This is a pre-print version. The final version is published in the Milan Journal of Mathematics
DOI: 10.1007/s00032-021-00326-x

A blow-up result for a generalized Tricomi equation with nonlinearity of derivative type

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Abstract

In this note, we prove a blow-up result for a semilinear generalized Tricomi equation with nonlinear term of derivative type, i.e., for the equation $\mathscr{T}_\ell u = |\partial_t u|^p$, where $\mathscr{T}_\ell = \partial_t^2 - t^{2\ell}\Delta$. Smooth solutions blow up in finite time for positive Cauchy data when the exponent p of the nonlinear term is below $\frac{\mathbb{Q}}{\mathbb{Q}-2}$, where $\mathbb{Q} = (\ell+1)n+1$ is the quasi-homogeneous dimension of the generalized Tricomi operator $\mathscr{T}_\ell$. Furthermore, we get also an upper bound estimate for the lifespan.

Keywords Generalized Tricomi operator, Glassey exponent, Blow-up, Lifespan

AMS Classification (2010) Primary: 35B44, 35L71; Secondary: 35B33, 35C15

1 Introduction

In the present work, we prove a blow-up result for the semilinear generalized Tricomi equation with nonlinearity of derivative type, namely,

$$\begin{cases} \partial_t^2 u - t^{2\ell}\Delta u = |\partial_t u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\ell > 0$, $p > 1$ and ε is a positive constant describing the size of Cauchy data.

The semilinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = f(u, \partial_t u), & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{2}$$

has been widely investigated for several nonlinear terms $f = f(u, \partial_t u)$ over the last decades.

For the power nonlinearity $f(u, \partial_t u) = |u|^p$, $p > 1$ the small data critical exponent is the so-called Strauss exponent $p_{\mathrm{Str}}(n)$, that is, the positive root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$. This exponent was named in [23] after the author's conjecture. Strauss' conjecture has been proved by several authors (we address the reader to [6, Section 20.1] for details. Here, by critical exponent we mean that for $1 < p \leqslant p_{\mathrm{Str}}(n)$ a blow-up result holds for local in time solutions under suitable sign assumptions on the data and regardless of their size, while for $p > p_{\mathrm{Str}}(n)$ the global in time existence of small data solutions holds in suitable function spaces.

On the other hand, for the defocusing nonlinearity $f(u, \partial_t u) = -u|u|^{p-1}$, $p > 1$ the critical exponent is the Sobolev exponent $p_{\mathrm{Sob}}(n) \doteq \frac{n+2}{n-2}$. In this case, by critical exponent we mean the existence of global in time solutions for $p < p_{\mathrm{Sob}}(n)$ without any assumption on the size of $\varepsilon > 0$ (in the literature this kind of solutions are called large data solutions). We refer to the introduction of [14] for further details.

Finally, for the nonlinearity of derivative type $f(u, \partial_t u) = |\partial_t u|^p$ the small data critical exponent is the so-called *Glassey exponent*

$$p_{\mathrm{Gla}}(n) \doteq \frac{n+1}{n-1}.$$

This exponent coincides with the weak solutions blow-up exponent determined in [11] for $f(u, \partial_t u) = |u|^p$. Coming back to $f(u, \partial_t u) = |\partial_t u|^p$, we refer to [10] and references therein for a summary of the known results. Up to our best knowledge the global existence in the supercritical case for the not radial symmetric case in high dimensions is still open. We point out that the Glassey exponent appears also in the study of models somehow related to the semilinear wave equation with nonlinearity of derivative type. For example, in [12] a blow-up result for $1 < p \leqslant p_{\mathrm{Gla}}(n)$ has been proved for a semilinear damped wave model in the scattering case (see [20] for the generalization to the case of a weakly coupled system). Moreover, in [1] the nonexistence of globally in time solutions is proved for the semilinear Moore-Gibson-Thompson equation in the conservative case with the same assumptions on data, kind of nonlinearity and range for the exponent.

One can try to study existence, blow up and critical exponents for weakly semilinear hyperbolic equations. A first step in this direction was made in [4] with $u_{tt} - a^2(t)\Delta u = -u|u|^{p-1}$, where $a(t)$ may vanish. In particular, this model includes the semilinear Cauchy problem for the *generalized Tricomi operator* $\mathscr{T}_\ell \doteq \partial_t^2 - t^{2\ell}\Delta$, $\ell > 0$, namely

$$\begin{cases} \partial_t^2 u - t^{2\ell}\Delta u = f(u, \partial_t u), & x \in \mathbb{R}^n, \ t > 0, \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{3}$$

Dealing with the above discussed three different kinds of nonlinearity, we expect that the corresponding critical exponents depend on ℓ.

For $f(u, \partial_t u) = |u|^p$ the semilinear generalized Tricomi equation with power nonlinearity has been studied in several papers over the last years. Although the global existence of small data solution in the supercritical case has been proved only for $\ell = 1$ and in space dimension $n = 1, 2$ (cf. [8, 9]), due to the blow-up results both in the subcritical case and in the critical case from [7, 13], it seems reasonable that the critical exponent for (3) should be given by the greatest root of the quadratic equation

$$((\ell+1)n - 1)p^2 - ((\ell+1)n + 1 - 2\ell)p - 2(\ell+1) = 0.$$

Note that for $\ell = 0$ the previous quadratic equation provides the Strauss exponent $p_{\mathrm{Str}}(n)$, so that this exponent is a natural generalization of the Strauss exponent.

Moreover, in [2] the nonexistence of global in time solutions (under suitable sign conditions) is proved by using a *test function type* approach for a smaller range for p, namely, $1 < p \leqslant p_{\mathrm{Gla}}((\ell+1)n)$.

On the other hand, many papers have been devoted to the study of semilinear generalized Tricomi equation with large data and solutions defocusing nonlinearity, that is, (3) for $f(u, \partial_t u) = -u|u|^{p-1}$. Also in this case, a scaling of the critical exponent appears: up to dimension $n \leqslant 4$ the global existence has been proved for $p < p_{\mathrm{Sob}}((\ell+1)n)$. For an overview on these results we quote the most recent work [16] and references therein.

The purpose of this paper is to investigate (3) with nonlinearity of derivative type $f(u, \partial_t u) = |\partial_t u|^p$, which seems not present in the literature currently. According to previous results, we would expect the critical exponent to be $p_{\mathrm{Gla}}\big((\ell+1)n\big)$. In the present paper, we prove the blow-up in finite time of a local in time solution to (1) under suitable sign assumptions for the Cauchy data when the exponent of the nonlinearity $|\partial_t u|^p$ satisfies

$$1 < p \leqslant p_{\mathrm{Gla}}\big((\ell+1)n\big) = \frac{(\ell+1)n + 1}{(\ell+1)n - 1}. \tag{4}$$

As byproduct of the comparison argument that will be employed to prove the blow-up result, we find an upper bound estimate for the lifespan in terms of ε.

Let us provide an explanation on the consistency and on the reasonableness of $p_{\mathrm{Gla}}\big((\ell+1)n\big)$ as critical exponent for the semilinear Cauchy problem (1). Although a global existence result for small data solutions in the supercritical case $p > p_{\mathrm{Gla}}\big((\ell+1)n\big)$, should be considered in order to prove that this exponent is actually sharp, there are some hints that would suggest the likelihood of our conjecture. This exponent is consistent with the result for the case of the wave equation ($\ell = 0$). Moreover, we might interpret the parameter for the Glassey type exponent in a significant way: if we denote the quasi-homogeneous dimension of the generalized Tricomi operator $\partial_t^2 - t^{2\ell}\Delta$ by $\mathbb{Q} = \mathbb{Q}(\ell) \doteq (\ell+1)n + 1$ (cf. [3, 2, 15]), then, the previous exponent may be rewritten as $p_{\mathrm{Gla}}\big(\mathbb{Q}(\ell) - 1\big)$. Note that, even though we are working with a nonlinearity of derivative type rather than with a power nonlinearity, this kind of exponent can be included in the class of Fujita-type critical exponents (cf. [15, Section 2]).

Let us state now the main result.

Theorem 1.1. *Let $n \geqslant 1$ and $\ell > 0$. We assume that $(u_0, u_1) \in \mathscr{C}_0^2(\mathbb{R}^n) \times \mathscr{C}_0^1(\mathbb{R}^n)$ are nonnegative and compactly supported in $B_R \doteq \{x \in \mathbb{R}^n : |x| < R\}$ functions. Let us assume that the exponent of the nonlinearity of derive type p satisfies (4). Then, there exists $\varepsilon_0 = \varepsilon_0(n, p, \ell, u_0, u_1, R) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ if $u \in \mathscr{C}^2([0,T) \times \mathbb{R}^n)$ is a local in time solution to (1) and $T = T(\varepsilon)$ is the lifespan of u, then, u blows up in finite time.*

Furthermore, the following upper bound estimate for the lifespan holds

$$T(\varepsilon) \leqslant \begin{cases} C\varepsilon^{-\left(\frac{1}{p-1}-\frac{(\ell+1)n-1}{2}\right)^{-1}} & \text{if } 1 < p < p_{\mathrm{Gla}}\big((\ell+1)n\big), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } p = p_{\mathrm{Gla}}\big((\ell+1)n\big), \end{cases} \tag{5}$$

where the positive constant C is independent of ε.

We will apply a generalization of Zhou's method (see [27]) for the proof of the analogous result for the semilinear wave equation with nonlinearity of derivative type.

In particular, instead of the classical d'Alembert's formula we shall employ from the series of papers [24, 25] Yagdjian's integral representation formulas (obtained via a "two-step Duhamel's principle") for solutions to the Cauchy problem for Tricomi type equations. We end up with a nonlinear ordinary integral inequality for a suitable functional related to a local solution of (1). Then, a comparison argument suffices to prove Theorem 1.1. A similar method has been applied very recently in [21] to study the blow-up dynamic for the semilinear wave equation with time-dependent scale-invariant coefficients for the damping and mass and with nonlinearity of derivative type.

Remark 1. The result obtained in this work can be improved in two directions. On the one hand, the validity of a global existence result for small data solutions in the supercritical case $p > p_{\mathrm{Gla}}\big(\mathbb{Q}(\ell) - 1\big)$ should be proved.

Secondly, the approach that we employed for the proof of Theorem 1.1 strongly relies on the assumption that the Cauchy data are compactly supported. Thus, the subcritical case with not compactly supported data is open as well.

2 Fundamental tools

2.1 Integral representation formula

In this section, we firstly recall an integral representation formula for the solution of the linear Cauchy problem for the generalized Tricomi equation in the one-dimensional case, namely,

$$\begin{cases} \partial_t^2 u - t^{2\ell}\partial_x^2 u = g(t,x), & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{6}$$

where ℓ is a positive constant. For the proof of the representation formula one can see [24, Theorem 3.1] when the data are identically 0 and [22] in the sourceless case. Furthermore, by following the main steps from [26, 19, 18] it is possible to derive the representation formula for the homogeneous case as a consequence of Yagdjian's integral formula in the inhomogeneous case with vanishing initial data.

Proposition 2.1. *Let $n = 1$ and $\ell > 0$. Let us assume $u_0 \in \mathscr{C}_0^2(\mathbb{R})$, $u_1 \in \mathscr{C}_0^1(\mathbb{R})$ and $g \in \mathscr{C}([0,\infty), \mathscr{C}^1(\mathbb{R}))$. Then, a representation formula for the solution of (6) is given by*

$$\begin{aligned} u(t,x) &= a_\ell\, \phi_\ell(t)^{1-2\gamma} \int_{x-\phi_\ell(t)}^{x+\phi_\ell(t)} u_0(y) \left(\phi_\ell(t)^2 - (y-x)^2\right)^{\gamma-1} \mathrm{d}y \\ &\quad + b_\ell \int_{x-\phi_\ell(t)}^{x+\phi_\ell(t)} u_1(y) \left(\phi_\ell(t)^2 - (y-x)^2\right)^{-\gamma} \mathrm{d}y \\ &\quad + c_\ell \int_0^t \int_{x-\phi_\ell(t)+\phi_\ell(b)}^{x+\phi_\ell(t)-\phi_\ell(b)} g(b,y) E(t,x;b,y;\ell)\, \mathrm{d}y\, \mathrm{d}b, \end{aligned} \tag{7}$$

where the parameter γ and the multiplicative constants a_ℓ, b_ℓ, c_ℓ are given by

$$\gamma \doteq \frac{\ell}{2(\ell+1)}, \quad a_\ell \doteq 2^{1-2\gamma}\frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)}, \quad b_\ell \doteq 2^{2\gamma-1}(\ell+1)^{1-2\gamma}\frac{\Gamma(2-2\gamma)}{\Gamma^2(1-\gamma)}, \quad c_\ell \doteq 2^{2\gamma-1}(\ell+1)^{-2\gamma}, \tag{8}$$

the distance function ϕ_ℓ is

$$\phi_\ell(\tau) \doteq \int_0^\tau s^\ell \mathrm{d}s = \frac{\tau^{\ell+1}}{\ell+1}, \tag{9}$$

and the kernel function is defined by

$$E(t,x;b,y;\ell) \doteq \left((\phi_\ell(t)+\phi_\ell(b))^2 - (y-x)^2\right)^{-\gamma} \mathsf{F}\left(\gamma,\gamma;1;\frac{(\phi_\ell(t)-\phi_\ell(b))^2-(y-x)^2}{(\phi_\ell(t)+\phi_\ell(b))^2-(y-x)^2}\right). \tag{10}$$

Here $\Gamma(z)$ and $\mathsf{F}(a,b;c;z)$ denote the gamma function and Gauss hypergeometric function, respectively.

In the next section, we will need to estimate from below the kernel function $E(t,x;b,y;\ell)$.

Lemma 2.2. *Let $a \in \mathbb{R}$ and $c > 0$. Then,*

$$\mathsf{F}(a,a;c;z) \geqslant 1 \qquad \text{for any} \ \ z \in [0,1).$$

The proof of the previous estimate is based on the series expansion for $\mathsf{F}(a,a;c;z)$ (see for example [17, Chapter 15]). By Lemma 2.2, it follows the lower bound estimates

$$E(t,x;b,y;\ell) \geqslant \big((\phi_\ell(t)+\phi_\ell(b))^2-(y-x)^2\big)^{-\gamma} \tag{11}$$

for any $b \in [0,t]$ and any $y \in [x-\phi_\ell(t)+\phi_\ell(b), x+\phi_\ell(t)-\phi_\ell(b)]$.

2.2 The curved light cone for the Tricomi equation

From Proposition 2.1 we see that if $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset B_R$ and $\operatorname{supp} g \subset \{(t,x) \in [0,\infty)\times\mathbb{R}^n : |x| \leqslant \phi_\ell(t)+R\}$, then, $\operatorname{supp} u(t,\cdot) \subset B_{R+\phi_\ell(t)}$ for any $t \geqslant 0$.

Hence, the forward light cone associated to the generalized Tricomi operator in the one dimensional case is $\{(t,x)\in[0,\infty)\times\mathbb{R} : |x| = \phi_\ell(t)\}$. This property holds still true in higher dimensions, see for example [24] where the representation formulae in higher dimensions are provided.

Following the same steps as in [5, Section 2.1], it is possible to prove a local in time existence result for (1), regardless the size of the Cauchy data.

Carrying out the chance of variables $v(t,x) = u(\psi(t),x)$ the semilinear equation in (1) can be rewritten as

$$v_{tt} = (\psi)^{2\ell}(\psi')^2\Delta v + \frac{\psi''}{\psi'}v_t + (\psi')^{2-p}|v_t|^p.$$

Therefore, choosing ψ so that $\psi' = \psi^{-\ell}$, that is, ψ is the inverse function of ϕ_ℓ, we have that v solves

$$v_{tt} - \Delta v + \frac{\mu_\ell}{t}\,\partial_t v = c_{\ell,p}t^{\mu_\ell(p-2)}|\partial_t v|^p,$$

where $\mu_\ell \doteq \frac{\ell}{\ell+1}$ and $c_{\ell,p} \doteq (\ell+1)^{\mu_\ell(p-2)}$, which is a semilinear Euler-Darboux-Poisson equation.

For $t > 0$ the equation is strictly hyperbolic and the classical theory applies. In particular, $v = v(t,x)$ has finite speed of propagation property. Furthermore, the light cone for v is given by $\{(t,x)\in[0,\infty)\times\mathbb{R}^n : |x| = t\}$, so, coming back to $u = u(t,x)$, if $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset B_R$ then,

$$\operatorname{supp} u(t,\cdot) \subset B_{R+\phi_\ell(t)} \quad \text{for any} \ \ t \geqslant 0. \tag{12}$$

Remark 2. According to the previously introduced change of variables, applying the techniques from our approach, we may also obtain a blow-up result for the semilinear Euler-Darboux-Poisson equation with nonlinearity of derivative type.

3 Proof of Theorem 1.1

Let u be a local (in time) solution to the Cauchy problem (1). We introduce an auxiliary function which depends on the time variable and only on the first space variable, by integrating u with respect to the remaining $(n-1)$ spatial variables. This means that, if we denote $x = (z,w)$ with $z \in \mathbb{R}$ and $w \in \mathbb{R}^{n-1}$, then, we deal with the function

$$\mathcal{U}(t,z) \doteq \int_{\mathbb{R}^{n-1}} u(t,z,w)\,\mathrm{d}w \qquad \text{for any} \ \ t>0, z\in\mathbb{R}.$$

Hereafter, we will deal formally only with the case $n \geqslant 2$ for the sake of brevity, although one can proceed exactly in the same way for $n = 1$ by working with u in place of $\mathcal{U}$. In order to describe the initial values of $\mathcal{U}$, we introduce

$$\mathcal{U}_0(z) \doteq \int_{\mathbb{R}^{n-1}} u_0(z,w)\,\mathrm{d}w, \quad \mathcal{U}_1(z) \doteq \int_{\mathbb{R}^{n-1}} u_1(z,w)\,\mathrm{d}w \qquad \text{for any} \ \ z\in\mathbb{R}.$$

Since we assume that u_0, u_1 are compactly supported with support contained in B_R, it follows that $\mathcal{U}_0, \mathcal{U}_1$ are compactly supported in $(-R,R)$. Similarly, due to the property of finite speed of propagation of perturbations for u, from (12) we have

$$\operatorname{supp} \mathcal{U}(t,\cdot) \subset (-(R+\phi_\ell(t)), R+\phi_\ell(t)) \qquad \text{for any} \ \ t>0. \tag{13}$$

Consequently, $\mathcal{U}$ solves the following Cauchy problem

$$\begin{cases} \partial_t^2 \mathcal{U} - t^{2\ell}\partial_z^2 \mathcal{U} = \int_{\mathbb{R}^{n-1}} |\partial_t u(t,z,w)|^p \, \mathrm{d}w, & z\in\mathbb{R}, \ t>0, \\ \mathcal{U}(0,z) = \varepsilon\, \mathcal{U}_0(z), & z\in\mathbb{R}, \\ \partial_t\mathcal{U}(0,z) = \varepsilon\, \mathcal{U}_1(z), & z\in\mathbb{R}. \end{cases}$$

By Proposition 2.1 it follows

$$\begin{aligned} \mathcal{U}(t,z) &= a_\ell\, \varepsilon\, \phi_\ell(t)^{1-2\gamma} \int_{z-\phi_\ell(t)}^{z+\phi_\ell(t)} \mathcal{U}_0(y) \left(\phi_\ell(t)^2 - (y-z)^2\right)^{\gamma-1} \mathrm{d}y \\ &\quad + b_\ell\, \varepsilon \int_{z-\phi_\ell(t)}^{z+\phi_\ell(t)} \mathcal{U}_1(y) \left(\phi_\ell(t)^2 - (y-z)^2\right)^{-\gamma} \mathrm{d}y \\ &\quad + c_\ell \int_0^t \int_{z-\phi_\ell(t)+\phi_\ell(b)}^{z+\phi_\ell(t)-\phi_\ell(b)} \int_{\mathbb{R}^{n-1}} |\partial_t u(b,y,w)|^p \, \mathrm{d}w\, E(t,z;b,y;\ell)\, \mathrm{d}y\, \mathrm{d}b, \end{aligned}$$

where the kernel function E is defined by (10).

Due to the sign assumption for u_0,u_1 it follows that $\mathcal{U}_0,\mathcal{U}_1$ are nonnegative functions. Therefore, using the fact that $\gamma\in(0,1)$ and estimating the kernel functions in the integrals containing $\mathcal{U}_0$ and $\mathcal{U}_1$ from below, we obtain

$$\begin{aligned} \mathcal{U}(t,z) &\geqslant 2^{\gamma-1} a_\ell\, \varepsilon\, \phi_\ell(t)^{-\gamma} \int_{z-\phi_\ell(t)}^{z+\phi_\ell(t)} \mathcal{U}_0(y) \big(\phi_\ell(t) - z + y\big)^{\gamma-1} \mathrm{d}y \\ &\quad + 2^{-\gamma} b_\ell\, \varepsilon\, \phi_\ell(t)^{-\gamma} \int_{z-\phi_\ell(t)}^{z+\phi_\ell(t)} \mathcal{U}_1(y) \big(\phi_\ell(t) - z + y\big)^{-\gamma} \mathrm{d}y \\ &\quad + c_\ell \int_0^t \int_{z-\phi_\ell(t)+\phi_\ell(b)}^{z+\phi_\ell(t)-\phi_\ell(b)} \int_{\mathbb{R}^{n-1}} |\partial_t u(b,y,w)|^p \, \mathrm{d}w\, E(t,z;b,y;\ell)\, \mathrm{d}y\, \mathrm{d}b. \end{aligned} \tag{14}$$

Let us investigate the behavior of the terms

$$\begin{aligned} J(t,z) &\doteq \phi_\ell(t)^{-\gamma} \int_{z-\phi_\ell(t)}^{z+\phi_\ell(t)} \Big(2^{\gamma-1} a_\ell \mathcal{U}_0(y) \big(\phi_\ell(t)-z+y\big)^{\gamma-1} + 2^{-\gamma} b_\ell \mathcal{U}_1(y) \big(\phi_\ell(t)-z+y\big)^{-\gamma} \Big)\, \mathrm{d}y, \\ I(t,z) &\doteq c_\ell \int_0^t \int_{z-\phi_\ell(t)+\phi_\ell(b)}^{z+\phi_\ell(t)-\phi_\ell(b)} \int_{\mathbb{R}^{n-1}} |\partial_t u(b,y,w)|^p \, \mathrm{d}w\, E(t,z;b,y;\ell)\, \mathrm{d}y\, \mathrm{d}b. \end{aligned}$$

On the characteristic line $\phi_\ell(t) - z = R$ and for $z\geqslant R$, it holds $[-R,R]\subset[z-\phi_\ell(t), z+\phi_\ell(t)]$. Since $\operatorname{supp}\mathcal{U}_0, \operatorname{supp}\mathcal{U}_1 \subset (-R,R)$, we may estimate

$$\begin{aligned} J(t,z) &\geqslant \phi_\ell(t)^{-\gamma} \int_{-R}^{R} \Big(2^{2(\gamma-1)} a_\ell R^{\gamma-1} \mathcal{U}_0(y) + 2^{-2\gamma} b_\ell R^{-\gamma} \mathcal{U}_1(y) \Big)\, \mathrm{d}y \\ &= (z+R)^{-\gamma} \int_{\mathbb{R}} \big(2^{2(\gamma-1)} a_\ell R^{\gamma-1} \mathcal{U}_0(y) + 2^{-2\gamma} b_\ell R^{-\gamma} \mathcal{U}_1(y) \big)\, \mathrm{d}y \\ &\geqslant K (z+R)^{-\gamma} \int_{\mathbb{R}^n} \big(u_0(y) + u_1(y) \big)\, \mathrm{d}y, \end{aligned}$$

where $K = K(R,\ell) = \min\{2^{2(\gamma-1)} a_\ell R^{\gamma-1}, 2^{-2\gamma} b_\ell R^{-\gamma}\}$.

Next we estimate the term $I(t,z)$. Due to the support condition $\operatorname{supp}\partial_t u(t,\cdot)\subset B_{R+\phi_\ell(t)}$ it results

$$\operatorname{supp}\partial_t u(t,z,\cdot) \subset \left\{ w\in\mathbb{R}^{n-1} : |w| \leqslant \left((R+\phi_\ell(t))^2 - z^2 \right)^{1/2} \right\} \quad \text{for any} \quad t>0, z\in\mathbb{R}.$$

Then, by Hölder's inequality, we have

$$\begin{aligned} |\partial_t \mathcal{U}(b,y)| &= \Big| \int_{\mathbb{R}^{n-1}} \partial_t u(b,y,w)\, \mathrm{d}w \Big| \\ &\leqslant \Big(\int_{\mathbb{R}^{n-1}} |\partial_t u(b,y,w)|^p \, \mathrm{d}w \Big)^{\frac{1}{p}} \big(\mathrm{meas}_{n-1} \left(\operatorname{supp} \partial_t u(b,y,\cdot) \right) \big)^{1-\frac{1}{p}} \\ &\lesssim \big((R+\phi_\ell(b))^2 - y^2 \big)^{\frac{n-1}{2}(1-\frac{1}{p})} \Big(\int_{\mathbb{R}^{n-1}} |\partial_t u(b,y,w)|^p \, \mathrm{d}w \Big)^{\frac{1}{p}}. \end{aligned}$$

Henceforth, the unexpressed multiplicative constants will depend on n, ℓ, R, p. Hence,

$$\int_{\mathbb{R}^{n-1}} |\partial_t u(b,y,w)|^p \, \mathrm{d}w \gtrsim \big((R+\phi_\ell(b))^2 - y^2\big)^{-\frac{n-1}{2}(p-1)} |\partial_t \mathcal{U}(b,y)|^p,$$

which implies in turn

$$\begin{aligned} I(t,z) &\gtrsim \int_0^t \int_{z-\phi_\ell(t)+\phi_\ell(b)}^{z+\phi_\ell(t)-\phi_\ell(b)} \big((R+\phi_\ell(b))^2 - y^2\big)^{-\frac{n-1}{2}(p-1)} |\partial_t \mathcal{U}(b,y)|^p E(t,z;b,y;\ell) \, \mathrm{d}y \, \mathrm{d}b \\ &= \int_{z-\phi_\ell(t)}^{z+\phi_\ell(t)} \int_0^{\phi_\ell^{-1}(\phi_\ell(t)-|z-y|)} \big((R+\phi_\ell(b))^2 - y^2\big)^{-\frac{n-1}{2}(p-1)} |\partial_t \mathcal{U}(b,y)|^p E(t,z;b,y;\ell) \, \mathrm{d}b \, \mathrm{d}y, \end{aligned}$$

where we used Fubini's theorem in the last equality.

Hereafter, we work on the characteristic line $\phi_\ell(t) - z = R$ and for $z \geqslant R$. Thus, shrinking the domain of integration, we get

$$\begin{aligned} I(\phi_\ell^{-1}(z+R), z) &\gtrsim \int_R^z \int_{\phi_\ell^{-1}(y-R)}^{\phi_\ell^{-1}(y+R)} \big((R+\phi_\ell(b))^2 - y^2\big)^{-\frac{n-1}{2}(p-1)} |\partial_t \mathcal{U}(b,y)|^p E(t,z;b,y;\ell) \, \mathrm{d}b \, \mathrm{d}y \\ &\gtrsim \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)} \int_{\phi_\ell^{-1}(y-R)}^{\phi_\ell^{-1}(y+R)} |\partial_t \mathcal{U}(b,y)|^p E(t,z;b,y;\ell) \, \mathrm{d}b \, \mathrm{d}y. \end{aligned}$$

Let us fix $(t,z;y,b)$ such that $\phi_\ell(t) - z = R$ and $z \geqslant R$, $y \in [R,z]$, $b \in [\phi_\ell^{-1}(y-R), \phi_\ell^{-1}(y+R)]$, then, from (11) we get

$$\begin{aligned} E(t,z;b,y;\ell) &\geqslant \big((\phi_\ell(t)+\phi_\ell(b))^2 - (z-y)^2\big)^{-\gamma} = \big((z+R+\phi_\ell(b))^2 - (z-y)^2\big)^{-\gamma} \\ &\geqslant \big((z+y+2R)^2 - (z-y)^2\big)^{-\gamma} = 2^{-2\gamma}(y+R)^{-\gamma}(z+R)^{-\gamma}. \end{aligned}$$

Consequently, we obtain

$$I(\phi_\ell^{-1}(z+R), z) \gtrsim (z+R)^{-\gamma} \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma} \int_{\phi_\ell^{-1}(y-R)}^{\phi_\ell^{-1}(y+R)} |\mathcal{U}_t(b,y)|^p \, \mathrm{d}b \, \mathrm{d}y. \tag{15}$$

Applying Jensen's inequality and $\phi_\ell^{-1}(\tau) = \big((\ell+1)\tau\big)^{\frac{1}{\ell+1}}$, we find

$$\begin{aligned} \left| \int_{\phi_\ell^{-1}(y-R)}^{\phi_\ell^{-1}(y+R)} \partial_t \mathcal{U}(b,y) \, \mathrm{d}b \right|^p &\leqslant \big(\phi_\ell^{-1}(y+R) - \phi_\ell^{-1}(y-R)\big)^{p-1} \int_{\phi_\ell^{-1}(y-R)}^{\phi_\ell^{-1}(y+R)} |\mathcal{U}_t(b,y)|^p \, \mathrm{d}b \\ &\leqslant (2R(\ell+1))^{\frac{p-1}{\ell+1}} \int_{\phi_\ell^{-1}(y-R)}^{\phi_\ell^{-1}(y+R)} |\partial_t \mathcal{U}(b,y)|^p \, \mathrm{d}b. \end{aligned} \tag{16}$$

Combining (15), (16) and the fundamental theorem of calculus, we arrive at

$$\begin{aligned} I(\phi_\ell^{-1}(z+R), z) &\gtrsim (z+R)^{-\gamma} \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma} \left| \int_{\phi_\ell^{-1}(y-R)}^{\phi_\ell^{-1}(y+R)} \partial_t \mathcal{U}(b,y) \, \mathrm{d}b \right|^p \mathrm{d}y \\ &= (z+R)^{-\gamma} \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma} |\mathcal{U}(\phi_\ell^{-1}(y+R), y)|^p \, \mathrm{d}y, \end{aligned}$$

where in the second step we used $\mathcal{U}(\phi_\ell^{-1}(y-R), y) = 0$ due to (13). Combining the lower bound estimates for J and I, on the characteristic $\phi_\ell(t) - z = R$ and for $z \geqslant R$, it results

$$(R+z)^\gamma \mathcal{U}(\phi_\ell^{-1}(z+R), z) \geqslant K\varepsilon \|u_0 + u_1\|_{L^1(\mathbb{R}^n)} + C \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma} |\mathcal{U}(\phi_\ell^{-1}(y+R), y)|^p \, \mathrm{d}y, \tag{17}$$

where $C = C(n, \ell, R, p) > 0$ is the unexpressed multiplicative constant appearing in the estimate from below of $I(\phi_\ell^{-1}(z+R), z)$. We define

$$U(z) \doteq (R+z)^\gamma \mathcal{U}(\phi_\ell^{-1}(z+R), z).$$

We shall use the dynamic of this function to prove the blow-up result. We may rewrite (17) as

$$U(z) \geqslant K\varepsilon \|u_0 + u_1\|_{L^1(\mathbb{R}^n)} + C \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma(p+1)} |U(y)|^p \, \mathrm{d}y \qquad \text{for } z \geqslant R. \tag{18}$$

Now we apply a comparison argument to U. We introduce the auxiliary function G as follows:

$$G(z) \doteq M\varepsilon + C\int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma(p+1)}\,|U(y)|^p\,\mathrm{d}y \qquad \text{for } z\geqslant R,$$

where $M \doteq K\|u_0+u_1\|_{L^1(\mathbb{R}^n)}$. Then, by (18) we get $U \geqslant G$. Moreover, G satisfies the differential inequality

$$\begin{aligned} G'(z) &= C\,(R+z)^{-\frac{n-1}{2}(p-1)-\gamma(p+1)}\,|U(z)|^p \\ &\geqslant C\,(R+z)^{-\frac{n-1}{2}(p-1)-\gamma(p+1)}\,(G(z))^p. \end{aligned}$$

Let us consider the initial value $G(R)=M\varepsilon$.

If

$$-\tfrac{n-1}{2}(p-1)-\gamma(p+1)=-1, \tag{19}$$

then,

$$(M\varepsilon)^{1-p}-G(z)^{1-p}\geqslant C(p-1)\log\left(\frac{R+z}{2R}\right).$$

Otherwise, being G a positive function, we have

$$(M\varepsilon)^{1-p}-G(z)^{1-p}\geqslant \frac{C(\ell+1)}{\frac{1}{p-1}-\frac{(\ell+1)n-1}{2}}\Big((R+z)^{1-2\gamma-\frac{n+2\gamma-1}{2}(p-1)}-(2R)^{1-2\gamma-\frac{n+2\gamma-1}{2}(p-1)}\Big). \tag{20}$$

We point out that (19) is equivalent to require $p=p_{\mathrm{Gla}}\big((\ell+1)n\big)$. Hence, if $p\in\big(1,p_{\mathrm{Gla}}\big((\ell+1)n\big)\big)$ the multiplicative factor on the right-hand side of (20) is positive and we can choose $\varepsilon_0=\varepsilon_0(n,p,\ell,u_0,u_1,R)$ sufficiently small such that for $\varepsilon\in(0,\varepsilon_0]$ we obtain

$$\begin{aligned} G(z) &\geqslant \left[(M\varepsilon)^{1-p}+\frac{C(\ell+1)}{\frac{1}{p-1}-\frac{(\ell+1)n-1}{2}}\Big((2R)^{1-2\gamma-\frac{n+2\gamma-1}{2}(p-1)}-(R+z)^{1-2\gamma-\frac{n+2\gamma-1}{2}(p-1)}\Big)\right]^{-\frac{1}{p-1}} \\ &\geqslant \left[2(M\varepsilon)^{1-p}-\frac{C(\ell+1)}{\frac{1}{p-1}-\frac{(\ell+1)n-1}{2}}(R+z)^{1-2\gamma-\frac{n+2\gamma-1}{2}(p-1)}\right]^{-\frac{1}{p-1}}. \end{aligned} \tag{21}$$

In the critical case $p=p_{\mathrm{Gla}}\big((\ell+1)n\big)$, we have that

$$U(z)\geqslant G(z)\geqslant \left[(M\varepsilon)^{1-p}-C(p-1)\log\left(\tfrac{R+z}{2R}\right)\right]^{-\frac{1}{p-1}}$$

implies the blow-up in finite time of $U(z)$ and the lifespan estimate

$$T(\varepsilon)\lesssim \exp\left(\widetilde{C}\varepsilon^{-(p-1)}\right).$$

On the other hand, in the subcritical case $p\in\big(1,p_{\mathrm{Gla}}\big((\ell+1)n\big)\big)$, in (21) the right-hand side blows up for

$$\phi_\ell(t)=R+z\simeq \varepsilon^{-(\ell+1)\left(\frac{1}{p-1}-\frac{(\ell+1)n-1}{2}\right)^{-1}},$$

where we used the relation

$$1-2\gamma-\tfrac{n+2\gamma-1}{2}(p-1)=\tfrac{1}{\ell+1}\left(1-\tfrac{(\ell+1)n-1}{2}(p-1)\right).$$

Being $U\geqslant G$, then, U blows up in finite time and the upper bound for the lifespan is given by

$$T(\varepsilon)\lesssim \varepsilon^{-\left(\frac{1}{p-1}-\frac{(\ell+1)n-1}{2}\right)^{-1}}.$$

Therefore, the proof of Theorem 1.1 is completed.

Acknowledgments

A. Palmieri is supported by the GNAMPA project 'Problemi stazionari e di evoluzione nelle equazioni di campo nonlineari dispersive'. S. Lucente is supported by the PRIN 2017 project 'Qualitative and quantitative aspects of nonlinear PDEs' and by the GNAMPA project 'Equazioni di tipo dispersivo: teoria e metodi'.

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