

Nonexistence of global solutions for the semilinear Moore – Gibson – Thompson equation in the conservative case

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Abstract. In this work, the Cauchy problem for the semilinear Moore – Gibson – Thompson (MGT) equation with power nonlinearity $|u|^p$ on the right – hand side is studied. Applying $L^2 - L^2$ estimates and a fixed point theorem, we obtain local (in time) existence of solutions to the semilinear MGT equation. Then, the blow – up of local in time solutions is proved by using an iteration method, under certain sign assumption for initial data, and providing that the exponent of the power of the nonlinearity fulfills $1 < p \leq p_{\text{Str}}(n)$ for $n \geq 2$ and $p > 1$ for $n = 1$. Here the Strauss exponent $p_{\text{Str}}(n)$ is the critical exponent for the semilinear wave equation with power nonlinearity. In particular, in the limit case $p = p_{\text{Str}}(n)$ a different approach with a weighted space average of a local in time solution is considered.

Mathematics Subject Classification (2010). Primary 35B44, 35L30, 35L76; Secondary 35B33, 35L25.

Keywords. Moore – Gibson – Thompson equation, semilinear third order equation, blow – up, Strauss exponent.

1. Introduction

In recent years, the Moore - Gibson - Thompson (MGT) equation, a linearization of a model for wave propagation in viscous thermally relaxing fluids, has caught a lot of attention (see [29, 43, 14, 20, 19, 28, 18, 25, 35, 3, 6, 24, 7, 23, 34, 2, 5] and references therein). This model is realized through the third order hyperbolic partial differential equation

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - b \Delta u_t = 0. \quad (1.1)$$

In the physical context of acoustic waves, the unknown function $u = u(t, x)$ denotes a scalar acoustic velocity, c denotes the speed of sound and τ denotes the thermal relaxation. Besides, the coefficient $b = \beta c^2$ is related to the diffusivity of the sound with $\tau \in (0, \beta]$. In particular, there is a transition from a linear model that can be described with an exponentially stable strongly continuous semigroup in the case $0 < \tau < \beta$ to the limit case $\beta = \tau$, where the exponential stability of a semigroup is lost and it holds the conservation of a suitable defined energy (see [20, 28]). For this reason, we shall call the limit case $\beta = \tau$ the *conservative case*.

In this paper, we consider the semilinear Cauchy problem associated to the MGT equation

$$\begin{cases} \beta u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t, u_{tt})(0, x) = (u_0, u_1, u_2)(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.2)$$

in the limit case $\tau = \beta > 0$, where $p > 1$ and, for the sake of simplicity, we normalized the speed of the sound by putting $c^2 = 1$. We are interested to the blow – up in finite time of local (in time) solutions under suitable sign assumptions for the Cauchy data regardless of their size and for suitable values of the exponent p .

Let us recall some results that are related to our model (1.2). By taking formally $\beta = 0$, we find the semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.3)$$

where $p > 1$. According to [17, 21, 39, 12, 13, 38, 37, 49, 27, 11, 42, 16, 45, 50, 22] the so – called *Strauss exponent* $p_{\text{Str}}(n)$ is the critical exponent of (1.3), where $p_{\text{Str}}(n)$ is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0. \quad (1.4)$$

The lifespan of solutions to (1.3) that blow up in finite time has been intensively considered. Here we refer to [26, 46, 47, 48, 27, 9, 41, 51, 40, 15]. According to these works, the sharp estimates for the lifespan $T(\varepsilon)$ are given by

$$T(\varepsilon) \approx \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{2+(n+1)p-(n-1)p^2}} & \text{if } 1 < p < p_{\text{Str}}(n), \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_{\text{Str}}(n), \end{cases}$$

for $n \geq 3$ and for $n = 2$ and $2 < p < p_{\text{Str}}(2)$, by

$$T(\varepsilon) \approx \begin{cases} \varepsilon^{-\frac{p-1}{3-p}} & \text{if } 1 < p < 2, \\ a(\varepsilon) & \text{if } p = 2, \end{cases}$$

for $n = 2$ and $1 < p \leq 2$, where $a = a(\varepsilon)$ satisfies $a^2\varepsilon^2 \log(1+a) = 1$, and by

$$T(\varepsilon) \approx \varepsilon^{-\frac{p-1}{2}}$$

for $n = 1$ and $p > 1$, where in each case ε is a sufficiently small positive quantity. Note that, for the sake of simplicity, in the previous lifespan estimates for low dimensions $n = 1, 2$ we restricted our considerations to the case in which the integral of u_1 is not zero.

Our main results Theorem 4.1 and Theorem 5.1, which are stated and proved in Section 4 and in Section 5, respectively, are blow – up results for the semilinear model (1.2) that hold for exponents of the nonlinearity such that $1 < p \leq p_{\text{Str}}(n)$ and under suitable sign assumptions for compactly supported initial data. Furthermore, we will obtain an upper bound estimate for the lifespan of local solutions to (1.2) which coincides in some cases with the optimal one for (1.3), as we have just recalled. The proof of Theorem 4.1 is based on an iteration argument, which allows us to show the blow – up in finite time of the space average of a local in time solution to (1.2), while in Theorem 5.1 a weighted version of the space average is employed as time – dependent functional.

Let us point out that in the subcritical case, i.e. for $1 < p < p_{\text{Str}}(n)$, the iteration argument is not just a straightforward generalization of the one for (1.3). Indeed, in the iteration procedure we have to deal with an unbounded exponential multiplier. For this purpose, we propose a slicing procedure of the domain of integration by taking inspiration from [1], even though the sequence of the parameters (cf. $\{L_j\}_{j \in \mathbb{N}}$ below in Subsection 4.3), that characterize the slicing of the domain of integration, has a quite different structure. Up to our best knowledge, our result is the first attempt to include an unbounded exponential multiplier in an iteration argument for proving a blow – up result for hyperbolic semilinear models.

In the critical case, i.e. for $p = p_{\text{Str}}(n)$ and $n \geq 2$, the approach with the space average of the solution is no longer suitable and it has to be refined. This is done by considering a weighted space average (with a weighted function depending on the time variable as well) as functional, whose dynamic is studied in the iteration procedure. In this case, we follow the approach developed in [44], which is based on the so – called *slicing method*, developed for the first time in [1]. Nonetheless, as in the subcritical case, we have to consider a sequence of parameters (cf. $\{\Omega_j\}_{j \in \mathbb{N}}$ in Section 5) which characterize the slicing procedure of the domain of integration that has a relatively different structure with respect to the one which usually used to deal with critical cases (see for example [1, 44, 31, 32, 33, 30]).

The present paper is organized as follows. In Section 2 we first derive $L^2 - L^2$ estimates and well – posedness for the linear MGT equation. In Section 3 combining Banach fixed point theorem with the derived $L^2 - L^2$ estimates, the local (in time) existence of solutions to the semilinear MGT equation is proved. Then, in Section 4 we apply an iteration method associated with the test function introduced in [45] to prove the blow – up of energy solutions in the subcritical case. Afterwards, in Section 5 we prove the blow – up of a local in time solution (under certain assumptions for the initial data) also in the critical case $p = p_{\text{Str}}(n)$ when $n \geq 2$. Finally, some concluding remarks in Section 6 complete the paper.

Notation: We give some notations to be used in this paper. We write $f \lesssim g$ when there exists a positive constant C such that $f \leq Cg$. We denote $g \lesssim f \lesssim g$ by $f \approx g$. Moreover, B_R denotes the ball around the origin with radius R in \mathbb{R}^n . As mentioned in the introduction, $p_{\text{Str}}(n)$ denotes the Strauss exponent.

2. Linear problem for the MGT equation

In this section, we will derive some qualitative properties of solutions to the corresponding linearized Cauchy problem to (1.2), which is advantageous for us in order to understand the semilinear problem. More precisely, we are interested in the following linear MGT equation:

$$\begin{cases} \beta u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = 0, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t, u_{tt})(0, x) = (u_0, u_1, u_2)(x), & x \in \mathbb{R}^n, \end{cases} \quad (2.5)$$

where β is a positive constant.

Remark 2.1. *The principal symbol of the equation in (2.5) is given by $\tau^3 - \tau|\eta|^2 = 0$ and has real and pairwise distinct roots $\tau = 0$, $\tau = |\eta|$ and $\tau = -|\eta|$. Thus, the linear MGT equation in (2.5) is strictly hyperbolic.*

According to [20, 24], a suitable defined energy for the MGT equation is conserved from the point of view of semigroups.

We now state the energy conservation result for the linear homogeneous Cauchy problem.

Proposition 2.1. *Let us introduce the following energy for a solution u to (2.5):*

$$E_{\text{MGT}}[u](t) \doteq \frac{1}{2} \|\partial_t(\beta u_t + u)(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \|\nabla_x(\beta u_t + u)(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2.$$

Then, this energy is conserved, i.e., $E_{\text{MGT}}[u](t) \equiv E_{\text{MGT}}[u](0)$ for any $t > 0$.

Proof. Indeed, the linear MGT equation in (2.5) can be rewritten as

$$\partial_t^2(\beta u_t + u) - \Delta(\beta u_t + u) = 0,$$

which implies that the unknown function $\beta u_t + u$ is the solution to free wave equation. From the energy conservation for the free wave equation, we immediately complete the proof. \square

Proposition 2.2. *Let $n \geq 1$. Let us consider $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then, there exists a uniquely determined solution*

$$u \in \mathcal{C}([0, T], H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T], H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0, T], L^2(\mathbb{R}^n))$$

to (2.5) for all $T > 0$. Moreover, the solution to (2.5) satisfies the following estimates for $\ell = 0, 1$:

$$\begin{aligned} \|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u_0\|_{L^2(\mathbb{R}^n)} + (1+t) (\|u_1\|_{L^2(\mathbb{R}^n)} + \|u_2\|_{L^2(\mathbb{R}^n)}), \\ \|\nabla_x^{1+\ell} u(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u_0\|_{H^{1+\ell}(\mathbb{R}^n)} + \|u_1\|_{H^\ell(\mathbb{R}^n)} + \|u_2\|_{L^2(\mathbb{R}^n)}, \\ \|\nabla_x^\ell u_t(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u_0\|_{H^\ell(\mathbb{R}^n)} + \|u_1\|_{H^\ell(\mathbb{R}^n)} + \|u_2\|_{L^2(\mathbb{R}^n)}, \\ \|u_{tt}(t, \cdot)\|_{L^2(\mathbb{R}^n)} &\lesssim \|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{H^1(\mathbb{R}^n)} + \|u_2\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Proof. Employing the partial Fourier transform with respect to spatial variables to (2.5), we get

$$\begin{cases} \beta \hat{u}_{ttt} + \hat{u}_{tt} + |\xi|^2 \hat{u} + \beta |\xi|^2 \hat{u}_t = 0, & \xi \in \mathbb{R}^n, t > 0, \\ (\hat{u}, \hat{u}_t, \hat{u}_{tt})(0, \xi) = (\hat{u}_0, \hat{u}_1, \hat{u}_2)(\xi), & \xi \in \mathbb{R}^n. \end{cases} \quad (2.6)$$

By direct calculations, the characteristic roots of (2.6) are

$$\lambda_{1,2} = \pm i|\xi| \quad \text{and} \quad \lambda_3 = -\frac{1}{\beta}.$$

Therefore, the solution to (2.6) is given by

$$\begin{aligned} \hat{u}(t, \xi) &= \left(\frac{\cos(|\xi|t)}{1+\beta^2|\xi|^2} + \frac{\beta|\xi|\sin(|\xi|t)}{1+\beta^2|\xi|^2} + \frac{\beta^2|\xi|^2}{2(1+\beta^2|\xi|^2)} e^{-t/\beta} \right) \hat{u}_0(\xi) + \frac{\sin(|\xi|t)}{|\xi|} \hat{u}_1(\xi) \\ &+ \left(\frac{\beta \sin(|\xi|t)}{|\xi|(1+\beta^2|\xi|^2)} - \frac{\beta^2 \cos(|\xi|t)}{1+\beta^2|\xi|^2} + \frac{\beta^2}{2(1+\beta^2|\xi|^2)} e^{-t/\beta} \right) \hat{u}_2(\xi). \end{aligned}$$

By applying the same approach used to prove the existence of solutions in the classical energy space to the Cauchy problem for free wave equation (e.g. Chapter 14 in [10]) and the mean value theorem, we may conclude the existence of solutions to the MGT equation (2.5).

The conservation of the energy stated in Proposition 2.1 leads immediately to the uniqueness of the solution to the Cauchy problem (2.5).

Finally, in order to get the desired estimates of solutions, we apply $|\sin(|\xi|t)| \leq |\xi|t$ for $|\xi| \leq \epsilon \ll 1$, $|\sin(|\xi|t)| \leq 1$ and $|\cos(|\xi|t)| \leq 1$. Thus, the proof is completed. \square

To conclude this section, we point out that the solution to the linear Cauchy problem for MGT equation (2.5) fulfills the inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = e^{-t/\beta}(u_2(x) - \Delta u_0(x)), & x \in \mathbb{R}^n, t > 0, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n. \end{cases} \quad (2.7)$$

Thus, we claim that $\text{supp } u(t, \cdot) \subset B_{R+t}$, if we assume $\text{supp } u_j \subset B_R$ for any $j = 0, 1, 2$ and for some $R > 0$. Indeed, the source $f(t, x) = e^{-t/\beta}(u_2(x) - \Delta u_0(x))$ has support contained in the forward cone $\{(t, x) : |x| \leq R + t\}$ under these assumptions and we can use the property of finite speed of propagation for the classical wave equation.

3. Existence of local (in time) solution

Theorem 3.1. *Let $n \geq 1$. Let us consider $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ compactly supported with $\text{supp } u_j \subset B_R$ for any $j = 0, 1, 2$ and for some $R > 0$. We assume $p > 1$ such that $p \leq n/(n-2)$ when $n \geq 3$. Then, there exists a positive T and a uniquely determined local (in time) mild solution*

$$u \in \mathcal{C}([0, T], H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T], H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0, T], L^2(\mathbb{R}^n))$$

to (1.2) satisfying $\text{supp } u(t, \cdot) \subset B_{R+t}$ for any $t \in [0, T]$.

Let us introduce some notations for the proof of the local (in time) existence of solutions. We denote by $K_0(t, x)$, $K_1(t, x)$ and $K_2(t, x)$ the fundamental solutions to the linear Cauchy problem (2.5) with initial data $(u_0, u_1, u_2) = (\delta_0, 0, 0)$, $(u_0, u_1, u_2) = (0, \delta_0, 0)$ and $(u_0, u_1, u_2) = (0, 0, \delta_0)$, respectively. Here δ_0 is the Dirac distribution in $x = 0$ with respect to spatial variables. Therefore, the solution to (2.5) is given by

$$u(t, x) = K_0(t, x) *_{(x)} u_0(x) + K_1(t, x) *_{(x)} u_1(x) + K_2(t, x) *_{(x)} u_2(x),$$

where the Fourier transforms of the kernels $K_0(t, x)$, $K_1(t, x)$ and $K_2(t, x)$ are given by

$$\begin{aligned} \widehat{K}_0(t, \xi) &= \frac{\cos(|\xi|t)}{1+\beta^2|\xi|^2} + \frac{\beta|\xi|\sin(|\xi|t)}{1+\beta^2|\xi|^2} + \frac{\beta^2|\xi|^2}{2(1+\beta^2|\xi|^2)} e^{-t/\beta}, \\ \widehat{K}_1(t, \xi) &= \frac{\sin(|\xi|t)}{|\xi|}, \\ \widehat{K}_2(t, \xi) &= \frac{\beta \sin(|\xi|t)}{|\xi|(1+\beta^2|\xi|^2)} - \frac{\beta^2 \cos(|\xi|t)}{1+\beta^2|\xi|^2} + \frac{\beta^2}{2(1+\beta^2|\xi|^2)} e^{-t/\beta}. \end{aligned}$$

Proof. Let us define the family of evolution spaces

$$X(T) \doteq \{u \in \mathcal{C}([0, T], H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T], H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0, T], L^2(\mathbb{R}^n)) \\ \text{with } \text{supp } u(t, \cdot) \subset B_{R+t} \text{ for any } t \in [0, T]\},$$

with the norm

$$\|u\|_{X(T)} \doteq \max_{t \in [0, T]} \sum_{\ell+j \leq 2, \ell, j \in \mathbb{N}_0} \left\| \nabla_x^\ell \partial_t^j u(t, \cdot) \right\|_{L^2(\mathbb{R}^n)}.$$

According to Duhamel's principle, we introduce the operator

$$N : u \in X(T) \rightarrow Nu(t, x) \doteq K_0(t, x) *_{(x)} u_0(x) + K_1(t, x) *_{(x)} u_1(x) + K_2(t, x) *_{(x)} u_2(x) \\ + \int_0^t K_2(t - \tau, x) *_{(x)} |u(\tau, x)|^p d\tau.$$

We will consider as mild local in time solutions to (1.2) the fixed points of the operator N . Therefore, with the aim of deriving the local (in time) existence and uniqueness of the solution in $X(T)$, we need to prove

$$\|Nu\|_{X(T)} \leq C_0(u_0, u_1, u_2) + C_1(u_0, u_1, u_2) T \|u\|_{X(T)}^p, \quad (3.8)$$

$$\|Nu - Nv\|_{X(T)} \leq C_2(u_0, u_1, u_2) T \|u - v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right). \quad (3.9)$$

First of all, from Proposition 2.2 it is clear that

$$u^{\text{ln}}(t, x) \doteq K_0(t, x) *_{(x)} u_0(x) + K_1(t, x) *_{(x)} u_1(x) + K_2(t, x) *_{(x)} u_2(x) \in X(T)$$

and

$$\|u^{\text{ln}}\|_{X(T)} \lesssim \|u_0\|_{H^2(\mathbb{R}^n)} + (1 + T) (\|u_1\|_{H^1(\mathbb{R}^n)} + \|u_2\|_{L^2(\mathbb{R}^n)}).$$

Next, to prove (3.8), we apply the classical Gagliardo – Nirenberg inequality. Thus, we get for any $\tau \in [0, T]$

$$\|u(\tau, \cdot)\|_{L^{2p}(\mathbb{R}^n)}^p \leq C \|u(\tau, \cdot)\|_{L^2(\mathbb{R}^n)}^{(1 - \frac{2}{p}(1 - \frac{1}{p}))p} \|u(\tau, \cdot)\|_{H^1(\mathbb{R}^n)}^{\frac{2}{p}(1 - \frac{1}{p})p} \leq C \|u\|_{X(T)}^p,$$

where $p > 1$ if $n = 1, 2$ and $1 < p \leq n/(n - 2)$ if $n \geq 3$.

Then, applying the previous inequality and using $L^2 - L^2$ estimates from Proposition 2.2, we derive

$$\left\| \int_0^t K_2(t - \tau, x) *_{(x)} |u(\tau, x)|^p d\tau \right\|_{L^2(\mathbb{R}^n)} \leq C \int_0^t (1 + t - \tau) \| |u(\tau, \cdot)|^p \|_{L^2(\mathbb{R}^n)} d\tau \\ \leq C(1 + t)t \|u\|_{X(T)}^p.$$

Analogously,

$$\left\| \nabla_x^\ell \partial_t^j \int_0^t K_2(t - \tau, x) *_{(x)} |u(\tau, x)|^p d\tau \right\|_{L^2(\mathbb{R}^n)} \leq C \int_0^t \| |u(\tau, \cdot)|^p \|_{L^2(\mathbb{R}^n)} d\tau \\ \leq Ct \|u\|_{X(T)}^p,$$

for any $\ell, j \in \mathbb{N}_0$ such that $1 \leq \ell + j \leq 2$.

Finally, Nu satisfies the support condition $\text{supp } Nu(t, \cdot) \subset B_{R+t}$ for any $t \in [0, T]$, since $w = Nu$ is a solution of the inhomogeneous Cauchy problem for the wave equation

$$\begin{cases} w_{tt} - \Delta w = e^{-t/\beta} (u_2(x) - \Delta u_0(x)) + \frac{1}{\beta} \int_0^t e^{(\tau-t)/\beta} |u(\tau, x)|^p d\tau, & x \in \mathbb{R}^n, t > 0, \\ (w, w_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases}$$

and u is supported in the forward cone due to $u \in X(T)$. Thus, we may conclude that N maps $X(T)$ into itself and (3.8).

To derive (3.9), we remark that

$$\|Nu - Nv\|_{X(t)} = \left\| \int_0^t K_2(t - \tau, x) *_{(x)} (|u(\tau, x)|^p - |v(\tau, x)|^p) d\tau \right\|_{X(t)}.$$

By employing

$$||u(\tau, x)|^p - |v(\tau, x)|^p| \leq C|u(\tau, x) - v(\tau, x)| (|u(\tau, x)|^{p-1} + |v(\tau, x)|^{p-1})$$

and Hölder's inequality, we conclude

$$\| |u(\tau, \cdot)|^p - |v(\tau, \cdot)|^p \|_{L^2(\mathbb{R}^n)} \leq C \|u(\tau, \cdot) - v(\tau, \cdot)\|_{L^{2p}(\mathbb{R}^n)} \left(\|u(\tau, \cdot)\|_{L^{2p}(\mathbb{R}^n)}^{p-1} + \|v(\tau, \cdot)\|_{L^{2p}(\mathbb{R}^n)}^{p-1} \right).$$

Finally, by using $L^2 - L^2$ estimates from Proposition 2.2 again, we immediately obtain the desired estimate (3.9). This completes the proof. \square

4. Blow – up result in the subcritical case

Let $u = u(t, x)$ be a local in time solution to the semilinear Cauchy problem

$$\begin{cases} \beta u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t, u_{tt})(0, x) = \varepsilon(u_0, u_1, u_2)(x), & x \in \mathbb{R}^n, \end{cases} \quad (4.10)$$

where $\varepsilon > 0$ is a parameter describing the smallness of initial data.

The aim of this section is to prove the blow – up of local (in time) solutions to (4.10) in the subcritical case, that is for $1 < p < p_{\text{Str}}(n)$, under suitable conditions for the Cauchy data, and to derive an upper bound estimate for the lifespan. To do this, we introduce first the definition of energy solutions to (4.10).

Definition 4.1. *Let $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. We say u is an energy solution to (4.10) on $[0, T)$ if*

$$u \in \mathcal{C}([0, T), H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0, T), L^2(\mathbb{R}^n)) \cap L_{\text{loc}}^p([0, T) \times \mathbb{R}^n)$$

satisfy $u(0, \cdot) = \varepsilon u_0$ in $H^2(\mathbb{R}^n)$ and the integral identity

$$\begin{aligned} & \beta \int_{\mathbb{R}^n} u_{tt}(t, x) \psi(t, x) dx + \int_{\mathbb{R}^n} u_t(t, x) \psi(t, x) dx - \beta \varepsilon \int_{\mathbb{R}^n} u_2(x) \psi(0, x) dx - \varepsilon \int_{\mathbb{R}^n} u_1(x) \psi(0, x) dx \\ & + \beta \int_0^t \int_{\mathbb{R}^n} (\nabla_x u_t(s, x) \cdot \nabla_x \psi(s, x) - u_{tt}(s, x) \psi_s(s, x)) dx ds \\ & + \int_0^t \int_{\mathbb{R}^n} (\nabla_x u(s, x) \cdot \nabla_x \psi(s, x) - u_t(s, x) \psi_s(s, x)) dx ds \\ & = \int_0^t \int_{\mathbb{R}^n} |u(s, x)|^p \psi(s, x) dx ds \end{aligned} \quad (4.11)$$

for any $\psi \in \mathcal{C}_0^\infty([0, T) \times \mathbb{R}^n)$ and any $t \in [0, T)$.

Applying a further step of integration by parts in (4.11), it results

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^n} (-\beta \psi_{sss}(s, x) + \psi_{ss}(s, x) - \Delta \psi(s, x) + \beta \Delta \psi_s(s, x)) u(s, x) \, dx \, ds \\
& + \beta \int_{\mathbb{R}^n} (\psi(t, x) u_{tt}(t, x) - \psi_t(t, x) u_t(t, x) + \psi_{tt}(t, x) u(t, x) - \Delta \psi(t, x) u(t, x)) \, dx \\
& - \beta \varepsilon \int_{\mathbb{R}^n} (\psi(0, x) u_2(x) - \psi_t(0, x) u_1(x) + \psi_{tt}(0, x) u_0(x) - \Delta \psi(0, x) u_0(x)) \, dx \\
& + \int_{\mathbb{R}^n} (\psi(t, x) u_t(t, x) - \psi_t(t, x) u(t, x)) \, dx - \varepsilon \int_{\mathbb{R}^n} (\psi(0, x) u_1(x) - \psi_t(0, x) u_0(x)) \, dx \\
& = \int_0^t \int_{\mathbb{R}^n} |u(s, x)|^p \psi(s, x) \, dx \, ds. \tag{4.12}
\end{aligned}$$

In particular, letting $t \rightarrow T$, we find that u fulfills the definition of weak solution to (4.10).

Theorem 4.1. *Let us consider $p > 1$ such that*

$$\begin{cases} p < \infty & \text{if } n = 1, \\ p < p_{\text{Str}}(n) & \text{if } n \geq 2. \end{cases}$$

Let $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be nonnegative and compactly supported functions with supports contained in B_R for some $R > 0$ such that u_0 is not identically zero. Let

$$u \in \mathcal{C}([0, T], H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T], H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0, T], L^2(\mathbb{R}^n)) \cap L^p_{\text{loc}}([0, T] \times \mathbb{R}^n)$$

be an energy solution on $[0, T)$ to the Cauchy problem (4.10) according to Definition 4.1 with lifespan $T = T(\varepsilon)$ such that

$$\text{supp } u(t, \cdot) \subset B_{R+t} \quad \text{for any } t \in (0, T). \tag{4.13}$$

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, u_2, n, p, R, \beta)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq C \varepsilon^{-\frac{2p(p-1)}{\theta(p, n)}}$$

holds, where C is an independent of ε , positive constant and

$$\theta(p, n) \doteq 2 + (n+1)p - (n-1)p^2. \tag{4.14}$$

4.1. Iteration frame

According to Theorem 4.1, we assume that u_0, u_1 and u_2 are nonnegative functions, with nontrivial u_0 , and compactly supported with support contained in B_R for some suitable $R > 0$.

Then, thanks to what we underlined in Section 3, we have

$$\text{supp } u(t, \cdot) \subset B_{R+t} \quad \text{for any } t \in (0, T). \tag{4.15}$$

We introduce now the following time – dependent functional:

$$U(t) \doteq \int_{\mathbb{R}^n} u(t, x) \, dx.$$

Choosing a test function ψ in (4.11) such that $\psi = 1$ on $\{(s, x) \in [0, t] \times \mathbb{R}^n : |x| \leq R + s\}$, due to (4.15) we have

$$\beta \int_{\mathbb{R}^n} u_{tt}(t, x) \, dx + \int_{\mathbb{R}^n} u_t(t, x) \, dx - \beta \varepsilon \int_{\mathbb{R}^n} u_2(x) \, dx - \varepsilon \int_{\mathbb{R}^n} u_1(x) \, dx = \int_0^t \int_{\mathbb{R}^n} |u(s, x)|^p \, dx \, ds.$$

Differentiating the previous relation with respect to t , we get

$$\beta U'''(t) + U''(t) = \int_{\mathbb{R}^n} |u(t, x)|^p \, dx. \tag{4.16}$$

By employing Hölder's inequality and (4.15), we may estimate

$$\int_{\mathbb{R}^n} |u(t, x)|^p dx \geq C(R+t)^{-n(p-1)} |U(t)|^p,$$

where $C = C(n, p) > 0$ is a constant that depends on the measure of the unitary ball. Hence, from (4.16) we obtain

$$\beta U'''(t) + U''(t) \geq C(R+t)^{-n(p-1)} |U(t)|^p. \quad (4.17)$$

The previous ordinary differential inequality for U allows us to derive the frame for our iteration argument. In other words, integrating twice, by (4.17) we have

$$\beta U'(s) + U(s) \geq \beta U'(0) + U(0) + (\beta U''(0) + U'(0))s + C \int_0^s \int_0^\sigma (R+\tau)^{-n(p-1)} |U(\tau)|^p d\tau d\sigma.$$

Then, multiplying the last inequality by $e^{s/\beta}$ and integrating over $[0, t]$, we arrive at

$$\begin{aligned} U(t) &\geq U(0) e^{-t/\beta} + (\beta U'(0) + U(0)) \left(1 - e^{-t/\beta}\right) + \left(U''(0) + \frac{1}{\beta} U'(0)\right) \left(\beta(t - \beta) + \beta^2 e^{-t/\beta}\right) \\ &\quad + \frac{C}{\beta} \int_0^t e^{(s-t)/\beta} \int_0^s \int_0^\sigma (R+\tau)^{-n(p-1)} |U(\tau)|^p d\tau d\sigma ds. \end{aligned} \quad (4.18)$$

From the integral inequality (4.18) we have a twofold consequence. Since we assume that initial data are nonnegative, then (4.18) implies $U(t) \gtrsim \varepsilon$ for any $t \geq 0$. So, in particular, U is a positive function.

On the other hand, if we neglect the terms involving $U(0), U'(0)$ and $U''(0)$ in (4.18), then, we find

$$U(t) \geq \frac{C}{\beta} \int_0^t e^{(s-t)/\beta} \int_0^s \int_0^\sigma (R+\tau)^{-n(p-1)} (U(\tau))^p d\tau d\sigma ds. \quad (4.19)$$

We point out explicitly that (4.19) will play a fundamental role in our iteration argument: this is, in fact, the frame which allows us to determine a sequence of lower bound estimates for the function U .

4.2. Lower bound for the functional

Even though we proved that $U(t) \gtrsim \varepsilon$ in the last subsection, this lower bound for U is too weak in order to start with the iteration procedure. For this reason we will improve this lower bound for U by introducing a second time – dependent functional. Let us consider the function

$$\begin{aligned} \Phi(x) &\doteq e^x + e^{-x} && \text{if } n = 1, \\ \Phi(x) &\doteq \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} d\sigma_\omega && \text{if } n \geq 2. \end{aligned} \quad (4.20)$$

This function has been introduced for the first time in the study of blow – up results for wave models in [45]. The function Φ is a positive smooth function that satisfies the following crucial properties:

$$\Delta \Phi = \Phi, \quad (4.21)$$

$$\Phi(x) \sim |x|^{-\frac{n-1}{2}} e^x \quad \text{as } |x| \rightarrow \infty. \quad (4.22)$$

Furthermore, we introduce the function with separate variables $\Psi = \Psi(t, x) = e^{-t} \Phi(x)$. Clearly, Ψ is a solution of the adjoint equation to the homogeneous linear MGT equation, namely,

$$-\beta \partial_t^3 \Psi + \partial_t^2 \Psi - \Delta \Psi + \beta \Delta \partial_t \Psi = 0. \quad (4.23)$$

We can introduce now the definition of the second functional U_1 as follows:

$$U_1(t) \doteq \int_{\mathbb{R}^n} u(t, x) \Psi(t, x) dx.$$

Since Ψ is a positive function, applying (4.12) with test function Ψ , we get

$$\begin{aligned}
0 &\leq \int_0^t \int_{\mathbb{R}^n} |u(s, x)|^p \Psi(s, x) \, dx \, ds \\
&= \int_0^t \int_{\mathbb{R}^n} \left(-\beta \Psi_{ttt}(s, x) + \Psi_{tt}(s, x) - \Delta \Psi(s, x) + \beta \Delta \Psi_t(s, x) \right) u(s, x) \, dx \, ds \\
&\quad + \beta \int_{\mathbb{R}^n} \left(u_{tt}(s, x) \Psi(s, x) - u_t(s, x) \Psi_t(s, x) + u(s, x) \Psi_{tt}(s, x) \right) dx \Big|_{s=0}^{s=t} \\
&\quad + \int_{\mathbb{R}^n} \left(u_t(s, x) \Psi(s, x) - u(s, x) \Psi_t(s, x) \right) dx \Big|_{s=0}^{s=t} - \beta \int_{\mathbb{R}^n} u(s, x) \Delta \Psi(s, x) \, dx \Big|_{s=0}^{s=t} \\
&= \int_{\mathbb{R}^n} \left(\beta u_{tt}(s, x) \Psi(s, x) + (\beta + 1) u_t(s, x) \Psi(s, x) + u(s, x) \Psi(s, x) \right) dx \Big|_{s=0}^{s=t} \\
&= \beta U_1''(t) + (3\beta + 1) U_1'(t) + (2\beta + 2) U_1(t) - (\beta U_1''(0) + (3\beta + 1) U_1'(0) + (2\beta + 2) U_1(0)),
\end{aligned}$$

where in the second last step we used that Ψ solves (4.23), while in the last step we used the obvious representations

$$\begin{aligned}
U_1'(t) &= \int_{\mathbb{R}^n} \left(u_t(s, x) \Psi(s, x) + u(s, x) \Psi_t(s, x) \right) dx, \\
U_1''(t) &= \int_{\mathbb{R}^n} \left(u_{tt}(s, x) \Psi(s, x) + 2u_t(s, x) \Psi_t(s, x) + u(s, x) \Psi_{tt}(s, x) \right) dx.
\end{aligned}$$

Note that we may employ Ψ as test function even though it has no compact support thanks to the support property for u in (4.13). As outcome of the previous chain of equalities, we get that

$$U_1''(t) + \left(3 + \frac{1}{\beta}\right) U_1'(t) + \left(2 + \frac{2}{\beta}\right) U_1(t) \geq U_1''(0) + \left(3 + \frac{1}{\beta}\right) U_1'(0) + \left(2 + \frac{2}{\beta}\right) U_1(0). \quad (4.24)$$

We can rewrite the left – hand side of (4.24) as

$$e^{-(1+1/\beta)t} \frac{d}{dt} \left\{ e^{(1+1/\beta)t} \left[e^{-2t} \frac{d}{dt} (e^{2t} U_1(t)) \right] \right\},$$

while the right – hand side depends only on initial data

$$\begin{aligned}
U_1''(0) + \left(3 + \frac{1}{\beta}\right) U_1'(0) + \left(2 + \frac{2}{\beta}\right) U_1(0) &= \varepsilon \int_{\mathbb{R}^n} \left(u_2(x) + \frac{\beta+1}{\beta} u_1(x) + \frac{1}{\beta} u_0(x) \right) \Phi(x) \, dx \\
&= \varepsilon I_\beta[u_0, u_1, u_2] > 0,
\end{aligned}$$

where

$$I_\beta[u_0, u_1, u_2] \doteq \int_{\mathbb{R}^n} \left(u_2(x) + \frac{\beta+1}{\beta} u_1(x) + \frac{1}{\beta} u_0(x) \right) \Phi(x) \, dx.$$

Multiplying (4.24) by $e^{(1+1/\beta)t}$ and integrating over $[0, t]$, we get

$$e^{-2t} \frac{d}{dt} (e^{2t} U_1(t)) \geq (U_1'(0) + 2U_1(0)) e^{-(1+1/\beta)t} + \varepsilon \frac{\beta}{\beta+1} I_\beta[u_0, u_1, u_2] \left(1 - e^{-(1+1/\beta)t}\right).$$

Analogously to the last step, we multiply the previous inequality by e^{2t} and we integrate over $[0, t]$, so that

$$\begin{aligned}
U_1(t) &\geq U_1(0) e^{-2t} + \frac{\beta}{\beta-1} (U_1'(0) + 2U_1(0)) \left(e^{-(1+1/\beta)t} - e^{-2t} \right) \\
&\quad + \frac{\beta}{\beta+1} \left(U_1''(0) + \left(3 + \frac{1}{\beta}\right) U_1'(0) + \left(2 + \frac{2}{\beta}\right) U_1(0) \right) \left(\frac{1}{2} (1 - e^{-2t}) - \frac{\beta}{\beta-1} \left(e^{-(1+1/\beta)t} - e^{-2t} \right) \right).
\end{aligned}$$

for $\beta \neq 1$, while for $\beta = 1$ we get

$$U_1(t) \geq U_1(0) e^{-2t} + (U_1'(0) + 2U_1(0)) t e^{-2t} + \frac{1}{2} \left(\frac{1}{2} (1 - e^{-2t}) - t e^{-2t} \right) (U_1''(0) + 4U_1'(0) + 4U_1(0)).$$

Therefore, thanks to the assumptions on u_0, u_1, u_2 , the previous estimates yield easily

$$U_1(t) \gtrsim \varepsilon, \quad (4.25)$$

where the unexpressed multiplicative constant depends on u_0, u_1, u_2 .

Let us show now how (4.25) provides a lower bound estimate for the spatial integral of the nonlinearity $|u|^p$. Applying Hölder's inequality, we have

$$\varepsilon \lesssim U_1(t) \leq \left(\int_{\mathbb{R}^n} |u(t, x)|^p dx \right)^{\frac{1}{p}} \left(\int_{B_{R+t}} \Psi(t, x)^{p'} dx \right)^{\frac{1}{p'}}$$

where p' denotes the conjugate exponent of p . In the literature, it is well – known that the p' power of the $L^{p'}(B_{R+T})$ – norm of $\Psi(t, \cdot)$ can be estimate in the following way:

$$\int_{B_{R+t}} \Psi(t, x)^{p'} dx \lesssim (R+t)^{n-1-\frac{n-1}{2}p'}$$

(cf. [45, Estimate (2.5)]), thus, we find

$$\int_{\mathbb{R}^n} |u(t, x)|^p dx \geq K \varepsilon^p (R+t)^{n-1-\frac{n-1}{2}p} \quad (4.26)$$

for a suitable positive constant K .

Finally, we combine (4.16) and (4.26) in order to get a lower bound for U which will allow us to start with the iteration argument. Repeating the same intermediate steps that we did in order to prove (4.19) starting from (4.17), from (4.16) we find

$$U(t) \geq \frac{1}{\beta} \int_0^t e^{(s-t)/\beta} \int_0^s \int_0^\sigma \int_{\mathbb{R}^n} |u(\tau, x)|^p dx d\tau d\sigma ds.$$

Next, we plug the lower bound (4.26) in the last estimate. Thus, we obtain

$$\begin{aligned} U(t) &\geq \frac{K}{\beta} \varepsilon^p \int_0^t e^{(s-t)/\beta} \int_0^s \int_0^\sigma (R+\tau)^{n-1-\frac{n-1}{2}p} d\tau d\sigma ds \\ &\geq \frac{K}{\beta} \varepsilon^p (R+t)^{-\frac{n-1}{2}p} \int_0^t e^{(s-t)/\beta} \int_0^s \int_0^\sigma \tau^{n-1} d\tau d\sigma ds \\ &= \frac{K}{n(n+1)} \frac{1}{\beta} \varepsilon^p (R+t)^{-\frac{n-1}{2}p} \int_0^t e^{(s-t)/\beta} s^{n+1} ds \\ &\geq \frac{K}{n(n+1)} \frac{1}{\beta} \varepsilon^p (R+t)^{-\frac{n-1}{2}p} \int_{t/2}^t e^{(s-t)/\beta} s^{n+1} ds \\ &\geq \frac{K}{2^{n+1}n(n+1)} \varepsilon^p (R+t)^{-\frac{n-1}{2}p} t^{n+1} \int_{t/2}^t \frac{1}{\beta} e^{(s-t)/\beta} ds \\ &= \frac{K}{2^{n+1}n(n+1)} \varepsilon^p (R+t)^{-\frac{n-1}{2}p} t^{n+1} \left(1 - e^{-t/(2\beta)}\right). \end{aligned}$$

In particular, for $t \geq \beta$ the factor containing the exponential function in the last line of the previous chain of inequalities can be estimate from below by a constant, namely,

$$U(t) \geq C_0 (R+t)^{-\alpha_0} t^{\gamma_0} \quad \text{for any } t \geq \beta, \quad (4.27)$$

where the multiplicative constant is

$$C_0 \doteq K 2^{-(n+1)} (n(n+1))^{-1} (1 - e^{-1/2}) \varepsilon^p$$

and the exponents are defined by

$$\alpha_0 \doteq \frac{n-1}{2}p \quad \text{and} \quad \gamma_0 \doteq n+1.$$

4.3. Iteration argument

In the previous subsection, we derived a first lower bound for U . Now we will derive a sequence of lower bounds for U by using the iteration frame (4.19). More precisely, we will show that

$$U(t) \geq C_j (R+t)^{-\alpha_j} (t - L_j \beta)^{\gamma_j} \quad \text{for any } t \geq L_j \beta, \quad (4.28)$$

where $\{C_j\}_{j \in \mathbb{N}}$, $\{\alpha_j\}_{j \in \mathbb{N}}$ and $\{\gamma_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers that we will determine throughout the proof and $\{L_j\}_{j \in \mathbb{N}}$ is the sequence of the partial products of the convergent infinite product

$$\prod_{k=0}^{\infty} \ell_k \quad \text{with } \ell_k \doteq 1 + p^{-k} \quad \text{for any } k \in \mathbb{N},$$

that is,

$$L_j \doteq \prod_{k=0}^j \ell_k \quad \text{for any } j \in \mathbb{N}.$$

Note that (4.27) implies (4.28) for $j = 0$. We are going to prove (4.28) by using an inductive argument. Therefore, it remains to prove just the inductive step. Let us assume the validity of (4.28) for $j \geq 0$. We will prove (4.28) for $j + 1$. After shrinking the domain of integration in (4.19), if we plug (4.28) in (4.19), we get

$$\begin{aligned} U(t) &\geq \frac{C}{\beta} \int_{L_j \beta}^t e^{(s-t)/\beta} \int_{L_j \beta}^s \int_{L_j \beta}^{\sigma} (R+\tau)^{-n(p-1)} (U(\tau))^p \, d\tau \, d\sigma \, ds \\ &\geq \frac{C}{\beta} C_j^p \int_{L_j \beta}^t e^{(s-t)/\beta} \int_{L_j \beta}^s \int_{L_j \beta}^{\sigma} (R+\tau)^{-n(p-1)-\alpha_j p} (\tau - L_j \beta)^{\gamma_j p} \, d\tau \, d\sigma \, ds \\ &\geq \frac{C}{\beta} C_j^p (R+t)^{-n(p-1)-\alpha_j p} \int_{L_j \beta}^t e^{(s-t)/\beta} \int_{L_j \beta}^s \int_{L_j \beta}^{\sigma} (\tau - L_j \beta)^{\gamma_j p} \, d\tau \, d\sigma \, ds \\ &\geq \frac{C}{\beta} (\gamma_j p + 1)^{-1} (\gamma_j p + 2)^{-1} C_j^p (R+t)^{-n(p-1)-\alpha_j p} \int_{L_j \beta}^t e^{(s-t)/\beta} (s - L_j \beta)^{\gamma_j p + 2} \, ds \\ &\geq \frac{C}{\beta} (\gamma_j p + 1)^{-1} (\gamma_j p + 2)^{-1} C_j^p (R+t)^{-n(p-1)-\alpha_j p} \int_{t/\ell_{j+1}}^t e^{(s-t)/\beta} (s - L_j \beta)^{\gamma_j p + 2} \, ds \end{aligned}$$

for $t \geq L_{j+1} \beta$. Note that in the last step we could restrict the domain of integration with respect to s from $[L_j \beta, t]$ to $[t/\ell_{j+1}, t]$ because $t \geq L_{j+1} \beta$ and $\ell_{j+1} > 1$ imply $L_j \beta \leq t/\ell_{j+1} < t$. Consequently,

$$\begin{aligned} U(t) &\geq \frac{C C_j^p}{(\gamma_j p + 1)(\gamma_j p + 2) \ell_{j+1}^{\gamma_j p + 2}} (R+t)^{-n(p-1)-\alpha_j p} (t - L_j \ell_{j+1} \beta)^{\gamma_j p + 2} \int_{t/\ell_{j+1}}^t \frac{1}{\beta} e^{(s-t)/\beta} \, ds \\ &= \frac{C C_j^p}{(\gamma_j p + 1)(\gamma_j p + 2) \ell_{j+1}^{\gamma_j p + 2}} (R+t)^{-n(p-1)-\alpha_j p} (t - L_{j+1} \beta)^{\gamma_j p + 2} \left(1 - e^{-(t/\beta)(1-1/\ell_{j+1})}\right) \end{aligned}$$

for $t \geq L_{j+1} \beta$. Finally, we remark that for $t \geq L_{j+1} \beta \geq \ell_{j+1} \beta$ we may estimate

$$\begin{aligned} 1 - e^{-(t/\beta)(1-1/\ell_{j+1})} &\geq 1 - e^{-(\ell_{j+1}-1)} \geq 1 - \left(1 - (\ell_{j+1} - 1) + \frac{1}{2}(\ell_{j+1} - 1)^2\right) \\ &= (\ell_{j+1} - 1) \left(1 - \frac{1}{2}(\ell_{j+1} - 1)\right) = p^{-(j+1)} (1 - 1/(2p^{j+1})) \\ &= p^{-2(j+1)} (p^{j+1} - 1/2) \geq (p - 1/2) p^{-2(j+1)}. \end{aligned} \quad (4.29)$$

Also, for $t \geq L_{j+1} \beta$ we have shown

$$U(t) \geq \frac{(p - 1/2) C C_j^p p^{-2(j+1)}}{(\gamma_j p + 1)(\gamma_j p + 2) \ell_{j+1}^{\gamma_j p + 2}} (R+t)^{-n(p-1)-\alpha_j p} (t - L_{j+1} \beta)^{\gamma_j p + 2},$$

which is exactly (4.28) for $j + 1$, provided that

$$C_{j+1} \doteq \frac{(p-1/2)CC_j^p p^{-2(j+1)}}{(\gamma_j p + 1)(\gamma_j p + 2)\ell_{j+1}^{\gamma_j p + 2}}, \quad \alpha_{j+1} \doteq n(p-1) + p\alpha_j, \quad \gamma_{j+1} \doteq 2 + p\gamma_j.$$

4.4. Upper bound estimate for the lifespan

In the last subsection, we determined the sequence of lower bound estimates in (4.28) for U . Now we want to show that the j -dependent lower bound in (4.28) for U blows up in finite time as $j \rightarrow \infty$. This will provide the desired blow-up result and an upper bound estimate for the lifespan as well. Let us get started by estimating the multiplicative constant C_j in a suitable way.

In order to estimate C_j from below we have to determine first the explicit representation for γ_j . Since $\alpha_j = n(p-1) + p\alpha_{j-1}$ and $\gamma_j = 2 + p\gamma_{j-1}$, applying recursively these relations, we get

$$\alpha_j = p^2 \alpha_{j-2} + n(p-1)(1+p) = \cdots = p^j \alpha_0 + n(p-1)(1+p+\cdots+p^{j-1}) = (\alpha_0 + n)p^j - n, \quad (4.30)$$

$$\gamma_j = p^2 \gamma_{j-2} + 2(1+p) = \cdots = p^j \gamma_0 + 2(1+p+\cdots+p^{j-1}) = \left(\gamma_0 + \frac{2}{p-1}\right) p^j - \frac{2}{p-1}. \quad (4.31)$$

Therefore, from (4.31) we have

$$(\gamma_{j-1}p + 1)(\gamma_{j-1}p + 2) \leq (\gamma_{j-1}p + 2)^2 = \gamma_j^2 \leq \left(\gamma_0 + \frac{2}{p-1}\right)^2 p^{2j}$$

which implies in turns

$$C_j \geq (p-1/2)C \left(\gamma_0 + \frac{2}{p-1}\right)^{-2} C_{j-1}^p p^{-4j} \ell_j^{-\gamma_j}.$$

Moreover, it holds

$$\lim_{j \rightarrow \infty} \ell_j^{\gamma_j} = \lim_{j \rightarrow \infty} \exp\left(\left(\gamma_0 + \frac{2}{p-1}\right) p^j \log(1+p^{-j})\right) = e^{\gamma_0 + 2/(p-1)},$$

so, in particular, we can find a suitable constant $M = M(n, p) > 0$ such that $\ell_j^{-\gamma_j} \geq M$ for any $j \in \mathbb{N}$. Hence,

$$C_j \geq \underbrace{(p-1/2)CM}_{\doteq D} \left(\gamma_0 + \frac{2}{p-1}\right)^{-2} C_{j-1}^p p^{-4j} \quad \text{for any } j \in \mathbb{N}.$$

Applying the logarithmic function to both sides of the inequality $C_j \geq DC_{j-1}^p p^{-4j}$ and using iteratively the resulting inequality, we get

$$\begin{aligned} \log C_j &\geq p \log C_{j-1} - 4j \log p + \log D \\ &\geq p^2 \log C_{j-2} - 4(j+(j-1)p) \log p + (1+p) \log D \\ &\geq \cdots \geq p^j \log C_0 - 4 \left(\sum_{k=0}^{j-1} (j-k)p^k \right) \log p + \left(\sum_{k=0}^{j-1} p^k \right) \log D. \end{aligned}$$

Using the identities

$$\sum_{k=0}^{j-1} (j-k)p^k = \frac{1}{p-1} \left(\frac{p^{j+1} - p}{p-1} - j \right) \quad \text{and} \quad \sum_{k=0}^{j-1} p^k = \frac{p^j - 1}{p-1}, \quad (4.32)$$

it follows

$$\log C_j \geq p^j \left(\log C_0 - \frac{4p \log p}{(p-1)^2} + \frac{\log D}{p-1} \right) + \frac{4j \log p}{p-1} + \frac{4p \log p}{(p-1)^2} - \frac{\log D}{p-1}$$

for any $j \in \mathbb{N}$. Let $j_0 = j_0(n, p) \in \mathbb{N}$ be the smallest nonnegative integer such that

$$j_0 \geq \frac{\log D}{4 \log p} - \frac{p}{p-1}.$$

Then, for any $j \geq j_0$ it results

$$\log C_j \geq p^j \left(\log C_0 - \frac{4p \log p}{(p-1)^2} + \frac{\log D}{p-1} \right) = p^j \log \left(D^{1/(p-1)} p^{-(4p)/(p-1)^2} C_0 \right) = p^j \log(E_0 \varepsilon^p) \quad (4.33)$$

for a suitable constant $E_0 = E_0(n, p) > 0$.

Let us denote

$$L \doteq \lim_{j \rightarrow \infty} L_j = \prod_{j=0}^{\infty} \ell_j \in \mathbb{R}.$$

Note that thanks to $\ell_j > 1$, it holds $L_j \uparrow L$ as $j \rightarrow \infty$. So, in particular, (4.28) holds for any $j \in \mathbb{N}$ and any $t \geq L\beta$.

Combining (4.28), (4.30), (4.31) and (4.33), we find

$$\begin{aligned} U(t) &\geq \exp \left(p^j \log(E_0 \varepsilon^p) \right) (R+t)^{-\alpha_j} (t-L\beta)^{\gamma_j} \\ &= \exp \left(p^j \left(\log(E_0 \varepsilon^p) - (\alpha_0 + n) \log(R+t) + \left(\gamma_0 + \frac{2}{p-1} \right) \log(t-L\beta) \right) \right) (R+t)^n (t-L\beta)^{-2/(p-1)} \end{aligned}$$

for any $j \geq j_0$ and any $t \geq L\beta$. Finally, for $t \geq \max\{R, 2L\beta\}$, since $R+t \leq 2t$ and $t-L\beta \geq t/2$, we have

$$U(t) \geq \exp \left(p^j \log \left(E_1 \varepsilon^p t^{\gamma_0 + \frac{2}{p-1} - (\alpha_0 + n)} \right) \right) (R+t)^n (t-L\beta)^{-2/(p-1)} \quad (4.34)$$

for any $j \geq j_0$, where $E_1 \doteq 2^{-(\alpha_0 + n + \gamma_0 + 2/(p-1))} E_0$. We can rewrite the exponent for t in the last inequality as follows:

$$\gamma_0 + \frac{2}{p-1} - (\alpha_0 + n) = -\frac{n-1}{2}p + 1 + \frac{2}{p-1} = \frac{1}{2(p-1)} (2 + (n+1)p - (n-1)p^2) = \frac{\theta(p, n)}{2(p-1)},$$

where $\theta(p, n)$ is defined in (4.14). Therefore, for $1 < p < p_{\text{Str}}(n)$ (respectively, for $1 < p$ when $n = 1$), the exponent for t is positive. Let us fix $\varepsilon_0 = \varepsilon_0(u_0, u_1, u_2, n, p, R, \beta) > 0$ such that

$$\varepsilon_0^{-\frac{2p(p-1)}{\theta(p, n)}} \geq E_1^{\frac{2(p-1)}{\theta(p, n)}} \max\{R, 2L\beta\}.$$

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > E_2 \varepsilon^{-\frac{2p(p-1)}{\theta(p, n)}}$, where $E_2 \doteq E_1^{-\frac{2(p-1)}{\theta(p, n)}}$, we obtain

$$t \geq \max\{R, 2L\beta\} \quad \text{and} \quad \log \left(E_1 \varepsilon^p t^{\frac{\theta(p, n)}{2(p-1)}} \right) > 0.$$

Also, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > E_2 \varepsilon^{-\frac{2p(p-1)}{\theta(p, n)}}$ letting $j \rightarrow \infty$ in (4.34) we see that the lower bound for $U(t)$ blows up. Thus, for any $\varepsilon \in (0, \varepsilon_0]$ the functional U has to blow up in finite time and, moreover, the lifespan of the local solution u can be estimated from above as follows:

$$T(\varepsilon) \lesssim \varepsilon^{-\frac{2p(p-1)}{\theta(p, n)}}.$$

In conclusion, the proof of Theorem 4.1 is complete.

5. Blow – result in the critical case

In this section, we shall prove a blow-up result for the semilinear MGT equation in the conservative case with power nonlinearity in the critical case, that is, we are interested in the Cauchy problem (4.10) in the case in which the exponent of the nonlinear term is $p = p_{\text{Str}}(n)$ (clearly, provided that $n \geq 2$).

Our approach is based on the technique developed in [44], where the slicing procedure is applied in order to get a blow – up result for the semilinear wave equation in the critical case. Nonetheless, the parameters which characterize the slicing procedure itself are chosen in a more suitable way for the MGT equation (cf. Section 5.4).

5.1. Auxiliary functions

Let us recall the definition of a pair of auxiliary functions from [44], which are necessary in order to introduce the time – dependent functional that will be considered for the iteration argument in the critical case $p = p_{\text{Str}}(n)$.

Let $r > -1$ be a real parameter. We consider the function Φ defined by (4.20). Then, we introduce the couple of auxiliary functions

$$\xi_r(t, x) \doteq \int_0^{\lambda_0} e^{-\lambda(t+R)} \cosh(\lambda t) \Phi(\lambda x) \lambda^r d\lambda, \quad (5.35)$$

$$\eta_r(t, s, x) \doteq \int_0^{\lambda_0} e^{-\lambda(t+R)} \frac{\sinh(\lambda(t-s))}{\lambda(t-s)} \Phi(\lambda x) \lambda^r d\lambda, \quad (5.36)$$

where λ_0 is a fixed positive parameter.

Some useful properties of ξ_r and η_r are stated in the following lemma, whose proof can be found in [44, Lemma 3.1].

Lemma 5.1. *Let $n \geq 2$. There exists $\lambda_0 > 0$ such that the following properties hold:*

(i) *if $r > -1$, $|x| \leq R$ and $t \geq 0$, then,*

$$\begin{aligned} \xi_r(t, x) &\geq A_0, \\ \eta_r(t, 0, x) &\geq B_0 \langle t \rangle^{-1}; \end{aligned}$$

(ii) *if $r > -1$, $|x| \leq s + R$ and $t > s \geq 0$, then,*

$$\eta_r(t, s, x) \geq B_1 \langle t \rangle^{-1} \langle s \rangle^{-r};$$

(iii) *if $r > (n-3)/2$, $|x| \leq t + R$ and $t > 0$, then,*

$$\eta_r(t, t, x) \leq B_2 \langle t \rangle^{-\frac{n-1}{2}} \langle t - |x| \rangle^{\frac{n-3}{2}-r}.$$

Here A_0 and B_k , with $k = 0, 1, 2$, are positive constants depending only on λ_0 , r and R and we denote $\langle y \rangle \doteq 3 + |y|$.

Remark 5.1. *Even though in [44] the previous lemma is stated by assuming $r > 0$ in (i) and (ii), the proof provided in that paper holds true for any $r > -1$ as well.*

5.2. Main result

Throughout Section 5 we will consider a slightly different notion of energy solutions to (4.10) with respect to the one given in Section 4 (cf. Definition 4.1). Before introducing this different notion of energy solutions to (4.10), let us recall that u solves the Cauchy problem (4.10) if and only if it solves the second order Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = \varepsilon e^{-t/\beta} (u_2(x) - \Delta u_0(x)) + \frac{1}{\beta} \int_0^t e^{-(t-s)/\beta} |u(s, x)|^p ds, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t)(0, x) = \varepsilon (u_0, u_1)(x), & x \in \mathbb{R}^n. \end{cases} \quad (5.37)$$

Having this fact in mind, it is quite natural to introduce also the following reasonable notion of energy solutions to (5.37) and, then, to (4.10).

Definition 5.1. *Let $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. We say that*

$$u \in \mathcal{C}([0, T], H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T], H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0, T], L^2(\mathbb{R}^n)) \cap L_{\text{loc}}^p([0, T] \times \mathbb{R}^n)$$

is an energy solution of (4.10) on $[0, T)$ if u fulfills $u(0, \cdot) = \varepsilon u_0$ in $H^2(\mathbb{R}^n)$ and the integral relation

$$\begin{aligned} & \int_{\mathbb{R}^n} u_t(t, x) \psi(t, x) \, dx - \varepsilon \int_{\mathbb{R}^n} u_1(x) \psi(0, x) \, dx \\ & + \int_0^t \int_{\mathbb{R}^n} (\nabla_x u(s, x) \cdot \nabla_x \psi(s, x) - u_t(s, x) \psi_s(s, x)) \, dx \, ds \\ & = \varepsilon \int_0^t e^{-s/\beta} \int_{\mathbb{R}^n} \psi(s, x) (u_2(x) - \Delta u_0(x)) \, dx \, ds \\ & + \frac{1}{\beta} \int_0^t \int_0^s e^{(\tau-s)/\beta} \int_{\mathbb{R}^n} \psi(s, x) |u(\tau, x)|^p \, dx \, d\tau \, ds \end{aligned} \quad (5.38)$$

for any $\psi \in \mathcal{C}_0^\infty([0, T) \times \mathbb{R}^n)$ and any $t \in [0, T)$.

Note that, performing a further step of integration by parts in (5.38), it results

$$\begin{aligned} & \int_{\mathbb{R}^n} (\psi(t, x) u_t(t, x) - \psi_s(t, x) u(t, x)) \, dx - \varepsilon \int_{\mathbb{R}^n} (\psi(0, x) u_1(x) - \psi_s(0, x) u_0(x)) \, dx \\ & + \int_0^t \int_{\mathbb{R}^n} (\psi_{ss}(s, x) - \Delta \psi(s, x)) u(s, x) \, dx \, ds \\ & = \varepsilon \int_0^t e^{-s/\beta} \int_{\mathbb{R}^n} \psi(s, x) (u_2(x) - \Delta u_0(x)) \, dx \, ds \\ & + \frac{1}{\beta} \int_0^t \int_0^s e^{(\tau-s)/\beta} \int_{\mathbb{R}^n} \psi(s, x) |u(\tau, x)|^p \, dx \, d\tau \, ds \end{aligned} \quad (5.39)$$

for any $\psi \in \mathcal{C}_0^\infty([0, T) \times \mathbb{R}^n)$ and any $t \in [0, T)$.

We may now state the main result in the critical case for (4.10).

Theorem 5.1. *Let $n \geq 2$ and $p = p_{\text{Str}}(n)$. Let us assume that $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ are nonnegative and compactly supported functions with supports contained in B_R for some $R > 0$ such that u_0 is not identically zero and $u_2 - \Delta u_0$ is nonnegative. Let*

$$u \in \mathcal{C}([0, T), H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0, T), L^2(\mathbb{R}^n)) \cap L_{\text{loc}}^p([0, T) \times \mathbb{R}^n)$$

be an energy solution on $[0, T)$ to the Cauchy problem (4.10) according to Definition 5.1 with lifespan $T = T(\varepsilon)$ such that

$$\text{supp } u(t, \cdot) \subset B_{R+t} \quad \text{for any } t \in (0, T). \quad (5.40)$$

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, R, \beta)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the energy solution u blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq \exp\left(C\varepsilon^{-p(p-1)}\right)$$

holds, where the constant $C > 0$ is independent of ε .

5.3. Iteration frame and first lower bound estimate

Before introducing the time – dependent functional whose dynamic will be examined in order to prove Theorem 5.1, let us prove a fundamental identity satisfied by local in time solutions to (4.10)

Proposition 5.1. *Let $n \geq 2$ and $r > -1$. Assume that $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ are compactly supported in B_R for some $R > 0$. Let u be an energy solution to (4.10) on $[0, T)$ according*

to Definition 5.1 satisfying (5.40). Then, the following integral identity holds:

$$\begin{aligned} \int_{\mathbb{R}^n} u(t, x) \eta_r(t, t, x) dx &= \varepsilon \int_{\mathbb{R}^n} u_0(x) \xi_r(t, x) dx + \varepsilon t \int_{\mathbb{R}^n} u_1(x) \eta_r(t, 0, x) dx \\ &+ \varepsilon \int_0^t (t-s) e^{-s/\beta} \int_{\mathbb{R}^n} (u_2(x) - \Delta u_0(x)) \eta_r(t, s, x) dx ds \\ &+ \frac{1}{\beta} \int_0^t (t-s) \int_0^s e^{-(s-\sigma)/\beta} \int_{\mathbb{R}^n} |u(\sigma, x)|^p \eta_r(t, s, x) dx d\sigma ds, \end{aligned} \quad (5.41)$$

for any $t \in (0, T)$, where ξ_r and η_r are defined in (5.35) and (5.36), respectively.

Proof. Since $u(t, \cdot)$ has compact support contained in B_{R+t} for any $t \geq 0$, according to (5.40), we can use the identity (5.39) even for a noncompactly supported test function. Let Φ be the function defined by (4.20). Since Φ satisfies $\Delta\Phi = \Phi$ and the function $y(t, s; \lambda) = \lambda^{-1} \sinh(\lambda(t-s))$ solves the parameter dependent ODE as follows:

$$(\partial_s^2 - \lambda^2) y(t, s; \lambda) = 0$$

with final conditions $y(t, t; \lambda) = 0$ and $\partial_s y(t, t; \lambda) = -1$, then, the test function $\psi(s, x) = y(t, s; \lambda) \Phi(\lambda x)$ is a solution of the free wave equation $\psi_{ss} - \Delta\psi = 0$ and, moreover, satisfies

$$\begin{aligned} \psi(t, x) &= 0, & \psi(0, x) &= \lambda^{-1} \sinh(\lambda t) \Phi(\lambda x), \\ \psi_s(t, x) &= -\Phi(\lambda x), & \psi_s(0, x) &= -\cosh(\lambda t) \Phi(\lambda x). \end{aligned}$$

Let us begin to prove (5.41). Employing in (5.39) the above defined test function ψ and its properties, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} u(t, x) \Phi(\lambda x) dx &= \varepsilon \cosh(\lambda t) \int_{\mathbb{R}^n} u_0(x) \Phi(\lambda x) dx + \varepsilon \frac{\sinh(\lambda t)}{\lambda} \int_{\mathbb{R}^n} u_1(x) \Phi(\lambda x) dx \\ &+ \varepsilon \int_0^t \frac{\sinh(\lambda(t-s))}{\lambda} e^{-s/\beta} \int_{\mathbb{R}^n} (u_2(x) - \Delta u_0(x)) \Phi(\lambda x) dx ds \\ &+ \frac{1}{\beta} \int_0^t \frac{\sinh(\lambda(t-s))}{\lambda} \int_0^s e^{-(s-\sigma)/\beta} \int_{\mathbb{R}^n} |u(\sigma, x)|^p \Phi(\lambda x) dx d\sigma ds. \end{aligned}$$

Multiplying both sides of the above equality by $e^{-\lambda(t+R)} \lambda^r$, integrating with respect to λ over the interval $[0, \lambda_0]$ and using Tonelli's theorem, we find (5.41). \square

Hereafter until the end of Section 5, we shall assume that u_0, u_1, u_2 satisfy the assumptions from the statement of Theorem 5.1, that is, these functions are nonnegative, compactly supported and satisfy $u_0 \not\equiv 0$ and $u_2 - \Delta u_0 \geq 0$. Let u be an energy solution of (4.10) on $[0, T)$ according to Definition 5.1. We introduce the following time - dependent functional:

$$\mathcal{U}(t) \doteq \int_{\mathbb{R}^n} u(t, x) \eta_r(t, t, x) dx, \quad (5.42)$$

where

$$r \doteq \frac{n-1}{2} - \frac{1}{p}.$$

From Proposition 5.1 it follows immediately the positiveness of the functional \mathcal{U} , thanks to the assumptions on the Cauchy data.

The next step is to derive an integral inequalities involving \mathcal{U} both in the left and in the right - hand side, which will set the iteration frame for \mathcal{U} in the iteration procedure.

Proposition 5.2. *Let us assume that $r = (n-1)/2 - 1/p$. Let \mathcal{U} be the functional defined by (5.42). Then, there exist positive constants C depending on $n, p, \beta, \lambda_0, R$ such that the estimate*

$$\mathcal{U}(t) \geq C \langle t \rangle^{-1} \int_0^t (t-s) \langle s \rangle^{-\frac{n-1}{2} + \frac{1}{p}} \int_0^s e^{-\frac{1}{\beta}(s-\sigma)} \langle \sigma \rangle^{(n-1)(1-\frac{p}{2})} (\log \langle \sigma \rangle)^{-(p-1)} (\mathcal{U}(\sigma))^p d\sigma ds \quad (5.43)$$

holds for any $t \geq 0$.

Proof. In the proof of this proposition we adapt the main ideas of Proposition 4.2 in [44] to our model. By using Hölder's inequality and the support property for $u(\sigma, \cdot)$, we find

$$\mathcal{U}(\sigma) \leq \left(\int_{\mathbb{R}^n} |u(\sigma, x)|^p \eta_r(t, s, x) dx \right)^{\frac{1}{p}} \left(\int_{B_{\sigma+R}} \frac{\eta_r(\sigma, \sigma, x)^{p'}}{\eta_r(t, s, x)^{\frac{p'}{p}}} dx \right)^{\frac{1}{p'}}. \quad (5.44)$$

We start by estimating the second factor on the right hand side in the last inequality.

According to our choice of r , both $r > (n-3)/2$ and $r > -1$ are always satisfied, so, by (ii) and (iii) in Lemma 5.1, since $|x| \leq \sigma + R$ implies $|x| \leq s + R$ for any $\sigma \in [0, s]$, we get

$$\begin{aligned} \int_{B_{\sigma+R}} \frac{\eta_r(\sigma, \sigma, x)^{p'}}{\eta_r(t, s, x)^{\frac{p'}{p}}} dx &\lesssim \langle t \rangle^{\frac{p'}{p}} \langle s \rangle^{\frac{rp'}{p}} \langle \sigma \rangle^{-\frac{n-1}{2}p'} \int_{B_{\sigma+R}} \langle \sigma - |x| \rangle^{(\frac{n-3}{2}-r)p'} dx \\ &\lesssim \langle t \rangle^{\frac{1}{p-1}} \langle s \rangle^{\frac{r}{p-1}} \langle \sigma \rangle^{-\frac{n-1}{2}p'} \int_{B_{\sigma+R}} \langle \sigma - |x| \rangle^{-1} dx \\ &\lesssim \langle t \rangle^{\frac{1}{p-1}} \langle s \rangle^{\frac{r}{p-1}} \langle \sigma \rangle^{-\frac{n-1}{2}p'+n-1} \log \langle \sigma \rangle, \end{aligned}$$

where in the second step we used the definition of r to get the exponent for the term in the x -dependent integral. Thanks to the sign assumptions on the Cauchy data, combining (5.41), (5.44) and the previous estimate, we arrive at

$$\begin{aligned} \mathcal{U}(t) &\gtrsim \int_0^t (t-s) \int_0^s e^{-\frac{1}{\beta}(s-\sigma)} \int_{\mathbb{R}^n} |u(\sigma, x)|^p \eta_r(t, s, x) dx d\sigma ds \\ &\gtrsim \int_0^t (t-s) \int_0^s e^{-\frac{1}{\beta}(s-\sigma)} \langle t \rangle^{-1} \langle s \rangle^{-r} \langle \sigma \rangle^{\frac{n-1}{2}p-(n-1)(p-1)} (\log \langle \sigma \rangle)^{-(p-1)} (\mathcal{U}(\sigma))^p d\sigma ds \\ &= \langle t \rangle^{-1} \int_0^t (t-s) \langle s \rangle^{-\frac{n-1}{2}+\frac{1}{p}} \int_0^s e^{-\frac{1}{\beta}(s-\sigma)} \langle \sigma \rangle^{\frac{n-1}{2}p-(n-1)(p-1)} (\log \langle \sigma \rangle)^{-(p-1)} (\mathcal{U}(\sigma))^p d\sigma ds, \end{aligned}$$

which is exactly (5.43). \square

Lemma 5.2. *Let us suppose that the assumptions from Theorem 5.1 are fulfilled. Let u be a solution to (4.10) according to Definition 5.1. Then, there exists a positive constant $C_0 = C_0(u_0, u_1, u_2, n, p, R)$ such that the following lower bound estimate:*

$$\int_{\mathbb{R}^n} |u(t, x)|^p dx \geq C_0 \varepsilon^p \langle t \rangle^{n-1-\frac{n-1}{2}p} \quad (5.45)$$

holds for any $t \geq 0$.

Proof. By using (5.41) and the sign assumptions on initial data, we get

$$\varepsilon \int_{\mathbb{R}^n} u_0(x) \xi_r(t, x) dx \leq \mathcal{U}(t) \leq \left(\int_{\mathbb{R}^n} |u(t, x)|^p dx \right)^{1/p} I(t)^{1/p'}, \quad (5.46)$$

where we put

$$I(t) \doteq \int_{B_{t+R}} \eta_r(t, t, x)^{p'} dx.$$

Repeating exactly the same proof of Lemma 5.1 in [44], we have

$$I(t) \lesssim \langle t \rangle^{n-1-\frac{n-1}{2}p'}.$$

Combining this upper bound estimate for $I(t)$, Lemma 5.1 (i) and (5.46), it follows immediately our desired estimate (5.45). \square

Remark 5.2. *Let us point out explicitly that although (4.26) and (5.45) are formally the same estimates, we had to prove this lower bound estimate twice as the energy solution u satisfies different integral relations in the subcritical and critical case (cf. Definition 4.1 and Definition 5.1).*

Proposition 5.3. *Let us assume that $r = (n - 1)/2 - 1/p$ and $p = p_{\text{Str}}(n)$. Let \mathcal{U} be the functional defined by (5.42) and let us consider a positive parameter $\omega_0 > 1$ such that $\beta\omega_0 > 1$. Then, there exist a positive constant M depending on $n, p, \beta, \lambda_0, R, u_0, u_1$ such that*

$$\mathcal{U}(t) \geq M\varepsilon^p \log\left(\frac{t}{\beta\omega_0}\right) \quad (5.47)$$

holds for any $t \geq \beta\omega_0$.

Proof. Combining the lower bound estimate (5.45) for the integral of the p -power of the solution u together with Lemma 5.1 (ii) and (5.41), we get

$$\begin{aligned} \mathcal{U}(t) &\gtrsim \int_0^t (t-s) \int_0^s e^{-\frac{1}{\beta}(s-\sigma)} \int_{\mathbb{R}^n} |u(\sigma, x)|^p \eta_r(t, s, x) dx d\sigma ds \\ &\gtrsim \langle t \rangle^{-1} \int_0^t (t-s) \langle s \rangle^{-\frac{n-1}{2} + \frac{1}{p}} \int_0^s e^{-\frac{1}{\beta}(s-\sigma)} \int_{\mathbb{R}^n} |u(\sigma, x)|^p dx d\sigma ds \\ &\gtrsim \varepsilon^p \langle t \rangle^{-1} \int_0^t (t-s) \langle s \rangle^{-\frac{n-1}{2} + \frac{1}{p}} \int_0^s e^{-\frac{1}{\beta}(s-\sigma)} \langle \sigma \rangle^{n-1 - \frac{n-1}{2}p} d\sigma ds, \end{aligned}$$

where we used again the sign assumptions on the Cauchy data. Therefore, for $t \geq 1$ by shrinking the domain of integration we find

$$\begin{aligned} \mathcal{U}(t) &\gtrsim \varepsilon^p \langle t \rangle^{-1} \int_0^t (t-s) \langle s \rangle^{-\frac{n-1}{2}p - \frac{n-1}{2} + \frac{1}{p}} \int_{s/2}^s e^{-\frac{1}{\beta}(s-\sigma)} \sigma^{n-1} d\sigma ds \\ &\gtrsim \varepsilon^p \langle t \rangle^{-1} \int_0^t (t-s) \langle s \rangle^{-\frac{n-1}{2}p - \frac{n-1}{2} + \frac{1}{p}} s^{n-1} \beta \left(1 - e^{-\frac{s}{2\beta}}\right) ds \\ &\gtrsim \varepsilon^p \beta \left(1 - e^{-\frac{1}{2\beta}}\right) \langle t \rangle^{-1} \int_1^t (t-s) \langle s \rangle^{-\frac{n-1}{2}p + \frac{n-1}{2} + \frac{1}{p}} ds. \end{aligned}$$

Due to $p = p_{\text{Str}}(n)$, from (1.4), we have

$$-\frac{n-1}{2}p + \frac{n-1}{2} + \frac{1}{p} = -1. \quad (5.48)$$

So, the power of $\langle s \rangle$ in the last integral is exactly -1 . Hence, for $t \geq \beta\omega_0$ it results

$$\begin{aligned} \mathcal{U}(t) &\gtrsim \varepsilon^p \langle t \rangle^{-1} \int_1^t (t-s) \langle s \rangle^{-1} ds \gtrsim \varepsilon^p \langle t \rangle^{-1} \int_1^t \frac{t-s}{s} ds \gtrsim \varepsilon^p \langle t \rangle^{-1} \int_1^t \log s ds \\ &\gtrsim \varepsilon^p \langle t \rangle^{-1} \int_{\frac{t}{\beta\omega_0}}^t \log s ds \gtrsim \varepsilon^p \log\left(\frac{t}{\beta\omega_0}\right). \end{aligned}$$

This completes the proof. \square

In this subsection, we established the iteration frame (5.43) for the functional \mathcal{U} and we determined a first lower bound estimate (5.47) for \mathcal{U} containing a logarithmic factors. In the next subsection we are going to determine a sequence of lower bound estimates for \mathcal{U} applying the slicing procedure. More specifically, we will combine the main ideas from [31, 33] concerning two step iteration procedures and from Section 4 concerning the treatment of an exponential multiplier, in order to make the slicing procedure suitable for the iteration frame (5.43).

5.4. Iteration argument via slicing method

Let us introduce the sequence $\{\omega_k\}_{k \in \mathbb{N}}$, where ω_0 has been introduced in the statement of Proposition 5.3 and $\omega_k \doteq 1 + 2^{-k}$ for any $k \geq 1$. Hence, the sequence of parameters that characterize the slicing procedure $\{\Omega_j\}_{j \in \mathbb{N}}$ is defined by

$$\Omega_j \doteq \prod_{k=0}^j \omega_k. \quad (5.49)$$

We point out that $\{\Omega_j\}_{j \in \mathbb{N}}$ is an increasing sequence of positive real number and the infinite product $\prod_{k=0}^{\infty} \omega_k$ is convergent. Thus, if we denote

$$\Omega \doteq \prod_{k=0}^{\infty} \omega_k,$$

then, in particular, $\Omega_j \uparrow \Omega$ as $j \rightarrow \infty$.

The goal of this part is to prove via an iteration method the family of estimates

$$\mathcal{U}(t) \geq M_j (\log \langle t \rangle)^{-b_j} \left(\log \left(\frac{t}{\beta \Omega_{2j}} \right) \right)^{a_j} \quad (5.50)$$

for $t \geq \beta \Omega_{2j}$ and for any $j \in \mathbb{N}$, where $\{M_j\}_{j \in \mathbb{N}}$, $\{a_j\}_{j \in \mathbb{N}}$ and $\{b_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers that will be determined recursively throughout the iteration procedure. For $j = 0$ we know that (5.50) is true thanks to Proposition 5.3 with

$$M_0 \doteq M \varepsilon^p, \quad a_0 \doteq 1 \quad \text{and} \quad b_0 \doteq 0.$$

We shall prove the validity of (5.50) for any $j \in \mathbb{N}$ by induction. Since we have already shown the validity of the base case, it remains to prove the inductive step. Therefore, we assume that (5.50) holds for $j \geq 1$ and we want to prove it for $j + 1$. Plugging (5.50) for j in (5.43), we find

$$\begin{aligned} \mathcal{U}(t) &\geq C M_j^p \langle t \rangle^{-1} \int_{\beta \Omega_{2j}}^t (t-s) \langle s \rangle^{-r} \int_{\beta \Omega_{2j}}^s e^{-\frac{1}{\beta}(s-\sigma)} \langle \sigma \rangle^{(n-1)(1-\frac{p}{2})} \\ &\quad \times (\log \langle \sigma \rangle)^{-(p-1)-b_j p} \left(\log \left(\frac{\sigma}{\beta \Omega_{2j}} \right) \right)^{a_j p} d\sigma ds \\ &\geq C M_j^p \langle t \rangle^{-1} (\log \langle t \rangle)^{-(p-1)-b_j p} \int_{\beta \Omega_{2j}}^t (t-s) \langle s \rangle^{-r-\frac{n-1}{2}p} \int_{\beta \Omega_{2j}}^s e^{-\frac{1}{\beta}(s-\sigma)} \\ &\quad \times \langle \sigma \rangle^{n-1} \left(\log \left(\frac{\sigma}{\beta \Omega_{2j}} \right) \right)^{a_j p} d\sigma ds \end{aligned}$$

for $t \geq \beta \Omega_{2j+2}$. Now we estimate for $s \geq \beta \Omega_{2j+1}$ from below the σ – integral in the last line as follows:

$$\begin{aligned} \int_{\beta \Omega_{2j}}^s e^{-\frac{1}{\beta}(s-\sigma)} \langle \sigma \rangle^{n-1} \left(\log \left(\frac{\sigma}{\beta \Omega_{2j}} \right) \right)^{a_j p} d\sigma &\geq \int_{\frac{\Omega_{2j} s}{\Omega_{2j+1}}}^s e^{-\frac{1}{\beta}(s-\sigma)} \sigma^{n-1} \left(\log \left(\frac{\sigma}{\beta \Omega_{2j}} \right) \right)^{a_j p} d\sigma \\ &\geq \left(\frac{\Omega_{2j}}{\Omega_{2j+1}} \right)^{n-1} s^{n-1} \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p} \int_{\frac{\Omega_{2j} s}{\Omega_{2j+1}}}^s e^{-\frac{1}{\beta}(s-\sigma)} d\sigma \\ &\geq \beta \left(\frac{\Omega_{2j}}{\Omega_{2j+1}} \right)^{n-1} \left(1 - e^{-\frac{1}{\beta} \left(1 - \frac{\Omega_{2j}}{\Omega_{2j+1}} \right) s} \right) s^{n-1} \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p} \\ &\geq \beta \left(\frac{\Omega_{2j}}{\Omega_{2j+1}} \right)^{n-1} \left(1 - e^{-(\Omega_{2j+1} - \Omega_{2j})} \right) s^{n-1} \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p}. \end{aligned}$$

Using the inequalities $\Omega_{2j}/\Omega_{2j+1} = 1/\omega_{2j+1} > 1/2$, the estimate $4s \geq \langle s \rangle$ for any $s \geq 1$ and the inequality

$$\begin{aligned} 1 - e^{-(\Omega_{2j+1} - \Omega_{2j})} &= 1 - e^{-\Omega_{2j}(\omega_{2j+1} - 1)} \geq 1 - e^{-(\omega_{2j+1} - 1)} \geq 1 - \left(1 - (\omega_{2j+1} - 1) + \frac{1}{2}(\omega_{2j+1} - 1)^2 \right) \\ &\geq (\omega_{2j+1} - 1) \left(1 - \frac{1}{2}(\omega_{2j+1} - 1) \right) = 2^{-2(2j+1)} \left(2^{2j+1} - \frac{1}{2} \right) \geq 2^{-2(2j+1)}, \end{aligned}$$

where we used $\Omega_{2j} > 1$ and Taylor's formula up to order 2 neglecting the positive remaining term, we obtain

$$\int_{\beta \Omega_{2j}}^s e^{-\frac{1}{\beta}(s-\sigma)} \langle \sigma \rangle^{n-1} \left(\log \left(\frac{\sigma}{\beta \Omega_{2j}} \right) \right)^{a_j p} d\sigma \geq 2^{-3(n-1)} \beta 2^{-2(2j+1)} \langle s \rangle^{n-1} \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p}.$$

So, plugging the lower bound estimate for the σ – integral in the lower bound estimate for $\mathcal{U}(t)$ and using again (5.48), for $t \geq \beta \Omega_{2j+2}$ it holds

$$\begin{aligned} \mathcal{U}(t) &\geq \widehat{C} 2^{-2(2j+1)} M_j^p (\log \langle t \rangle)^{-(p-1)-b_j p} \langle t \rangle^{-1} \int_{\beta \Omega_{2j+1}}^t (t-s) \langle s \rangle^{-\frac{n-1}{2}p + \frac{n-1}{2} + \frac{1}{p}} \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p} ds \\ &\geq \widehat{C} 2^{-2(2j+1)} M_j^p (\log \langle t \rangle)^{-(p-1)-b_j p} \langle t \rangle^{-1} \int_{\beta \Omega_{2j+1}}^t (t-s) \langle s \rangle^{-1} \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p} ds \\ &\geq \widehat{C} 2^{-2(2j+2)} M_j^p (\log \langle t \rangle)^{-(p-1)-b_j p} \langle t \rangle^{-1} \int_{\beta \Omega_{2j+1}}^t \frac{t-s}{s} \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p} ds, \end{aligned}$$

where $\widehat{C} \doteq 2^{-3(n-1)} \beta C$. Let us estimate the s – dependent integral in the last line. Integration by parts and a further shrinking of the domain of integration lead to

$$\begin{aligned} \langle t \rangle^{-1} \int_{\beta \Omega_{2j+1}}^t \frac{t-s}{s} \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p} ds &= (a_j p + 1)^{-1} \langle t \rangle^{-1} \int_{\beta \Omega_{2j+1}}^t \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p + 1} ds \\ &\geq (a_j p + 1)^{-1} \langle t \rangle^{-1} \int_{\frac{\Omega_{2j+1} t}{\Omega_{2j+2}}}^t \left(\log \left(\frac{s}{\beta \Omega_{2j+1}} \right) \right)^{a_j p + 1} ds \\ &\geq (a_j p + 1)^{-1} \langle t \rangle^{-1} \left(\log \left(\frac{t}{\beta \Omega_{2j+2}} \right) \right)^{a_j p + 1} \left(1 - \frac{\Omega_{2j+1}}{\Omega_{2j+2}} \right) t \\ &\geq 2^{-3} 2^{-(2j+2)} (a_j p + 1)^{-1} \left(\log \left(\frac{t}{\beta \Omega_{2j+2}} \right) \right)^{a_j p + 1} \end{aligned}$$

for $t \geq \beta \Omega_{2j+2}$, where in the last estimate we used $4t \geq \langle t \rangle$ and

$$1 - \frac{\Omega_{2j+1}}{\Omega_{2j+2}} = \frac{\omega_{2j+2} - 1}{\omega_{2j+2}} \geq 2^{-1} (\omega_{2j+2} - 1) = 2^{-1-(2j+2)}.$$

Therefore, combining this time the lower bound estimate for $\mathcal{U}(t)$ with the estimate from below of the s – integral, we conclude

$$\mathcal{U}(t) \geq 2^{-3} \widehat{C} (a_j p + 1)^{-1} 2^{-3(2j+2)} M_j^p (\log \langle t \rangle)^{-(p-1)-b_j p} \left(\log \left(\frac{t}{\beta \Omega_{2j+2}} \right) \right)^{a_j p + 1}$$

for $t \geq \beta \Omega_{2j+2}$. Also, we proved (5.50) for $j+1$ provided that

$$M_{j+1} \doteq 2^{-3n} \beta C (a_j p + 1)^{-1} 2^{-6(j+1)} M_j^p, \quad a_{j+1} \doteq a_j p + 1, \quad b_{j+1} \doteq (p-1) + b_j p.$$

As next step, we determine a lower bound for the term M_j which is more easy to handle. For this reason, we determine an explicit representation of the exponents a_j and b_j . By using recursively the relations $a_j = 1 + p a_{j-1}$ and $b_j = (p-1) + p b_{j-1}$ and the initial exponents $a_0 = 1$, $b_0 = 0$, we get

$$a_j = a_0 p^j + \sum_{k=0}^{j-1} p^k = \frac{p^{j+1} - 1}{p-1} \quad \text{and} \quad b_j = p^j b_0 + (p-1) \sum_{k=0}^{j-1} p^k = p^j - 1. \quad (5.51)$$

In particular, $a_{j-1} p + 1 = a_j \leq p^{j+1}/(p-1)$ implies that

$$M_j \geq \widehat{D} (2^6 p)^{-j} M_{j-1}^p \quad (5.52)$$

for any $j \geq 1$, where $\widehat{D} \doteq 2^{-3n} \beta C(p-1)/p$. Applying the logarithmic function to both sides of (5.52) and using iteratively the resulting inequality, we have

$$\begin{aligned} \log M_j &\geq p \log M_{j-1} - j \log(2^6 p) + \log \widehat{D} \\ &\geq p^j \log M_0 - \left(\sum_{k=0}^{j-1} (j-k)p^k \right) \log(2^6 p) + \left(\sum_{k=0}^{j-1} p^k \right) \log \widehat{D} \\ &= p^j \left(\log M_0 - \frac{p \log(2^6 p)}{(p-1)^2} + \frac{\log \widehat{D}}{p-1} \right) + \left(\frac{j}{p-1} + \frac{p}{(p-1)^2} \right) \log(2^6 p) - \frac{\log \widehat{D}}{p-1}, \end{aligned}$$

where we used again (4.32). Let us define $j_1 = j_1(n, p, \beta)$ as the smallest nonnegative integer such that

$$j_1 \geq \frac{\log \widehat{D}}{\log(2^6 p)} - \frac{p}{p-1}.$$

Then, for any $j \geq j_1$ we may estimate

$$\log M_j \geq p^j \left(\log M_0 - \frac{p \log(2^6 p)}{(p-1)^2} + \frac{\log \widehat{D}}{p-1} \right) = p^j \log(N_0 \varepsilon^p), \quad (5.53)$$

where $N_0 \doteq M(2^6 p)^{-p/(p-1)^2} \widehat{D}^{1/(p-1)}$. Consequently, combining (5.50), (5.51) and (5.53), it results

$$\begin{aligned} \mathcal{U}(t) &\geq \exp(p^j \log(N_0 \varepsilon^p)) (\log \langle t \rangle)^{-p^j+1} \left(\log \left(\frac{t}{\beta \Omega} \right) \right)^{(p^{j+1}-1)/(p-1)} \\ &= \exp \left(p^j \log \left(N_0 \varepsilon^p (\log \langle t \rangle)^{-1} \left(\log \left(\frac{t}{\beta \Omega} \right) \right)^{p/(p-1)} \right) \right) \log \langle t \rangle \left(\log \left(\frac{t}{\beta \Omega} \right) \right)^{-1/(p-1)} \end{aligned}$$

for $t \geq \beta \Omega$ and any $j \geq j_1$.

For $t \geq t_0 \doteq \max \left\{ 3, (\beta \Omega)^{\frac{\beta \Omega}{\beta \Omega - 1}} \right\}$ the inequalities

$$\log \langle t \rangle \leq \log(2t) \leq 2 \log t \quad \text{and} \quad \log \left(\frac{t}{\beta \Omega} \right) \geq \frac{1}{\beta \Omega} \log t$$

hold true, so,

$$\mathcal{U}(t) \geq \exp \left(p^j \log \left(N_1 \varepsilon^p (\log t)^{1/(p-1)} \right) \right) \log \langle t \rangle \left(\log \left(\frac{t}{\beta \Omega} \right) \right)^{-1/(p-1)} \quad (5.54)$$

for $t \geq t_0$ and any $j \geq j_1$, where $N_1 \doteq 2^{-1} (\beta \Omega)^{-p/(p-1)} N_0$. Let us introduce the function

$$J(t, \varepsilon) \doteq N_1 \varepsilon^p (\log t)^{1/(p-1)}.$$

We can choose $\varepsilon_0 = \varepsilon_0(n, p, \beta, \lambda_0, R, u_0, u_1)$ sufficiently small so that

$$\exp \left(N_1^{1-p} \varepsilon_0^{-p(p-1)} \right) \geq t_0.$$

Hence, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp \left(N_1^{1-p} \varepsilon^{-p(p-1)} \right)$ the two conditions $t \geq t_0$ and $J(t, \varepsilon) > 1$ are always fulfilled. Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp \left(N_1^{1-p} \varepsilon^{-p(p-1)} \right)$ taking the limit as $j \rightarrow \infty$ in (5.54) we find that the lower bound for $\mathcal{U}(t)$ blows up. Then, $\mathcal{U}(t)$ cannot be finite for this t . Thus, we proved that \mathcal{U} blows up in finite time and, furthermore, we provided the upper bound estimate for the lifespan

$$T(\varepsilon) \leq \exp \left(N_1^{1-p} \varepsilon^{-p(p-1)} \right).$$

This completes the proof of Theorem 5.1.

6. Concluding remarks

Let us consider the general case of the Cauchy problem for the semilinear MGT equation

$$\begin{cases} \tau u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t, u_{tt})(0, x) = (u_0, u_1, u_2)(x), & x \in \mathbb{R}^n, \end{cases} \quad (6.55)$$

where $0 < \tau < \beta$ (the dissipative case) or $\tau = \beta$ (the conservative case) and $p > 1$. By introducing

$$w \doteq \tau u_t + u,$$

we may transform (6.55) to the following semilinear second order evolution equation:

$$\begin{cases} w_{tt} - \frac{\beta}{\tau} \Delta w + \frac{\beta - \tau}{\tau^2} G * \Delta w = H(u_0, w; p, \beta, \tau), & x \in \mathbb{R}^n, t > 0, \\ (w, w_t)(0, x) = (\tau u_1 + u_0, \tau u_2 + u_1)(x), & x \in \mathbb{R}^n, \end{cases} \quad (6.56)$$

where the kernel function $G = G(t)$ is given by $G(t) \doteq e^{-t/\tau}$ and the right – hand side is

$$H(u_0, w; p, \beta, \tau)(t, x) \doteq -\frac{\beta - \tau}{\tau} e^{-t/\tau} \Delta u_0(x) + \left| e^{-t/\tau} u_0(x) + \frac{1}{\tau} (G * w)(t, x) \right|^p.$$

Here the convolution term is defined by

$$(G * w)(t, x) \doteq \int_0^t G(t - s) w(s, x) ds.$$

We now may understand the equation in (6.56) in the two cases above mentioned. In the conservative case $\tau = \beta$, we may interpret the model (6.56) as a wave equation with power source nonlinearity, which includes a memory term with an exponential decaying kernel function. On the other hand, in the dissipative case $0 < \tau < \beta$, the dissipation generated by the memory term comes into play even in the linear part, therefore, the model (6.56) can be interpreted as a semilinear viscoelastic equation (see for example [8, 36]). Up to the knowledge of the authors, the blow – up of solutions to this kind of semilinear viscoelastic equations is still an open problem.

In this paper, we considered the Cauchy problem for the semilinear MGT equation with power nonlinearity $|u|^p$ and proved the blow – up of local energy solutions in the sub – Strauss case, i.e., for $1 < p \leq p_{\text{Str}}(n)$. Concerning the Cauchy problem for the semilinear MGT equation with nonlinearity of derivative type in the conservative case, namely,

$$\begin{cases} \beta u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = |u_t|^p, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t, u_{tt})(0, x) = (u_0, u_1, u_2)(x), & x \in \mathbb{R}^n, \end{cases} \quad (6.57)$$

with $\beta > 0$, in the forthcoming paper [4], we shall study the blow – up of local in time solutions to (6.57) and the corresponding lifespan estimates under suitable assumptions for initial data. More specifically, the blow – up in finite time of energy solutions to (6.57) is going to be proved providing that the power p of the nonlinearity satisfies

$$1 < p \leq p_{\text{Gla}}(n) \doteq \frac{n + 1}{n - 1},$$

for $n \geq 2$ and $p > 1$ for $n = 1$. We underline that the Glassey exponent $p_{\text{Gla}}(n)$ is the critical exponent for the corresponding semilinear wave equation with nonlinearity of derivative type.

Acknowledgments. The Ph.D. study of the first author is supported by Sächsisches Landesgraduiertenstipendium. The second author is supported by the University of Pisa, Project PRA 2018 49.

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