General properties of slant submanifolds in contact metric manifolds*

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1 Slant submanifolds of almost contact manifolds

B.Y. Chen' s concept of a slant submanifold can be translated into the context of contact metric geometry in a very natural fashion. In this chapter we shall discuss the basic facts concerning this variant of the theory.

Our standard reference for contact geometry is Blair's book [2], to which we refer the reader for the terminology, the notation and the relevant facts.

Let M be an almost contact metric manifold with structure (φ, ξ, η, g). By a *slant submanifold* of M, we shall mean an immersed submanifold N such that for any $x \in N$ and for any tangent vector $X \in T_xN$, linearly independent on ξ, the angle between φX and T_xN is a constant $\theta \in [0, \frac{\pi}{2}]$, called the *slant angle* of N in M.

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Like in complex geometry, when the ambient manifold is a *contact metric* manifold, for $\theta = \frac{\pi}{2}$ one recovers the notion of *anti-invariant* submanifold. For $\theta = 0$, this class coincides with that of *invariant* submanifolds, i.e. those for which each tangent space of the submanifold is invariant under φ. We remark that it is known that such a submanifold must be tangent to the Reeb vector field ξ (see [2, Section 8.1], p. 152). We shall see below that this property also holds in the larger class of non anti-invariant slant submanifolds.

We shall denote by $\bar{\nabla}$ the Levi-Civita connection of the ambient manifold, by ∇ the corresponding connection relative to the metric induced on a submanifold, while the second fundamental form will be denoted by α.

We shall also denote the kernel of η by D, which is a distribution on M of rank $\dim(M) - 1$. Using the same notation as in the complex case, for every tangent vector $X \in TN$ we write

$$\varphi X = PX + FX$$

where PX is tangential and FX is normal to the submanifold. Then $P : TN \to TN$ is a skew-symmetric $(1,1)$ tensor field on N with respect to the induced metric. We shall also denote by Q the symmetric operator P^2.

We begin by discussing the following basic result showing that the class of non anti-invariant slant submanifolds of a given almost contact metric manifold splits into two sub-classes, characterized by the position of the characteristic vector field ξ with respect to the submanifold (cf. [11]):

Theorem 1.1 *Let N be an immersed slant submanifold of the almost contact metric manifold M with structure tensors (φ, ξ, η, g). Let $n = \dim(N)$. Assume that N is not anti-invariant. Then*

$$n \textit{ is odd} \iff \xi \textit{ is tangent to } N$$

$$n \textit{ is even} \iff \xi \textit{ is normal to } N.$$

Proof For every point $x \in N$, the orthogonal complement $H \subset T_xN$ of $\mathrm{Ker}(Q_x)$ is even dimensional. Observe that if $X \in \mathrm{Ker}(Q_x)$, then φX is normal to N. By definition of a slant submanifold, this forces that X be a scalar multiple of ξ_x, because we are assuming that the slant angle $\theta \neq \frac{\pi}{2}$. Thus we have proved that

$$\mathrm{Ker}(Q_x) \subset \mathbb{R}\xi_x.$$

Now, if n is odd, then $\mathrm{Ker}(Q_x) \neq \{0\}$ for every $x \in N$, which yields that ξ is everywhere tangent to N. If n is even, we must have $\mathrm{Ker}(Q_x) = \{0\}$ for all $x \in N$. Fix x and consider an eigenspace H of Q_x relative to the eigenvalue λ. By definition of the slant angle, for every non null $X \in H$ we have:

$$\cos\theta = \frac{||PX||}{||\varphi X||} = \sqrt{-\lambda}\frac{||X||}{||\varphi X||}. \tag{1}$$

On the other hand, since $\dim(H) \geq 2$, H contains some non null $X \in H$ belonging to D_x, for which $||\varphi X|| = ||X||$. Subsituting in (1) yields $\lambda = -\cos^2\theta$. We have thus showed that $Q_x = -\cos^2\theta\,\mathrm{Id}$ and moreover, coming back again to (1) we have that $||\varphi X|| = ||X||$ for every $X \in T_xN$, which implies that $T_xN \subset D_x$. We conclude that ξ is everywhere orthogonal to N. □

We remark that the assumption that N is not anti-invariant in the above result is essential. A significant example is provided by a well-known result of Blair, who classified the contact metric manifolds of dimension at least five, whose curvature tensor annihilates ξ (i.e. $R(X,Y)\xi = 0$ for every tangent vector fields X, Y). The simply connected, complete ones are the Riemannian products

$$M = \mathbb{R}^{n+1} \times \mathbb{S}^n(4), \quad n > 1$$

where $\mathbb{S}^n(4)$ denotes a sphere endowed with a standard metric of constant sectional curvature 4. See [2], Theorem 7.5. Then, it turns out that both the standard immersions of $\mathbb{R}^{n+1}$ and of $\mathbb{S}^n(4)$ are anti-invariant in M. Moreover, for the first one ξ is everywhere tangent, while for the second one ξ is always normal, so that letting $n > 1$ vary, we provide a series of counterexamples to the equivalences in Theorem 1.1.

For a submanifold N tangent to ξ, as a further application of formula (1) the following characterization is readily verified, involving the symmetric tensor Q and the normal bundle valued 1-form $F : TN \to TN^\perp$ (see [3]):

Theorem 1.2 *Let N be a submanifold of an almost contact metric manifold M. Assume that ξ is tangent to N. Then the following are equivalent:*

a) N is slant in M with slant angle θ;

b) $Q = -\cos^2\theta(I - \eta \otimes \xi)$;

c) For every unit vector tangent to N and orthogonal to ξ one has

$$||PX|| = \cos\theta;$$

d) For every unit vector tangent to N and orthogonal to ξ one has

$$||FX|| = \sin\theta.$$

We remark that, in the general context of almost contact metric manifolds, one can provide simple examples showing that both possibilities regarding the position of ξ with respect to a slant submanifold can occur. Namely, given any almost Hermitian manifold (M, J, g_o), the product $M \times \mathbb{R}$ carries a standard almost contact metric structure (φ, ξ, η, g), where:

$$\varphi(X, a\frac{d}{dt}) = (JX, 0),\ \xi = (0, \frac{d}{dt}),\ \eta = dt,$$

g being the product metric of g_o and the standard metric on the real line.

Now, given any θ-slant submanifold N of M, it is not difficult to verify that $N \times \{0\}$ and $N \times \mathbb{R}$ are both θ-slant in $M \times \mathbb{R}$ (cf. [11]). More generally, the same is true if instead of the product metric one considers a warped product metric g on $M \times I$, where $I \subset \mathbb{R}$ is an open interval, namely $g = \lambda^2\pi_1^* g_o + \pi_2^* dt \otimes dt$, where $\lambda : I \to \mathbb{R}$ is a smooth positive function and $\pi_1 : M \times I \to M$ and $\pi_2 : M \times I \to I$ are the canonical projections (see [6]).

Explicit examples of slant submanifolds (most of them in the Sasakian space form $\mathbb{R}^5$) are exhibited in [3]. Other examples and some general results concerning stant submanifolds of some particular classes of almost contact metric manifolds can be found in the recent papers [6], [7] by de Candia and Falcitelli.

We report here the following fact concerning even dimensional submanifolds (it is proved in [7] for the class of $C_5 \oplus C_{12}$-almost contact metric manifolds according to Chinea-Gonzalez classification scheme [5]):

Theorem 1.3 *Let $(M, \varphi, \xi, \eta, g)$ an almost contact metric manifold and assume that φ is η-parallel, i.e.*

$$g((\bar{\nabla}_X \varphi)Y, Z) = 0$$

for every X, Y, Z vector fields orthogonal to ξ.

Let N be an even dimensional θ-slant submanifold of M, $\theta \neq \frac{\pi}{2}$. Then M induces on N an almost Kähler structure (J, g) where $J = \sec\theta P$.

Proof We know that ξ is normal to N. Moreover, $Q = -\cos^2\theta \mathrm{Id}$. Hence $J = \sec\theta P$ is an almost complex structure on N, which is Hermitian with respect to the induced metric. Moreover, by the η-parallelism of φ, for every X, Y, Z vector fields tangent to N we get:

$$g(\bar{\nabla}_X PY, Z) + g(\bar{\nabla}_X FY, Z) - g(\varphi\nabla_X Y, Z) - g(\varphi\alpha(X, Y), Z) = 0,$$

yielding

$$g((\nabla_X P)Y, Z) = g(A_{FY}X, Z) - g(A_{FZ}X, Y).$$

It follows that

$$\mathfrak{S}_{X,Y,Z} g((\nabla_X J)Y, Z) = 0,$$

where $\mathfrak{S}$ is a cyclic sum, and this ensures that the almost Hermitian structure (J, g) is almost Kähler. $\square$

2 Slant submanifolds of contact metric manifolds

From now on we shall consider the case when the ambient manifold is a contact metric manifold.

Theorem 2.1 *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold. Every non anti-invariant slant submanifold N of M is tangent to ξ. Moreover, the restriction of η to N is again a contact form and N inherits canonically a contact metric structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$, where*

$$\bar{\varphi} := \sec\theta P,\ \bar{\xi} := \sec\theta\xi,\ \bar{\eta} := \cos\theta\,\eta,\ \bar{g} := \cos^2\theta\, g. \tag{2}$$

Proof If ξ were normal to N, from the formula

$$\bar{\nabla}_X \xi = -\varphi X - \varphi h X, \quad h := \frac{1}{2}\mathcal{L}_\xi \varphi \tag{3}$$

which is valid in the ambient manifold (cf. [2, Lemma 6.2]), for every X and Y vector fields tangent to N we would have:

$$g(A_\xi X, Y) = g(\varphi X, Y) + g(\varphi h X, Y),$$

A_ξ being the Weingarten operator in the direction of ξ. But since φh and A_ξ are both symmetric operators, this would imply $g(\varphi X, Y) = 0$ identically, yielding that N is anti-invariant against the assumption. Hence according to Theorem 1.1, ξ must be tangent to N. Concerning the last statement, denoting by the same symbols the restrictions of η, ξ and g to the submanifold, setting $\bar\varphi := \sec\theta\, P$, it is easy to check that $(\bar\varphi, \xi, \eta, g)$ is an almost contact metric structure satisfying

$$d\eta = \cos\theta\bar\Phi,$$

where $\bar\Phi$ is its fundamental 2-form, i.e. it is a $\cos\theta$-homothetic contact metric structure on N. This implies the last claims. $\square$

The next proposition provides a formula linking the operator h of the ambient manifold and the analogous operator $\bar h$ relative to the induced contact metric structure.

Proposition 2.2 *Let N be a θ-slant, non anti-invariant submanifold of a contact metric manifold $(M, \varphi, \xi, \eta, g)$. Then for every X, Y vectors tangent to N and orthogonal to ξ we have:*

$$g(hX, Y) = \cos^2\theta g(\bar hX, Y) - \sin^2\theta g(X, Y) - g(\alpha(X, \xi), FY).$$

In particular,

$$g(\alpha(X, \xi), FY) = g(\alpha(Y, \xi), FX)$$

holds with the same assumption on X, Y.

Proof We shall use (3) and the analogous formula for the induced contact metric structure (2) on N. Observing that the Levi-Civita connections of $g|_N$ and $\bar g$ coincide, we have:

$$\begin{aligned} g(hX, Y) &= g(\varphi\bar\nabla_X\xi, Y) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) - g(\nabla_X\xi, PY) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) - \cos\theta g(\nabla_X\bar\xi, PY) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) + \cos\theta g((\bar\varphi + \bar\varphi\bar h)X, PY) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) + g(PX, PY) + g(P\bar hX, PY) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) + \cos^2\theta g(\bar hX, Y) - \sin^2\theta g(X, Y), \end{aligned}$$

where to deduce the last equality we used c) in Theorem 1.2. The last claim follows since h and $\bar h$ are both symmetric operators. $\square$

As another application of Theorem 1.2, we prove a result concerning contact totally umbilical submanifolds. Recall that a submanifold, tangent to ξ, of a contact metric manifold, is said to be *contact totally umbilical* provided the second fundamental form satisfies (cf. e.g. [15]):

$$\alpha(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}H + \eta(X)\alpha(Y,\xi) + \eta(Y)\alpha(X,\xi). \tag{4}$$

Here H is the mean curvature normal vector field. If in addition $H = 0$, one speaks of a *contact totally geodesic* submanifold.

Given a proper slant submanifold, we shall consider the orthogonal splitting

$$TN^{\perp} = F(TN) \oplus E = F(\bar{D}) \oplus E$$

of the normal bundle, where $\bar{D}$ denotes the induced contact distribution on the slant submanifold. Of course this is meaningful because $F_x : \bar{D}_x \to T_xN^{\perp}$ is injective for each point of the submanifold.

Theorem 2.3 *Let* (M,φ,ξ,η,g) *be a contact metric manifold of dimension* $2m+1$*, whose almost* CR *structure* $(D,\varphi|_D)$ *is integrable, i.e.* M *is a strongly pseudoconvex* CR *manifold. Let* N *be a contact totally umbilical proper slant submanifold of* M*.*

Then N *is contact totally geodesic provided* $\dim(N) = m+1$ *or* $D_XH \in \Gamma(E)$ *for every vector field tangent to* N *and orthogonal to* ξ*.*

Proof First of all, we recall that for contact metric manifolds, the integrability condition for $(D,\varphi|_D)$ is equivalent to φ being η-parallel; indeed, one has the following formula for the covariant derivative of φ:

$$(\bar{\nabla}_X\varphi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX),$$

see [2, Theorem 6.7]. This implies that for every X, Y vector fields tangent to N and orthogonal to ξ (i.e. sections of $\bar{D}$), the vector field $(\bar{\nabla}_X\varphi)Y$ is tangent to N. Using this, by a standard computation similar to the case of submanifolds of Kähler manifolds, one can derive the following formula for the covariant derivative of F, with the same assumption on the vector fields X, Y:

$$(\nabla_XF)Y = f\alpha(X,Y) - \alpha(X,PY). \tag{5}$$

Here, as usual, for every normal vector field ν to N we set

$$\varphi\nu = t\nu + f\nu$$

with $t\nu$ tangent resp. $f\nu$ normal to N.

Now, assuming (4), (5) yields, for any local unit vector field X tangent to N and orthogonal to ξ:

$$(\nabla_{PX}F)X = -g(PX, PX)H = -\cos^2\theta\, H. \tag{6}$$

Observe now that the left hand side of this equation is orthogonal to FX; indeed, by the condition c) in Thorem 1.2 we have, for every Z, W tangent to N and orthogonal to ξ:

$$g(FZ, FW) = \sin^2\theta g(Z, W);$$

using this, we get, assuming $||X|| = 1$:

$$\begin{aligned} g((\nabla_{PX}F)X, FX) = g(D_{PX}FX, FX) - g(F\nabla_{PX}X, FX) = \\ = -\sin^2\theta g(\nabla_{PX}X, X) = 0. \end{aligned}$$

As a consequence, we obtain from (6) that

$$g(H, FX) = 0$$

for every X orthogonal to ξ and tangent to N, showing that $H \in \Gamma E$. If $\dim(N) = m + 1$, this suffices to prove the result, since in this case the subbundle E is trivial, because N has codimension m. Now assume that $D_X H \in \Gamma(E)$ for all sections of $\bar{D}$. Taking the inner product of both sides of (6) with H, one gets

$$-\cos^2\theta g(H, H) = g(D_{PX}FX, H) - g(F(\nabla_{PX}X), H) = -g(FX, D_{PX}H) = 0$$

due to our assumption. $\Box$

In [8] a similar result has been proved by Gupta in the case where M is a Kenmotsu manifold.

Corollary 2.4 *Let M be a Sasakian space form with φ-sectional curvature c. Then every totally contact umbilical proper slant submanifold N of M is totally contact geodesic, provided* $\dim(N) > 3$ *or* $c = 1$*.*

PROOF In this case we shall verify that actually $D_X H = 0$ for every X orthogonal to ξ. This can be proved by using the Codazzi equation. Indeed, observe first that for every X, Y, Z tangent to N and orthogonal to ξ:

$$\alpha(\nabla_X Y, Z) = \{g(\nabla_X Y, Z) - \eta(\nabla_X Y)\eta(Z)\}H = g(\nabla_X Y, Z)H$$

hence the Codazzi equation reads:

$$g(Y,Z)D_XH - g(X,Z)D_YH = (\bar{R}(X,Y)Z)^{\perp}$$

where $\bar{R}$ is the curvature tensor of M. Using the explicit expression of $\bar{R}$ (cf. [2, Theorem 7.19]) we thus obtain:

$$g(Y,Z)D_XH - g(X,Z)D_YH = \frac{c-1}{4}\{g(Z,PY)FX - g(Z,PX)FY + 2g(X,PY)FZ\}.$$

Choosing now $Y = Z$ of length one and orthogonal to X yields:

$$D_XH = \frac{3}{4}(c-1)g(X,PY)FY.$$

Hence the claim follows in the case $c = 1$. If $c \neq 1$, assuming $\dim(N) > 3$ we can choose Y so that Y is also orthogonal to PX, and the same formula yields $D_XH = 0$. $\square$

3 The K-contact case

In this section we consider submanifolds of K-contact metric manifolds, i.e. contact metric manifolds whose Reeb vector field ξ is Killing. This is equivalent to requiring that the tensor field h in (3) vanishes. This class contains in particular the class of Sasakian manifolds.

We shall prove the following characterization of slant submanifolds purely in terms of curvature (cf. [11]):

Theorem 3.1 *Let N be a submanifold of a K-contact metric manifold M with structure (φ, ξ, η, g). Assume that N is tangent to ξ. Fix $\theta \in [0, \frac{\pi}{2}]$; then the following conditions are equivalent:*

a) N is slant with slant angle θ,

b) For every $x \in N$ the sectional curvature, with respect to the induced metric, of every 2-plane containing ξ_x equals $\cos^2\theta$.

Moreover, every non anti-invariant slant submanifold of M is itself a K-contact metric manifold with respect to the induced contact metric structure.

Proof For every $x \in N$, any 2-plane containing ξ_x is spanned by ξ_x and some unit vector vector X orthogonal to ξ; the corresponding sectional curvature $K(X,\xi)$ is related to the sectional curvature $\bar{K}(X,\xi)$ of the same 2-plane computed in the ambient manifold M by the Gauss equation:

$$K(X,\xi) = \bar{K}(X,\xi) + g(\alpha(X,X),\alpha(\xi_x,\xi_x)) - ||\alpha(X,\xi_x)||^2.$$

Now, being M a K-contact metric manifold, it is known that $\bar{K}(X,\xi_x) = 1$; moreover, we also have $\alpha(\xi,\xi) = 0$, because $\bar{\nabla}_\xi \xi = 0$. Hence the above formula can be rewritten

$$K(X,\xi) = 1 - ||\alpha(X,\xi_x)||^2.$$

On the other hand, remembering (3), in this case we have:

$$\alpha(X,\xi_x) = -FX.$$

In conclusion:

$$K(X,\xi) = 1 - ||FX||^2.$$

Now the equivalence of a) and b) is clear taking into account the characterization of slant submanifolds provided by Theorem 1.2. Finally, concerning the last claim, observe that (3) also yields

$$\nabla_X \xi = -PX$$

for every vector field tangent to N, which implies that the restriction of ξ to N is again a Killing vector field, since P is skew-symmetric (alternatively, one can infer that the flow of ξ on N consists of local isometries). Hence, assuming that N is slant, the same is true for the Reeb vector field $\bar{\xi}$ of the induced contact metric structure, which is thus K-contact. □

Corollary 3.2 *Any torus admits no slant, non anti-invariant, isometric immersions into any K-contact metric manifold.*

This is due to the fact that a torus cannot carry a K-contact metric structure [14].

Next we consider regular K-contact manifolds. Recall that a contact manifold (M,η) is called regular provided the Reeb vector field is, i.e. it determines a regular 1-dimensional foliation on M, so that the space $B =$

M/ξ of maximal integral curves of ξ is a manifold. When M carries a K-contact metric g associated to η, then being $\mathcal{L}_\xi\varphi = \mathcal{L}_\xi g = 0$, g induces in a natural way a metric g' on M/ξ and φ also descends to an almost complex structure J.

Denoting by $\pi : M \to B$ the canonical projection, it turns out by construction that π is a Riemannian submersion with $Ker(d\pi)_x = \mathbb{R}\xi_x$ for every $x \in N$, and

$$d\pi \circ \varphi = J \circ d\pi$$

(see also [13]). Moreover, it is proved by Ogiue in [12] that (B, J, g') is an almost Kähler manifold. If M is Sasakian, then B is Kähler.

A remarkable case is when $M = M(c)$ is a simply connected, complete Sasakian space form; then B is either a flat Euclidean space $\mathbb{C}^m$ (when $c = -3$), a complex hyperbolic space $\mathbb{C}H_m$ with negative constant holomorphic sectional curvature ($c < -3$), or a complex projective space $\mathbb{C}P_m$ with positive constant holomorphic sectional curvature ($c > -3$) (see [2] or [9] for details).

The following result relates slant submanifolds of M with slant submanifolds of B. In particular, it provides a natural way to produce examples of slant submanifolds of the Sasakian space forms, by "lifting up" slant submanifolds of the corresponding complex space form. Another approach for constructing examples in this context has been developed by Cabrerizo, Carriazo, L. M. Fernandez and M. Fernandez in [4], who established a general existence and uniqueness result for slant immersions in Sasakian space forms, along the lines of the corresponding result of Chen-Vrancken for complex space forms.

Theorem 3.3 *Let M be a regular K-contact manifold canonically fibering onto the almost Kähler manifold B, with projection $\pi : M \to B$. Fix $\theta \in [0, \frac{\pi}{2})$. Then:*

a) If S is an embedded θ-slant submanifold of B, then $\pi^{-1}(B)$ is a θ-slant submanifold of M.

b) If N is a compact θ-slant submanifold of M, then $\pi(N)$ is a θ-slant submanifold of B.

PROOF a) Since π is a surjective submersion, it is known that $N = \pi^{-1}(S)$ is an embedded submanifold of M, having the same codimension as

S. Clearly, N is tangent to ξ, because at each point $x \in N$ the tangent space T_xN is $(d\pi)_x^{-1}(T_{\pi(x)}S)$. Moreover, observe that for every normal vector $\nu \in T_xN^{\perp}$, we have that $(d\pi)_x(\nu)$ is normal to S, because for every $X \in T_xN$ orthogonal to ξ, from $g(\nu, X) = 0$ it follows $g'((d\pi)_x\nu, (d\pi)_xX) = 0$, being π a Riemannian submersion.

Now, let X a unit vector tangent to T_xN and orthogonal to ξ. Then from

$$\varphi X = PX + FX$$

we get

$$J(d\pi)_xX = (d\pi)_x(PX) + (d\pi)_xFX$$

where $(d\pi)_x(PX)$ is tangent and $(d\pi)_xFX$ is orthogonal to S, yielding

$$||PX|| = ||(d\pi)_x(PX)|| = \cos\theta$$

where the last equality holds because S is θ-slant.

b) Since N is tangent to ξ, and π is a submersion satisfying $Ker(d\pi_x) = \mathbb{R}\xi_x$, for every $x \in M$, we have that the restriction of $d\pi$ to T_xN has rank $\dim(N) - 1$. Hence $\pi : N \to B$ is a smooth map of constant rank; being N compact, it is known that its image $\pi(N)$ is a submanifold of B (cf. [1, Theorem 3.5.18]). The verification that $S = \pi(N)$ is θ-slant is based on the same argument used in the proof of a), taking into account that at each point $\pi(x)$ of S we have $T_{\pi(x)}S = (d\pi)_x(T_xN)$. $\square$

Observe that, under the assumption of b), one deduces that N is also a regular contact manifold. This provides a generalization of a result by Harada [10] concerning invariant submanifolds of regular Sasakian manifolds.

Corollary 3.4 *For every $m \geq 2$, the Sasakian space form $\mathbb{R}^{2m+1}$ of φ-sectional curvature -3 admits no compact proper slant submanifold.*

This holds since $\mathbb{R}^{2m+1}$ fibers onto the flat complex Euclidean space $\mathbb{C}^m$, and Chen-Tazawa's non compactness result for slant submanifolds of $\mathbb{C}^m$ applies.

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