European Journal of Mathematics
https://doi.org/10.1007/s40879-023-00604-5

RESEARCH ARTICLE

Contractibility of homogeneous Kenmotsu manifolds

Antonio Lotta[1]

Received: 27 April 2022 / Revised: 5 August 2022 / Accepted: 18 September 2022

Abstract
We prove that every homogeneous Kenmotsu manifold is a contractible space.

Keywords Almost contact metric manifold · Homogeneous Kenmotsu manifold

Mathematics Subject Classification 53C25 · 53C15 · 53C30

1 Introduction

Kenmotsu manifolds constitute a relevant class of almost contact metric manifolds, introduced in 1972 in [5]; the original motivation was Tanno's classification of almost contact metric manifolds having largest automorphism group [12]. In Tanno's list some warped products of the complex Euclidean space and the real line appear, being particular cases of Kenmotsu manifolds. Recall that an almost contact metric structure (φ, ξ, η, g) on an odd dimensional manifold M consists in a $(1,1)$-tensor field φ, a vector field ξ, a one form η and a Riemannian metric g satisfying:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where X, Y are arbitrary vector fields. For more information and background on this notion we refer the reader to Blair's monograph [2].

In terms of these structure tensors, the analytic condition by which the class in object is defined is the following:

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{1.1}$$

where ∇ is the Levi-Civita connection.

✉ Antonio Lotta
antonio.lotta@uniba.it

1 Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via E. Orabona 4, 70125 Bari, Italy

Published online: 13 February 2023

Kenmotsu elucidated the structure of these manifolds, proving that locally they are a warped product of a Kähler manifold with the real line, with warping function of the form $f(t) = ce^t$, and obtained some fundamental curvature properties. Accordingly, there is a wide literature focusing on results of local nature, concerning for instance generalizations of the Einstein condition (e.g. η-Einstein manifolds, almost η-Ricci solitons) and the differential geometry of relevant kind of submanifolds (e.g. like warped product submanifolds); see for instance [4, 8, 10, 13]. The interested reader can find a discussion of recent results about Kenmotsu manifolds in [1, Section 2], including a rich bibliography on the subject.

On the other hand, to the author's knowledge, up to now no results of global nature have been discussed and no relevant information emerges from the literature about the topological structure of Kenmotsu manifolds (apart from the fact that such a manifold cannot be compact, already treated in [5]).

In particular, a global classification of the homogeneous models is missing, with the exception of the 3-dimensional case, which has been examined recently by Wang [14]. It turns out that the unique simply connected, three-dimensional homogeneous Kenmotsu manifold is the 3-dimensional hyperbolic space form $\mathbb{H}^3$. Of course, homogeneity means that the automorphism group $\mathrm{Aut}(M)$ of M acts transitively; this is the closed Lie subgroup of the isometry group of (M, g), consisting of those isometries which preserve the structure tensors φ and η.

It is an open question if this rigidity result can be extended in higher dimension. In this note we make a first step in approaching this problem, proving the following fact concerning the topological structure of homogeneous Kenmotsu manifolds:

Theorem 1.1 *Let $(M, \varphi, \xi, \eta, g)$ be a homogeneous Kenmotsu manifold. Then M is a contractible space.*

Clearly, this implies that Wang's classification can be refined as follows:

Corollary 1.2 *Up to equivalence, the 3-dimensional hyperbolic space form $\mathbb{H}^3$ is the unique homogeneous 3-dimensional Kenmotsu manifold.*

We also remark that our result might be useful if combined with the recent characterization of contractible homogeneous Kähler manifolds obtained by Loi and Mossa in [7, Theorem 1.2], in order to investigate further the classification of homogeneous Kenmotsu manifolds.

2 Proof of the result

We start by recalling the following relevant fact, which we state as a lemma:

Lemma 2.1 *For each point p of a non-contractible homogeneous Riemannian manifold M there exists a periodic geodesic starting from p.*

Even though this fact is well-known, we sketch here the proof for convenience. By a general result of Serre, given a point p of a complete, non-contractible Riemannian manifold M, there always exists a geodesic loop $\gamma\colon [0, 1] \to M$ such that $\gamma(0) =$

$\gamma(1) = p$ (see [11] or [3]); extend γ to a geodesic $\gamma\colon \mathbb{R} \to M$. Then, if M is homogeneous, γ must be periodic, being non-injective (cf. e.g. [9, p. 321]).

Now we proceed with the proof of our result. Let $(M, \varphi, \xi, \eta, g)$ be a homogeneous Kenmotsu manifold. Recall that, as a consequence of (1.1), one has

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.1}$$

for every vector field X. We first show that each integral curve $\gamma\colon \mathbb{R} \to M$ of ξ is not periodic. Let $p = \gamma(0)$ and fix a tangent vector $v \in T_pM$ orthogonal to ξ_p, $v \neq 0$. Then, by homogeneity, v can be extended to a Killing vector field V such that $[V, \xi] = 0$. Indeed, denoting by $\mathfrak{g}$ the Lie algebra of the automorphism group $\mathrm{Aut}(M)$ of M, since the action of $\mathrm{Aut}(M)$ is transitive the mapping

$$Z \in \mathfrak{g} \mapsto Z^*_p \in T_pM$$

is surjective. Here Z^* denotes the fundamental vector field generated by Z by means of the action, cf. e.g. [6, Chapter I]. So it suffices to take $V = Z^*$, where $Z \in \mathfrak{g}$ is chosen so that $Z^*_p = v$. Since by construction the flow of V preserves ξ, we have $[V, \xi] = 0$.

We remark that, since γ is a geodesic, V remains orthogonal to ξ along γ, i.e. $\eta(V_{\gamma(t)}) = 0$ for all t. Consider the function

$$F\colon \mathbb{R} \to \mathbb{R}, \quad F := g(V, V) \circ \gamma.$$

Then, taking (2.1) into account, we have

$$F'(t) = 2g(\nabla_\xi V, V)(\gamma(t)) = 2g(\nabla_V \xi, V)(\gamma(t)) = 2g(V, V)(\gamma(t)),$$

i.e.

$$F' = 2F.$$

If γ were periodic, F would also be periodic, and this would force $F = 0$, leading to a contradiction since $F(0) = g(v, v) \neq 0$.

Now, assume that M is non-contractible and consider a periodic geodesic $\gamma\colon \mathbb{R} \to M$, parametrized by arc length (this is possible in accordance with the above lemma). Set $p := \gamma(0)$. Let $G\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$G(t) := \eta(\dot\gamma(t)).$$

Clearly, G is periodic. Again by (2.1), we have

$$G' = g(\dot\gamma(t) - \eta(\dot\gamma(t))\xi, \dot\gamma(t)),$$

whence

$$G' = 1 - G^2.$$

Since $|G| \leqslant 1$, it follows that $G' \geqslant 0$, therefore G must be constant. In particular,

$$0 = G'(0) = 1 - g(\dot{\gamma}(0), \xi_p)^2,$$

yielding

$$\dot{\gamma}(0) = \pm \xi_p.$$

But this is impossible, since the integral curves of ξ and of $-\xi$ starting from p are non-periodic geodesics, according to the above discussion. This contradiction concludes our proof.

Funding Open access funding provided by Universitá degli Studi di Bari Aldo Moro within the CRUI-CARE Agreement.

References

1. Banaru, M.B., Kirichenko, V.F.: Almost contact metric structures on the hypersurface of almost Hermitian manifolds. J. Math. Sci. (N.Y.) **207**(4), 513–537 (2015)
2. Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds. Progress in Mathematics, vol. 203, 2nd edn. Birkhäuser, Boston (2010)
3. Flores, J.L.: Locally extremal geodesic loops on a Riemannian manifold. Proc. Amer. Math. Soc. **146**(9), 4029–4033 (2018)
4. Ghosh, A.: Cotton tensor, Bach tensor and Kenmotsu manifolds. Afr. Mat. **31**(7–8), 1193–1205 (2020)
5. Kenmotsu, K.: A class of almost contact Riemannian manifolds. Tohoku Math. J. **24**, 93–103 (1972)
6. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. I. Wiley, New York (1963)
7. Loi, A., Mossa, R.: Some remarks on homogeneous Kähler manifolds. Geom. Dedicata **179**, 377–383 (2015)
8. Lee, C.W., Lee, J.W., Vîlcu, G.-E.: Optimal inequalities for the normalized δ-Casorati curvatures of submanifolds in Kenmotsu space forms. Adv. Geom. **17**(3), 355–362 (2017)
9. O'Neill, B.: Semi-Riemannian Geometry: With Applications to Relativity. Pure and Applied Mathematics, vol. 103. Academic Press, New York (1983)
10. Patra, D., Rovenski, V.: Almost η-Ricci solitons on Kenmotsu manifolds. Eur. J. Math. **7**(4), 1753–1766 (2021)
11. Serre, J.-P.: Homologie singuliere des espaces fibrés. Applications. Ann. Math. **54**, 425–505 (1951)
12. Tanno, S.: The automorphism groups of almost contact Riemannian manifolds. Tohoku Math. J. **21**, 21–38 (1969)
13. Uddin, S., Khan, V.A., Khan, K.A.: Warped product submanifolds of a Kenmotsu manifold. Turkish. J. Math. **36**(2), 319–330 (2012)
14. Wang, Y.: Homogeneity and symmetry on almost Kenmotsu 3-manifolds. J. Korean Math. Soc. **56**(4), 917–934 (2019)