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A note on a conjecture for the critical curve of a weakly coupled system of semilinear wave equations with scale-invariant lower order terms

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Abstract

In this note two blow-up results are proved for a weakly coupled system of semilinear wave equations with distinct scale-invariant lower order terms both in the subcritical case and in the critical case, when the damping and the mass terms make both equations in some sense "wave-like". In the proof of the subcritical case an iteration argument is used. This approach is based on a coupled system of nonlinear ordinary integral inequalities and lower bound estimates for the spatial integral of the nonlinearities. In the critical case we employ a test function type method, that has been developed recently by Ikeda-Sobajima-Wakasa and relies strongly on a family of certain self-similar solutions of the adjoint linear equation. Therefore, as critical curve in the p - q plane of the exponents of the power nonlinearities for this weakly coupled system we conjecture a shift of the critical curve for the corresponding weakly coupled system of semilinear wave equations.

Keywords: Semilinear weakly coupled system; Blow-up; Scale-invariant lower order terms; Critical curve; Self-similar solutions; Test function method.
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1. Introduction

In this paper we consider the weakly coupled system of wave equations with scale-invariant damping and mass terms with different multiplicative constants in the lower order terms and with power nonlinearities, namely,

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t + \frac{\nu_1^2}{(1+t)^2} u = |v|^p, & x \in \mathbb{R}^n,\ t > 0, \\ v_{tt} - \Delta v + \frac{\mu_2}{1+t} v_t + \frac{\nu_2^2}{(1+t)^2} v = |u|^q, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t, v, v_t)(0, x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x) & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\mu_1, \mu_2, \nu_1^2, \nu_2^2$ are nonnegative constants, ε is a positive parameter describing the size of initial data and $p, q > 1$.

Recently, the Cauchy problem for a semilinear wave equation with scale-invariant damping and mass

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = |u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{2}$$

where μ, ν^2 are nonnegative constants and $p > 1$, has attracted a lot of attention. The value of $\delta \doteq (\mu - 1)^2 - 4\nu^2$ has a strong influence on some properties of solutions to (2) and to the corresponding

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homogeneous linear equation. According to [3, 40, 5, 4, 39, 22, 30, 27, 21, 13, 31, 37, 38, 28, 29, 6, 35, 16, 20] for $\delta \geqslant 0$ the model in (2) is somehow an intermediate model between the semilinear free wave equation and the semilinear classical damped equation, whose critical exponent is $p_{\mathrm{Fuj}}(n + \frac{\mu-1}{2} - \frac{\sqrt{\delta}}{2})$ for $\delta \geq (n+1)^2$ and seems reasonably to be $p_0(n+\mu)$ for small and nonnegative values of delta, where $p_{\mathrm{Fuj}}(n)$ and $p_0(n)$ denote the Fujita exponent and the Strauss exponent, respectively.

As for the single semilinear wave equation with scale-invariant damping and mass term, the quantities

$$\delta_j \doteq (\mu_j - 1)^2 - 4\nu_j^2, \qquad j = 1, 2, \tag{3}$$

play a fundamental role in the description of some of the properties of the solutions to (1) as, for example, the critical curve. In particular, in [2] the critical curve for (1) is proved to be

$$\max\left\{\frac{p+1}{pq-1} - \frac{1}{2}\left(\frac{\mu_1 - 1}{2} - \frac{\sqrt{\delta_1}}{2}\right), \frac{q+1}{pq-1} - \frac{1}{2}\left(\frac{\mu_2 - 1}{2} - \frac{\sqrt{\delta_2}}{2}\right)\right\} = \frac{n}{2} \tag{4}$$

in the case $\delta_1, \delta_2 \geqslant (n+1)^2$. Let us remark that (4) is a shift of the critical curve in the p - q plane for the weakly coupled system of semilinear classical damped equation with power nonlinearities, which is (cf. [36, 23, 24, 25, 26])

$$\frac{\max\{p,q\}+1}{pq-1} = \frac{n}{2}.$$

This paper is devoted to the proof of a blow-up results for (1) in the case $\delta_1, \delta_2 \geqslant 0$ both in the subcritical case and on the critical curve. Analogously to what happens in the case of single equations, when δ_1, δ_2 are small the model is somehow "wave-like". Therefore, the blow-up result that we will prove may be optimal only for small values of δ_1, δ_2 according to the above mentioned papers, where (2) is considered. This is reasonable since we obtain as "critical curve"

$$\max\left\{\frac{p+2+q^{-1}}{pq-1} - \frac{\mu_1}{2}, \frac{q+2+p^{-1}}{pq-1} - \frac{\mu_2}{2}\right\} = \frac{n-1}{2}$$

which is a generally asymmetric shift of the critical curve for the weakly coupled system of semilinear wave equation with power nonlinearities (see also [7, 9, 8, 1, 18, 17, 11, 19]), namely,

$$\max\left\{\frac{p+2+q^{-1}}{pq-1}, \frac{q+2+p^{-1}}{pq-1}\right\} = \frac{n-1}{2}. \tag{5}$$

Before stating the main results of this paper, let us introduce a suitable notion of energy solutions according to [21].

Definition 1.1. *Let $u_0, v_0 \in H^1(\mathbb{R}^n)$ and $u_1, v_1 \in L^2(\mathbb{R}^n)$. We say that (u,v) is an energy solution of (1) on $[0,T)$ if*

$$u \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n)) \cap L^q_{loc}(\mathbb{R}^n \times [0,T)),$$
$$v \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n)) \cap L^p_{loc}(\mathbb{R}^n \times [0,T))$$

satisfy $u(0,x) = \varepsilon u_0(x)$ and $v(0,x) = \varepsilon v_0(x)$ in $H^1(\mathbb{R}^n)$ and the equalities

$$\begin{aligned}
&\int_{\mathbb{R}^n} u_t(t,x)\phi(t,x)\,dx - \int_{\mathbb{R}^n} u_t(0,x)\phi(0,x)\,dx - \int_0^t\int_{\mathbb{R}^n} u_t(s,x)\phi_t(s,x)\,dx\,ds \\
&+ \int_0^t\int_{\mathbb{R}^n} \nabla u(s,x)\cdot\nabla\phi(s,x)\,dx\,ds + \int_0^t\int_{\mathbb{R}^n}\left(\frac{\mu_1}{1+s}u_t(s,x) + \frac{\nu_1^2}{(1+s)^2}u(s,x)\right)\phi(s,x)\,dx\,ds \\
&= \int_0^t\int_{\mathbb{R}^n} |v(s,x)|^p\phi(s,x)\,dx\,ds
\end{aligned} \tag{6}$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n} v_t(t,x)\psi(t,x)\,dx - \int_{\mathbb{R}^n} v_t(0,x)\psi(0,x)\,dx - \int_0^t\int_{\mathbb{R}^n} v_t(s,x)\psi_t(s,x)\,dx\,ds \\
&+ \int_0^t\int_{\mathbb{R}^n} \nabla v(s,x)\cdot\nabla\psi(s,x)\,dx\,ds + \int_0^t\int_{\mathbb{R}^n}\left(\frac{\mu_2}{1+s}v_t(s,x) + \frac{\nu_2^2}{(1+s)^2}v(s,x)\right)\psi(s,x)\,dx\,ds \\
&= \int_0^t\int_{\mathbb{R}^n} |u(s,x)|^q\psi(s,x)\,dx\,ds \qquad (7)
\end{aligned}
$$

for any $\phi,\psi\in\mathscr{C}_0^\infty([0,T)\times\mathbb{R}^n)$ and any $t\in[0,T)$.

After a further integration by parts in (6) and (7), letting $t\to T$, we find that (u,v) fulfills the definition of weak solution to (1).

We can now state the main theorem in the subcritical case.

Theorem 1.2. *Let $\mu_1,\mu_2,\nu_1^2,\nu_2^2$ be nonnegative constants such that $\delta_1,\delta_2\geqslant 0$. Let us consider $p,q>1$ satisfying*

$$\max\left\{\frac{p+2+q^{-1}}{pq-1}-\frac{\mu_1}{2},\frac{q+2+p^{-1}}{pq-1}-\frac{\mu_2}{2}\right\}>\frac{n-1}{2}. \qquad (8)$$

Assume that $u_0,v_0\in H^1(\mathbb{R}^n)$ and $u_1,v_1\in L^2(\mathbb{R}^n)$ are compactly supported in $B_R\doteq\{x\in\mathbb{R}^n:|x|\leqslant R\}$ and satisfy

$$u_0(x)\geqslant 0 \quad \textit{and} \quad u_1(x)+\tfrac{\mu_1-1-\sqrt{\delta_1}}{2}u_0(x)\geqslant 0, \qquad (9)$$

$$v_0(x)\geqslant 0 \quad \textit{and} \quad v_1(x)+\tfrac{\mu_2-1-\sqrt{\delta_2}}{2}v_0(x)\geqslant 0. \qquad (10)$$

Let (u,v) be an energy solution of (1) with lifespan $T=T(\varepsilon)$ according to Definition 1.1. Then, there exists a positive constant $\varepsilon_0=\varepsilon_0(u_0,u_1,v_0,v_1,n,p,q,\mu_1,\mu_2,\nu_1^2,\nu_2^2,R)$ such that for any $\varepsilon\in(0,\varepsilon_0]$ the solution (u,v) blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon)\leqslant C\varepsilon^{-\max\{F(n+\mu_1,p,q),F(n+\mu_2,q,p)\}^{-1}} \qquad (11)$$

holds, where C is an independent of ε, positive constant and

$$F(n,p,q)\doteq\frac{p+2+q^{-1}}{pq-1}-\frac{n-1}{2}. \qquad (12)$$

Remark 1.3. *For $\mu_1=\mu_2=0$ the previous upper bound for the lifespan coincides with the sharp estimate for the lifespan of local solutions to the weakly coupled system of semilinear wave equations with power nonlinearities in the subcritical case. However, as we do not deal with global in time existence results for (1) in the present work, we do not derive a lower bound estimate for $T(\varepsilon)$. Let us underline that the shift in the first argument of $F=F(n,p,q)$ corresponds to the shift in the critical curve.*

Let us state the main result in the critical case.

Theorem 1.4. *Let $\mu_1,\mu_2,\nu_1^2,\nu_2^2$ be nonnegative constants such that $\delta_1,\delta_2\geqslant 0$. Let us consider $p,q>1$ satisfying*

$$\max\left\{\frac{p+2+q^{-1}}{pq-1}-\frac{\mu_1}{2},\frac{q+2+p^{-1}}{pq-1}-\frac{\mu_2}{2}\right\}=\frac{n-1}{2} \qquad (13)$$

and

$$\tfrac{1}{p}<\tfrac{n-\sqrt{\delta_2}}{2},\quad \tfrac{1}{q}<\tfrac{n-\sqrt{\delta_1}}{2}. \qquad (14)$$

Assume that $u_0, v_0 \in H^1(\mathbb{R}^n)$ and $u_1, v_1 \in L^2(\mathbb{R}^n)$ are nonnegative, pairwise nontrivial and compactly supported in B_{r_0}, with $r_0 \in (0,1)$.

Let (u,v) be an energy solution of (1) with lifespan $T = T(\varepsilon)$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, \mu_1, \mu_2, \nu_1^2, \nu_2^2, r_0)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution (u,v) blows up in finite time. Moreover, the upper bound estimates for the lifespan

$$T(\varepsilon) \leqslant \begin{cases} \exp\left(C\varepsilon^{-q(pq-1)}\right) & \text{if } \ 0 = F(n+\mu_1, p, q) > F(n+\mu_2, q, p), \\ \exp\left(C\varepsilon^{-p(pq-1)}\right) & \text{if } \ 0 = F(n+\mu_2, q, p) > F(n+\mu_1, p, q), \\ \exp\left(C\varepsilon^{-(pq-1)}\right) & \text{if } \ 0 = F(n+\mu_1, p, q) = F(n+\mu_2, q, p), \\ \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{if } \ p = q = p_0(n+\mu_1) \ \text{ and } \ \mu_1 = \mu_2, \, \nu_1^2 = \nu_2^2 \end{cases} \tag{15}$$

hold, where C is an independent of ε, positive constant and $F = F(n,p,q)$ is defined by (12).

Remark 1.5. *In (15) the last case corresponds to the case in which $0 = F(n+\mu_1, p, q) = F(n+\mu_2, q, p)$ and the scale-invariant terms in (1) have the same coefficients (same partial differential operator on the left hand sides).*

Remark 1.6. *As we will see in the proof of Theorem 1.4, the conditions (14) are technical requirements, which guarantee the nonemptiness of the ranges for certain parameters. Nonetheless, in dimension $n \geqslant 3$ and for $0 \leqslant \delta_1, \delta_2 \leqslant (n-2)^2$ the assumption on the exponents p, q given by (14) is trivially satisfied for any $p, q > 1$.*

The remaining part of this paper is organized as follows: in Section 2 we present a solution to the corresponding adjoint linear homogeneous system, whose components have separated variables, and we derive some lower bounds for certain functionals related to a local solution; then, in Section 3 we prove Theorem 1.2 using the preliminary results proved in Section 2. In Section 4 we introduce the notion of super-solutions of the wave equation with scale-invariant damping and mass and we derive some estimates for them. A family of self-similar solutions of the adjoint equation of the linear wave equation with scale-invariant damping and mass and their properties are shown in Section 5. Finally, Theorem 1.4 is proved in Section 6. Let us underline explicitly that besides the notations that have been introduced in this introduction, the notations in Sections 2-3 (subcritical case) and the notations in Sections 4-5-6 (critical case) are mutually independent and they should be not compared or overlapped by the reader.

Notations

Throughout this paper we will use the following notations: B_R denotes the ball around the origin with radius R; $f \lesssim g$ means that there exists a positive constant C such that $f \leqslant Cg$ and, similarly, for $f \gtrsim g$; moreover, $f \approx g$ means $f \lesssim g$ and $f \gtrsim g$; finally, as in the introduction, $p_0(n)$ denotes the Strauss exponent.

2. Solution of the adjoint linear problem and preliminaries

The arguments used in this section are the generalization for a weakly coupled system of those used in [35, Section 2] for a single equation.

Before starting with the construction of a solution to the adjoint system to homogeneous system of scale-invariant wave equations, that is, a solution of the system

$$\begin{cases} \Phi_{tt} - \Delta\Phi - \partial_t\left(\frac{\mu_1}{1+t}\Phi\right) + \frac{\nu_1^2}{(1+t)^2}\Phi = 0, & x \in \mathbb{R}^n, \ t > 0, \\ \Psi_{tt} - \Delta\Psi - \partial_t\left(\frac{\mu_2}{1+t}\Psi\right) + \frac{\nu_2^2}{(1+t)^2}\Psi = 0, & x \in \mathbb{R}^n, \ t > 0, \end{cases} \tag{16}$$

we recall the definition of the modified Bessel function of the second kind of order ς

$$K_\varsigma(t) = \int_0^\infty \exp(-t\cosh z)\cosh(\varsigma z)dz, \quad \varsigma \in \mathbb{R}$$

which is a solution of the equation

$$\left(t^2\frac{d^2}{dt^2}+t\frac{d}{dt}-(t^2+\varsigma^2)\right)K_\varsigma(t)=0,\quad t>0.$$

We collect some important properties concerning $K_\varsigma(t)$ in the case in which ς is a real parameter. Interested reader may refer to [10]. On the one hand, the following asymptotic behavior of $K_\varsigma(t)$ holds:

$$K_\varsigma(t)=\sqrt{\frac{\pi}{2t}}\,e^{-t}[1+O(t^{-1})]\quad\text{as}\quad t\to\infty. \tag{17}$$

On the other hand, the following derivative identity holds:

$$\frac{d}{dt}K_\varsigma(t)=-K_{\varsigma+1}(t)+\frac{\varsigma}{t}K_\varsigma(t). \tag{18}$$

As we will construct a solution (Φ,Ψ) with separated variables, firstly, we set the auxiliary functions with respect to the time variable, namely,

$$\lambda_j(t)\doteq(1+t)^{\frac{\mu_j+1}{2}}K_{\varsigma_j}(1+t)\quad\text{for}\quad t\geqslant 0\quad\text{and}\quad j=1,2,$$

where $\varsigma_j=\frac{\sqrt{\delta_j}}{2}$ for $j=1,2$. It is clear by direct computations that λ_1,λ_2 satisfy

$$\left(\frac{d^2}{dt^2}-\frac{\mu_j}{1+t}\frac{d}{dt}+\frac{\mu_j+\nu_j^2}{(1+t)^2}-1\right)\lambda_j(t)=0\quad\text{for}\quad t>0\quad\text{and}\quad j=1,2. \tag{19}$$

Following [44], let us introduce the function

$$\varphi(x)\doteq\begin{cases}\int_{\mathbb{S}^{n-1}}e^{x\cdot\omega}d\omega & \text{if}\ \ n\geqslant 2,\\ e^x+e^{-x} & \text{if}\ \ n=1.\end{cases}$$

The function φ satisfies

$$\Delta\varphi(x)=\varphi(x)\quad\text{for}\quad x\in\mathbb{R}^n$$

and the asymptotic estimate

$$\varphi(x)\sim C_n|x|^{-\frac{n-1}{2}}e^{|x|}\quad\text{as}\quad |x|\to\infty. \tag{20}$$

We may introduce now the functions Φ,Ψ

$$\Phi(t,x)\doteq\lambda_1(t)\,\varphi(x),$$
$$\Psi(t,x)\doteq\lambda_2(t)\,\varphi(x),$$

which constitute a solution to the adjoint system (16).

The remaining part of the section is devoted to determine lower bounds for $\int_{\mathbb{R}^n}|v(x,t)|^p dx$ and $\int_{\mathbb{R}^n}|u(x,t)|^q dx$.

Lemma 2.1. *Let us assume that u_0,u_1,v_0,v_1 are compactly supported in B_R for some $R>0$ and that* (9), (10) *are fulfilled. Then, a local energy solution (u,v) satisfies*

$$\operatorname{supp}u,\ \operatorname{supp}v\subset\{(t,x)\in[0,T)\times\mathbb{R}^n:|x|\leqslant t+R\}$$

and there exists a large T_0, which is independent of u_0,u_1,v_0,v_1 and ε, such that for any $t>T_0$ and $p,q>1$, the following estimates hold:

$$\int_{\mathbb{R}^n}|u(t,x)|^q dx\geqslant C_1\varepsilon^q(1+t)^{n-1-\frac{n+\mu_1-1}{2}q}, \tag{21}$$

$$\int_{\mathbb{R}^n}|v(t,x)|^p dx\geqslant K_1\varepsilon^p(1+t)^{n-1-\frac{n+\mu_2-1}{2}p}, \tag{22}$$

where $C_1=C_1(u_0,u_1,\varphi,q,R)>0$ and $K_1=K_1(v_0,v_1,\varphi,p,R)>0$ are independent of ε and t.

Proof. We begin with (21). Let us define the functional

$$F(t) \doteq \int_{\mathbb{R}^n} u(t,x)\Phi(t,x)dx$$

with Φ defined as above. Then, by Hölder inequality, we have

$$\int_{\mathbb{R}^n} |u(t,x)|^q dx \geqslant |F(t)|^q \bigg(\int_{|x|\leqslant t+R} \Phi^{q'}(t,x)dx \bigg)^{-(q-1)}, \tag{23}$$

where q' denotes the conjugate exponent of q.

The next step is to determine a lower bound for $F(t)$ and an upper bound for $\int_{|x|\leqslant t+R} \Phi^{q'}(t,x)dx$, respectively. Due to the support property for u, we can apply the definition of weak solution with test function Φ. So, for any $t \in (0,T)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(u_t(s,x)\Phi(s,x) - u(s,x)\Phi_t(s,x) + \frac{\mu_1}{1+s} u(s,x)\Phi(s,x) \Big) dx \, \bigg|_{s=0}^{s=t} \\
&\quad + \int_0^t \int_{\mathbb{R}^n} u(s,x) \Big(\Phi_{ss}(s,x) - \Delta\Phi(s,x) - \partial_s\Big(\frac{\mu_1}{1+s}\Phi(s,x)\Big) + \frac{\nu_1^2}{(1+s)^2}\Phi(s,x) \Big) dxds \\
&= \int_0^t \int_{\mathbb{R}^n} |v(s,x)|^p \Phi(s,x)\, dxds.
\end{aligned}$$

As the product $|v|^p\,\Phi$ is nonnegative and Φ solves the first equation in (16), from the previous equality we obtain

$$F'(t) + \bigg(\frac{\mu_1}{1+t} - 2\frac{\lambda_1'(t)}{\lambda_1(t)} \bigg) F(t) \geqslant \varepsilon \int_{\mathbb{R}^n} \Big(\lambda_1(0)\, u_1(x) + \big(\mu_1\lambda_1(0) - \lambda_1'(0) \big)\, u_0(x) \Big) \varphi(x)\, dx.$$

Using (18), we have

$$\begin{aligned}
\lambda_1'(t) &= \tfrac{\mu_1+1}{2}(1+t)^{\frac{\mu_1-1}{2}} K_{\varsigma_1}(1+t) + (1+t)^{\frac{\mu_1+1}{2}} K_{\varsigma_1}'(1+t) \\
&= \tfrac{\mu_1+1}{2}(1+t)^{\frac{\mu_1-1}{2}} K_{\varsigma_1}(1+t) + (1+t)^{\frac{\mu_1+1}{2}} \Big(-K_{\varsigma_1+1}(1+t) + \tfrac{\varsigma_1}{(1+t)} K_{\varsigma_1}(1+t) \Big) \\
&= \tfrac{\mu_1+1+\sqrt{\delta_1}}{2}(1+t)^{\frac{\mu_1-1}{2}} K_{\varsigma_1}(1+t) - (1+t)^{\frac{\mu_1+1}{2}} K_{\varsigma_1+1}(1+t).
\end{aligned}$$

Also,

$$\begin{aligned}
\lambda_1'(0) &= \tfrac{\mu_1+1+\sqrt{\delta_1}}{2} K_{\varsigma_1}(1) - K_{\varsigma_1+1}(1), \\
\mu_1\lambda_1(0) - \lambda_1'(0) &= \tfrac{\mu_1-1-\sqrt{\delta_1}}{2} K_{\varsigma_1}(1) + K_{\varsigma_1+1}(1).
\end{aligned}$$

Consequently,

$$\lambda_1(0)\, u_1(x) + \big(\mu_1\lambda_1(0) - \lambda_1'(0) \big)\, u_0(x) = K_{\varsigma_1}(1) \Big(u_1(x) + \tfrac{\mu_1-1-\sqrt{\delta_1}}{2} u_0(x) \Big) + K_{\varsigma_1+1}(1)\, u_0(x).$$

If we denote

$$C(u_0,u_1) \doteq \int_{\mathbb{R}^n} \Big(\lambda(0)\, u_1(x) + \big(\mu_1\lambda(0) - \lambda'(0) \big)\, u_0(x) \Big) \varphi(x)\, dx,$$

then, since we assume that u_0 and u_1 are compactly supported and satisfy (9), $C(u_0,u_1)$ is finite and positive. Therefore, we conclude that F satisfies the differential inequality

$$F'(t) + \bigg(\frac{\mu_1}{1+t} - 2\frac{\lambda_1'(t)}{\lambda_1(t)} \bigg) F(t) \geqslant \varepsilon\, C(u_0,u_1).$$

Multiplying by $\frac{(1+t)^{\mu_1}}{\lambda_1^2(t)}$ both sides of the previous inequality and then integrating over $[0,t]$, we derive

$$F(t) \geqslant \varepsilon\, C(u_0,u_1)\,\frac{\lambda_1^2(t)}{(1+t)^{\mu_1}}\int_0^t \frac{(1+s)^{\mu_1}}{\lambda_1^2(s)}\,ds.$$

Inserting $\lambda_1(t)=(1+t)^{\frac{\mu_1+1}{2}}K_{\varsigma_1}(1+t)$, we obtain as lower bound for F

$$F(t) \geqslant \varepsilon\, C(u_0,u_1)\int_0^t \frac{(1+t)K^2_{\varsigma_1}(1+t)}{(1+s)K^2_{\varsigma_1}(1+s)}\,ds \geqslant 0. \tag{24}$$

The integral involving $\Phi^{q'}$ in the right-hand side of (23) can be estimated as in [44, estimate (2.5)], namely,

$$\begin{aligned}\int_{|x|\leqslant t+R}\Phi^{q'}(t,x)\,dx &\leqslant \lambda_1^{\frac{q}{q-1}}(t)\int_{|x|\leqslant t+R}\varphi^{q'}(x)\,dx\\ &\leqslant C_{\varphi,R}(1+t)^{n-1+\left(\frac{\mu_1+1}{2}-\frac{n-1}{2}\right)\frac{q}{q-1}}\,e^{\frac{q}{q-1}(t+R)}K_{\varsigma_1}^{\frac{q}{q-1}}(1+t),\end{aligned} \tag{25}$$

where $C_{\varphi,R}$ is a suitable positive constant.

Combing the estimate (24), (25) and (23), we find

$$\begin{aligned}&\int_{\mathbb{R}^n}|u(x,t)|^q dx\\ &\quad\geqslant C(u_0,u_1)^q C_{\varphi,R}^{1-q}\varepsilon^q(1+t)^{q-(n-1)(q-1)-\left(\frac{\mu_1+1}{2}-\frac{n-1}{2}\right)q}e^{-q(t+R)}K^q_{\varsigma_1}(1+t)\left(\int_0^t\frac{ds}{(1+s)K^2_{\varsigma_1}(1+s)}\right)^q\\ &\quad\geqslant C(u_0,u_1)^q C_{\varphi,R}^{1-q}e^{q(1-R)}\varepsilon^q(1+t)^{(2-n-\mu_1)\frac{q}{2}+(n-1)}e^{-q(1+t)}K^q_{\varsigma_1}(1+t)\left(\int_0^t\frac{ds}{(1+s)K^2_{\varsigma_1}(1+s)}\right)^q.\end{aligned}$$

Due to (17), for a sufficiently large T_0 (which is independent of u_0,u_1,ε) and $t>T_0$, we have

$$K^q_{\varsigma_1}(1+t)\sim\left(\frac{\pi}{2(1+t)}\right)^{\frac{q}{2}}e^{-q(t+1)}$$

and

$$\int_0^t\frac{1}{(1+s)K^2_{\varsigma_1}(1+s)}ds \geqslant \frac{2}{\pi}\int_{t/2}^t e^{2(1+s)}ds=\frac{1}{\pi}\left(e^{2(1+t)}-e^{2+t}\right)\geqslant\frac{1}{2\pi}e^{2(1+t)}.$$

Consequently,

$$\int_{\mathbb{R}^n}|u(t,x)|^q dx \geqslant C_1\varepsilon^q(1+t)^{\frac{q}{2}(1-n-\mu_1)+(n-1)} \quad \text{for} \quad t>T_0,$$

where $C_1 \doteq 2^{-\frac{3q}{2}}C(u_0,u_1)^qC_{\varphi,R}^{1-q}e^{q(1-R)}\pi^{-\frac{q}{2}}$. The proof of (22) is analogous, as one has to consider the functional

$$G(t)\doteq\int_{\mathbb{R}^n}v(t,x)\Psi(t,x)dx$$

instead of F and to use the assumption (10) in place of (9). This concludes the proof. □

3. Subcritical case: Proof of Theorem 1.2

Let us consider a local solution (u,v) of (1) on $[0,T)$ and define the couple of time-dependent functionals

$$U(t)\doteq\int_{\mathbb{R}^n}u(t,x)\,dx,\quad V(t)\doteq\int_{\mathbb{R}^n}v(t,x)\,dx.$$

The proof of Theorem 1.2 is dived in two step. The first step consists in the determination a coupled system of nonlinear ordinary integral inequalities for U and V (iteration frame), while in the second one an iteration argument is used to show the blow-up of (U,V) in finite time.

Determination of the iteration frame

Let us begin with the first step.

Choosing $\phi = \phi(s,x)$ and $\psi = \psi(s,x)$ in (6) and in (7), respectively, that satisfy $\phi \equiv 1 \equiv \psi$ on $\{(x,s) \in [0,t]\times\mathbb{R}^n : |x| \leqslant s+R\}$, we obtain

$$\int_{\mathbb{R}^n} u_t(t,x)\,dx - \int_{\mathbb{R}^n} u_t(0,x)\,dx + \int_0^t\int_{\mathbb{R}^n}\left(\frac{\mu_1 u_t(s,x)}{1+s} + \frac{\nu_1^2 u(s,x)}{(1+s)^2}\right)dx\,ds = \int_0^t\int_{\mathbb{R}^n} |v(s,x)|^p dx\,ds,$$
$$\int_{\mathbb{R}^n} v_t(t,x)\,dx - \int_{\mathbb{R}^n} v_t(0,x)\,dx + \int_0^t\int_{\mathbb{R}^n}\left(\frac{\mu_2 v_t(s,x)}{1+s} + \frac{\nu_2^2 v(s,x)}{(1+s)^2}\right)dx\,ds = \int_0^t\int_{\mathbb{R}^n} |u(s,x)|^q dx\,ds$$

which means that

$$U'(t) - U'(0) + \int_0^t \frac{\mu_1}{1+s}U'(s)\,ds + \int_0^t \frac{\nu_1^2}{(1+s)^2}U(s)\,ds = \int_0^t\int_{\mathbb{R}^n} |v(s,x)|^p dx\,ds,$$
$$V'(t) - V'(0) + \int_0^t \frac{\mu_2}{1+s}V'(s)\,ds + \int_0^t \frac{\nu_2^2}{(1+s)^2}V(s)\,ds = \int_0^t\int_{\mathbb{R}^n} |u(s,x)|^q dx\,ds.$$

Differentiating with repect to t the previous equalities, we get

$$U''(t) + \frac{\mu_1}{1+t}U'(t) + \frac{\nu_1^2}{(1+t)^2}U(t) = \int_{\mathbb{R}^n} |v(t,x)|^p dx, \tag{26}$$
$$V''(t) + \frac{\mu_2}{1+t}V'(t) + \frac{\nu_2^2}{(1+t)^2}V(t) = \int_{\mathbb{R}^n} |u(t,x)|^q dx. \tag{27}$$

Let us consider the quadratic equations

$$r^2 - (\mu_1 - 1)r + \nu_1^2 = 0, \quad \rho^2 - (\mu_2 - 1)\rho + \nu_2^2 = 0.$$

Since $\delta_1, \delta_2 \geqslant 0$ there exit two pair of real roots,

$$r_{1,2} \doteq \frac{\mu_1 - 1 \mp \sqrt{\delta_1}}{2}, \quad \rho_{1,2} \doteq \frac{\mu_2 - 1 \mp \sqrt{\delta_2}}{2}.$$

Clearly, if $\mu_1 > 1$ and $\mu_2 > 1$, then, $r_{1,2}$ and $\rho_{1,2}$ are positive. Else, if $0 \leqslant \mu_1 < 1$ or $0 \leqslant \mu_2 < 1$, then, $r_{1,2}$ or $\rho_{1,2}$ are negative. When $\mu_1 = 1$, then, $\nu_1 = 0$ as $\delta_1 \geqslant 0$ and, hence, $r_1 = r_2 = 0$. Similarly, if $\mu_2 = 1$. Moreover, in all cases

$$r_{1,2} + 1 > 0 \ \text{ and } \ \rho_{1,2} + 1 > 0.$$

We may rewrite (26) as

$$\left(U'(t) + \frac{r_1}{1+t}U(t)\right)' + \frac{r_2+1}{1+t}\left(U'(t) + \frac{r_1}{1+t}U(t)\right) = \int_{\mathbb{R}^n} |v(t,x)|^p dx.$$

Multiplying by $(1+t)^{r_2+1}$ and integrating over $[0,t]$, we obtain

$$(1+t)^{r_2+1}\left(U'(t) + \frac{r_1}{1+t}U(t)\right) - \Big(U'(0) + r_1 U(0)\Big) = \int_0^t (1+s)^{r_2+1}\int_{\mathbb{R}^n} |v(s,x)|^p dx\,ds.$$

Using (9), we have

$$U'(t) + \frac{r_1}{1+t}U(t) > (1+t)^{-r_2-1}\int_0^t (1+s)^{r_2+1}\int_{\mathbb{R}^n} |v(s,x)|^p dx\,ds. \tag{28}$$

Multiplying the above inequality by $(1+t)^{r_1}$ and integrating over $[0,t]$, we arrive at

$$(1+t)^{r_1}U(t) - U(0) > \int_0^t (1+\tau)^{r_1-r_2-1}\int_0^\tau (1+s)^{r_2+1}\int_{\mathbb{R}^n} |v(s,x)|^p dx\,ds\,d\tau.$$

Since u_0 is nonnegative, we have

$$U(t) \geqslant \int_0^t \left(\frac{1+\tau}{1+t}\right)^{r_1} \int_0^\tau \left(\frac{1+s}{1+\tau}\right)^{r_2+1} \int_{\mathbb{R}^n} |v(s,x)|^p dx\, ds\, d\tau. \tag{29}$$

Furthermore, using Hölder inequality and the compactness of the support of solution with respect to x, we get from (29)

$$U(t) \geqslant C_0 \int_0^t \left(\frac{1+\tau}{1+t}\right)^{r_1} \int_0^\tau \left(\frac{1+s}{1+\tau}\right)^{r_2+1} (1+s)^{-n(p-1)} |V(s)|^p ds\, d\tau, \tag{30}$$

where

$$C_0 \doteq (\text{meas}(B_1))^{1-p} R^{-n(p-1)} > 0.$$

In a similar way, from (27) we may derive

$$V(t) \geqslant \int_0^t \left(\frac{1+\tau}{1+t}\right)^{\rho_1} \int_0^\tau \left(\frac{1+s}{1+\tau}\right)^{\rho_2+1} \int_{\mathbb{R}^n} |u(s,x)|^q dx\, ds\, d\tau \tag{31}$$

$$\geqslant K_0 \int_0^t \left(\frac{1+\tau}{1+t}\right)^{\rho_1} \int_0^\tau \left(\frac{1+s}{1+\tau}\right)^{\rho_2+1} (1+s)^{-n(q-1)} |U(s)|^q ds\, d\tau, \tag{32}$$

where

$$K_0 \doteq (\text{meas}(B_1))^{1-q} R^{-n(q-1)} > 0.$$

Iteration argument

Now we can proceed with the second step. We shall apply an iteration method based on lower bound estimates (21), (22) and on the iteration frame (29)-(32). In comparison to the iteration method for a single semilinear wave equation with scale-invariant damping and mass (cf. [35, Section 3]), as the system is weakly coupled, we will combine the lower bounds for U and V.

By using an induction argument, we will prove that

$$U(t) \geqslant D_j (1+t)^{-a_j} (t-T_0)^{b_j} \quad \text{for} \quad t \geqslant T_0, \tag{33}$$

$$V(t) \geqslant \Delta_j (1+t)^{-\alpha_j} (t-T_0)^{\beta_j} \quad \text{for} \quad t \geqslant T_0, \tag{34}$$

where $\{a_j\}_{j\geqslant 1}$, $\{b_j\}_{j\geqslant 1}$, $\{D_j\}_{j\geqslant 1}$, $\{\alpha_j\}_{j\geqslant 1}$, $\{\beta_j\}_{j\geqslant 1}$ and $\{\Delta_j\}_{j\geqslant 1}$ are suitable sequences of positive real numbers that we shall determine throughout the iteration procedure.

Let us begin with the base case $j=1$ in (33) and (34). Plugging (22) in (29) and shrinking the domain of integration, we find for $t \geqslant T_0$

$$\begin{aligned} U(t) &\geqslant K_1 \varepsilon^p (1+t)^{-r_1} \int_{T_0}^t (1+\tau)^{r_1-r_2-1} \int_{T_0}^\tau (1+s)^{r_2+n-(n+\mu_2-1)\frac{p}{2}} ds\, d\tau \\ &\geqslant K_1 \varepsilon^p (1+t)^{-r_1} \int_{T_0}^t (1+\tau)^{r_1-r_2-1-(n+\mu_2-1)\frac{p}{2}} \int_{T_0}^\tau (1+s)^{r_2+n} ds\, d\tau \\ &\geqslant K_1 \varepsilon^p (1+t)^{-r_2-1-(n+\mu_2-1)\frac{p}{2}} \int_{T_0}^t \int_{T_0}^\tau (s-T_0)^{r_2+n} ds\, d\tau \\ &= K_1 \varepsilon^p (1+t)^{-r_2-1-(n+\mu_2-1)\frac{p}{2}} \frac{(t-T_0)^{r_2+n+2}}{(r_2+n+1)(r_2+n+2)}, \end{aligned}$$

which is the desired estimate, if we put

$$D_1 \doteq \frac{K_1 \varepsilon^p}{(r_2+n+1)(r_2+n+2)}, \qquad a_1 \doteq r_2 + 1 + (n+\mu_2-1)\frac{p}{2}, \quad b_1 \doteq r_2+n+2.$$

Analogously, we can prove (34) for $j = 1$ combining (31) and (21), provided that

$$\Delta_1 \doteq \frac{C_1 \varepsilon^q}{(\rho_2 + n + 1)(\rho_2 + n + 2)}, \qquad \alpha_1 \doteq \rho_2 + 1 + (n + \mu_1 - 1)\frac{q}{2}, \quad \beta_1 \doteq \rho_2 + n + 2.$$

Let us proceed with the inductive step: (33) and (34) are assumed to be true for $j \geqslant 1$, we prove them for $j + 1$. Let us plug (34) in (30). Then, shrinking the domain of integration and using the positiveness of α_j and β_j and the condition $r_2 + 1 > 0$, for $t \geqslant T_0$ we get

$$\begin{aligned} U(t) &\geqslant C_0\, \Delta_j^p\, (1+t)^{-r_1} \int_{T_0}^t (1+\tau)^{r_1 - r_2 - 1} \int_{T_0}^\tau (1+s)^{r_2 + 1 + n(1-p) - \alpha_j p} (s - T_0)^{\beta_j p}\, ds\, d\tau \\ &\geqslant C_0\, \Delta_j^p\, (1+t)^{-r_1} \int_{T_0}^t (1+\tau)^{r_1 - r_2 - 1 - n(p-1) - \alpha_j p} \int_{T_0}^\tau (s - T_0)^{r_2 + 1 + \beta_j p}\, ds\, d\tau \\ &\geqslant C_0\, \Delta_j^p\, (1+t)^{-r_2 - 1 - n(p-1) - \alpha_j p} \int_{T_0}^t \int_{T_0}^\tau (s - T_0)^{r_2 + 1 + \beta_j p}\, ds\, d\tau \\ &= \frac{C_0\, \Delta_j^p}{(r_2 + 2 + \beta_j p)(r_2 + 3 + \beta_j p)} (1+t)^{-r_2 - 1 - n(p-1) - \alpha_j p} (t - T_0)^{r_2 + 3 + \beta_j p}, \end{aligned}$$

that is, (33) for $j + 1$ provided that

$$D_{j+1} \doteq \frac{C_0\, \Delta_j^p}{(r_2 + 2 + \beta_j p)(r_2 + 3 + \beta_j p)}, \qquad a_{j+1} \doteq r_2 + 1 + n(p-1) + \alpha_j p, \quad b_{j+1} \doteq r_2 + 3 + \beta_j p.$$

Similarly, we can prove (34) for $j + 1$ combining (32) and (33), in the case in which

$$\Delta_{j+1} \doteq \frac{K_0\, D_j^q}{(\rho_2 + 2 + b_j q)(\rho_2 + 3 + b_j q)} \qquad \alpha_{j+1} \doteq \rho_2 + 1 + n(q-1) + a_j q, \quad \beta_{j+1} \doteq \rho_2 + 3 + b_j q.$$

Let us determine explicitly the expression for $a_j, b_j, \alpha_j, \beta_j$ at least for odd j. Let us start with a_j. Using the previous relations, we have

$$\begin{aligned} a_j &= r_2 + 1 + n(p-1) + \alpha_{j-1}\, p = r_2 + 1 + n(p-1) + \big(\rho_2 + 1 + n(q-1) + a_{j-2}\, q\big)p \\ &= \underbrace{r_2 + 1 - n + (\rho_2 + 1)p + npq}_{\doteq A} + pq\, a_{j-2}. \end{aligned}$$

Applying iteratively the previous relation, for odd j we get

$$\begin{aligned} a_j &= A + pq\, a_{j-2} = A + A\, pq\, a_{j-2} + (pq)^2 a_{j-4} = \quad \cdots \\ &= A \sum_{k=0}^{(j-3)/2} (pq)^k + (pq)^{\frac{j-1}{2}} a_1 = A \frac{(pq)^{\frac{j-1}{2}} - 1}{pq - 1} + (pq)^{\frac{j-1}{2}} a_1 \\ &= \left(\frac{A}{pq - 1} + a_1 \right) (pq)^{\frac{j-1}{2}} - \frac{A}{pq - 1}. \end{aligned} \tag{35}$$

In a similar way, for odd j we get

$$\alpha_j = \left(\frac{\widetilde{A}}{pq - 1} + \alpha_1 \right) (pq)^{\frac{j-1}{2}} - \frac{\widetilde{A}}{pq - 1}, \tag{36}$$

where $\widetilde{A} \doteq \rho_2 + 1 - n + (r_2 + 1)q + npq$. For the sake of simplicity we do not derive the representations of a_j and α_j for even j, as it is unnecessary to prove the theorem.

Analogously, for odd j we have, combining the definitions of b_j and β_j,

$$b_j = r_2 + 3 + \beta_{j-1}\, p = r_2 + 3 + \big(\rho_2 + 3 + b_{j-2}\, q\big)p = \underbrace{r_2 + 3 + (\rho_2 + 3)p}_{\doteq B} + pq\, b_{j-2}, \tag{37}$$

$$\beta_j = \rho_2 + 3 + b_{j-1}\, q = \rho_2 + 3 + \big(r_2 + 3 + \beta_{j-2}\, p\big)q = \underbrace{\rho_2 + 3 + (r_2 + 3)q}_{\doteq \widetilde{B}} + pq\, \beta_{j-2}. \tag{38}$$

Also,

$$b_j = \left(\frac{B}{pq-1} + b_1\right)(pq)^{\frac{j-1}{2}} - \frac{B}{pq-1} \quad \text{and} \quad \beta_j = \left(\frac{\widetilde{B}}{pq-1} + \beta_1\right)(pq)^{\frac{j-1}{2}} - \frac{\widetilde{B}}{pq-1}. \tag{39}$$

The next step is to derive lower bounds for D_j and Δ_j. From the definition of D_j and Δ_j it follows immediately

$$D_j \geqslant \frac{C_0}{b_j^2}\Delta_{j-1}^p \quad \text{and} \quad \Delta_j \geqslant \frac{K_0}{\beta_j^2} D_{j-1}^q. \tag{40}$$

Therefore, the next step is to determine upper bounds for b_j and for β_j, respectively. If j is odd, plugging the first equation from (39) for $j-2$ in (37) and using the definition of B, it follows

$$\begin{aligned} b_j &= r_2 + 3 + (\rho_2 + 3)p + pq\left[\left(\frac{B}{pq-1} + b_1\right)(pq)^{\frac{j-3}{2}} - \frac{B}{pq-1}\right] \\ &= r_2 + 3 + \left(\rho_2 + 3 - \frac{Bq}{pq-1}\right)p + \left(\frac{B}{pq-1} + b_1\right)(pq)^{\frac{j-1}{2}} \\ &= -\frac{(r_2+3) + (\rho_2+3)p}{pq-1} + \left(\frac{B}{pq-1} + b_1\right)(pq)^{\frac{j-1}{2}} < B_0\,(pq)^{\frac{j-1}{2}}, \end{aligned}$$

where

$$B_0 \doteq \frac{B}{pq-1} + b_1 = n - 1 + \frac{(r_2+3)pq + (\rho_2+3)p}{pq-1} > 0.$$

Similarly, for odd j, employing (38) and the second equation in (39), one finds

$$\beta_j < \widetilde{B}_0\,(pq)^{\frac{j-1}{2}},$$

where

$$\widetilde{B}_0 \doteq \frac{\widetilde{B}}{pq-1} + \beta_1 = n - 1 + \frac{(\rho_2+3)pq + (r_2+3)q}{pq-1} > 0.$$

It is possible to derive similar estimates also for b_{j-1} and β_{j-1}. Indeed, from (37) and (39) we get

$$\begin{aligned} b_{j-1} &= r_2 + 3 + \beta_{j-2}\, p = r_2 + 3 - \frac{\widetilde{B}p}{pq-1} + \frac{1}{q}\left(\frac{\widetilde{B}}{pq-1} + \beta_1\right)(pq)^{\frac{j-1}{2}} < \widetilde{B}_0\,(pq)^{\frac{j-1}{2}}, \\ \beta_{j-1} &= \rho_2 + 3 + b_{j-2}\, q = \rho_2 + 3 - \frac{Bq}{pq-1} + \frac{1}{p}\left(\frac{B}{pq-1} + b_1\right)(pq)^{\frac{j-1}{2}} < B_0\,(pq)^{\frac{j-1}{2}}. \end{aligned}$$

Hence, due to the above derived upper bounds for $b_j, b_{j-1}, \beta_j, \beta_{j-1}$, from (40) it follows

$$D_j \geqslant \frac{C_0}{B_0^2}\frac{\Delta_{j-1}^p}{(pq)^{j-1}} \geqslant \frac{C_0 K_0^p}{B_0^2}\frac{D_{j-2}^{pq}}{(pq)^{j-1}\beta_{j-1}^{2p}} \geqslant \frac{\widetilde{C} D_{j-2}^{pq}}{\big((pq)^{p+1}\big)^{j-1}}, \tag{41}$$

$$\Delta_j \geqslant \frac{K_0}{\widetilde{B}_0^2}\frac{D_{j-1}^q}{(pq)^{j-1}} \geqslant \frac{K_0 C_0^q}{\widetilde{B}_0^2}\frac{\Delta_{j-2}^{pq}}{(pq)^{j-1} b_{j-1}^{2q}} \geqslant \frac{\widetilde{K}\Delta_{j-2}^{pq}}{\big((pq)^{q+1}\big)^{j-1}}, \tag{42}$$

where $\widetilde{C} \doteq C_0 K_0^p / B_0^{2(p+1)}$ and $\widetilde{K} \doteq K_0 C_0^q / \widetilde{B}_0^{2(q+1)}$.

From (41), if j is odd, then, it follows

$$\begin{aligned}
\log D_j &\geqslant pq \log D_{j-2} - (j-1)(p+1)\log(pq) + \log \widetilde{C} \\
&\geqslant (pq)^2 \log D_{j-4} - \big((j-1) + (j-3)pq\big)(p+1)\log(pq) + \big(1+pq\big)\log\widetilde{C} \\
&\geqslant \cdots \\
&\geqslant (pq)^{\frac{j-1}{2}} \log D_1 - \left(\sum_{k=1}^{(j-1)/2} (j+1-2k)\,(pq)^{k-1}\right)(p+1)\log(pq) + \left(\sum_{k=0}^{(j-3)/2} (pq)^k\right)\log\widetilde{C}.
\end{aligned}$$

Using an inductive argument, the following formulas can be shown:

$$\sum_{k=0}^{(j-3)/2} (pq)^k = \frac{(pq)^{\frac{j-1}{2}} - 1}{pq-1}$$

and

$$\sum_{k=1}^{(j-1)/2} (j+1-2k)\,(pq)^{k-1} = \frac{1}{pq-1}\left(2(pq)\,\frac{(pq)^{\frac{j-1}{2}}-1}{pq-1} - j + 1\right).$$

Consequently,

$$\begin{aligned}
\log D_j \geqslant (pq)^{\frac{j-1}{2}} &\left[\log D_1 - \frac{2(pq)(p+1)}{(pq-1)^2}\log(pq) + \frac{\log\widetilde{C}}{pq-1}\right] + \frac{2(pq)(p+1)}{(pq-1)^2}\log(pq) \\
&+ (j-1)\frac{(p+1)}{pq-1}\log(pq) - \frac{\log\widetilde{C}}{pq-1}.
\end{aligned}$$

Thus, for an odd j such that $j > \frac{\log\widetilde{C}}{(p+1)\log(pq)} - \frac{2(pq)}{pq-1} + 1$, it holds

$$\log D_j \geqslant (pq)^{\frac{j-1}{2}}\big(\log D_1 - S_{p,q}(\infty)\big), \tag{43}$$

where $S_{p,q}(\infty) \doteq \frac{2(pq)(p+1)}{(pq-1)^2}\log(pq) - \frac{\log\widetilde{C}}{pq-1}$.

In a similar way, one can show for an odd j the validity of

$$\begin{aligned}
\log \Delta_j \geqslant (pq)^{\frac{j-1}{2}} &\left[\log \Delta_1 - \frac{2(pq)(q+1)}{(pq-1)^2}\log(pq) + \frac{\log\widetilde{K}}{pq-1}\right] + \frac{2(pq)(q+1)}{(pq-1)^2}\log(pq) \\
&+ (j-1)\frac{(q+1)}{pq-1}\log(pq) - \frac{\log\widetilde{K}}{pq-1},
\end{aligned}$$

and, then, for $j > \frac{\log\widetilde{K}}{(q+1)\log(pq)} - \frac{2(pq)}{pq-1} + 1$ this yields

$$\log \Delta_j \geqslant (pq)^{\frac{j-1}{2}}\big(\log \Delta_1 - \widetilde{S}_{p,q}(\infty)\big), \tag{44}$$

where $\widetilde{S}_{p,q}(\infty) \doteq \frac{2(pq)(q+1)}{(pq-1)^2}\log(pq) - \frac{\log\widetilde{K}}{pq-1}$.

For the sake of brevity, we denote $j_0 \doteq \big\lceil \frac{1}{\log(pq)} \max\{\frac{\widetilde{C}}{p+1}, \frac{\widetilde{K}}{q+1}\} - \frac{2pq}{pq-1} + 1\big\rceil$.

Let us combine now (33) and (43). For an odd $j > j_0$ and $t \geqslant T_0$, using (35) and (39), we get

$$\begin{aligned}
U(t) &\geqslant \exp\Big((pq)^{\frac{j-1}{2}}\big(\log D_1 - S_{p,q}(\infty)\big)\Big)(1+t)^{-a_j}(t-T_0)^{b_j} \\
&= \exp\Big((pq)^{\frac{j-1}{2}}\big(\log D_1 - S_{p,q}(\infty)\big)\Big)(1+t)^{-\left(\frac{A}{pq-1}+a_1\right)(pq)^{\frac{j-1}{2}} + \frac{A}{pq-1}}(t-T_0)^{\left(\frac{B}{pq-1}+b_1\right)(pq)^{\frac{j-1}{2}} - \frac{B}{pq-1}} \\
&= \exp\Big((pq)^{\frac{j-1}{2}}\big(\log D_1 - \big(\tfrac{A}{pq-1}+a_1\big)\log(1+t) + \big(\tfrac{B}{pq-1}+b_1\big)\log(t-T_0) - S_{p,q}(\infty)\big)\Big) \\
&\quad \times (1+t)^{\frac{A}{pq-1}}(t-T_0)^{-\frac{B}{pq-1}}.
\end{aligned}$$

Also, for $t \geqslant 2T_0 + 1$ from the previous estimate it follows

$$U(t) \geqslant \exp\left((pq)^{\frac{j-1}{2}} J(t)\right)(1+t)^{\frac{A}{pq-1}}(t-T_0)^{-\frac{B}{pq-1}}, \tag{45}$$

where

$$\begin{aligned} J(t) &\doteq \log D_1 + \left(\tfrac{B-A}{pq-1} + b_1 - a_1\right)\log(t-T_0) - \left(\tfrac{A}{pq-1} + a_1\right)\log 2 - S_{p,q}(\infty) \\ &= \log\left(D_1 (t-T_0)^{\frac{B-A}{pq-1}+b_1-a_1}\right) - \left(\tfrac{A}{pq-1} + a_1\right)\log 2 - S_{p,q}(\infty). \end{aligned}$$

Let us calculate more precisely the power of $(t-T_0)$ in the last line:

$$\begin{aligned} \frac{B-A}{pq-1} + b_1 - a_1 &= \frac{2p+2+n-npq}{pq-1} + n + 1 - (n+\mu_2-1)\frac{p}{2} \\ &= \frac{pq+1+2p}{pq-1} - (n+\mu_2-1)\frac{p}{2} = p\left(\frac{q+p^{-1}+2}{pq-1} - \frac{n+\mu_2-1}{2}\right). \end{aligned}$$

So, $\frac{B-A}{pq-1} + b_1 - a_1 > 0$ if and only if $F(n+\mu_2, q, p) > 0$.

In an analogous way, from (34), (44), (36) and (39) we obtain for $t \geqslant 2T_0 + 1$ and for an odd $j > j_0$

$$V(t) \geqslant \exp\left((pq)^{\frac{j-1}{2}} \widetilde{J}(t)\right)(1+t)^{\frac{\widetilde{A}}{pq-1}}(t-T_0)^{-\frac{\widetilde{B}}{pq-1}}, \tag{46}$$

where

$$\begin{aligned} \widetilde{J}(t) &\doteq \log \Delta_1 + \left(\tfrac{\widetilde{B}-\widetilde{A}}{pq-1} + \beta_1 - \alpha_1\right)\log(t-T_0) - \left(\tfrac{\widetilde{A}}{pq-1} + \alpha_1\right)\log 2 - \widetilde{S}_{p,q}(\infty) \\ &= \log\left(\Delta_1 (t-T_0)^{\frac{\widetilde{B}-\widetilde{A}}{pq-1}+\beta_1-\alpha_1}\right) - \left(\tfrac{\widetilde{A}}{pq-1} + \alpha_1\right)\log 2 - \widetilde{S}_{p,q}(\infty). \end{aligned}$$

In this case, $\frac{\widetilde{B}-\widetilde{A}}{pq-1} + \beta_1 - \alpha_1 = q\left(\frac{p+2+q^{-1}}{pq-1} - \frac{n+\mu_1-1}{2}\right) > 0$ if and only if $F(n+\mu_1, p, q) > 0$.

If $F(n+\mu_2, q, p) > 0$, since $D_1 = K_2\varepsilon^p$, where $K_2 \doteq K_1(r_2+n+1)^{-1}(r_2+n+2)^{-1}$, then, $J(t) > 0$ is equivalent to require

$$t > T_0 + E\varepsilon^{-F(n+\mu_2,q,p)^{-1}}, \qquad \text{where } \; E \doteq \left(e^{\left(\frac{A}{pq-1}+a_1\right)\log 2 + S_{p,q}(\infty)} K_2^{-1}\right)^{\frac{1}{pF(n+\mu_2,q,p)}}.$$

If we choose $\varepsilon_0 > 0$ sufficiently small so that

$$2E\varepsilon_0^{-F(n+\mu_2,q,p)^{-1}} > 2T_0 + 1,$$

then, for any $\varepsilon \in (0, \varepsilon_0]$ and $t > 2E\varepsilon^{-F(n+\mu_2,q,p)^{-1}}$ we have $t > 2T_0 + 1$ and $J(t) > 0$. Thus, letting $j \to \infty$ in (45), the lower bound for U blows up and, hence, U can be finite only for $t \leqslant 2E\varepsilon^{-F(n+\mu_2,q,p)^{-1}}$.

Analogously, in the case $F(n+\mu_1, p, q) > 0$, as $\Delta_1 = C_2\varepsilon^q$, where $C_2 \doteq C_1(\rho_2+n+1)^{-1}(\rho_2+n+2)^{-1}$, we get that $\widetilde{J}(t) > 0$ is equivalent to

$$t > T_0 + \widetilde{E}\varepsilon^{-F(n+\mu_1,p,q)^{-1}}, \qquad \text{where } \; \widetilde{E} \doteq \left(e^{\left(\frac{\widetilde{A}}{pq-1}+\alpha_1\right)\log 2 + \widetilde{S}_{p,q}(\infty)} C_2^{-1}\right)^{\frac{1}{qF(n+\mu_1,p,q)}}.$$

Also, in this case we may choose $\varepsilon_0 > 0$ sufficiently small so that

$$2\widetilde{E}\varepsilon_0^{-F(n+\mu_1,p,q)^{-1}} > 2T_0 + 1.$$

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and $t > 2\widetilde{E}\varepsilon^{-F(n+\mu_1,p,q)^{-1}}$ we have $t > 2T_0 + 1$ and $\widetilde{J}(t) > 0$ and, then, taking the limit as $j \to \infty$ in (46) the lower bound for $V(t)$ diverges. Hence, V may be finite just for $t \leqslant 2\widetilde{E}\varepsilon^{-F(n+\mu_1,p,q)^{-1}}$. Summarizing, we proved that if (8) holds, then, (u, v) blows up in finite time and (11) is satisfied. This completes the proof.

4. Super-solutions of the scale-invariant wave equations and their properties

Henceforth we deal with the critical case and the proof of Theorem 1.4. In this section we introduce the notion of super-solutions of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = H, & x \in \mathbb{R}^n, \ t \in (0,T), \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{47}$$

and, then, we derive some estimates related to super-solutions.

Definition 4.1. *Let $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be compactly supported and $H \in L^1_{\text{loc}}([0,T) \times \mathbb{R}^n)$. We say that $u \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n))$ is a super-solution of (47) on $[0,T)$ if $u(0,x) = \varepsilon u_0(x)$ in $H^1(\mathbb{R}^n)$ and*

$$\begin{aligned} \varepsilon \int_{\mathbb{R}^n} u_1(x)\Psi(0,x)\,dx + \int_0^T \int_{\mathbb{R}^n} H(t,x)\Psi(t,x)\,dx\,dt &\leqslant \int_0^T \int_{\mathbb{R}^n} -u_t(t,x)\Psi_t(t,x)\,dx\,dt \\ + \int_0^T \int_{\mathbb{R}^n} \Big(\nabla u(t,x) \cdot \nabla \Psi(t,x) &+ \frac{\mu}{1+t} u_t(t,x)\Psi(t,x) + \frac{\nu^2}{(1+t)^2} u(t,x)\Psi(t,x) \Big)\,dx\,dt \end{aligned} \tag{48}$$

for any nonnegative test function $\Psi \in \mathscr{C}_0^\infty([0,T) \times \mathbb{R}^n)$.

Lemma 4.2. *Let u be a super-solution of (47) with (u_0, u_1) in the classical energy space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and compactly supported in B_{r_0}, H locally summable and* $\operatorname{supp} u \subset \{(t,x) \in [0,T) \times \mathbb{R}^n : |x| \leqslant r_0 + t\}$. *Then, it holds*

$$\begin{aligned} \varepsilon \int_{\mathbb{R}^n} \big(u_1(x)\Psi(0,x) + u_0(x)(\mu\Psi(0,x) - \Psi_t(0,x)\big)\,dx + \int_0^T \int_{\mathbb{R}^n} H(t,x)\Psi(t,x)\,dx\,dt \\ \leqslant \int_0^T \int_{\mathbb{R}^n} u(t,x) \Big(\Psi_{tt}(t,x) - \Delta\Psi(t,x) - \frac{\mu}{1+t}\Psi_t(t,x) + \frac{\mu+\nu^2}{(1+t)^2}\Psi(t,x) \Big)\,dx\,dt \end{aligned} \tag{49}$$

for any nonnegative test function $\Psi \in \mathscr{C}_0^\infty([0,T) \times \mathbb{R}^n)$.

Proof. Using the support condition for u, integration by parts provides

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \Big(- u_t(t,x)\Psi_t(t,x) + \nabla u(t,x) \cdot \nabla\Psi(t,x) + \frac{\mu}{1+t} u_t(t,x)\Psi(t,x) \Big)\,dx\,dt \\ &= \int_0^T \frac{d}{dt} \Big[\int_{\mathbb{R}^n} (-u(t,x)\Psi_t(t,x) + \frac{\mu}{1+t} u(t,x)\Psi(t,x))\,dx \Big]\,dt \\ &\quad + \int_0^T \int_{\mathbb{R}^n} u(t,x) \Big(\Psi_{tt}(t,x) - \Delta\Psi(t,x) - \partial_t \Big(\frac{\mu}{1+t}\Psi(t,x) \Big) \Big)\,dx\,dt \\ &= \varepsilon \int_{\mathbb{R}^n} u_0(x) \big(\Psi_t(0,x) - \mu\Psi(0,x) \big)\,dx + \int_0^T \int_{\mathbb{R}^n} u(t,x) \Big(\Psi_{tt}(t,x) - \Delta\Psi(t,x) - \partial_t \Big(\frac{\mu}{1+t}\Psi(t,x) \Big) \Big)\,dx\,dt. \end{aligned}$$

Substituting this relation in (48), we get (49). This concludes the proof. □

In the next result, we will employ the following solution of the adjoint equation to the homogeneous linear equation related to (47), which is a particular solution among the self-similar solutions that we will introduce in Section 5:

$$\begin{aligned} V(t,x) &\doteq (1+t)^{\frac{\mu+1+\sqrt{\delta}}{2}} ((1+t)^2 - |x|^2)^{-\frac{n+\sqrt{\delta}}{2}} \\ &= (1+t)^{-n+\frac{\mu+1-\sqrt{\delta}}{2}} \Big(1 - \frac{|x|^2}{(1+t)^2} \Big)^{-\frac{n+\sqrt{\delta}}{2}} \qquad \text{for } (t,x) \in Q, \end{aligned} \tag{50}$$

where $Q \doteq \{(t,x) \in [0,T) \times \mathbb{R}^n : |x| \leqslant 1+t\}$.

Moreover, we introduce a parameter dependent bump function. Let $\psi \in \mathscr{C}_0^\infty([0,\infty))$ be a nonincreasing function such that $\psi = 1$ on $[0, \frac{1}{2}]$ and $\operatorname{supp}\psi \subset [0,1)$. Besides, we denote

$$\psi^*(t) = \begin{cases} 0 & \text{if } \ t \in [0, \frac{1}{2}), \\ \psi(t) & \text{if } \ t \in [\frac{1}{2}, \infty). \end{cases}$$

Clearly, ψ^* is not smooth. We will use this bounded function only to keep trace of the support property of derivatives of ψ. More precisely, if $\psi_R(t) \doteq \psi(\frac{t}{R}), \psi^*_R(t) \doteq \psi^*(\frac{t}{R})$ for any $t \geqslant 0$ with $R > 0$, then, the following estimates hold (see, for example Lemma 3.1 in [15])

$$|\partial_t \psi_R(t)| \lesssim R^{-1}[\psi^*_R(t)]^{1-\frac{1}{k}} \qquad \text{for any } \ k \geqslant 1, \tag{51}$$

$$|\partial_t^2 \psi_R(t)| \lesssim R^{-2}[\psi^*_R(t)]^{1-\frac{2}{k}} \qquad \text{for any } \ k \geqslant 2. \tag{52}$$

Now we can prove a lower bound estimate, which is somehow related to (21) and (22).

Lemma 4.3. *Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ be nonnegative functions such that $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset B_{r_0}$ with $r_0 \in (0,1)$. Let u be a super-solution of*

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = 0, & x \in \mathbb{R}^n, \ t \in (0,T), \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{53}$$

such that $\operatorname{supp} u \subset Q_{r_0} \doteq \{(t,x) \in [0,T) \times \mathbb{R}^n : |x| \leqslant r_0 + t\}$. Then, for any $p > 1$ and any $R \in (1,T)$ it holds

$$\big(I_{\mu,\nu^2}[u_0,u_1]\varepsilon\big)^p R^{n - \frac{n+\mu-1}{2}p} \lesssim \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p \psi^*_R(t)\, dx\, dt, \tag{54}$$

where the multiplicative constant in (54) is independent of ε and R and

$$I_{\mu,\nu^2}[u_0,u_1] \doteq \int_{\mathbb{R}^n} \Big(u_1(x) V(0,x) + u_0(x) \big(\mu V(0,x) - V_t(0,x) \big) \Big)\, dx.$$

Remark 4.4. *In the previous statement the nonnegativity of u_0, u_1 can be relaxed by requiring simply that u_0, u_1 satisfy $I_{\mu,\nu^2}[u_0,u_1] > 0$.*

Proof. Let us consider $\Psi(t,x) = \psi_R(t) V(t,x) \chi(x)$, where $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ satisfies $\chi = 1$ on B_{r_0+T}. Applying (49) to this Ψ, we get

$$\begin{aligned} &\varepsilon \int_{\mathbb{R}^n} \big(u_1(x)\psi_R(0)V(0,x) + u_0(x)(\mu\psi_R(0)V(0,x) - \psi_R'(0)V(0,x) - \psi_R(0)V_t(0,x) \big)\, dx \\ &\quad \leqslant \int_0^T \int_{\mathbb{R}^n} u(t,x) \Big(\partial_t^2(\psi_R(t)V(t,x)) - \Delta(\psi_R(t)V(t,x)) - \tfrac{\mu}{1+t}\partial_t(\psi_R(t)V(t,x)) + \tfrac{\mu+\nu^2}{(1+t)^2}\psi_R(t)V(t,x) \Big)\, dx\, dt \end{aligned}$$

and, then,

$$\begin{aligned} \varepsilon \int_{\mathbb{R}^n} & \big(u_1(x) V(0,x) + u_0(x)(\mu V(0,x) - V_t(0,x) \big)\, dx = \varepsilon\, I_{\mu,\nu^2}[u_0,u_1] \\ &\leqslant \int_0^T \int_{\mathbb{R}^n} u(t,x) \Big(\partial_t^2 \psi_R(t) V(t,x) + 2\partial_t \psi_R(t) V_t(t,x) - \tfrac{\mu}{1+t} \partial_t \psi_R(t) V(t,x) \Big)\, dx\, dt \\ &\quad + \int_0^T \int_{\mathbb{R}^n} u(t,x) \psi_R(t) \Big(V_{tt}(t,x) - \Delta V(t,x) - \tfrac{\mu}{1+t} V_t(t,x) + \tfrac{\mu+\nu^2}{(1+t)^2} V(t,x) \Big)\, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}^n} u(t,x) \Big(\partial_t^2 \psi_R(t) V(t,x) + 2\partial_t \psi_R(t) V_t(t,x) - \tfrac{\mu}{1+t} \partial_t \psi_R(t) V(t,x) \Big)\, dx\, dt, \end{aligned}$$

where in last step we used the fact that V solves the adjoint equation of the homogeneous wave equation with scale-invariant damping and mass.

Let us remark that

$$V_t(t,x) = -(1+t)^{-n-\frac{\mu-1-\sqrt{\delta}}{2}}\Big(1-\tfrac{|x|^2}{(1+t)^2}\Big)^{-\frac{n+\sqrt{\delta}}{2}-1}\Big(n+\tfrac{\sqrt{\delta}-(\mu+1)}{2}+\tfrac{\mu+1+\sqrt{\delta}}{2}\tfrac{|x|^2}{(1+t)^2}\Big),$$

so that

$$\mu V(0,x) - V_t(0,x) = (1-|x|^2)^{-\frac{n+\sqrt{\delta}}{2}-1}\Big(n+\tfrac{\mu-1+\sqrt{\delta}}{2}+\tfrac{1-\mu+\sqrt{\delta}}{2}|x|^2\Big) \geqslant 0 \qquad \text{for any } \ x \in B_{r_0}.$$

In particular, for nonnegative and nontrivial u_0, u_1 the last estimate yields $I_{\mu,\nu^2}[u_0,u_1] > 0$.

If we employ now (51) and (52) for $k = p'$ and $k = 2p'$, respectively, then, we arrive at

$$\begin{aligned}
\varepsilon\, I_{\mu,\nu^2}[u_0,u_1] &\lesssim \int_0^T\int_{\mathbb{R}^n} |u(t,x)|\Big(\tfrac{|V(t,x)|}{R^2}+\tfrac{|V_t(t,x)|}{R}+\tfrac{\mu}{1+t}\tfrac{|V(t,x)|}{R}\Big)[\psi_R^*(t)]^{\frac{1}{p}}\,dx\,dt \\
&\lesssim \int_0^T\int_{\mathbb{R}^n} |u(t,x)|\Big(\tfrac{|V(t,x)|}{R^2}+\tfrac{|V_t(t,x)|}{R}\Big)[\psi_R^*(t)]^{\frac{1}{p}}\,dx\,dt \\
&\lesssim \Big(\int_0^T\int_{\mathbb{R}^n} |u(t,x)|^p\,\psi_R^*(t)\,dx\,dt\Big)^{\frac{1}{p}}\Big(\int_{\frac{R}{2}}^R\int_{B_{r_0+t}}\Big(\tfrac{|V(t,x)|}{R^2}+\tfrac{|V_t(t,x)|}{R}\Big)^{p'}dx\,dt\Big)^{\frac{1}{p'}}.
\end{aligned}$$

For $t \in [\frac{R}{2}, R]$ and $|x| \leqslant r_0 + t$ it holds

$$|V(t,x)| \lesssim R^{-n+\frac{\mu+1-\sqrt{\delta}}{2}}\Big(1-\tfrac{|x|}{1+t}\Big)^{-\frac{n+\sqrt{\delta}}{2}}, \qquad |V_t(t,x)| \lesssim R^{-n+\frac{\mu-1-\sqrt{\delta}}{2}}\Big(1-\tfrac{|x|}{1+t}\Big)^{-\frac{n+\sqrt{\delta}}{2}-1}.$$

Therefore,

$$\begin{aligned}
\varepsilon I_{\mu,\nu^2}[u_0,u_1] &\lesssim R^{-n-2+\frac{\mu+1-\sqrt{\delta}}{2}}\Big(\int_0^T\int_{\mathbb{R}^n}|u(t,x)|^p\,\psi_R^*(t)\,dx\,dt\Big)^{\frac{1}{p}}\Big(\int_{\frac{R}{2}}^R\int_{B_{r_0+t}}\Big(1-\tfrac{|x|}{1+t}\Big)^{-\big(\frac{n+\sqrt{\delta}}{2}+1\big)p'}dx\,dt\Big)^{\frac{1}{p'}} \\
&\lesssim R^{-\frac{n}{p}+\frac{n+\mu-1}{2}}\Big(\int_0^T\int_{\mathbb{R}^n}|u(t,x)|^p\,\psi_R^*(t)\,dx\,dt\Big)^{\frac{1}{p}}, \qquad (55)
\end{aligned}$$

where in the second inequality we used

$$\begin{aligned}
\int_{\frac{R}{2}}^R\int_{B_{r_0+t}}\Big(1-\tfrac{|x|}{1+t}\Big)^{-\big(\frac{n+\sqrt{\delta}}{2}+1\big)p'}dx\,dt &\lesssim \int_{\frac{R}{2}}^R (1+t)^{\big(\frac{n+\sqrt{\delta}}{2}+1\big)p'}(r_0+t)^{n-1}\int_0^{r_0+t}(1+t-r)^{-\big(\frac{n+\sqrt{\delta}}{2}+1\big)p'}dr\,dt \\
&\lesssim \int_{\frac{R}{2}}^R (1+t)^{\big(\frac{n+\sqrt{\delta}}{2}+1\big)p'}(r_0+t)^{n-1}\,dt \lesssim R^{n+\big(\frac{n+\sqrt{\delta}}{2}+1\big)p'}.
\end{aligned}$$

From (55) it follows easily (54). The proof is complete. □

5. Self-similar solutions related to Gauss hypergeometric functions

In the critical case of blow-up phenomena for semilinear wave equations with scale-invariant damping and mass, it is important to have a precise description of the behavior of solutions to the adjoint equation to the corresponding linear homogeneous equation. According to this purpose, in this section we will introduce a family of self-similar solutions to this equation, that can be represented by using Gauss hypergeometric functions (see also [45, 46, 12, 13, 15, 35]). In particular, we refer to [35, Section 4] for the proofs of results which are not proved here.

Hence, our goal is to provide a family of solutions on Q to the adjoint equation

$$\partial_t^2\Phi - \Delta\Phi - \partial_t\left(\frac{\mu}{1+t}\,\Phi\right) + \frac{\nu^2}{(1+t)^2}\,\Phi = 0. \tag{56}$$

Let β be a real parameter. If we make the following ansatz:

$$\Phi_\beta(t,x) \doteq (1+t)^{-\beta+1}\phi_\beta\left(\frac{|x|^2}{(1+t)^2}\right),$$

where $\psi_\beta \in \mathscr{C}^2([0,1))$, then, Φ_β solves (56) if and only if ϕ_β solves

$$z(1-z)\phi_\beta''(z) + \left(\tfrac{n}{2} - \left(\beta + \tfrac{\mu+1}{2}\right)z\right)\phi_\beta'(z) - \tfrac{1}{4}\left(\beta(\beta+\mu-1)+\nu^2\right)\phi_\beta(z) = 0. \tag{57}$$

Choosing

$$a_\beta = a_\beta(\mu,\nu^2) \doteq \tfrac{\beta}{2} + \tfrac{\mu-1}{4} + \tfrac{\sqrt{\delta}}{4} \qquad \text{and} \qquad b_\beta = b_\beta(\mu,\nu^2) \doteq \tfrac{\beta}{2} + \tfrac{\mu-1}{4} - \tfrac{\sqrt{\delta}}{4},$$

we have

$$a_\beta + b_\beta + 1 = \beta + \tfrac{\mu+1}{2} \qquad \text{and} \qquad a_\beta b_\beta = \tfrac{1}{4}\left(\beta(\beta+\mu-1)+\nu^2\right).$$

Therefore, (57) coincides with the hypergeometric equation with parameters $(a_\beta, b_\beta; \frac{n}{2})$, namely,

$$z(1-z)\phi_\beta''(z) + \left(\tfrac{n}{2} - \left(a_\beta + b_\beta + 1\right)z\right)\phi_\beta'(z) - a_\beta b_\beta\,\phi_\beta(z) = 0.$$

Also, we may choose ϕ_β as the Gauss hypergeometric function

$$\phi_\beta(z) \doteq \mathsf{F}(a_\beta, b_\beta; \tfrac{n}{2}; z) = \sum_{k=0}^{\infty} \frac{(a_\beta)_k\,(b_\beta)_k}{(n/2)_k}\frac{z^k}{k!} \qquad |z|<1,$$

where $(m)_k$ denotes Pochhammer's symbol, which is defined by

$$(m)_k \doteq \begin{cases} 1 & \text{if} \;\; k=0, \\ \prod_{j=1}^k (m+j-1) & \text{if} \;\; k>0. \end{cases}$$

Definition 5.1. *Let β a real parameter such that $\beta > \frac{\sqrt{\delta}+1-\mu}{2}$. Then, we define*

$$\begin{aligned} \Phi_{\beta,\mu,\nu^2}(t,x) &\doteq (1+t)^{-\beta+1}\phi_\beta\left(\tfrac{|x|^2}{(1+t)^2}\right) \\ &= (1+t)^{-\beta+1}\mathsf{F}\left(a_\beta(\mu,\nu^2)\,,b_\beta(\mu,\nu^2)\,;\tfrac{n}{2}\,;\tfrac{|x|^2}{(1+t)^2}\right) \qquad \text{for} \;\; (t,x)\in Q. \end{aligned} \tag{58}$$

According to the construction we explained until now in this section, it is clear that $\{\Phi_{\beta,\mu,\nu^2}\}_\beta$ is a family of solutions to (56). In the next lemma, we discuss some properties of this family of self-similar solutions.

Lemma 5.2. *The function Φ_{β,μ,ν^2} satisfies the following properties:*

(i) *Φ_{β,μ,ν^2} is a solution of (16) on Q.*

(ii) *$|\partial_t\Phi_{\beta,\mu,\nu^2}| \lesssim \Phi_{\beta+1,\mu,\nu^2}$ on Q.*

(iii) *If $\beta \in \left(\frac{\sqrt{\delta}+1-\mu}{2}, \frac{n+1-\mu}{2}\right)$, then,*

$$\Phi_{\beta,\mu,\nu^2}(t,x) \approx (1+t)^{-\beta+1}$$

for any $(t,x)\in Q$.

(iv) *If* $\beta > \frac{n+1-\mu}{2}$, *then,*

$$\Phi_{\beta,\mu,\nu^2}(t,x) \approx (1+t)^{-\beta+1}\Big(1-\tfrac{|x|^2}{(1+t)^2}\Big)^{\frac{n-\mu+1}{2}-\beta}$$

for any $(t,x)\in Q$.

Proof. Let us prove (ii). If we denote $z \doteq \frac{|x|^2}{(1+t)^2}$, then,

$$\begin{aligned}\partial_t\Phi_{\beta,\mu,\nu^2}(t,x) &= (1+t)^{-\beta}\big[(1-\beta)\mathsf{F}(a_\beta,b_\beta;\tfrac{n}{2};z) - 2z\,\mathsf{F}'(a_\beta,b_\beta;\tfrac{n}{2};z)\big]\\ &= (1+t)^{-\beta}\big[(1-\beta)\mathsf{F}(a_\beta,b_\beta;\tfrac{n}{2};z) - 4\tfrac{a_\beta b_\beta}{n} z\,\mathsf{F}(a_\beta+1,b_\beta+1;\tfrac{n}{2}+1;z)\big].\end{aligned} \tag{59}$$

Moreover,

$$\Phi_{\beta+1,\mu,\nu^2}(t,x) = (1+t)^{-\beta}\mathsf{F}(a_{\beta+1},b_{\beta+1};\tfrac{n}{2};z) = (1+t)^{-\beta}\mathsf{F}(a_\beta+\tfrac12,b_\beta+\tfrac12;\tfrac{n}{2};z).$$

Since for real parameters $(a,b;c)$ the hypergeometric function has the following behavior for $z\in[0,1)$

$$\mathsf{F}(a,b;c;z) \approx \begin{cases} 1 & \text{if } \ c > a+b,\\ -\log(1-z) & \text{if } \ c = a+b,\\ (1-z)^{c-(a+b)} & \text{if } \ c < a+b,\end{cases} \tag{60}$$

and $\frac{n}{2}+1-(a_\beta+b_\beta+2) = \frac{n}{2}-(a_{\beta+1}+b_{\beta+1})$, as the second term in (59) is the dominant one, we get immediately the desired property. By using (60), we find (iii) and (iv) as well. □

Remark 5.3. *If we consider* β *such that* $b_\beta = \frac{n}{2}$, *i.e.* $\beta = n + \frac{\sqrt{\delta}+1-\mu}{2}$, *then,* $a_\beta = b_\beta + \frac{\sqrt{\delta}}{2}$ *and*

$$\Phi_{\beta,\mu,\nu}(t,x) = (1+t)^{-n+\frac{\mu+1-\sqrt{\delta}}{2}}\mathsf{F}\Big(\tfrac{n+\sqrt{\delta}}{2},\tfrac{n}{2};\tfrac{n}{2};\tfrac{|x|^2}{(1+t)^2}\Big) = V(t,x)$$

with V *defined by* (50). *In the previous equality, we used the relation* $\mathsf{F}(\alpha,\gamma;\gamma;z) = (1-z)^{-\alpha}$.

Lemma 5.4. *Let us assume* $u_0\in H^1(\mathbb{R}^n)$ *and* $u_1\in L^2(\mathbb{R}^n)$ *satisfying* $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset B_{r_0}$ *for some* $r_0\in(0,1)$ *and*

$$J_{\beta,\mu,\nu^2}[u_0,u_1] \doteq \int_{\mathbb{R}^n}\Big(u_1(x)\Phi_{\beta,\mu,\nu^2}(0,x) + u_0(x)\big(\mu\Phi_{\beta,\mu,\nu^2}(0,x) - \partial_t\Phi_{\beta,\mu,\nu^2}(0,x)\big)\Big)\,dx > 0,$$

where $\beta > \frac{\sqrt{\delta}+1-\mu}{2}$ *is a parameter. Let* u *be a super-solution of* (47) *such that* $\operatorname{supp} u\subset Q_{r_0}$.

Then, for any $p>1$ *and any* $R\in(1,T)$ *it holds*

$$\begin{aligned}J_{\beta,\mu,\nu^2}[u_0,u_1]\,\varepsilon + \int_0^T\int_{\mathbb{R}^n} H(t,x)\,\psi_R(t)\,\Phi_{\beta,\mu,\nu^2}(t,x)\,dx\,dt&\\ \lesssim R^{-1}\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\,\Phi_{\beta+1,\mu,\nu^2}(t,x)\,[\psi_R^*(t)]^{\frac1p}\,dx\,dt,&\end{aligned} \tag{61}$$

where the multiplicative constant in (61) *is independent of* ε *and* R.

Proof. Let us consider $\Psi(t,x) = \psi_R(t)\Phi_{\beta,\mu,\nu^2}(t,x)\chi(x)$, where $\chi\in\mathscr{C}_0^\infty(\mathbb{R}^n)$ satisfies $\chi=1$ on B_{r_0+T}. Applying (49) to the test function Ψ, we get

$$\begin{aligned}&\varepsilon J_{\beta,\mu,\nu^2}[u_0,u_1] + \int_0^T\int_{\mathbb{R}^n} H(t,x)\,\psi_R(t)\,\Phi_{\beta,\mu,\nu^2}(t,x)\,dx\,dt\\ &\quad\leqslant \int_0^T\int_{\mathbb{R}^n} u(t,x)\Big(\partial_t^2\psi_R(t)\Phi_{\beta,\mu,\nu^2}(t,x) + 2\partial_t\psi_R(t)\partial_t\Phi_{\beta,\mu,\nu^2}(t,x) - \tfrac{\mu}{1+t}\partial_t\psi_R(t)\Phi_{\beta,\mu,\nu^2}(t,x)\Big)\,dx\,dt\\ &\qquad + \int_0^T\int_{\mathbb{R}^n} u(t,x)\psi_R(t)\Big(\partial_t^2 - \Delta - \tfrac{\mu}{1+t}\partial_t + \tfrac{\mu+\nu^2}{(1+t)^2}\Big)\Phi_{\beta,\mu,\nu^2}(t,x)\,dx\,dt\\ &\quad\lesssim \int_0^T\int_{\mathbb{R}^n}|u(t,x)|\Big(R^{-2}|\Phi_{\beta,\mu,\nu^2}(t,x)| + R^{-1}|\partial_t\Phi_{\beta,\mu,\nu^2}(t,x)|\Big)[\psi_R^*(t)]^{\frac1p}\,dx\,dt,\end{aligned} \tag{62}$$

where in last inequality we used the fact that Φ_{β,μ,ν^2} solves (56) and (51), (52). We note that for $t \in [\frac{R}{2}, R]$ and $|x| \leqslant r_0 + t$, it holds

$$\begin{aligned}\Phi_{\beta,\mu,\nu^2}(t,x) &= (1+t)^{-\beta+1}\mathsf{F}\Big(\tfrac{\beta}{2} + \tfrac{\mu-1+\sqrt{\delta}}{4}, \tfrac{\beta}{2} + \tfrac{\mu-1-\sqrt{\delta}}{4}; \tfrac{n}{2}; \tfrac{|x|^2}{(1+t)^2}\Big) \\ &\lesssim R(1+t)^{-\beta}\mathsf{F}\Big(\tfrac{\beta}{2} + \tfrac{\mu-1+\sqrt{\delta}}{4}, \tfrac{\beta}{2} + \tfrac{\mu-1-\sqrt{\delta}}{4}; \tfrac{n}{2}; \tfrac{|x|^2}{(1+t)^2}\Big) \\ &\lesssim R(1+t)^{-\beta}\mathsf{F}\Big(\tfrac{\beta+1}{2} + \tfrac{\mu-1+\sqrt{\delta}}{4}, \tfrac{\beta+1}{2} + \tfrac{\mu-1-\sqrt{\delta}}{4}; \tfrac{n}{2}; \tfrac{|x|^2}{(1+t)^2}\Big) = R\,\Phi_{\beta+1,\mu,\nu^2}(t,x)\end{aligned}$$

and, then, combining the previous estimate with Lemma 5.2 (ii), we get

$$R^{-2}|\Phi_{\beta,\mu,\nu^2}(t,x)| + R^{-1}|\partial_t\Phi_{\beta,\mu,\nu^2}(t,x)| \lesssim R^{-1}\Phi_{\beta+1,\mu,\nu^2}(t,x).$$

Thus, if we use the last estimate in the right hand side of (62) we get (61). This completes the proof. □

Remark 5.5. *Let us rewrite the function that multiplies u_0 in $J_{\beta,\mu,\nu^2}[u_0,u_1]$ in a more explicit way:*

$$\mu\Phi_{\beta,\mu,\nu^2}(0,x) - \partial_t\Phi_{\beta,\mu,\nu^2}(0,x) = (\beta+\mu-1)\mathsf{F}\big(a_\beta, b_\beta; \tfrac{n}{2}; |x|^2\big) + \frac{4a_\beta b_\beta}{n}\mathsf{F}\big(a_{\beta+1}, b_{\beta+1}; \tfrac{n+1}{2}; |x|^2\big).$$

Then, if $\beta \geqslant 1-\mu$, in order to get a strictly positive $J_{\beta,\mu,\nu^2}[u_0,u_1]$, it is sufficient to consider nonnegative and nontrivial u_0, u_1. Since in our treatment either $\beta \in \big(\frac{\sqrt{\delta}+1-\mu}{2}, \frac{n-\mu+1}{2}\big)$ or $\beta \geqslant \frac{n-\mu+1}{2}$, we may assume without loss of regularity that $\beta \geqslant 1-\mu$ thanks to $1-\mu < \frac{n-\mu+1}{2}$.

Lemma 5.6. *Let $\beta > \frac{\sqrt{\delta}+1-\mu}{2}$ be a real number such that $\beta \neq \frac{n-\mu+1}{2}$. Then, the following estimate holds for $R \geqslant R_0 > 0$:*

$$\int_{\frac{R}{2}}^R \int_{B_{r_0+t}} (\Phi_{\beta,\mu,\nu^2}(t,x))^{p'}\,dx\,dt \lesssim \begin{cases} R^{n+1+(1-\beta)p'} & \text{if } \beta < \frac{n-\mu+1}{2} + 1 - \frac{1}{p}, \\ R^{-\frac{n-\mu-1}{2}p'+n}\log R & \text{if } \beta = \frac{n-\mu+1}{2} + 1 - \frac{1}{p}, \\ R^{-\frac{n-\mu-1}{2}p'+n} & \text{if } \beta > \frac{n-\mu+1}{2} + 1 - \frac{1}{p}. \end{cases}$$

Proof. Let us begin with the case $\beta < \frac{n-\mu+1}{2}$. Using Lemma 5.2 (iii), we get

$$\begin{aligned}\int_{\frac{R}{2}}^R \int_{B_{r_0+t}} (\Phi_{\beta,\mu,\nu^2}(t,x))^{p'}\,dx\,dt &\approx \int_{\frac{R}{2}}^R \int_{B_{r_0+t}} (1+t)^{(-\beta+1)p'}\,dx\,dt \\ &\approx \int_{\frac{R}{2}}^R (1+t)^{(-\beta+1)p'}(r_0+t)^n\,dt \lesssim R^{n+1+(-\beta+1)p'}.\end{aligned}$$

When $\beta > \frac{n-\mu+1}{2}$, from Lemma (5.2) (iv) it follows

$$\begin{aligned}\int_{\frac{R}{2}}^R \int_{B_{r_0+t}} (\Phi_{\beta,\mu,\nu^2}(t,x))^{p'}\,dx\,dt &\lesssim \int_{\frac{R}{2}}^R (1+t)^{(-\beta+1)p'} \int_0^{r_0+t} \Big(1 - \tfrac{r}{(1+t)}\Big)^{\frac{n-\mu+1}{2}p'-\beta p'} r^{n-1}\,dr\,dt \\ &\lesssim \int_{\frac{R}{2}}^R (1+t)^{-\frac{n-\mu-1}{2}p'}(r_0+t)^{n-1} \int_0^{r_0+t} (1+t-r)^{\frac{n-\mu+1}{2}p'-\beta p'}\,dr\,dt \\ &\lesssim \begin{cases} \int_{\frac{R}{2}}^R (1+t)^{-\frac{n-\mu-1}{2}p'}(r_0+t)^{n-1}\,dt & \text{if } \beta > \frac{n-\mu+1}{2} + 1 - \frac{1}{p}, \\ \int_{\frac{R}{2}}^R (1+t)^{-\frac{n-\mu-1}{2}p'}(r_0+t)^{n-1}\log(1+t)\,dt & \text{if } \beta = \frac{n-\mu+1}{2} + 1 - \frac{1}{p}, \\ \int_{\frac{R}{2}}^R (1+t)^{(-\beta+1)p'+1}(r_0+t)^{n-1}\,dt & \text{if } \beta < \frac{n-\mu+1}{2} + 1 - \frac{1}{p} \end{cases} \\ &\lesssim \begin{cases} R^{-\frac{n-\mu-1}{2}p'+n} & \text{if } \beta > \frac{n-\mu+1}{2} + 1 - \frac{1}{p}, \\ R^{-\frac{n-\mu-1}{2}p'+n}\log R & \text{if } \beta = \frac{n-\mu+1}{2} + 1 - \frac{1}{p}, \\ R^{n+1+(-\beta+1)p'} & \text{if } \beta < \frac{n-\mu+1}{2} + 1 - \frac{1}{p}. \end{cases}\end{aligned}$$

Combining the two cases, we find the desired estimate. □

6. Critical case: Proof of Theorem 1.4

This section is organized as follows: firstly, we recall some technical lemmas from [14, 15]; then, in the last two subsections we prove the blow-up results and the corresponding upper bounds for the lifespan in the critical case $p = p_0(n+\mu)$ for (2) and on the critical curve $\max\{F(n+\mu_1,p,q); F(n+\mu_2,q,p)\} = 0$ for (1), respectively.

6.1. Lemmas on the blow-up dynamic in critical cases

The results stated in this section are already know in the literature (see [14, 15]). Nonetheless, for the ease of the reader they will be recalled. The upcoming lemmas will play a fundamental role in determining the upper bound lifespan estimate of exponential type, whenever we are in a critical case.

Definition 6.1. *Let $w \in L^1_{\mathrm{loc}}([0,T), L^1(\mathbb{R}^n))$ be a nonnegative function. We set*

$$Y[w](R) \doteq \int_0^R \left(\int_0^T \int_{\mathbb{R}^n} w(t,x)\, \psi^*_\sigma(t)\, dx\, dt \right) \sigma^{-1}\, d\sigma, \qquad \textit{for any} \ \ R \in (0,T).$$

The functional $Y[w]$ satisfies the properties stated in the next lemma.

Lemma 6.2. *Let $w \in L^1_{\mathrm{loc}}([0,T), L^1(\mathbb{R}^n))$ be a nonnegative function. Then, $Y[w] \in \mathscr{C}^1((0,T))$ and for any $R \in (0,T)$*

$$\frac{d}{dR} Y[w](R) = R^{-1} \int_0^T \int_{\mathbb{R}^n} w(t,x)\, \psi^*_R(t)\, dx\, dt,$$
$$Y[w](R) \leqslant \int_0^T \int_{\mathbb{R}^n} w(t,x)\, \psi_R(t)\, dx\, dt.$$

For the proof of the above lemma, one can see [14, Proposition 2.1].

Lemma 6.3. *Let $y \in \mathscr{C}^1([R_0,T))$ be a nonnegative function, where $2 < R_0 < T$. Moreover, there exist $\theta, K_1, K_2 > 0$ and $p_1, p_2 > 1$ such that*

$$\begin{cases} R\, y'(R) \geqslant K_1 \theta & \textit{for} \ \ R \in (R_0,T), \\ R\,(\log R)^{p_2-1} y'(R) \geqslant K_2 (y(R))^{p_1} & \textit{for} \ \ R \in (R_0,T). \end{cases}$$

If $p_2 < p_1 + 1$, then, there exist $\theta_0, K > 0$ such that

$$T \leqslant \exp\left(K \theta^{-\frac{p_1-1}{p_1-p_2+1}} \right) \qquad \textit{for any} \ \ \theta \in (0,\theta_0).$$

See [15, Lemma 3.10] for the proof of Lemma 6.3.

6.2. Critical case for the single semilinear equation

In this section we derive upper bound estimates for the lifespan of super-solutions of the semilinear wave equation with scale-invariant damping and mass in the critical case. Even though the result has been already proved for solutions in [35, Theorem 1.3], we need to use this generalization to super-solutions in Section 6.3.

Let us introduce the notion of super solutions for the semilinear model.

Definition 6.4. *Let c, ε be positive real constants. Let $(u_0,u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be compactly supported and $p > 1$. We say that $u \in \mathscr{C}([0,T), H^1(\mathbb{R}^n)) \cap \mathscr{C}^1([0,T), L^2(\mathbb{R}^n)) \cap L^p_{\mathrm{loc}}([0,T) \times \mathbb{R}^n)$ is a super-solution on $[0,T)$ of*

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = c|u|^p, & x \in \mathbb{R}^n, \ t \in (0,T), \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{63}$$

if $u(0,x) = \varepsilon u_0(x)$ *in* $H^1(\mathbb{R}^n)$ *and*

$$\begin{aligned}\varepsilon \int_{\mathbb{R}^n} u_1(x)\Psi(0,x)\,dx + \int_0^T \int_{\mathbb{R}^n} c\,|u(t,x)|^p \Psi(t,x)\,dx\,dt &\leqslant \int_0^T \int_{\mathbb{R}^n} -u_t(t,x)\Psi_t(t,x)\,dx\,dt \\ + \int_0^T \int_{\mathbb{R}^n} \Big(\nabla u(t,x)\cdot\nabla\Psi(t,x) + \frac{\mu}{1+t} u_t(t,x)\Psi(t,x) &+ \frac{\nu^2}{(1+t)^2} u(t,x)\Psi(t,x)\Big)\,dx\,dt \end{aligned} \tag{64}$$

for any nonnegative test function $\Psi \in \mathscr{C}_0^\infty([0,T)\times\mathbb{R}^n)$.

Proposition 6.5. *Let* $u_0 \in H^1(\mathbb{R}^n)$ *and* $u_1 \in L^2(\mathbb{R}^n)$ *be nonnegative and compactly supported functions such that* $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset B_{r_0}$ *with* $r_0 \in (0,1)$. *Let* $p = p_0(n+\mu)$ *and let* u *be a super-solution of* (63) *on* $[0, T(\varepsilon))$ *such that* $\operatorname{supp} u \subset Q_{r_0}$. *Then, there exist two positive and independent of* ε *constants* ε_0, C *such that*

$$T(\varepsilon) \leqslant \exp(C\varepsilon^{-p(p-1)}) \qquad \textit{for any} \quad \varepsilon \in (0,\varepsilon_0).$$

Proof. In order to prove the proposition, we consider $Y = Y[|u|^p \Phi_{\beta_p,\mu,\nu^2}]$ with $\beta_p = \frac{n-\mu+1}{2} - \frac{1}{p}$. Since $p = p_0(n+\mu)$, then,

$$-\beta_p + 1 + n - \tfrac{n+\mu-1}{2}p = \tfrac{1}{p}\big(1 + \tfrac{n+\mu+1}{2}p - \tfrac{n+\mu-1}{2}p^2\big) = 0. \tag{65}$$

Therefore, using (54), we find

$$\begin{aligned} Y'(R)R &= \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p\, \Phi_{\beta_p,\mu,\nu^2}(t,x)\, \psi_R^*(t)\,dx\,dt \gtrsim R^{-\beta_p+1} \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p\, \psi_R^*(t)\,dx\,dt \\ &\gtrsim R^{-\beta_p+1+n-\frac{n+\mu-1}{2}p}\varepsilon^p = \varepsilon^p. \end{aligned}$$

Furthermore, from Lemmas 6.2, 5.4 and 5.6 we obtain

$$\begin{aligned} (Y(R))^p &\lesssim \Big(\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p\, \Phi_{\beta_p,\mu,\nu^2}(t,x)\, \psi_R(t)\,dx\,dt\Big)^p \\ &\lesssim R^{-p}\Big(\int_0^T \int_{\mathbb{R}^n} |u(t,x)|\, \Phi_{\beta_p+1,\mu,\nu^2}(t,x)\, [\psi_R^*(t)]^{\frac{1}{p}}\,dx\,dt\Big)^p \\ &\lesssim R^{-p}\Big(\int_{\frac{R}{2}}^R \int_{B_{r_0+t}} (\Phi_{\beta_p+1,\mu,\nu^2}(t,x))^{p'}\,dx\,dt\Big)^{p-1} \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p\, \psi_R^*(t)\,dx\,dt \\ &\lesssim R^{-p-\frac{n-\mu-1}{2}p+n(p-1)}(\log R)^{p-1} \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p\, \psi_R^*(t)\,dx\,dt \\ &\lesssim R^{\frac{n+\mu-1}{2}p-n}(\log R)^{p-1} R^{\beta_p-1} \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p\, \Phi_{\beta_p,\mu,\nu^2}(t,x)\, \psi_R^*(t)\,dx\,dt \\ &= (\log R)^{p-1} \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p\, \Phi_{\beta_p,\mu,\nu^2}(t,x)\, \psi_R^*(t)\,dx\,dt = (\log R)^{p-1} Y'(R)R \end{aligned}$$

where in the last inequality we employed (65). Setting $\theta = \varepsilon^p$, $p_1 = p_2 = p$, by Lemma 6.3 it follows the upper bound for the lifespan $T(\varepsilon) \leqslant \exp\big(C\varepsilon^{-p(p-1)}\big)$ for a suitable constant C. □

6.3. Critical case for the weakly coupled system

This section is devoted to the proof of Theorem 1.4, but, before proving it, we will derive some estimates for the weakly coupled system (1) in the general case.

Let (u,v) be a solution to (1) in the sense of Definition 1.1. As the nonlinear terms in (1) are nonnegative, in particular, u, v are super-solutions of (53) for $(\mu,\nu^2) = (\mu_j,\nu_j^2)$, $j = 1,2$, respectively. Moreover, $\operatorname{supp} u, \operatorname{supp} v \subset Q_{r_0}$ due to the property of finite speed of propagation for hyperbolic equations.

Therefore, Lemma 4.3 implies

$$\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q \psi_R^*(t)\, dx\, dt \gtrsim \big(I_{\mu_1,\nu_1^2}[u_0,u_1]\varepsilon\big)^q R^{n-\frac{n+\mu_1-1}{2}q}, \tag{66}$$

$$\int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p \psi_R^*(t)\, dx\, dt \gtrsim \big(I_{\mu_2,\nu_2^2}[v_0,v_1]\varepsilon\big)^p R^{n-\frac{n+\mu_2-1}{2}p}. \tag{67}$$

Let us consider $\hat{\beta} \in (\frac{\sqrt{\delta_2}+1-\mu_2}{2}, \frac{n-\mu_2+1}{2} - \frac{1}{p})$. Note that the nonemptiness of the interval for $\hat{\beta}$ is guaranteed by the first condition in (14). Using Lemma 5.2 (iii), Lemma 5.4 and Lemma 5.6, we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R(t)\, dx\, dt \\
&\quad \lesssim R^{\hat{\beta}-1} \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \Phi_{\hat{\beta},\mu_2,\nu_2^2}(t,x)\, \psi_R(t)\, dx\, dt \\
&\quad \lesssim R^{\hat{\beta}-2} \int_0^T \int_{\mathbb{R}^n} |v(t,x)|\, \Phi_{\hat{\beta}+1,\mu_2,\nu_2^2}(t,x)\, [\psi_R^*(t)]^{\frac{1}{p}}\, dx\, dt \\
&\quad \lesssim R^{\hat{\beta}-2} \bigg(\int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R^*(t)\, dx\, dt \bigg)^{\frac{1}{p}} \bigg(\int_{\frac{R}{2}}^R \int_{B_{r_0+t}} \big(\Phi_{\hat{\beta}+1,\mu_2,\nu_2^2}(t,x)\big)^{p'} dx\, dt \bigg)^{\frac{1}{p'}} \\
&\quad \lesssim R^{-2+\frac{n+1}{p'}} \bigg(\int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R^*(t)\, dx\, dt \bigg)^{\frac{1}{p}}.
\end{aligned}$$

Raising to the p power both sides of the last inequality, we obtain

$$\bigg(\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R(t)\, dx\, dt \bigg)^p \lesssim R^{-2+(n-1)(p-1)} \int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R^*(t)\, dx\, dt. \tag{68}$$

In a similar way, choosing $\tilde{\beta} \in (\frac{\sqrt{\delta_1}+1-\mu_1}{2}, \frac{n-\mu_1+1}{2} - \frac{1}{q})$ and using Lemma 5.2 (iii) to estimate $\Phi_{\tilde{\beta},\mu_1,\nu_1^2}$, one can prove

$$\bigg(\int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R(t)\, dx\, dt \bigg)^q \lesssim R^{-2+(n-1)(q-1)} \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R^*(t)\, dx\, dt. \tag{69}$$

Remark 6.6. *Using* (66), (67), (68) *and* (69), *it is possible to prove the blow-up result and the corresponding upper bound for the lifespan in the subcritical case in a simpler way than using the iteration argument. Nonetheless, additional technical restrictions on* (p,q), *namely* (14), *have to be considered, making the result obtained with the iteration argument sharper.*

Indeed, combining (68) *and* (69) *and the trivial inequality* $\psi_R^* \leqslant \psi_R$, *we find*

$$\begin{aligned}
\bigg(\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R(t)\, dx\, dt \bigg)^{pq} &\lesssim R^{[-2+(n-1)(p-1)]q} \bigg(\int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R^*(t)\, dx\, dt \bigg)^q \\
&\lesssim R^{[-2+(n-1)(p-1)]q-2+(n-1)(q-1)} \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R^*(t)\, dx\, dt.
\end{aligned}$$

Rearranging the previous estimate, we arrive at

$$\bigg(\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R(t)\, dx\, dt \bigg)^{pq-1} \lesssim R^{-2(q+1)+(n-1)(pq-1)},$$

that implies in turn

$$\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R(t)\, dx\, dt \lesssim R^{-\frac{2(q+1)}{pq-1}+n-1} = R^{n-\frac{pq+2q+1}{pq-1}}.$$

Combining the previous inequality with (66), *in the case* $F(n+\mu_1,p,q)>0$ *it follows*

$$\varepsilon^q R^{n-\frac{n+\mu_1-1}{2}q} \lesssim \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q \psi_R^*(t)\, dx\, dt \lesssim R^{n-\frac{pq+2q+1}{pq-1}}.$$

Comparing the lower bound and the upper bound for the integral in the last estimate, we obtain

$$R^{\frac{pq+2q+1}{pq-1}-\frac{n+\mu_1-1}{2}q} \lesssim \varepsilon^{-q},$$

which implies $R \lesssim \varepsilon^{-F(n+\mu_1,p,q)^{-1}}$. *Analogously,*

$$\begin{aligned}\left(\int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R(t)\, dx\, dt\right)^{pq} &\lesssim R^{[-2+(n-1)(q-1)]p}\left(\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R^*(t)\, dx\, dt\right)^p \\ &\lesssim R^{[-2+(n-1)(q-1)]p-2+(n-1)(p-1)} \int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R^*(t)\, dx\, dt.\end{aligned}$$

and (67) *imply*

$$\varepsilon^p R^{n-\frac{n+\mu_2-1}{2}p} \lesssim \int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R(t)\, dx\, dt \lesssim R^{n-\frac{pq+2p+1}{pq-1}}.$$

Proceeding as in the previous case, we have $R \lesssim \varepsilon^{-F(n+\mu_2,q,p)^{-1}}$ *in the case* $F(n+\mu_2,q,p)>0$. *Therefore, letting* $R\to T$, *we obtained* (11) *provided that* $p,q>1$ *satisfy* $\max\{F(n+\mu_1,p,q),F(n+\mu_2,q,p)\}>0$.

By using the estimates that we proved in this section, we can now prove Theorem 1.4. We will consider four subcases as in (15).

6.3.1. *Case* $F(n+\mu_1,p,q)=0>F(n+\mu_2,q,p)$

Differently from the treatment of the subcritical case (cf. Remark 6.6), in this case we study the blow-up dynamic of the function $Y = Y\left[|v|^p \Phi_{\beta_q,\mu_1,\nu_1^2}\right]$, where $\beta_q \doteq \frac{n-\mu_1+1}{2} - \frac{1}{q}$.

By (68) and (66), we get

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R^*(t)\, dx\, dt &\gtrsim R^{2-(n-1)(p-1)}\left(\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R(t)\, dx\, dt\right)^p \\ &\gtrsim R^{2-(n-1)(p-1)+np-\frac{n+\mu_1-1}{2}pq}\, \varepsilon^{pq} = R^{\frac{n-\mu_1-1}{2}-\frac{1}{q}}\, \varepsilon^{pq},\end{aligned}$$

where in the last step we employed $F(n+\mu_1,p,q)=0$. Due to the definition of β_q, the last estimates implies

$$Y'(R)R = \int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \Phi_{\beta_q,\mu_1,\nu_1^2}(t,x)\, \psi_R^*(t)\, dx\, dt \gtrsim R^{-\beta_q+1} \int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \psi_R^*(t)\, dx\, dt \gtrsim \varepsilon^{pq}. \quad (70)$$

By Lemma 5.4 and Lemma 5.6 in the logarithmic case, we find

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p\, \Phi_{\beta_q,\mu_1,\nu_1^2}(t,x)\, \psi_R(t)\, dx\, dt \\ &\quad\lesssim R^{-1} \int_0^T \int_{\mathbb{R}^n} |u(t,x)|\, \Phi_{\beta_q+1,\mu_1,\nu_1^2}(t,x)\, [\psi_R^*(t)]^{\frac{1}{q}}\, dx\, dt \\ &\quad\lesssim R^{-1}\left(\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R^*(t)\, dx\, dt\right)^{\frac{1}{q}} \left(\int_{\frac{R}{2}}^R \int_{B_{r_0+t}} \left(\Phi_{\beta_q+1,\mu_1,\nu_1^2}(t,x)\right)^{q'} dx\, dt\right)^{\frac{1}{q'}} \\ &\quad\lesssim R^{-1-\frac{n-\mu_1-1}{2}+\frac{n}{q'}} (\log R)^{\frac{1}{q'}} \left(\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q\, \psi_R^*(t)\, dx\, dt\right)^{\frac{1}{q}}.\end{aligned}$$

Raising to the pq power both sides of the previous inequality and using (68) and again the condition $F(n+\mu_1,p,q)=0$, we obtain

$$
\begin{aligned}
\Big(\int_0^T\int_{\mathbb{R}^n}|v(t,x)|^p\,\Phi_{\beta_q,\mu_1,\nu_1^2}(t,x)\,\psi_R(t)\,dx\,dt\Big)^{pq}& \\
&\lesssim R^{\frac{n+\mu_1-1}{2}pq-np}(\log R)^{p(q-1)}\Big(\int_0^T\int_{\mathbb{R}^n}|u(t,x)|^q\,\psi_R^*(t)\,dx\,dt\Big)^p \\
&\lesssim R^{\frac{n+\mu_1-1}{2}pq-np-2+(n-1)(p-1)}(\log R)^{p(q-1)}\int_0^T\int_{\mathbb{R}^n}|v(t,x)|^p\,\psi_R^*(t)\,dx\,dt \\
&= R^{1-\beta_q}(\log R)^{p(q-1)}\int_0^T\int_{\mathbb{R}^n}|v(t,x)|^p\,\psi_R^*(t)\,dx\,dt \\
&\lesssim (\log R)^{p(q-1)}\int_0^T\int_{\mathbb{R}^n}|v(t,x)|^p\,\Phi_{\beta_q,\mu_1,\nu_1^2}(t,x)\,\psi_R^*(t)\,dx\,dt. \qquad (71)
\end{aligned}
$$

Thanks to Lemma 6.2, from the last inequality we may derive the inequality

$$(\log R)^{p(q-1)}Y'(R)R\gtrsim (Y(R))^{pq}.$$

Setting $\theta=\varepsilon^{pq}$, $p_1=pq$ and $p_2=1+p(q-1)$, from Lemma 6.3 we have $T(\varepsilon)\leqslant\exp\big(C\varepsilon^{-q(pq-1)}\big)$ for a suitable positive constant C.

Remark 6.7. *Let us remark explicitly that from the condition $0=F(n+\mu_1,p,q)>F(n+\mu_2,q,p)$ does not follow in general, as for the weakly coupled system of free wave equations, that $p>q$, due to the presence of different shifts in the first argument of F.*

6.3.2. Case $F(n+\mu_1,p,q)<0=F(n+\mu_2,q,p)$

Proceeding as in the previous section but choosing now $Y=Y\big[|u|^q\Phi_{\beta_p,\mu_2,\nu_2^2}\big]$, where $\beta_p\doteq\frac{n-\mu_2+1}{2}-\frac{1}{p}$, it is possible to prove in the case $F(n+\mu_2,q,p)=0$ the upper bound estimate $T\leqslant\exp\big(C\varepsilon^{-p(pq-1)}\big)$ for a suitable positive constant C.

6.3.3. Case $F(n+\mu_1,p,q)=F(n+\mu_2,q,p)=0$

In this case, combining the results of Sections 6.3.1 and 6.3.2, it follows immediately the upper bound $T\leqslant\exp\big(C\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}\big)$ for the lifespan. However, we can further improve this estimate.

First, we prove that $F(n+\mu_1,p,q)=F(n+\mu_2,q,p)=0$ implies

$$
\begin{aligned}
\beta_q-1&=n-\frac{n+\mu_2-1}{2}p\,,\\
\beta_p-1&=n-\frac{n+\mu_1-1}{2}q\,.
\end{aligned} \qquad (72)
$$

Let us introduce the quantities

$$A\doteq n+1-\beta_p-\tfrac{n+\mu_1-1}{2}q,\quad B\doteq n+1-\beta_q-\tfrac{n+\mu_2-1}{2}p.$$

Straightforward computations show that

$$Ap+B=(pq-1)F(n+\mu_1,p,q)=0,\quad A+Bq=(pq-1)F(n+\mu_2,q,p)=0.$$

Hence, since $pq>1$, we have immediately $A=B=0$, that implies in turn the validity of (72).

Let us consider $Y=Y\big[|v|^p\Phi_{\beta_q,\mu_1,\nu_1^2}\big]$ as in Section 6.3.1. Due to the assumption $F(n+\mu_1,p,q)=0$, it holds (71) as in Section 6.3.1. The next step is to improve (70). Using (72), we may rewrite (66) and (67) as follows

$$\int_0^T\int_{\mathbb{R}^n}|u(t,x)|^q\psi_R^*(t)\,dx\,dt\gtrsim\varepsilon^qR^{\beta_p-1},\quad \int_0^T\int_{\mathbb{R}^n}|v(t,x)|^p\psi_R^*(t)\,dx\,dt\gtrsim\varepsilon^pR^{\beta_q-1}.$$

Consequently,

$$\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^q \, \Phi_{\beta_p,\mu_2,\nu_2^2}(t,x)\, \psi_R^*(t)\, dx\, dt \gtrsim \varepsilon^q, \qquad \int_0^T \int_{\mathbb{R}^n} |v(t,x)|^p \, \Phi_{\beta_q,\mu_1,\nu_1^2}(t,x)\, \psi_R^*(t)\, dx\, dt \gtrsim \varepsilon^p.$$

Also, we proved

$$\begin{cases} Y'(R)R \gtrsim \varepsilon^p, \\ (\log R)^{p(q-1)} Y'(R)R \gtrsim (Y(R))^{pq}. \end{cases}$$

Applying Lemma 6.3 with $\theta = \varepsilon^p$ and $p_1 = pq, p_2 = 1+p(q-1)$, we get the estimate $T(\varepsilon) \leqslant \exp\big(C\varepsilon^{-(pq-1)}\big)$.

6.3.4. Case $F(n+\mu_1,p,q) = F(n+\mu_2,q,p) = 0$ with the same scale-invariant coefficients in the linear part

In this last case we assume that $\mu_1 = \mu_2 \doteq \mu$ and $\nu_1^2 = \nu_2^2 \doteq \nu^2$. As we have the same shift, then, the condition $F(n+\mu,p,q) = F(n+\mu,q,p) = 0$ implies $p = q = p_0(n+\mu)$. Therefore, $w = u+v$ is a super-solution of (63) with $c = 2^{-p}$. Hence, Proposition 6.5 implies $T(\varepsilon) \leqslant \exp\big(C\varepsilon^{-p(p-1)}\big)$. This completes the proof of Theorem 1.4.

Remark 6.8. *Let us underline that the sign assumptions on u_0, u_1, v_0, v_1 in Theorem 1.4 can be weakened. Indeed, instead of assuming the nonnegativity of these functions, it is sufficient to require that*

$$I_{\mu_1,\nu_1^2}[u_0,u_1], I_{\mu_2,\nu_2^2}[v_0,v_1] > 0 \qquad \text{and} \qquad J_{\beta_q,\mu_1,\nu_1^2}[u_0,u_1], J_{\beta_p,\mu_2,\nu_2^2}[v_0,v_1] > 0,$$

as we have seen throughout the proof.

7. Final remarks

According to the blow-up results that are proved in this paper, it is natural to conjecture that for nonnegative and small δ_1, δ_2 (for example, at least for $n \geqslant 3$ and $0 \leqslant \delta_1, \delta_2 \leqslant (n-2)^2$ when (14) is always fulfilled) the critical curve for (1) is given by (13). Even though the existence of global in time small data solutions in the supercritical case is an open problem, some partial results for the single semilinear equation (2) in the case $\delta = 1$ (cf. [28, 29]) suggest the likelihood and plausibility of this conjecture.

In the case in which instead of scale-invariant damping terms (in the massless case though) we consider time-dependent coefficients for the damping terms in the scattering case (see [41, 42, 43] for the classification of a damping term for a wave model with time-dependent coefficient), the presence of these damping terms has no influence on the critical curve. Indeed, in a series of forthcoming papers [32, 33, 34] several blow-up results for weakly coupled systems of damped wave equations in the scattering case with different type of nonlinearities are proved. In particular, in the case of power nonlinearities the corresponding critical curve will be exactly the same one as for the weakly coupled system of semilinear not-damped wave equations with the same nonlinearities, that is, (5). This fact proves, once again, how the time-dependent and scale-invariant coefficients for lower order terms in a wave model make it a threshold model between "parabolic-like" and "hyperbolic-like" models. A further peculiar characteristic of scale-invariant models is that the multiplicative constants in the time-dependent coefficients (that is, $\mu_1, \mu_2, \nu_1^2, \nu_2^2$ for the weakly coupled system in (1)) determine the analytic expression of the critical condition for the exponents of the nonlinear terms with the presence of shifts in comparison to the corresponding critical condition for the related semilinear wave or damped wave model.

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