extension is needed to write h. The phenomenon is similar to that described in (7). The continuation of the coefficients to and through the horizon in Fig. 2 is studied in the Supplemental Material [28]. The Hamiltonian beyond the horizons reads

$$h = \alpha'(\omega,\eta)h_+ + \beta'(\omega,\eta)h_- + \gamma'(\omega,\eta)h_H + \delta\, q, \tag{14}$$

where the novel coefficients are primed and $\delta = \pi/2$ is a constant. Notice the presence of the additional operator

$$q = 2 - (-1)^{\hat{n}} = 1 + 2P_{\text{odd}}, \tag{15}$$

with $\hat{n} = \hat{a}^\dagger\hat{a}$ the number operator. Observe that q is related to the parity operator, with P_{odd} the projection operator onto the odd-number states. The operator q does not belong to the original algebra (4) and commutes with all its elements. It is needed in order to properly express the Hamiltonian beyond the horizons in Fig. 2. For instance, for $\omega = 2\pi$ and $\eta = t$, we get

$$e^{\pi i\hat{a}^\dagger\hat{a}}e^{t(\hat{a}^{\dagger 2}-\hat{a}^2)/4} = e^{t(\hat{a}^{\dagger 2}-\hat{a}^2)/4+\pi i P_{\text{odd}}}. \tag{16}$$

This is the counterpart of (7). Note that P_{odd} is not quadratic in $\hat{a}$ or $\hat{a}^\dagger$.

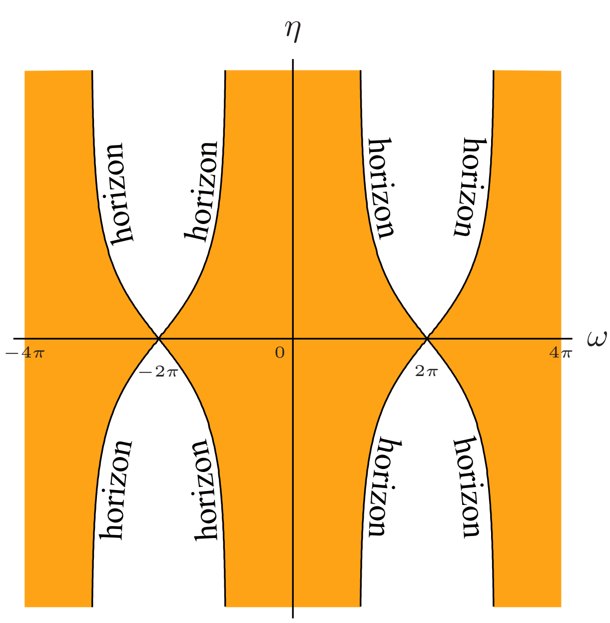

FIG. 2. Horizons in the (ω, η) plane. Along these critical curves, defined by (13), the Hamiltonian h in (9) diverges. Beyond the horizons it becomes impossible to express the Hamiltonian h as a linear combination of the elements of the algebra (4) and a central extension is needed.

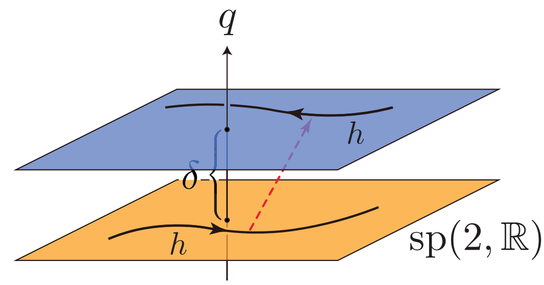

FIG. 3. Manifold spanned by the symplectic algebra $\mathrm{sp}(2,\mathbb{R})$ in (4) and the additional direction q. By approaching the horizon, the expression (9) of the Hamiltonian h in terms of the coefficients α, β, and γ diverges, ceases to be valid, and must be replaced by (14), which includes the central charge q. The upper manifold in the figure contains the regions beyond the horizons in Fig. 2. In the upper manifold, the product of two unitaries $e^{i\omega h_H}e^{i\eta h_-}$ cannot be expressed as a single exponential of a linear combination of operators in the original algebra $\mathrm{sp}(2,\mathbb{R})$.

One may get an alternative pictorial representation of the phenomenon; see Fig. 3, in which we represent the manifold spanned by the (three-dimensional) symplectic algebra $\mathrm{sp}(2,\mathbb{R})$ in (4) and an additional direction along q. Let us follow the evolution of a given trajectory of the Hamiltonian h in (9), parametrized by ω and η. By approaching the horizon (at finite values of ω and η), the Hamiltonian h and its coefficients α, β, and γ diverge and the expression (9) ceases to be valid and must be replaced by (14), which includes the additional operator (central charge) q. This is the upper manifold displayed in Fig. 3 and contains the regions beyond the horizons in Fig. 2.

Actually, one can confirm that there are two solutions of (10)–(12) coexisting in the region $-1 < \cosh\frac{\eta}{2}\cos\frac{\omega}{2} < 1$ [28]. One of them is just a smooth extension from the region $\cosh\frac{\eta}{2}\cos\frac{\omega}{2} > 1$ with smaller ω, connected with the identity 1 at the origin $(\omega,\eta) = (0,0)$, and faces a singularity at the horizon $\cosh\frac{\eta}{2}\cos\frac{\omega}{2} = -1$, as already pointed out in (13). The other solution, on the other hand, remains finite at the horizon $\cosh\frac{\eta}{2}\cos\frac{\omega}{2} = -1$, but faces a singularity at the curve $\cosh\frac{\eta}{2}\cos\frac{\omega}{2} = 1$, and is unable to reach the origin

$(\omega, \eta) = (0, 0)$. We stress that the point $\omega = \pi$ is unique, in the sense that the exponents h of the two solutions coincide only at $\omega = \pi$, where we can choose an alternative branch, keeping the exponent h continuous.

Experimental proposal. The phenomenon we described is interesting from an abstract point of view, but it would be very interesting if one could conceive an experiment able to bring it to light. One possible option is the following. Consider an interferometric experiment, in which one branch wave interacts with a squeezer h_- and then a phase shifter h_H, as on the left-hand side of (8), while the other branch wave interacts with a Trotterized version of Hamiltonian h in (14).

Consider the point $(\omega, \eta) = (2\pi - \epsilon, \eta)$ on the (ω, η) plane in Fig. 2 with $\epsilon = 0.1$ and $\eta = 0.2$, so that we are in the white region of parameters beyond the first horizon, close to the crossing point $(\omega, \eta) = (2\pi, 0)$ on the horizontal axis. Incidentally, observe that the squeezing is quite small compared to available technology [29]. The exponential of the Hamiltonian h in (14) without the parity operator q can be Trotterized as

$$(e^{i\alpha'(\omega,\eta)h_+/N} e^{i\beta'(\omega,\eta)h_-/N} e^{i\gamma'(\omega,\eta)h_H/N})^N, \tag{17}$$

where $(\alpha', \beta', \gamma')$ are explicitly given by $(\alpha_2, \beta_2, \gamma_2)$ in (60) and (61) of the SM [expressions valid beyond the first horizon, close to the point $(\omega, \eta) = (2\pi, 0)$]. We numerically computed the operator-norm difference between the Trotter product (17) and $e^{-i(\pi/2)q} e^{ih} = e^{-i(\pi/2)q} e^{i\omega h_H} e^{i\eta h_-}$ in the symplectic representation, to check the error of the Trotter approximation (17). Observe that for small ϵ and η the coefficients α', β', and γ' are small and we expect the Trotterization to work efficiently. We numerically found that for $\epsilon = 0.1$ and $\eta = 0.2$ the error of the Trotter approximation (17) is approximately equal to 0.5% surprisingly already for $N = 1$. It becomes approximately equal to 0.25% for $N = 2$ and approximately equal to 0.1% for $N = 5$, in agreement with the expectation that this difference should scale like $1/N$ [3,30,31].

The experiment then consists in comparing the Trotter product (17) (e.g., for $N = 1$) with $e^{ih} = e^{i\omega h_H} e^{i\eta h_-}$, in an interferometer. This extracts the exponential $e^{i(\pi/2)q}$ of the central charge q. As outlined after (15), q is essentially the parity operator. Thus, depending on the parity of the input photon number $\hat{n}$, we observe different interference at the output of the interferometer. This allows us to measure the central charge q.

Conclusion. We have considered a particular case of (1) with $A \propto ih_H$ and $B \propto ih_-$, as in (8). However, since the operators $ih_\pm$ and h_H satisfy the commutation relations of angular momentum and the strategy adopted in this Letter to derive ih depends only on algebraic relations, our results and conclusions can be easily generalized to other cases, with different A and B proportional to different elements of the algebra, just by performing an appropriate rotation among $ih_\pm$ and h_H together with the necessary change (or analytic continuation) of the parameters. In this sense, our outcomes are not limited to the particular case considered in (8), but have a general character.

We also observe that by further increasing (or decreasing) ω (and/or η) in Fig. 2, one can explore other regions of the central extension. It would be interesting to analyze whether the presence of a number of horizons or manifolds entails physical consequences and how they could be experimentally verified.

Observe that the phenomenon we described does not occur if all the operators $ih_\pm$ and h_H are Hermitian. In such a case, they are essentially angular momenta and no central extension appears. The singular behavior discussed in this Letter is solely due to the appearance of different signs in the commutation relations (4), reflecting the noncompactness of the algebra.

Central extensions have long been known in physics. They provide a powerful framework for understanding some fundamental aspects of quantum field theories, symmetries, anomalies, and their applications in various branches of high-energy physics. They are also crucial in unveiling the features of the topological, symmetry-protected, and fractionalized phases of matter in condensed-matter physics. The beautiful discussion by Weinberg [10] elucidates this mathematical phenomenon and its physical significance. We have shown here that the presence of central extensions can be brought to light in a simple quantum-optical setting and is amenable to direct experimental verification.

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