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THE LOGARITHMIC CHOQUARD EQUATION : SHARP ASYMPTOTICS AND NONDEGENERACY OF THE GROUNDSTATE

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ABSTRACT. We derive the asymptotic decay of the unique positive, radially symmetric solution to the logarithmic Choquard equation

$$-\Delta u + au = \frac{1}{2\pi}\left[\ln\frac{1}{|x|} * |u|^2\right] u \qquad \text{in } \mathbb{R}^2$$

and we establish its *nondegeneracy*. For the corresponding three-dimensional problem, the nondegeneracy property of the positive ground state to the Choquard equation was proved by E. Lenzmann (Analysis & PDE, 2009).

1. INTRODUCTION

We consider the nonlocal model equation

$$-\Delta u + au = \left[\Phi_N * |u|^2\right] u \qquad \text{in } \mathbb{R}^N \tag{1.1}$$

where a is a constant and $\Phi : \mathbb{R}^N \to \mathbb{R}$ is the Newton kernel, that is the fundamental solution of the Laplace equation in $\mathbb{R}^N$, namely

$$\Phi_N(x) = \frac{\Gamma(\frac{N-2}{2})}{4\pi^{N/2}|x|^{N-2}} \quad \text{if } N \geq 3, \quad \text{and} \quad \Phi(x) = \frac{1}{2\pi}\ln\frac{1}{|x|} \quad \text{if } N = 2.$$

In dimension $N = 3$, the integro-differential equation (1.1) has been introduced to study the quantum physics of *electrons in an ionic crystal* (Pekar's polaron model) [23]. It has later also been proposed as a *coupling of quantum physics with Newtonian gravitation* [9, 12, 24]. E. H. Lieb has proved the existence of a unique ground state solution of (1.1) in dimension $N = 3$, which is positive and radially symmetric [14] (see also [5, 16, 17, 21, 28]). Successively, E. Lenzmann has shown the nondegeneracy of the unique positive ground state solution to the three-dimensional equation (1.1) [13].

In this paper we focus on the planar integro-differential equation corresponding to (1.1)

$$-\Delta u + au = \frac{1}{2\pi}\left[\ln\frac{1}{|\cdot|} * |u|^2\right] u \qquad \text{in } \mathbb{R}^2. \tag{1.2}$$

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We refer to it as the logarithmic Choquard equation (or planar Schrödinger–Newton system).

This two-dimensional problem has remained for a long time a quite open field of study. While Lieb's existence proof has a straightforward extensions to the higher dimensions $N = 4$ and $N = 5$ and the existence of finite energy solutions is forbidden for $N \geq 6$ by a Pohozhaev identity (see for example [6, Lemma 2.1; 10, (56); 19, (2.8); 21, Proposition 3.1]), the situation is less clear for lower dimensions due to the *lack of positivity of the Coulomb interaction energy term.* For $N = 1$, this difficulty has been overcome recently and the existence of a unique ground state has been shown by solving a minimization problem [3].

Back to our planar case $N = 2$, after numerical studies suggesting the existence of bound states [11, §6], Ph. Choquard, J. Stubbe and M. Vuffray have proved the existence of a unique positive radially symmetric solution to (1.2) by applying a shooting method to the associated system of two ordinary differential equations [4].

In contrast with the higher-dimensional case $N \geq 3$, the applicability of variational methods is not straightforward for $N = 2$. Although (1.2) has, at least formally, a variational structure related to the energy functional

$$u \mapsto I(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + au^2) + \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|^2) |u(x)|^2 |u(y)|^2 \,\mathrm{d}x \,\mathrm{d}y$$

this energy functional is *not well-defined on the natural Sobolev space* $H^1(\mathbb{R}^2)$.

J. Stubbe has tackled that problem [26] by setting a variational framework for (1.2) within the functional space

$$X := \Big\{ u \in H^1(\mathbb{R}^2) \;:\; \int_{\mathbb{R}^2} \ln(1 + |x|) |u(x)|^2 \,\mathrm{d}x < \infty \Big\},$$

endowed with a norm defined for each function $u \in X$ by

$$\|u\|_X^2 := \int_{\mathbb{R}^2} |\nabla u(x)|^2 + |u(x)|^2 \big(a + \ln_+ |x|\big) \,\mathrm{d}x,$$

where, for each $s \in (0, +\infty)$ $\ln_+ s = (\ln s)_+$. This functional I is well-defined and continuously differentiable on the space X. Critical points $u \in X$ of I are strong solutions in $W^{2,p}(\mathbb{R}^2)$, for all $p \geq 1$, and classical solutions in $C^2(\mathbb{R}^2)$ of (1.2).

Even if X provides a variational framework for (1.2), some difficulties arise. First, the norm of X *is not invariant under translations* whereas the functional I is invariant under translations of $\mathbb{R}^2$. Second, the quadratic part of the functional I is never coercive on X, whatever the value of $a \in \mathbb{R}$.

By using strict rearrangement inequalities, J. Stubbe has proved that there exists, for any $a \geq 0$, a unique ground state, which is a positive spherically symmetric decreasing function [26]. In addition, he proved that there exists a negative number $a^* < 0$ such that for any $a \in (a^*, 0)$ there are two ground states with different L^2 norm and that in the limiting case $a = a^*$, there is again a unique ground state. T. Weth and the second author [7] recently constructed a sequence of solution pairs $(\pm u_n)_{n \in \mathbb{N}} \subset X$ of the equation (1.2) such that $I(u_n) \to \infty$ as $n \to +\infty$. They also provided a variational characterization of the least energy solution. Namely, they proved that the restriction of the functional I to the associated Nehari manifold

$\mathcal{N} := \{u \in X \setminus \{0\} \,:\, I'(u)u = 0\}$ attains a global minimum and that every minimizer $u \in \mathcal{N}$ of $I|_{\mathcal{N}}$ is a solution of (1.2) which does not change sign and obeys the variational characterization

$$I(u) = \inf_{u \in X} \sup_{t \in \mathbb{R}} I(tu).$$

In addition, the following uniqueness result was proved by T. Weth and the second author.

Theorem 1 ([7, Theorem 1.3])**.** *For every $a > 0$, every positive solution $u \in X$ of (1.2) is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center. Moreover u is unique up to translation in $\mathbb{R}^2$.*

Our first result is a description of the asymptotic behaviour of this unique positive solution of the logarithmic Choquard equation (1.2).

Theorem 2. *If $a > 0$ and if $u \in X$ is a radially symmetric positive solution of (1.2), then there exists $\mu \in (0, +\infty)$ such that, as $|x| \to \infty$,*

$$u(x) = \frac{(\mu + o(1))}{\sqrt{|x|}(\ln|x|)^{1/4}} \exp\Bigl(-\sqrt{M} e^{-a/M} \int_1^{e^{a/M}|x|} \sqrt{\ln s}\,\mathrm{d}s\Bigr),$$

where

$$M = \frac{1}{2\pi} \int_{\mathbb{R}^2} |u|^2.$$

The integral does not seem to have an explicit asymptotic equivalent at the order $o(1)$ as $|x| \to \infty$ in terms of elementary functions; roughly speaking it behaves as

$$\int_1^{e^{a/M}|x|} \sqrt{\ln s}\,\mathrm{d}s = |x|\sqrt{\ln|x|}(1 + o(1)),$$

as $|x| \to \infty$. This integral can be reexpressed in terms of classical special functions (imaginary error function or Dawson function, see Remark 3.1 below).

We obtain this decay rate by studying the decay rate of solutions to the *linear problem*

$$-\Delta u + Vu = 0,$$

when $V(x) \equiv M \ln|x|$ as $|x| \to \infty$.

The asymptotic behaviour of u is a key ingredient to derive the precise description of the kernel of the linear operator $\mathcal{L}(u)$ defined by

$$\mathcal{L}(u) : \tilde{X} \to L^2(\mathbb{R}^2) : \varphi \mapsto -\Delta\varphi + (a - w)\varphi + 2u\Bigl(\frac{\ln}{2\pi} * (u\varphi)\Bigr), \tag{1.3}$$

where

$$w : \mathbb{R}^2 \to \mathbb{R} : x \mapsto \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \frac{1}{|x - y|} |u(y)|^2 \,\mathrm{d}y \tag{1.4}$$

and

$$\tilde{X} := \Bigl\{\varphi \in X \,:\, \text{there exists } f \in L^2(\mathbb{R}^2) \text{ such that} \\ \text{for every } \psi \in C_c^\infty(\mathbb{R}^2) \int_{\mathbb{R}^2} \varphi \mathcal{L}(u)\psi = \int_{\mathbb{R}^2} f\psi\Bigr\}. \tag{1.5}$$

By standard arguments, one easily shows that $\mathcal{L}(u)$ is a self adjoint operator acting on $L^2(\mathbb{R}^2)$ with domain $\tilde{X}$. Also, differentiating the equation (1.2), it is clear that $\gamma \cdot \nabla u \in \ker \mathcal{L}(u)$ for every $\gamma \in \mathbb{R}^2$. Our main result is the nondegeneracy of the positive solution u of Theorem 1. Namely, the kernel of the operator $\mathcal{L}(u)$ is exactly the vector space spanned by the partial derivatives of u.

Theorem 3. *If $a > 0$ and $u \in X$ is a positive solution of* (1.2)*, then*

$$\ker \mathcal{L}(u) = \{\gamma \cdot \nabla u : \gamma \in \mathbb{R}^2\}.$$

The paper is organized as follows. In Section 2, we set up the variational framework and establish useful preliminary estimates. In Section 3, we study the asymptotic decay and prove Theorem 2. Section 4 is devoted to the proof of Theorem 3. Assuming without loss of generality that u is radial, we prove, as a first step, the nondegeneracy of the linearized operator $\mathcal{L}(u)$ restricted to the subspace of radial functions of $\tilde{X}$, that is, we show the triviality of its kernel on that subspace. As a second step, using the fact that u and w are radial, we describe by means of an angular decomposition, how the operator $\mathcal{L}(u)$ acts on each subspace $\tilde{X} \cap L^2_k(\mathbb{R}^2, \mathbb{C})$, where

$$L^2_k(\mathbb{R}^2;\mathbb{C}) := \{f \in L^2(\mathbb{R}^2;\mathbb{C}) \ : \ \text{for almost every } z \in \mathbb{R}^2 \simeq \mathbb{C} \text{ and } \theta \in \mathbb{R}, \\ f(e^{i\theta} z) = e^{ik\theta} f(z)\}.$$

Our proof relies on the multipole expansion of the logarithm kernel [25, §IV.5.7], which is an identity related to the generating function of the Chebyshev polynomials and is also known as the cylindrical multipole expansion (see formula (4.1)). The corresponding multipole expansion of the Newtonian kernel was already used in the proof of the nondegeneracy of the groundstate solution for the three-dimensional Choquard equation, see [13].

Finally we emphasize that the nondegeneracy of the groundstate is an important spectral assumption in a series of papers on effective solitary waves motion and semi-classical limit for Hartree type equations (see for instance [2, 8, 29]). In a forthcoming paper we use our nondegeneracy result for proving existence result of semiclassical states for the planar Schrödinger–Newton system.

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2. Variational framework

We begin by showing that the planar Choquard equation (1.2) can be derived from the Newton–Schrödinger system by a formal inversion.

Let us consider a classical solutions (u, w) of the planar Schrödinger–Newton system

$$\begin{cases} -\Delta u + au = wu & \text{in } \mathbb{R}^2, \\ \quad -\Delta w = u^2 & \text{in } \mathbb{R}^2, \end{cases} \tag{2.1}$$

where a is positive constant, subject to the conditions

$$u \in L^\infty(\mathbb{R}^2) \qquad \text{and} \qquad w(x) \to -\infty \quad \text{as } |x| \to \infty. \tag{2.2}$$

By Agmon's Theorem (see [1]), (2.1) and (2.2) imply that

$$u(x) = o(e^{-\alpha|x|}) \qquad \text{as } |x| \to \infty \text{ for every } \alpha > 0. \tag{2.3}$$

Moreover, since every semibounded harmonic function $\mathbb{R}^2 \to \mathbb{R}$ is constant, we have

$$w(x) = c + \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} |u(y)|^2 \,\mathrm{d}y, \tag{2.4}$$

for every $x \in \mathbb{R}^2$ and some constant $c \in \mathbb{R}$.

We recall that the solutions for which u is positive are known to enjoy symmetry properties [7, Theorem 6.1] up to the symmetries of the problem. Precisely, the following result holds. We write $u_\lambda(\cdot)$ to denote $\lambda^2 u(\lambda \cdot)$ for $\lambda \neq 0$.

Theorem 4. *Let $a > 0$. If (u, w) is a classical solution of (2.1) and (2.2) with $u > 0$ in $\mathbb{R}^2$, then, up to translation, the functions u and w are radially symmetric and strictly radially decreasing. Moreover, if $(\tilde{u}, \tilde{w})$ is another classical solution of (2.1) and (2.2) with $\tilde{u} > 0$ in $\mathbb{R}^2$, then the exists $x_0 \in \mathbb{R}^2$ and $\lambda > 0$ such that for each $x \in \mathbb{R}^2$,*

$$\begin{cases} \tilde{u}(x) = u_\lambda(x - x_0) \\ \tilde{w}(x) = w_\lambda(x - x_0) + a(1 - \lambda^2). \end{cases}$$

It follows from Theorem 4, that the solution (u, w) is unique up to translations, under a *suitable additional condition at infinity* on w. Indeed, we know from (2.4) that there exists $c \in \mathbb{R}$ such that

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} |u(y)|^2 \,\mathrm{d}y - w(x) = c.$$

Therefore, if $\rho > 0$ and $(\tilde{u}, \tilde{w})$ is another classical solution of (2.1) and (2.2), then, with $\lambda > 0$ given by Theorem 4,

$$\begin{aligned} &\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \frac{\rho}{|x-y|} |\tilde{u}(y)|^2 \,\mathrm{d}y - \tilde{w}(x) \\ &= \lambda^2 \Big(\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \frac{1}{|\lambda x - y|} |u(y)|^2 \,\mathrm{d}y - w(\lambda x) \Big) + \frac{\lambda^2 \ln \lambda\rho}{2\pi} \int_{\mathbb{R}^2} |u|^2 + a(\lambda^2 - 1) \\ &= \lambda^2 c + \frac{\lambda^2 \ln \lambda\rho}{2\pi} \int_{\mathbb{R}^2} |u|^2 + a(\lambda^2 - 1). \end{aligned}$$

Since the right hand side is an increasing continuous function of λ that takes $-a$ as a limit at 0 and diverges to $+\infty$ at $+\infty$, there exists a unique solution $\tilde{u}$ such that

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln \frac{\rho}{|x-y|} |\tilde{u}(y)|^2 \,\mathrm{d}y - \tilde{w}(x) = 0.$$

In particular, the asymptotic boundary condition

$$\lim_{x\to\infty} w(x) - \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln\frac{\rho}{|x-y|}|u(y)|^2\,\mathrm{d}y = 0$$

implies uniqueness of the solution up to translations. Because of the decay of u at infinity, this is equivalent with requiring the asymptotic condition

$$\lim_{x\to\infty} w(x) - \frac{1}{2\pi}\ln\frac{\rho}{|x|}\int_{\mathbb{R}^2}|u|^2 = 0.$$

Fixing thus $\rho > 0$, we are reduced to consider the integro-differential equation

$$-\Delta u + au = \frac{1}{2\pi}\Big[\ln\frac{1}{|\cdot|} * |u|^2\Big]\, u \qquad \text{in } \mathbb{R}^2. \tag{2.5}$$

Solutions of (2.5) are formally critical points of the functional

$$\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2 + a|u|^2 - \frac{1}{8\pi}\int_{\mathbb{R}^2}\ln\frac{1}{|x-y|}|u(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y.$$

The first integral is the norm on the Sobolev space $H^1(\mathbb{R}^2)$ induced by the scalar product

$$(u|v) = \int_{\mathbb{R}^2}(\nabla u\cdot\nabla v + a\,uv), \qquad \text{for } u, v \in H^1(\mathbb{R}^2),$$

by $\|u\|^2 := (u|u)$ for each $u \in H^1(\mathbb{R}^2)$. The second integral is not continuous on $H^1(\mathbb{R}^2)$ [7, 26]. However, an adequate functional setting that we now recall has been introduced by J. Stubbe [26] who used a smaller space with a stronger norm. One first defines for the functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, the three symmetric bilinear forms

$$\begin{aligned}
B^+(f,g) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\ln_+\frac{1}{|x-y|}f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y,\\
B^-(f,g) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\ln_+|x-y|f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y,\\
B(f,g) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\ln\frac{1}{|x-y|}f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

whenever the integrand is Lebesgue measurable, so that in particular,

$$B(f,g) = B^+(f,g) - B^-(f,g).$$

The classical Young convolution inequality, see for example [15, theorem 4.2], implies

$$|B^+(f,g)| \le \frac{1}{2\pi}\Big(\int_{\mathbb{R}^2}\Big(\ln_+\frac{1}{|x|}\Big)^{\frac{p}{p-2}}\mathrm{d}x\Big)^{2-\frac{2}{p}}\|f\|_{L^p(\mathbb{R}^2)}\|g\|_{L^p(\mathbb{R}^2)}$$

for every $p \in (2, +\infty)$. On the other hand, we have for every $x, y \in \mathbb{R}^2$

$$\ln_+|x-y| \le \ln_+(|x|+|y|) \le \ln_+|x| + \ln_+|y|,$$

so that

$$\begin{aligned}
|B^-(f,g)| &= \frac{1}{2\pi}\Big|\iint_{\mathbb{R}^2\times\mathbb{R}^2}\ln_+|x-y|f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\Big|\\
&\le \frac{1}{2\pi}\left(\|g\|_{L^1(\mathbb{R}^2)}\int_{\mathbb{R}^2}|f(x)|\ln_+|x|\,\mathrm{d}x + \|f\|_{L^1(\mathbb{R}^2)}\int_{\mathbb{R}^2}|g(x)|\ln_+|x|\,\mathrm{d}x\right).
\end{aligned}$$

We have thus proved

Proposition 2.1. *For every $p > 2$, the bilinear form B is well defined and bounded on $Y \times Y$, where the space*

$$Y = \Big\{ f : \mathbb{R}^2 \to \mathbb{R} \;:\; \int_{\mathbb{R}^2} \Big(|f(x)|^p + |f(x)|(1 + \ln_+|x|) \Big) \,\mathrm{d}x < \infty \Big\},$$

is endowed with the norm defined for $f \in Y$ by

$$\|f\|_Y = \|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} |f(x)| \ln_+|x| \,\mathrm{d}x.$$

In order to go back to our original functional, we first note that the multiplication map $(u, v) \mapsto uv$ is a bilinear map which is bounded from $Z \times Z$ to Y, where the space Z is defined by

$$Z = \Big\{ u : \mathbb{R}^2 \to \mathbb{R} \;:\; \int_{\mathbb{R}^2} |u(x)|^{2p} + (1 + \ln_+|x|)|u(x)|^2 \,\mathrm{d}x < \infty \Big\}$$

is endowed with the norm defined for $u \in Z$ by

$$\|u\|_Z = \Big(\int_{\mathbb{R}^2} |u|^p \Big)^{\frac{1}{2p}} + \Big(\int_{\mathbb{R}^2} (1 + \ln_+|x|)|u(x)|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}}.$$

We now define the functional space

$$X := \Big\{ u \in H^1(\mathbb{R}^2) \;:\; \int_{\mathbb{R}^2} \ln_+|x| \, |u(x)|^2 \,\mathrm{d}x < \infty \Big\}$$

endowed with norm defined through

$$\|u\|_X^2 := \int_{\mathbb{R}^2} |\nabla u(x)|^2 + (1 + \ln_+|x|)|u(x)|^2 \,\mathrm{d}x,$$

on which we consider the functional

$$I(u) = \int_{\mathbb{R}^2} (|\nabla u|^2 + a|u|^2) \,\mathrm{d}x - \frac{1}{4} B(u^2, u^2).$$

Since the second term of the functional I is the composition of the continuous linear embedding of X into Z, a continuous bilinear map from $Z \times Z$ to Y and a continuous bilinear map from $Y \times Y$ to $\mathbb{R}$, it follows that the functional I is *smooth* on the space X. Moreover, its first two derivatives are given by

$$I'(u)[\varphi] = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla \varphi + a\, u\varphi) - B(u^2, u\varphi),$$

and

$$I''(u)[\varphi, \psi] = \int_{\mathbb{R}^2} (\nabla \varphi \cdot \nabla \psi + a\, \varphi\psi) - B(u^2, \varphi\psi) - 2B(u\varphi, u\psi),$$

for each $u, \varphi, \psi \in X$.

3. Asymptotic behaviour of the groundstate solution

The goal of the present section is to study the asymptotics of the groundstate solution to (1.2) and prove Theorem 2.

3.1. **Rough asymptotics for linear Schrödinger operators with a logarithmic potential.** We first construct upper and lower solutions to a linear problem related to (1.2). These estimates are too rough to deduce Theorem 2. We state them because the proof is quite elementary and might help the reader to understand the more sophisticated construction in the proof of Lemma 3.3 below. Moreover, the reader can verify that these rough estimates would be sufficient to obtain the proof of Theorem 2 concerning the nondegeneracy which is given in Section 4.

Lemma 3.1. *Let $V \in C(\mathbb{R}^N)$ be such that*

$$\lim_{|x|\to\infty} \frac{V(x)}{\ln|x|} = \lambda > 0.$$

For every $\varepsilon > 0$, there exist $\underline{W}_\varepsilon$, $\overline{W}_\varepsilon$ and $R_\varepsilon > 0$ such that

$$\begin{aligned}
-\Delta \underline{W}_\varepsilon + V \underline{W}_\varepsilon &\le 0 && \text{in } \mathbb{R}^2 \setminus B_{R_\varepsilon},\\
-\Delta \overline{W}_\varepsilon + V \overline{W}_\varepsilon &\ge 0 && \text{in } \mathbb{R}^2 \setminus B_{R_\varepsilon},\\
\lim_{|x|\to\infty} \frac{\underline{W}_\varepsilon(x)}{\exp\bigl(-(1+\varepsilon)|x|\sqrt{\lambda \ln|x|}\,\bigr)} &= 1,\\
\lim_{|x|\to\infty} \frac{\overline{W}_\varepsilon(x)}{\exp\bigl(-(1-\varepsilon)|x|\sqrt{\lambda \ln|x|}\,\bigr)} &= 1.
\end{aligned}$$

Proof. We define, for every $\tau \in \mathbb{R}$, the function

$$w_\tau : (1, +\infty) \to \mathbb{R} : r \mapsto \exp(-\tau r \sqrt{\ln r}).$$

We compute directly for each $r \in (1 + \infty)$,

$$w_\tau'(r) = -\tau\Bigl(\sqrt{\ln r} + \frac{1}{2\sqrt{\ln r}}\Bigr) w_\tau(r)$$

and

$$w_\tau''(r) = \tau\Biggl(\tau\Bigl(\sqrt{\ln r} + \frac{1}{2\sqrt{\ln r}}\Bigr)^2 - \frac{1}{2r\sqrt{\ln r}} + \frac{1}{4r\sqrt{(\ln r)^3}}\Biggr) w_\tau(r).$$

If we define the function $W_\tau : \mathbb{R}^2 \setminus B_1$ for each $x \in \mathbb{R}^2 \setminus B_1$ by $W_\tau(x) = w_\tau(|x|)$, we have

$$-\Delta W_\tau(x) + V(x) W_\tau(x) = \Bigl(\ln|x|\Bigl(\frac{V(x)}{\ln|x|} - \tau^2\Bigr) + O(1)\Bigr) W_\tau(x).$$

We obtain the conclusion by taking $R_\varepsilon > 1$ sufficiently large and by choosing $\underline{W}_\varepsilon = W_{\sqrt{\lambda}(1-\varepsilon)}$ and $\overline{W}_\varepsilon = W_{\sqrt{\lambda}(1+\varepsilon)}$. □

We immediately deduce from Lemma 3.1 some rough asymptotic decay estimates on the solutions of linear equations with a potential growing logarithmically at infinity.

Corollary 3.2. *Assume that $V \in C(\mathbb{R}^N)$ and that there exists $\lambda > 0$ such that*

$$\lim_{|x|\to\infty} \frac{V(x)}{\ln|x|} = \lambda.$$

If u is a positive solution of

$$-\Delta u + V u = 0, \quad x \in \mathbb{R}^2,$$

then

$$\lim_{|x|\to\infty} \frac{\ln u(x)}{|x|\sqrt{\ln|x|}} = -\sqrt{\lambda}.$$

3.2. **Refined asymptotics for linear Schrödinger operators with a logarithmic potential.** In order to obtain fine asymptotics, we rely on the following construction of upper and lower solutions.

Lemma 3.3. *Assume $V \in C(\mathbb{R}^N)$ is such that for some $\beta \in (0,1]$,*

$$\frac{V'(r)}{V(r)} = \frac{1}{r\ln r} + o\Big(\frac{1}{r^{\beta+1}}\Big)$$

as $r \to \infty$. Then, for $R > 0$ sufficiently large, there exist radial functions $\underline{W}_\pm \in C^2(\mathbb{R}^2 \setminus B_R)$ and $\overline{W}_\pm \in C^2(\mathbb{R}^2 \setminus B_R)$ such that $\underline{W}_\pm > 0$ and $\overline{W}_\pm > 0$ in $\mathbb{R}^2 \setminus B_R$,

$$-\Delta\underline{W}_\pm(x) + V(|x|)\underline{W}_\pm(x) \le 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus B_R,$$
$$-\Delta\overline{W}_\pm(x) + V(|x|)\overline{W}_\pm(x) \ge 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus B_R,$$

$$\underline{W}_\pm(x) = (1 + O(|x|^{-\beta}))\frac{\exp\Big(\pm\displaystyle\int_R^{|x|}\sqrt{V(s)}\,\mathrm{d}s\Big)}{|x|^{1/2}\,(\ln|x|)^{1/4}}$$

$$\overline{W}_\pm(x) = (1 + O(|x|^{-\beta}))\frac{\exp\Big(\pm\displaystyle\int_R^{|x|}\sqrt{V(s)}\,\mathrm{d}s\Big)}{|x|^{1/2}\,(\ln|x|)^{1/4}},$$

and

$$\overline{W}_\pm(x) = \underline{W}_\pm(x)\Big(1 + O\Big(\frac{1}{|x|^{\beta+1}}\Big)\Big),$$

as $|x| \to \infty$.

Proof of Lemma 3.1. We define the functions $w_{\tau,+}$ and $w_{\tau,-}$ for every $\tau \in \mathbb{R}$ and $r \in (0,+\infty)$ by

$$w_{\tau,\pm}(r) = \exp\Big(\pm\int_0^r \sqrt{V(s)}\,\mathrm{d}s\Big) r^{-1/2}(\ln(r/R))^{-1/4}\Big(1 - \frac{\tau}{r^\beta}\Big).$$

We compute directly

$$w'_{\tau,\pm}(r) = \Big(\pm\sqrt{V(r)} - \frac{1}{2r} - \frac{1}{4r\ln r} + \frac{\beta\tau}{r^{\beta+1} - \tau r}\Big) w_{\tau,\pm}(r)$$

and

$$\begin{aligned} w''_{\tau,\pm}(r) = &\Big(\Big(\pm\sqrt{V(r)} - \frac{1}{2r} - \frac{1}{4r\ln r} + \frac{\beta\tau}{r^{\beta+1} - \tau r}\Big)^2 \\ &\pm \frac{V'(r)}{2\sqrt{V(r)}} + \frac{1}{2r^2} + \frac{1}{4r^2\ln r} + \frac{1}{4r^2(\ln r)^2} \\ &- \frac{\beta\tau\big((\beta+1)r^\beta - \tau\big)}{(r^{\beta+1} - \tau r)^2}\Big) w_{\tau,\pm}(r). \end{aligned}$$

If we set $W_{\tau,\pm}(x) = w_{\tau,\pm}(|x|)$, we have

$$\begin{aligned} &-\Delta W_{\tau,\pm}(x) + V(x) W_{\tau,\pm}(x) \\ &\quad = \Big(\pm\sqrt{V(r)}\Big(\frac{V'(r)}{2V(r)} - \frac{1}{2r\ln r} + \frac{2\beta\tau}{r^{\beta+1} - \tau r}\Big) + O\Big(\frac{1}{r^2}\Big)\Big) W_{\tau,\pm}(x). \end{aligned}$$

We therefore conclude by taking $\underline{W}^{\pm} = w_{\mp 1,\pm}$ and $\overline{W}^{\pm} = w_{\pm 1,\pm}$. $\square$

If $V(x) = \tilde{V}(x)\ln|x|$, the assumption of Lemma 3.3 can be written
$$\frac{\tilde{V}'(x)}{\tilde{V}(x)} = o\Bigl(\frac{1}{|x|^{\beta+1}}\Bigr)$$
as $|x| \to \infty$. As we derived Corollary 3.2 from Lemma 3.1, we are able to improve the asymptotics of the solutions of linear equations thanks to Lemma 3.3.

Corollary 3.4. *Assume $V \in C(\mathbb{R}^N)$ is such that for some $\beta \in (0,1]$,*
$$\frac{V'(|x|)}{V(|x|)} = \frac{1}{|x|\ln|x|} + o\Bigl(\frac{1}{|x|^{\beta+1}}\Bigr)$$
as $|x| \to \infty$. Let u be a positive radial solution of
$$-\Delta u + Vu = 0.$$
If
$$u(x) = o\left(\frac{\exp\Bigl(\int_0^{|x|}\sqrt{V(s)}\,\mathrm{d}s\Bigr)}{|x|^{1/2}(\ln|x|)^{1/4}}\right),$$
then there exists $\mu \in (0,+\infty)$ such that, as $|x| \to \infty$,
$$u(x) = \frac{(\mu + O(r^{-\beta}))\exp\Bigl(-\int_0^{|x|}\sqrt{V(s)}\,\mathrm{d}s\Bigr)}{|x|^{1/2}(\ln|x|)^{1/4}}.$$

Proof. Let $R > 0$, $\underline{W}_{\pm}$ and $\overline{W}_{\pm}$ be given by Lemma 3.3. We now consider the function
$$v_{\varepsilon,r} = (1+\varepsilon)\frac{u(r)}{\underline{W}_{-}(r)}\underline{W}_{-} - \varepsilon\frac{u(r)}{\overline{W}_{+}(r)}\overline{W}_{+} - u,$$
for every $r \ge R$. By the growth assumptions on u and by the known growth of $\overline{W}_{-}$ and $\underline{W}_{+}$, the set
$$\Omega_{\varepsilon,r} = \{x \in \mathbb{R}^2 \setminus B_r \;:\; v_{\varepsilon,r}(x) > 0\}$$
is bounded. Moreover
$$-\Delta v_{\varepsilon,r} + V v_{\varepsilon,r} \le 0,$$
and $v_{\varepsilon,r} \le 0$ on ∂B_r, so that we infer from the weak maximum principle for second order linear operators on bounded sets that $v_{\varepsilon,r} \le 0$ in $\Omega_{\varepsilon,r}$. Henceforth, we conclude that $v_{\varepsilon,r} \le 0$ in $\mathbb{R}^2 \setminus B_r$. Since $\varepsilon > 0$ is arbitrary, we have
$$u(s) \ge \underline{W}_{-}(s)\frac{u(r)}{\underline{W}_{-}(r)}$$
for all $s \ge r \ge R$. Arguing similarly with the function
$$w_{\varepsilon,r} = (1+\varepsilon)u - \frac{u(r)}{\overline{W}_{-}(r)}\overline{W}_{-} - \varepsilon\frac{u(r)}{\underline{W}_{+}(r)}\underline{W}_{+},$$
we deduce that
$$u(s) \le \overline{W}_{-}(s)\frac{u(r)}{\overline{W}_{-}(r)},$$

for all $s \geq r \geq R$. By the asymptotic equivalence of $\underline{W}_-$ and $\overline{W}_-$, we have thus proved

$$1 \leq \frac{u(s)/\underline{W}_-(s)}{u(r)/\underline{W}_-(r)} \leq 1 + O\Big(\frac{1}{r^{\beta+1}}\Big),$$

and it follows that the function $u/\underline{W}_-$ has a limit at infinity by the Cauchy criterion of convergence. □

3.3. **Asymptotics on the logarithmic potential.** In order to use our previous asymptotic estimates, we need to understand the logarithmic term $\ln * |u|^2$. A naïve approach gives the following inequality.

Proposition 3.5. *If f is radial and $f \in L^1(\mathbb{R}^2)$, then for each $x \in \mathbb{R}^2$*

$$\Big|\int_{\mathbb{R}^2} \ln|x-y|\, f(y)\,\mathrm{d}y - \ln|x| \int_{\mathbb{R}^2} f(y)\,\mathrm{d}y\Big| \leq \int_{\mathbb{R}^2\setminus B_{|x|}} \ln\frac{|y|}{|x|} |f(y)|\,\mathrm{d}y.$$

Proof. Since f is a radial function, we obtain by Newton's shell theorem, see for instance [15, Theorem 9.7], for each $x \in \mathbb{R}^2$,

$$\begin{aligned}\int_{\mathbb{R}^2} \ln|x-y|\, f(y)\,\mathrm{d}y &= \ln|x| \int_{B_{|x|}} f(y)\,\mathrm{d}y + \int_{\mathbb{R}^2\setminus B_{|x|}} \ln|y|\, f(y)\,\mathrm{d}y \\ &= \ln|x| \int_{\mathbb{R}^2} f(y)\,\mathrm{d}y + \int_{\mathbb{R}^2\setminus B_{|x|}} \ln\frac{|y|}{|x|} f(y)\,\mathrm{d}y. \qquad \square\end{aligned}$$

3.4. **Asymptotics for the groundstate.** We now go back to the logarithmic Choquard problem (1.2) for which we derive the sharp asymptotics at infinity of groundstates, that is, Theorem 2.

Proof of Theorem 2. We first observe that since $\lim_{|x|\to\infty}(\frac{\ln}{2\pi} * |u|^2)(x) = -\infty$, we have for each $\lambda > 0$,

$$-\Delta u + \lambda u \leq 0, \quad x \in \mathbb{R}^2 \setminus B_R$$

for some $R > 0$ large enough depending on λ. It follows therefrom that

$$\lim_{|x|\to\infty} e^{\lambda|x|} u(x) = 0$$

for every $\lambda > 0$. In view of Proposition 3.5, this leads to

$$\lim_{|x|\to\infty} \Big(w(x) + M \ln|x|\Big) e^{\lambda|x|} = 0,$$

where w has been defined in (1.4) and

$$M = \frac{1}{2\pi} \int_{\mathbb{R}^2} |u|^2.$$

This implies that, as $r \to \infty$,

$$\begin{aligned}&\int_{e^{-a/M}}^{r} \sqrt{a - w(s)}\,\mathrm{d}s \\ &= \int_{e^{-a/M}}^{r} \sqrt{a + M \ln s}\,\mathrm{d}s + \int_{e^{-a/M}}^{\infty} \Big(\sqrt{a - w(s)} - \sqrt{a + M \ln(s)}\Big)\,\mathrm{d}s + o(1),\end{aligned}$$

where the second integral on the right-hand side is finite and does not depend on the variable r. We observe now by the elementary change of variable $s = e^{-a/M} t$ that

$$\int_{e^{-a/M}}^{r} \sqrt{a + M \ln s}\,\mathrm{d}s = \sqrt{M} e^{-a/M} \int_{1}^{e^{a/M} r} \sqrt{\ln t}\,\mathrm{d}t.$$

The asymptotic estimate now follows from the linear asymptotics of Corollary 3.4. $\square$

Remark 3.1. The integral appearing in the statement of Theorem 2 can be expressed in terms of classical special functions. Indeed, by integration by parts and by a change of variable $s = \exp(\sigma^2)$,

$$\begin{aligned}\int_1^\lambda \sqrt{\ln s}\,\mathrm{d}s &= \lambda\sqrt{\ln \lambda} - \int_1^\lambda \frac{1}{2\sqrt{\ln s}}\,\mathrm{d}s = \lambda\sqrt{\ln\lambda} - \int_0^{\sqrt{\ln\lambda}} e^{\sigma^2}\,\mathrm{d}\sigma\\ &= \lambda\big(\sqrt{\ln\lambda} - F(\sqrt{\ln\lambda})\big) = \lambda\sqrt{\ln\lambda} - \frac{\sqrt{\pi}}{2}\operatorname{erfi}(\sqrt{\ln\lambda}),\\ &= \lambda\sqrt{\ln\lambda} - \frac{\gamma(\frac{1}{2}, -\ln\lambda)}{2(-1)^{1/2}} = \frac{\gamma(\frac{3}{2}, -\ln\lambda)}{(-1)^{3/2}}\\ &= \sqrt{\ln\lambda}\,\Gamma(\tfrac{3}{2})\gamma^*(\tfrac{3}{2}, -\ln\lambda)\,.\end{aligned}$$

where F is *Dawson's integral*, erfi is the *imaginary error function* (defined for $z \in \mathbb{C}$ by $\operatorname{erfi}(z) = -i \operatorname{erfi}(zi)$) and γ is the *lower incomplete gamma function* (using the same branch in its computation than for $(-1)^{1/2}$) [27] and γ^* is its entire part [22].

4. Nondegeneracy of the positive solutions

Let $u \in X$ be a solution of the planar logarithmic Choquard equation (1.2) which does not change sign and satisfies the variational characterization

$$I(u) = \inf_{u\in X} \sup_{t\in\mathbb{R}} I(tu).$$

Let $w : \mathbb{R}^2 \to \mathbb{R}$ be the function defined for each $x \in \mathbb{R}^2$ by

$$w(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}|u(y)|^2\,\mathrm{d}y.$$

By Theorem 4, up to translation, this solution u is unique and radially symmetric. We can therefore assume without loss of generality that the functions u and w are *radially symmetric*. We consider the linear operator $\mathcal{L}(u)$ defined by (1.3), that is,

$$\mathcal{L}(u)\varphi = -\Delta\varphi + (a - w)\varphi + 2u\Big(\frac{\ln}{2\pi} * (u\varphi)\Big)$$

on the space $\tilde{X}$ defined by (1.5). In the sequel, we just write $\mathcal{L}$ to shorten the notation.

4.1. **Closedness of the operator $\mathcal{L}$.** We show that the operator $\mathcal{L}$ is closed.

Proposition 4.1. *The operator $\mathcal{L}$ is a closed operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.*

The proof will rely on the following estimate.

Lemma 4.2. *For each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for each $\varphi, \psi \in H^1(\mathbb{R}^2)$,*

$$\Big|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u(x)\varphi(x)u(y)\psi(y)\,\mathrm{d}x\,\mathrm{d}y\Big| \le \varepsilon\Big(\int_{\mathbb{R}^2}|\nabla\varphi|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2}|\nabla\psi|^2\Big)^{\frac{1}{2}} + C_\varepsilon\Big(\int_{\mathbb{R}^2}|\varphi|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2}|\psi|^2\Big)^{\frac{1}{2}}.$$

Proof. Let $\delta > 0$, and let $A_\delta = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \ :\ |x-y| \le \delta\}$. Using the Young convolution inequality and the Sobolev inequality for some fixed $p \in (2, +\infty)$, we have then

$$\begin{aligned}\Big|\int_{A_\delta} \ln|x-y| u(x)\varphi(x)u(y)\psi(y)\,\mathrm{d}x\,\mathrm{d}y\Big| &\le \Big(\int_{|z|\le\delta} |\ln|z||^{\frac{p}{2(p-1)}}\Big)^{2-\frac{2}{p}}\Big(\int_{\mathbb{R}^2}|u|^p|\varphi|^p\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2}|u|^p|\psi|^p\Big)^{\frac{1}{2}}\\ &\le \varepsilon\Big(\int_{\mathbb{R}^2}|\nabla\varphi|^2 + |\varphi|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2}|\nabla\psi|^2 + |\psi|^2\Big)^{\frac{1}{2}},\end{aligned}$$

provided that $\delta > 0$ is sufficiently small. On the other hand, we observe that if $|x-y| \ge \delta$, $|x| \ge \delta$ and $|y| \ge \delta$, then

$$0 \le \ln\frac{|x-y|}{\delta} \le \ln_+\frac{|x|}{\delta} + \ln_+\frac{|y|}{\delta}$$

so that, because of the exponential decay of u (Theorem 2),

$$\begin{aligned}\Big|\int_{\mathbb{R}^2\times\mathbb{R}^2\setminus A_\delta} \ln|x-y| u(x)\varphi(x)u(y)\psi(y)\,\mathrm{d}x\,\mathrm{d}y\Big| &\le \int_{\mathbb{R}^2\times\mathbb{R}^2}\Big(|\ln\delta| + \ln_+\frac{|x|}{\delta} + \ln_+\frac{|y|}{\delta}\Big)|u(x)|\,|u(y)|\,|\varphi(x)|\,|\psi(y)|\,\mathrm{d}x\,\mathrm{d}y\\ &\le C\Big(\int_{\mathbb{R}^2}|\varphi|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2}|\psi|^2\Big)^{\frac{1}{2}}. \qquad \square\end{aligned}$$

Proof of Proposition 4.1. Let $(\varphi)_{n\in\mathbb{N}}$ be a sequence in $\tilde{X}$ that converges strongly in $L^2(\mathbb{R}^2)$ to $\varphi \in L^2(\mathbb{R}^2)$ and such that $(\mathcal{L}\varphi_n)_{n\in\mathbb{N}}$ converges strongly in $L^2(\mathbb{R}^2)$ to some $f \in L^2(\mathbb{R}^2)$. We observe that, in view of Lemma 4.2, for each $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^2} \varphi_n \mathcal{L}\varphi_n \ge \frac{1}{2}\int_{\mathbb{R}^2}|\nabla\varphi|^2 + \int_{\mathbb{R}^2}(-w)|\varphi|^2 - C\int_{\mathbb{R}^2}|\varphi|^2.$$

By the convergences of the sequences $(\varphi_n)_{n\in\mathbb{N}}$ and $(\mathcal{L}\varphi_n)_{n\in\mathbb{N}}$ and by the asymptotic behaviour of the function w, it follows that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is bounded in X, and thus it converges weakly to φ in the space X.

If we now take $\psi \in C^1_c(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} \psi f = \lim_{n\to\infty} \int_{\mathbb{R}^2} \psi \mathcal{L} \varphi_n = \lim_{n\to\infty} \int_{\mathbb{R}^2} \varphi_n \mathcal{L}\psi = \int_{\mathbb{R}^2} \varphi \mathcal{L}\psi.$$

Since $\varphi \in X$, we have by the Hölder inequality and by the exponential decay of u we have for $x \in \mathbb{R}^2$ large enough, $\lvert \ln * (u\varphi)(x)\rvert \le C \ln\lvert x\rvert$. In particular $u\,(\ln *(u\varphi)) \in L^2(\mathbb{R}^2)$. ☐

4.2. **Angular splitting of the operator $\mathcal{L}$.** Since the solution u is radial, the operator $\mathcal{L}$ commutes with rotations acting on $L^2(\mathbb{R}^2)$. This suggests to use the orthogonal splitting [25, §IV.2]

$$L^2(\mathbb{R}^2;\mathbb{C}) = \bigoplus_{k\in\mathbb{Z}} L^2_k(\mathbb{R}^2;\mathbb{C}),$$

where the summands

$$L^2_k(\mathbb{R}^2;\mathbb{C}) = \{f \in L^2(\mathbb{R}^2;\mathbb{C}) \;:\; \text{for almost every } z \in \mathbb{R}^2 \simeq \mathbb{C} \text{ and } \theta \in \mathbb{R},\\ f(e^{i\theta}z) = e^{ik\theta} f(z)\},$$

are closed mutually orthogonal subspaces of $L^2(\mathbb{R}^2;\mathbb{C})$. In particular $L^2_0(\mathbb{R}^2;\mathbb{C})$ is the subspace of radial functions of $L^2(\mathbb{R}^2;\mathbb{C})$. In general, if $f \in L^2_k(\mathbb{R}^2;\mathbb{C})$, then there exists a function $g : (0,+\infty) \to \mathbb{C}$ such that

$$\int_0^\infty \lvert g(r)\rvert^2 r \,\mathrm{d}r < \infty$$

and for each $r \in (0,+\infty)$ and $\theta \in \mathbb{R}$

$$f(re^{i\theta}) = g(r)e^{ik\theta}.$$

In order to describe how the linear operator $\mathcal{L}$ acts on each of the subspaces $L^2_k(\mathbb{R}^2;\mathbb{C})$, we rely on the multipole expansion of the logarithm kernel [25, §IV.5.7]: for each $x = re^{i\theta} \in \mathbb{R}^2 \simeq \mathbb{C}$ and $y = se^{i\eta} \in \mathbb{R}^2 \simeq \mathbb{C}$, if $r > s$, then

$$\ln\lvert x-y\rvert = \ln r + \sum_{k=1}^{\infty} \Bigl(\frac{s}{r}\Bigr)^k \frac{\cos(k(\theta-\eta))}{k}. \tag{4.1}$$

This identity is related to the generating function of the Chebyshev polynomials and is also known as the cylindrical multipole expansion. The formula (4.1) follows directly from the convergence of the Taylor series of the complex logarithm, that is,

$$\ln\lvert x-y\rvert = \ln r + \operatorname{Re}\Bigl(\ln\Bigl(1 - \frac{s}{r}e^{i(\eta-\theta)}\Bigr)\Bigr) = \ln r + \sum_{k=1}^{\infty} \Bigl(\frac{s}{r}\Bigr)^k \frac{\cos(k(\theta-\eta))}{k}.$$

The corresponding multipole expansion of the Newtonian kernel was used in the proof of the nondegeneracy of the groundstate solution for the three-dimensional Choquard equation [13].

If $\varphi \in \tilde{X} \cap L^2_k(\mathbb{R}^2;\mathbb{C})$, then we can write $\varphi(re^{i\theta}) = \psi(r)e^{ik\theta}$ for some $\psi : (0,+\infty) \to \mathbb{C}$, and

$$(\mathcal{L}\varphi)(re^{i\theta}) = (\mathcal{L}_k\psi)(r)e^{ik\theta},$$

where for each $k \in \mathbb{Z} \setminus \{0\}$, the operator $\mathcal{L}_k$ is defined by

$$\mathcal{L}_k\psi(r) = -\psi''(r) - \frac{1}{r}\psi'(r) + \Big(a + \frac{k^2}{r^2} - w(r)\Big)\psi(r) \\ - \frac{u(r)}{|k|}\int_0^\infty \psi(s)u(s)\Big(\frac{\min(r,s)}{\max(r,s)}\Big)^k s\,\mathrm{d}s,$$

whereas for $k = 0$, the operator $\mathcal{L}_0$ is defined by

$$\mathcal{L}_0\psi(r) = -\psi''(r) - \frac{1}{r}\psi'(r) + (a - w(r))\psi(r) \\ - 2u(r)\int_0^\infty \psi(s)u(s)\ln\frac{1}{\min(r,s)}s\,\mathrm{d}s.$$

This last formula can also be obtained by Newton's shell theorem. We also have

$$w(r) = \int_0^\infty |u(s)|^2 s \ln\frac{1}{\max(r,s)}\,\mathrm{d}s.$$

Observe in particular that if $\varphi \in \tilde{X} \cap L^2_k(\mathbb{R}^2;\mathbb{C})$, we have $\mathcal{L}\varphi \in L^2_k(\mathbb{R}^2;\mathbb{C})$ and therefore

$$\ker\mathcal{L} = \bigoplus_{k\in\mathbb{Z}}\big(\ker\mathcal{L} \cap L^2_k(\mathbb{R}^2;\mathbb{C})\big).$$

This allows to study separately the kernels of the operators $\mathcal{L}_k$ which is our aim in the next subsections.

4.3. **Radial eigenfunctions.** We show that the kernel of the operator $\mathcal{L}_0$ is trivial. As in [13], we first decompose the operator $\mathcal{L}_0$ as follows

$$\mathcal{L}_0\psi(r) = \hat{\mathcal{L}}_0\psi(r) + 2u(r)\int_0^\infty u(s)\psi(s)\,s\ln s\,\mathrm{d}s,$$

where the operator $\hat{\mathcal{L}}_0$ is defined by

(4.2)

$$\hat{\mathcal{L}}_0\psi(r) = -\psi''(r) - \frac{1}{r}\psi'(r) + (a - w(r))\psi(r) + 2u(r)\int_0^r u(s)\psi(s)\,s\ln\frac{r}{s}\,\mathrm{d}s$$

and we prove the exponential growth of solutions v to the linear equation $\hat{\mathcal{L}}_0 v = 0$.

Lemma 4.3. *If $\psi \in C^2([0,+\infty);\mathbb{C})$ satisfies $\hat{\mathcal{L}}_0\psi = 0$, then, either $\psi = 0$ or there exists $C_1, C_2 > 0$ such that*

$$|\psi(r)| \geq C_1 u(r)\exp\Big(C_2\int_1^r \frac{1}{s|u(s)|^2}\,\mathrm{d}s\Big),$$

for each $r \geq 1$. In this last case, we have $\lim_{r\to\infty}|\psi(r)| = \infty$.

Proof. As u is a solution of (1.2), it satisfies for each $r \in (0,+\infty)$,

$$-u''(r) - \frac{1}{r}u'(r) + (a - w(r))u(r) = 0. \tag{4.3}$$

Since $\hat{\mathcal{L}}_0\psi = 0$, it follows from (4.3) that for every $r \in (0, +\infty)$,

$$\begin{aligned}
&\frac{d}{dr}\bigl(r(u(r)\psi'(r) - u'(r)\psi(r))\bigr)\\
&\qquad = r\Bigl(u(r)\psi''(r) + \frac{1}{r}u(r)\psi'(r) - u''(r)\psi(r) - \frac{1}{r}u'(r)\psi(r)\Bigr)\\
&\qquad = 2r|u(r)|^2 \int_0^r \psi(s)u(s)\, s \ln \frac{r}{s}\,\mathrm{d}s.
\end{aligned}$$

Integrating, this implies that if $\eta = \psi/u$,

$$\begin{aligned}
\eta'(r) &= \frac{r\bigl(u(r)\psi'(r) - \psi(r)u'(r)\bigr)}{r|u(r)|^2}\\
&= 2\int_0^r \frac{s|u(s)|^2}{r|u(r)|^2} \int_0^s \eta(t)|u(t)|^2\, t \ln \frac{s}{t}\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

Integrating again, we finally obtain, by exchanging the order of integration

$$\begin{aligned}
\eta(r) &= \eta(0) + 2\int_0^r \int_0^s \frac{t|u(t)|^2}{s|u(s)|^2} \int_0^t \eta(q)\, q|u(q)|^2 \ln \frac{t}{q}\,\mathrm{d}q\,\mathrm{d}t\,\mathrm{d}s\\
&= \eta(0) + \int_0^r \eta(s)b(s)\,\mathrm{d}s
\end{aligned} \tag{4.4}$$

where the function $b : (0, +\infty) \to \mathbb{R}$ is defined for $s \in (0, +\infty)$ by

$$b(s) = 2\int_s^r \int_t^r \frac{s|u(s)|^2\, t|u(t)|^2}{q|u(q)|^2} \ln \frac{t}{s}\,\mathrm{d}q\,\mathrm{d}t.$$

The formula (4.4) is in fact the integral form of a first-order homogeneous linear differential equation. Such an equation has an explicit solution, given by

$$\begin{aligned}
\eta(r) &= \eta(0)\exp\Bigl(\int_0^r b(s)\,\mathrm{d}s\Bigr)\\
&= \eta(0)\exp\Bigl(2\int_0^r \int_s^r \int_t^r \frac{s|u(s)|^2\, t|u(t)|^2}{q|u(q)|^2} \ln \frac{t}{s}\,\mathrm{d}q\,\mathrm{d}t\,\mathrm{d}s\Bigr).
\end{aligned} \tag{4.5}$$

We observe now that if $r \ge 1$,

$$\begin{aligned}
&\int_0^r \int_s^r \int_t^r \frac{s|u(s)|^2\, t|u(t)|^2}{q|u(q)|^2} \ln \frac{t}{s}\,\mathrm{d}q\,\mathrm{d}t\,\mathrm{d}s\\
&\qquad \ge \int_0^1 \int_s^1 s|u(s)|^2\, t|u(t)|^2 \ln \frac{t}{s}\,\mathrm{d}t\,\mathrm{d}s \int_1^r \frac{1}{q|u(q)|^2}\,\mathrm{d}q\\
&\qquad\qquad \ge c\int_1^r \frac{1}{q|u(q)|^2}\,\mathrm{d}q,
\end{aligned}$$

from which the conclusion follows. □

To go on, in view of (4.3), we compute

$$\hat{\mathcal{L}}_0 u(r) = 2u(r)\int_0^r |u(s)|^2\, s \ln \frac{r}{s}\,\mathrm{d}s,$$

for each $r \in (0, +\infty)$. If we now set $z(r) = ru'(r)$, we get

$$\begin{aligned}\hat{\mathcal{L}}_0 z(r) = -ru'''(r) - 3u''(r) - \frac{1}{r}u'(r)\\ + (a - w(r))ru'(r) + 2u(r)\int_0^r u(s)u'(s)\, s^2 \ln\frac{r}{s}\,\mathrm{d}s.\end{aligned}$$

On the other hand, by differentiating the equation (4.3) satisfied by u, we have, for each $r \in (0, +\infty)$

$$-u'''(r) - \frac{1}{r}u''(r) + \frac{1}{r^2}u'(r) + (a - w(r))u'(r) - w'(r)u(r) = 0, \tag{4.6}$$

where

$$w'(r) = -\int_0^r |u(s)|^2 \frac{s}{r}\,\mathrm{d}s. \tag{4.7}$$

Combining (4.3) and (4.6), we deduce that

$$\begin{aligned}\hat{\mathcal{L}}_0 z(r) &= u(r)\Big(-\int_0^r |u(s)|^2\, s\,\mathrm{d}s - 2a + 2\int_0^\infty |u(s)|^2\, s \ln\frac{1}{\max(r,s)}\,\mathrm{d}s\\ &\qquad - 2\int_0^r |u(s)|^2\, s\ln\frac{r}{s}\,\mathrm{d}s + \int_0^r |u(s)|^2\, s\,\mathrm{d}s\Big)\\ &= u(r)\Big(-2a + 2\int_0^\infty |u(s)|^2\, s\ln\frac{1}{s}\,\mathrm{d}s - 4\int_0^r |u(s)|^2\, s\ln\frac{r}{s}\,\mathrm{d}s\Big).\end{aligned}$$

If we now define the function $\zeta : (0, +\infty) \to \mathbb{R}$ for each $r \in (0, +\infty)$ by

$$\zeta(r) = 2u(r) + z(r) = 2u(r) + ru'(r),$$

we obtain

$$\hat{\mathcal{L}}_0 \zeta(r) = -2\Big(a + \int_0^\infty |u(s)|^2\, s\ln s\,\mathrm{d}s\Big)u(r).$$

We claim that

$$a + \int_0^\infty |u(s)|^2\, s\ln s\,\mathrm{d}s \neq 0.$$

Otherwise, since by integration by parts, we have for each $r \in (0, +\infty)$,

$$\int_0^r \zeta(s)\,\mathrm{d}s = \int_0^r 2u(s) + su'(s)\,\mathrm{d}s = ru(r) + \int_0^r u(s)\,\mathrm{d}s,$$

by the decay properties of u and by Lemma 4.3, this would only be possible if $\zeta = 0$ on $(0, +\infty)$. This in turn would imply that for each $r \in (0, +\infty)$, $u(r) = u(1)/r^2$, which is impossible since the function u also has to satisfy the equation (4.3).

We are now in position to prove that there are no radial eigenfunctions.

Lemma 4.4. *We have*

$$\ker \mathcal{L} \cap L^2_0(\mathbb{R}^2; \mathbb{C}) = \ker \mathcal{L}_0 = \{0\}.$$

Proof. Let us assume that $\varphi \in L^2_0(\mathbb{R}^2; \mathbb{C}) \setminus \{0\}$ and

$$\mathcal{L}_0 \varphi = 0.$$

We then deduce that

$$\hat{\mathcal{L}}_0 \varphi = -2\Big(\int_0^\infty u(s)\varphi(s)\, s\ln s\,\mathrm{d}s\Big)u$$

We define

$$\psi = \varphi - \zeta \frac{\displaystyle\int_0^\infty u(s)\varphi(s)\, s \ln s \,\mathrm{d}s}{\displaystyle\int_0^\infty |u(s)|^2\, s \ln s \,\mathrm{d}s + a}.$$

By construction, $\psi \in L^2_0(\mathbb{R}^2;\mathbb{C})$. By Lemma 4.3, this implies $\psi(r) = 0$ for each $r \in (0, +\infty)$, that is, for every $r \in (0, +\infty)$,

$$\varphi(r) = \zeta(r) \frac{\displaystyle\int_0^\infty u(s)\varphi(s)\, s \ln s \,\mathrm{d}s}{\displaystyle\int_0^\infty |u(s)|^2\, s \ln s \,\mathrm{d}s + a}.$$

Integrating, we infer that

$$\int_0^\infty u(s)\varphi(s)\, s \ln s \,\mathrm{d}s = \int_0^\infty u(s)\zeta(s)\, s \ln s \,\mathrm{d}s \frac{\displaystyle\int_0^\infty u(s)\varphi(s)\, s \ln s \,\mathrm{d}s}{\displaystyle\int_0^\infty |u(s)|^2\, s \ln s \,\mathrm{d}s + a}$$

and thus, by the definition of ζ

$$\begin{aligned}\int_0^\infty |u(s)|^2\, s \ln s \,\mathrm{d}s + a &= \int_0^\infty 2|u(s)|^2\, s \ln s \,\mathrm{d}s + \int_0^\infty u(s)u'(s)\, s^2 \ln s \,\mathrm{d}s\\ &= \int_0^\infty |u(s)|^2\, s \ln s \,\mathrm{d}s - \frac{1}{2}\int_0^\infty |u(s)|^2\, s \,\mathrm{d}s,\end{aligned}$$

which is impossible since $a > 0$. $\square$

4.4. **Nonradial eigenfunctions.** For every $k \in \mathbb{Z} \setminus \{0\}$, we have the variational definition of the eigenvalues

$$\lambda_0(\mathcal{L}_k) = \inf\Big\{\mathcal{Q}_k(\psi) \;:\; \psi e^{ik\theta} \in X \text{ and } \int_0^\infty |\psi(r)|^2\, r \,\mathrm{d}r = 1\Big\},$$

where

$$\begin{aligned}\mathcal{Q}_k(\psi) = \int_0^\infty \big(|\psi'(r)|^2 + (1 - w(r))|\psi(r)|^2\big)\, r \,\mathrm{d}r + k^2 \int_0^\infty \frac{|\psi(r)|^2}{r} \,\mathrm{d}r&\\ - \operatorname{Re} \int_0^\infty \int_0^\infty \psi(r)\overline{\psi(s)} K_k(r, s) \,\mathrm{d}s \,\mathrm{d}r&,\end{aligned}$$

with

$$K_k(r, s) = \frac{1}{|k|} u(r)u(s)\Big(\frac{\min(r, s)}{\max(r, s)}\Big)^k rs,$$

is defined for $\psi \in W^{1,1}_{\mathrm{loc}}((0, +\infty);\mathbb{C})$ such that

$$\int_0^\infty \big(|\psi'(r)|^2 + |\psi(r)|^2\big)\, r + \frac{|\psi(r)|^2}{r} + |\psi(r)|^2 r \ln_+(r) \,\mathrm{d}r < \infty.$$

Thanks to the logarithmic weight in the definition of the functional space X and the logarithmic growth of w at infinity, the eigenvalue $\lambda_0(\mathcal{L}_k)$ is achieved. We now observe that

$$\mathcal{Q}_k(|\psi|) = \mathcal{Q}_k(\psi) - \int_0^\infty \int_0^\infty \big(|\psi(r)||\psi(s)| - \operatorname{Re}(\psi(r)\overline{\psi(s)})\big)\, K_k(r, s) \,\mathrm{d}s,$$

Since $K_k(r, s) > 0$ on $(0, +\infty) \times (0, +\infty)$, it follows that

$$\mathcal{Q}_k(|\psi|) \geq \mathcal{Q}_k(\psi),$$

with equality if and only if $\mathrm{Re}(\psi(r)\overline{\psi(s)}) = |\psi(s)||\psi(r)|$ for almost every $(r,s) \in (0,+\infty) \times (0,+\infty)$, which implies that there exists $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and for each $r \in (0,+\infty)$,

$$\psi(r) = \alpha|\psi(r)|.$$

In particular, the space of eigenvectors corresponding to $\lambda_0(\mathcal{L}_k)$ is spanned by nonnegative eigenvectors. If ψ_1 and ψ_2 are two nonnegative eigenvectors, then the quantity $D : (0,+\infty)^2 \to \mathbb{R}$

$$D(r,s) = \psi_1(r)\psi_2(s) - \psi_2(r)\psi_1(s)$$

has constant sign, since $D(\cdot, s)$ and $D(r, \cdot)$ are real eigenvectors corresponding to the eigenvalue $\lambda_0(\mathcal{L}_k)$. Since $D(r,s) = -D(s,r)$, it follows that $D(r,s) = 0$. As a consequence, we deduce that $\lambda_0(\mathcal{L}_k)$ is a simple eigenvalue and we can assume that the associated eigenfunction is a nonnegative function $\psi_k : (0,+\infty) \to \mathbb{R}$.

Note that if $|\ell| \le |k|$, then $K_k \le K_\ell$ pointwize. Therefore, we have, since $\psi_k \ge 0$,

$$\begin{aligned}\mathcal{Q}_k(\psi_k) &= \mathcal{Q}_\ell(\psi_k) + (k^2 - \ell^2) \int_0^\infty \frac{|\psi_k(r)|^2}{r}\,\mathrm{d}r \\ &\qquad + \int_0^\infty \psi_k(r)\psi_k(s)\big(K_\ell(r,s) - K_k(r,s)\big)\,\mathrm{d}s\,\mathrm{d}r \\ &> \mathcal{Q}_\ell(\psi_k),\end{aligned}$$

from which it follows that $\lambda_0(\mathcal{L}_k) > \lambda_0(\mathcal{L}_\ell)$. We can now conclude our proof of Theorem 3 by showing that $\lambda_0(\mathcal{L}_1) = 0$.

Proof of Theorem 3. We observe that

$$\begin{aligned}\mathcal{L}_1(u')(r) = -u'''(r) - \frac{u''(r)}{r} + \frac{u'(r)}{r^2} + (a - w(r))u'(r) \\ - u(r)\int_0^\infty u(s)u(s)\frac{\min(r,s)}{\max(r,s)} s\,\mathrm{d}s.\end{aligned}$$

By (4.6) and by integration by parts, we deduce that

$$\mathcal{L}_1(u')(r) = -u(r)\Big(\int_0^r |u(s)|^2 \frac{s}{r}\,\mathrm{d}s + \int_0^\infty u'(s)u(s)\frac{\min(r,s)}{\max(r,s)} s\,\mathrm{d}s\Big) = 0.$$

This implies that u' is an eigenfunction of the operator $\mathcal{L}_1$. Assume by contradiction that $\lambda_0(\mathcal{L}_1) < 0$. Then by orthogonality of eigenfunctions, we have

$$\int_0^\infty \psi_1(r)u'(r)\,r\,\mathrm{d}r = 0,$$

which cannot hold since the function u is decreasing and the function ψ_1 is nonnegative. Therefore we have shown $\lambda_0(\mathcal{L}_1) = 0$. Since we already proved that the eigenvalue $\lambda_0(\mathcal{L}_1)$ is simple, it follows that $\ker \mathcal{L}_1 = \mathrm{span}\langle u'\rangle$. By the discussion preceding the proof, we also know that $\ker \mathcal{L}_k = \{0\}$ whenever $|k| > 1$ and Lemma 4.4 implies $\ker \mathcal{L}_0 = \{0\}$. We have thus proved that

$$\ker \mathcal{L} = \bigoplus_{k\in\mathbb{Z}} (\ker \mathcal{L}_k) = \mathcal{L}_1 = \{u'(r)e^{i\theta} : \theta \in [0, 2\pi)\}. \qquad \square$$

Remark 4.1. Alternatively, we could have used a nonlocal groundstate representation formula [20, Proposition 2.1], to write

$$\mathcal{Q}_1(\psi) = \int_0^\infty (\psi/u')' r \,\mathrm{d}r + \frac{1}{2} \int_0^\infty K_1(r,s) u'(r) u'(s) \Big(\frac{\psi(r)}{u'(r)} - \frac{\psi(s)}{u'(s)}\Big)^2 \,\mathrm{d}r \,\mathrm{d}s$$

and obtain that the eigenvalue $\lambda_0(\mathcal{L}_1)$ is equal to 0 and is simple.

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