Expected vs. Real Transaction Costs in European Option Pricing: a Possible Model

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Abstract

As an application and extension of some previous results contained in [1], we face up the problem of the option pricing in presence of transaction costs and hence in the framework of incomplete markets. The model proposed herein passes through defining properly the expected transaction costs, opposite to the real transaction costs in trading. The analysis is carried out both in the discrete and the continuous case and leads to suitable modifications of Cox-Ross-Rubinstein and Black-Scholes formulas. An application to a specific case referred to real market data at the end of the paper seems to validate our approach.

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1. Introduction

In a recent paper [1] the authors have carried out a detailed analysis about the possibility of expressing classical formulas in option pricing theory by using probability measures different from the risk-neutral measure $\mathbb{Q}$, though linked profoundly to it through suitable Radon-Nikodym derivatives.
The advantage of this procedure, successfully performed both in the discrete and

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the continuous case and essentially based upon a change of numèrarie, is twofold: on one hand, from a theoretical point of view, it allows a deeper understanding of the financial meaning of some terms appearing in classical pricing formulas; on the other hand and in view of a more applicative and numerical approach, concerning mainly the Cox-Ross-Rubinstein model, it provides a closed non iterative formula which prices the value of a derivative security at each node of the binomial tree, with a considerable reduction of computations when compared with the standard "complete backward procedure" in the binomial tree.

In the present paper we show that, quite surprisingly, adopting the new measures offers an additional benefit in the more challenging framework of incomplete markets when dealing with the thorny issue of the transaction costs, allowing to set up a possible approach to this model.

It seems important, in this respect, to remind that the transaction costs are often taken into account in pricing theory: see, for instance, the seminal papers [3], [5] and [17] and some additional contributions in [8], [6], [10], [13], [14], [15], [9], [19] and [22].

However, as far as we know, our approach seems to be new, not falling, for instance, within the arguments and methods adopted in the above quoted papers.

We remark that inserting transaction costs in a pricing model causes non trivial drawbacks in the consistent of the model itself; to borrow H. Leland's words in [15]: "Transactions costs invalidate the Black-Scholes arbitrage argument for option pricing, since continuous revision implies infinite trading. Discrete revision using Black-Scholes deltas generates errors which are correlated with the market, and do not approach zero with more frequent revision when transactions costs are included".

Put another way, transaction costs cause dynamic incompleteness; generally, they consist of commissions (and similar payments) and ask/bid spreads. The ask/bid spread is the difference beetween the ask and bid price, i.e., between the highest price a buyer will pay for a commodity and the lowest price a seller will, in turn, accept for a commodity. The average of bid and ask price is the market price of the stock, while the half of the ask/bid spread is the trading cost of the unit share.

The impact of the transaction costs plays a crucial rule in liquidity and future prices of a stock traded, as explained in [10]: "liquidity is an important issue in stock markets. In fact, a liquidity of a stock traded on the stock exchange is measured by the cost of its trading. For the purposes of market participants, the correct way to view liquidity should imply the possibility of sufficiently accurate forecasting the stock price change caused by the trade initiator and estimating the transaction cost.

Transaction costs (trading costs) are widely recognized as an important factor which determines the financial investment performance, being, in addition, a substantial

component of realistic models of the stock market microstructure".
Trading costs can be explicit, e.g., broker commissions, or implicit, i.e., depending on the trading rate, the transaction volume, the drift and the volatility of the stock price, etc.
By virtue of these relationships, the expectation of transaction costs helps the forecasting of future stock prices and viceversa.
Turning back to our paper, we proceed along the following lines: Section 2 presents a rather complete survey of the main results stated in [1] and which are fully involved in the new analysis throught the paper; at the same time, some notation and properties coming from stochastic calculus and largely concerning conditional expectations under a change of measure are conveniently recalled in order to make the treatment as self-contained as possible.
Section 3 is devoted to build up a possible option pricing theory with transaction costs in the discrete case, starting, for simplicity, from the one-period model and, later, passing to the more general case of the binomial tree with an arbitrarily large number of steps.
Crucial is the definition of the expected endogenous transaction cost coefficient (opposite to an exogenous similar coefficient) which comes to the fore quite naturally if one wants to fulfill the martingale property of the discounted stock price, net of transaction costs, with respect to a suitable probability measure (see (17)). Subsequently, the standard hedging technique is properly applied (with the necessary changes somewhere) in order to price options, with the doubtless advantage of having at disposal a closed non iterative formula, valid at each node of the binomial tree.
As the reader may easily figure out, the passage to the continuous case, as discussed in Section 4, does not present particular difficulties: after introducing the continuous version of the expected transaction cost coefficient (endogenous/exogenous), the martingale property with respect to a suitable probability measure and the routine hedging procedure leads, with the help of Ito formula, to a modified Black-Scholes equation whose solution may be achieved through classical results about parabolic PDEs or through a version of the Feynman-Kac formula, as shown at the end of the Section. In both cases (discrete and continuous), it is also proved that options are more valuable than their no transaction cost counterparts, as expected.
Lastly, in Section 5, in order to test our methodology, we turn to consider a concrete case involving real market data: the results obtained herein (also in comparison with those coming out from Leland model) seem to validate our approach, stimulating, in this way, further investigations in this direction.
As already said, the notation used throughout the paper is quite standard and, any-

way, encompassed in Section 2 with one exception: $\widetilde{c}_t = \widetilde{c}(t, S_t)$ and $\widetilde{c}_n$ will denote the price of an European call option in presence of transaction costs in the continuous and discrete setting, respectively, opposite to the standard $c_t = c(t, S_t)$ and c_n, when such costs are overlooked. Occasionally, some new symbol or terminology will be clarified at each occurrence.

2. Notations and preliminar results

In this Section, for the reader's convenience, we summarize the main results of our recent work [1]. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and consider a simple model of market where a risky asset S and a bond B with risk-free interest rate $r \geq 0$ are available. Consider the binomial model ($N = 1$) in which the stock price, starting from an initial value $S_0 > 0$, evolves randomly at time 1 according to

$$S_1 = \begin{cases} S_1^u := uS_0, \\ S_1^d := dS_0, \end{cases} \tag{1}$$

where d, u are the up and down factors with $d < 1 + r < u$. The main goal of the model is to provide a portfolio replication strategy by using a derivative security. In particular we consider a call option, i.e., a contract that pays $(S_1 - X)^+ := max(S_1 - X, 0)$ at maturity 1 for a fixed strike price X. By means of an hedging procedure we can compute the initial derivative price (see, e.g., [20] or [5]) obtaining

$$c_0 = \frac{1}{1+r}(c_1^u p^{\mathbb{Q}} + c_1^d(1 - p^{\mathbb{Q}})), \tag{2}$$

where $\mathbb{Q} := (p^{\mathbb{Q}}, 1 - p^{\mathbb{Q}})$ is the risk-neutral measure defined by

$$p^{\mathbb{Q}} = \frac{1 + r - d}{u - d}, \tag{3}$$

and $c_1^u = (S_1^u - X)^+$, $c_1^d = (S_1^d - X)^+$. In the interesting case $dS_0 < X < uS_0$, the above formula reduces to

$$c_0 = \frac{1}{1+r}(S_0 p^{\mathbb{Q}} u - X p^{\mathbb{Q}}),$$

which, in turn, may be written as

$$c_0 = S_0 p^{\widetilde{\mathbb{Q}}} - \frac{1}{1+r} X p^{\mathbb{Q}} \tag{4}$$

after introducing a new measure $\widetilde{\mathbb{Q}} = (p^{\widetilde{\mathbb{Q}}}, 1 - p^{\widetilde{\mathbb{Q}}})$ (see [1, Section 3.1]) defined by

$$p^{\widetilde{\mathbb{Q}}} := \frac{S_1^u}{(1+r)S_0}\, p^{\mathbb{Q}} = \frac{u}{1+r}\, p^{\mathbb{Q}}, \tag{5}$$

or, equivalently, by the following Radon-Nikodym derivative

$$\frac{\widetilde{\mathbb{Q}}}{\mathbb{Q}} = \frac{S_1}{(1+r)S_0}.$$

The multi-period model with $N \geq 2$ may be treated analogously with the necessary modification thereof: to set the stage, for a fixed (once and for all) $n \leq N$, we have $n+1$ distinct realizations of S_n, namely

$$S_n^{\rho^{(i,n-i)}(u,d)} := u^i d^{n-i} S_0, \tag{6}$$

where $i \in \{0, 1, ..., n\}$ and $\rho^{(i,n-i)}(u,d)$ denotes any permutation with repetition of the n-tuple

$$(\underbrace{u, ..., u}_{i}, \underbrace{d, ..., d}_{n-i}).$$

In [1, Section 3] we show that the potential price c_k of the call at a fixed time $k \leq n$ is given by the random variable

$$c_k = S_k \widetilde{\mathbb{Q}}\{S_n > X | \mathcal{F}_k\} - \frac{X}{(1+r)^{n-k}} \mathbb{Q}\{S_n > X | \mathcal{F}_k\}. \tag{7}$$

In this framework the hedging strategy leads to a closed (non iterative) formula for the call option price corresponding to the initial time $k \leq n$ at each node i of the binomial tree; indeed, in [1, formula (36)] we prove the following identity

$$c_k^{\rho^{(i,k-i)}(u,d)} = S_k^{\rho^{(i,k-i)}(u,d)} \Phi(p^{\widetilde{\mathbb{Q}}}, n-k, m_k^i) - \frac{X}{(1+r)^{n-k}} \Phi(p^{\mathbb{Q}}, n-k, m_k^i), \tag{8}$$

where $\Phi(p^{\widetilde{\mathbb{Q}}}, n-k, m_k^i)$ and $\Phi(p^{\mathbb{Q}}, n-k, m_k^i)$ are the complementary binomial distributions with parameters $p^{\widetilde{\mathbb{Q}}}$ and $p^{\mathbb{Q}}$, respectively. In the equality above, for any $i \leq k$, m_k^i is a realization of the random variable m_k which denotes the minimum number of upward moves necessary for the option to end in the money at time n starting from the step k; more formally

$$m_k = \begin{cases} m_0, & if\, k = 0, \\ m_k^i \in \{0, ..., (m_0 - i)^+\} \cup \{+\infty\}, & if\, k > 0,\ 0 \leq i \leq k; \end{cases} \tag{9}$$

obviously $c_k^{\rho^{(i,k-i)}(u,d)} = 0$ if $m_k^i = +\infty$. All the details in this respect may be found in [1, Section 3.2].
In this multi-step background the right measure $\widehat{\mathbb{Q}}$ which does the job and equals $\widetilde{\mathbb{Q}}$ when $N = 1$ is given by

$$\widehat{\mathbb{Q}} := \frac{S_N}{(1+r)^N S_0}\,\mathbb{Q}, \tag{10}$$

or, again, by the following Radon-Nikodym derivative

$$Z := \frac{\widehat{\mathbb{Q}}}{\mathbb{Q}} = \frac{S_N}{(1+r)^N S_0},$$

as explained in [1, (39)] and in the subsequent discussion.
Accordingly, the stochastic process $(Z_k)_{0\leq k\leq N}$ where $Z_N = Z$ and $Z_k := \mathbb{E}_k[Z] = \frac{S_k}{(1+r)^k S_0}$ $(0 \leq k \leq N)$ is called a Radon-Nikodym process (see, e.g., [20, pp. 65-70]); it will play a fundamental role while computing conditional expectations under a change of measure. The continuous version $(Z_t)_{0\leq t\leq T}$ of this process in the time interval $[0,T]$ with $Z = \frac{e^{-rT}S_T}{S_0}$ and $Z_t = \frac{e^{-rt}S_t}{S_0}$ $(0 \leq t \leq T)$, defined in [1, Section 2], will be a helpful tool, as well.
In relation with the (discounted) stock price process the newly defined measure $\widehat{\mathbb{Q}}$ satisfies the next proposition which resumes Proposition 3.5 and properties *i*)-*iii*) of Proposition 3.2 of [1] as a particular case.

Proposition 2.1. *The following properties hold true:*

i) the discounted stock price process $\left(\frac{S_i}{(1+r)^i}\right)_{0\leq i\leq N}$ *is a* $\widehat{\mathbb{Q}}$*-submartingale;*

ii) if $r \geq 0$*, the stock price process* $(S_i)_{0\leq i\leq N}$ *is a* $\widehat{\mathbb{Q}}$*-submartingale;*

iii) for any $n \leq N$ *the (unconditional)* $\widehat{\mathbb{Q}}$*-expectation of* S_n *is given by*

$$\mathbb{E}^{\widehat{\mathbb{Q}}}[S_n] = \frac{\mathbb{E}^{\mathbb{Q}}[S_n^2]}{\mathbb{E}^{\mathbb{Q}}[S_n]},$$

i.e., by the ratio between the second and the first moment of S_n *under* $\mathbb{Q}$*.*

In the continuous setting, starting from a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, a riskless bond process $(B_t)_{0\leq t\leq T}$ (T being the corresponding maturity) and a risky asset process $(S_t)_{0,\leq t\leq T}$ satisfying

$$dB_t = rB_t dt, \quad B_0 = 1$$

and

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0$$

(as usual, $\mu \geq 0$ and $\sigma > 0$ denote respectively the drift and the volatility and $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion), the routine hedging procedure (see, e.g., [21]) leads to the classical Black-Scholes formula for the price $c_t = c(t, S_t)$ of a call option, namely to

$$c_t = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[S_T \cdot \mathbf{1}_{\{S_T>X\}}|\mathcal{F}_t] - e^{-r\tau}X\mathbb{Q}\{S_T > X|\mathcal{F}_t\} =$$

$$S_t N(d_1(\tau, S_t)) - e^{-r\tau}XN(d_2(\tau, S_t)); \tag{11}$$

here $\tau = T - t$, $\mathbb{Q}$ is the risk-neutral measure satisfying

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := exp\Big(-\int_0^T \theta\, dW_u - \frac{1}{2}\int_0^T \theta^2\, du\Big), \qquad \theta := \frac{\mu - r}{\sigma},$$

$N(\cdot)$ denotes the cumulative distribution function of a standard normal and

$$d_1(\tau, S_t) := \frac{1}{\sigma\sqrt{\tau}}\Big(ln\frac{S_t}{X} + \big(r + \frac{\sigma^2}{2}\big)\tau\Big), \qquad d_2(\tau, S_t) := d_1(\tau, S_t) - \sigma\sqrt{\tau}, \tag{12}$$

(see, e.g., [3]). As performed in [1, Section 2], passing to a new measure $\widetilde{\mathbb{Q}}$ related to $\mathbb{Q}$ by the following Radon-Nikodym derivative

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} := e^{-rT}\frac{S_T}{S_0} = exp\Big(-\frac{\sigma^2}{2}T + \sigma W_T^{\mathbb{Q}}\Big), \tag{13}$$

we can rewrite (11) in the form

$$c_t = S_t\, \widetilde{\mathbb{Q}}\{S_T > X|\mathcal{F}_t\} - Xe^{-r\tau}\, \mathbb{Q}\{S_T > X|\mathcal{F}_t\}. \tag{14}$$

The main properties of $\widetilde{\mathbb{Q}}$ with respect to the (discounted) stock price process are listed in the next proposition, some of them being analogous to those enjoined by $\widehat{\mathbb{Q}}$ in the discrete case. For further details we refer to [1, Proposition 2.1]).

Proposition 2.2. *The following properties hold true:*

i) the discounted process $(e^{-rt}S_t)_{0\leq t\leq T}$ *is a* $\widetilde{\mathbb{Q}}$*-submartingale;*

ii) if $r \geq 0$*, the process* $(S_t)_{0\leq t\leq T}$ *is a* $\widetilde{\mathbb{Q}}$*-submartingale;*

iii) the Ito process

$$W_t^{\widetilde{\mathbb{Q}}} := W_t^{\mathbb{Q}} - \int_0^t \sigma \, du \qquad (0 \le t \le T) \tag{15}$$

is a $\widetilde{\mathbb{Q}}$-Brownian motion. In particolar, the process $(S_t)_{0 \le t \le T}$ obeys the following $\widetilde{\mathbb{Q}}$-dynamics

$$dS_t = (r + \sigma^2) S_t dt + \sigma S_t dW_t^{\widetilde{\mathbb{Q}}}; \tag{16}$$

iv) for any $t \in [0, T]$ the (unconditional) $\widetilde{\mathbb{Q}}$-expectation of S_t is given by

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t] = \frac{\mathbb{E}^{\mathbb{Q}}[S_t^2]}{\mathbb{E}^{\mathbb{Q}}[S_t]},$$

i.e., by the ratio between the second and the first moment of S_t under $\mathbb{Q}$;

v) $\widetilde{\mathbb{Q}}\{S_T > X | \mathcal{F}_t\} = N(d_1(\tau, S_t))$.

The formulation (14) allows a better understanding of the terms $N(d_1(\tau, S_t))$ and $N(d_2(\tau, S_t))$, as already remarked and developed in [18]. In the approach addressed in [1], however, the change of measure from the risk-neutral probability $\mathbb{Q}$ to $\widetilde{\mathbb{Q}}$ passes through a change of numèraire (see, e.g., [21, Chapter IX] and [11]); specifically, when dealing with $\mathbb{Q}$, the numèraire is the money market account e^{rt}, whereas, for the measure $\widetilde{\mathbb{Q}}$, the numèraire is intended to be the stock price itself S_t. For further details we remand to [1, Remark 2.2 and 2.3].

3. The Binomial Model with Transaction Costs

Consider the stock price process $(S_n)_{0 \le n \le N}$ defined by (6) in the multi-period binomial model with $N \ge 2$. Let us recall that, according to [1, Appendix A], we may write down

$$S_n := S_0 \, U_1 \cdots U_n \qquad (1 \le n \le N),$$

where the U_i's are i.i.d. random variables with Bernoulli distribution with parameter $p^{\mathbb{Q}}$ and range $\{u, d\}$. Henceforth we shall always denote by U a random variable with such features. The following result is crucial in our investigation.

Theorem 3.1. *Let $\widehat{\mathbb{Q}}$ be the probability measure defined in (10) and set*

$$\beta := \frac{\mathbb{E}^{\mathbb{Q}}[U^2]}{(\mathbb{E}^{\mathbb{Q}}[U])^2} = \frac{u^2 p^{\mathbb{Q}} + d^2(1 - p^{\mathbb{Q}})}{(1+r)^2}. \tag{17}$$

Then the process $\left(\frac{S_n}{\beta^n (1+r)^n}\right)_{0 \le n \le N}$ is a $\widehat{\mathbb{Q}}$-martingale.

Proof. Let us fix k and n such that $0 \le k \le n \le N$; according to [20, Lemma 3.2.6] and adopting the notation just after formula (10), one has

$$\mathbb{E}_k^{\widehat{\mathbb{Q}}}\left[\frac{S_n}{\beta(1+r)^n}\right] = \frac{\mathbb{E}_k^{\mathbb{Q}}\left[\frac{S_n\,Z_n}{\beta(1+r)^n}\right]}{Z_k} = \frac{\mathbb{E}_k^{\mathbb{Q}}[S_n^2]}{(1+r)^{2n-k}S_k} = \frac{S_k^2(\mathbb{E}^{\mathbb{Q}}[U^2])^{n-k}}{(1+r)^k S_k (1+r)^{2(n-k)}} =$$

$$\frac{S_k}{(1+r)^k}\cdot\left(\frac{\mathbb{E}^{\mathbb{Q}}[U^2]}{(1+r)^2}\right)^{n-k} = \frac{S_k}{(1+r)^k}\cdot\left(\frac{\mathbb{E}^{\mathbb{Q}}[U^2]}{(\mathbb{E}^{\mathbb{Q}}[U])^2}\right)^{n-k} = \frac{S_k}{(1+r)^k}\cdot\beta^{n-k},$$

and the proof is complete. □

The role of the factor β is highlighted in the next proposition.

Proposition 3.2. *The following properties hold true:*

i) $1 \le \beta < \frac{u}{1+r}$*;*

ii) $\beta = \frac{\mathbb{E}_{n-1}^{\widehat{\mathbb{Q}}}[S_n]}{\mathbb{E}_{n-1}^{\mathbb{Q}}[S_n]} \qquad (1 \le n \le N)$*;*

iii) $\beta^n = \frac{\mathbb{E}^{\widehat{\mathbb{Q}}}[S_n]}{\mathbb{E}^{\mathbb{Q}}[S_n]} \qquad (1 \le n \le N)$*;*

iv) $\beta = \frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{n+1}^2}{S_n^2}\right]}{\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{n+1}}{S_n}\right]\right)^2} \qquad (0 \le n \le N-1).$

Proof. i) It is quite obvious that $1 \le \beta$ since $\mathbb{E}^{\mathbb{Q}}[U^2] \ge (E^{\mathbb{Q}}[U])^2$; furthermore the equality $up^{\mathbb{Q}} + d(1-p^{\mathbb{Q}}) = 1+r$ together with the definition (17) implies easily the upper estimate, as well.

ii) By virtue of [20, Lemma 3.2.6] and from the definition of the (conditional) $\mathbb{Q}$-expectation, for any n satisfying $1 \le n \le N$ we plainly compute

$$\frac{\mathbb{E}_{n-1}^{\widehat{\mathbb{Q}}}[S_n]}{\mathbb{E}_{n-1}^{\mathbb{Q}}[S_n]} = \frac{\mathbb{E}_{n-1}^{\mathbb{Q}}[S_n Z_n]}{Z_{n-1}(1+r)S_{n-1}} = \frac{\mathbb{E}_{n-1}^{\mathbb{Q}}[S_n^2]}{(1+r)^2 S_{n-1}^2} = \frac{\mathbb{E}^{\mathbb{Q}}[U^2]}{(1+r)^2} = \beta,$$

as desired.

iii) Our assertion soon follows from [20, Lemma 3.2.5], since, if $1 \le n \le N$, then

$$\frac{\mathbb{E}^{\widehat{\mathbb{Q}}}[S_n]}{\mathbb{E}^{\mathbb{Q}}[S_n]} = \frac{\mathbb{E}^{\mathbb{Q}}[S_n Z_n]}{(1+r)^n S_0} = \frac{\mathbb{E}^{\mathbb{Q}}[S_n^2]}{(1+r)^{2n} S_0^2} = \left(\frac{\mathbb{E}^{\mathbb{Q}}[U^2]}{(1+r)^2}\right)^n = \beta^n.$$

Alternatively, simply apply *iii*) of [1, Proposition 3.5].

iv) The property is trivially true being $S_{n+1} = S_n U$ $(0 \le n \le N-1)$.

$\square$

A possible interpretation of the economic meaning of the factor β pertains to transaction costs of the stock trading. Indeed, if we consider the process $\frac{S}{\beta}$ as the stock price net of payments, the *expected transaction cost coefficient* is given by

$$c_\beta = \left(1 - \frac{1}{\beta}\right),$$

that, in view of *ii*) of the above proposition, may be written more significantly as

$$c_\beta = \frac{\mathbb{E}_{n-1}^{\widehat{\mathbb{Q}}}[S_n] - \mathbb{E}_{n-1}^{\mathbb{Q}}[S_n]}{\mathbb{E}_{n-1}^{\widehat{\mathbb{Q}}}[S_n]} \tag{18}$$

in terms of S_n at time $n-1$. Moreover, the n-period actualized cost becomes $\left(1 - \frac{1}{\beta^n}\right)$ according to *iii*). Quite clearly, the expression of c_β is somewhat consistent with the usual definition of (expected) transaction cost coefficient, which is generally given by the ratio of the ask/bid spread to the average of the price of the stock (or, alternatively, to the average of the ask and bid prices); in this direction, at time $n-1$, $\mathbb{E}_{n-1}^{\widehat{\mathbb{Q}}}[S_n](> S_{n-1})$ stands for the ask price, knowing S_{n-1}, whereas $\mathbb{E}_{n-1}^{\mathbb{Q}}[S_n] = (1+r)S_{n-1}$ stands for the bid price (equivalent to a bank investment of S_{n-1} at the interest rate r).
Due to the definition of β given in (17), β itself and c_β may be rightly considered as endogenous variables of the binomial model, meaning that they are correlated with other factors inside the model itself.
Of course, there is the possibility of evaluating an exogenous transaction cost coefficient c_γ corresponding to an external factor γ, which looks beyond the internal model of pricing, being correlated with outside variables that affect the stock price. Though in general different and having no correlation to each other, β and γ must however share the same lower and upper estimate, i.e., $1 \le \beta, \gamma < \frac{u}{1+r}$: this aspect will be more evident at a later stage in this Section.
Now we are in a position to describe the portfolio hedging behaviour for the binomial model taking into account the transaction costs. Consider, for simplicity, the one-period model (i.e., $N = 1$) and let S_1 be the random variable defined by (1), describing the price at time 1.
Assume that the usual no-arbitrage condition, i.e., $d < 1 + r < u$, holds true and consider a derivative security, for instance a call option, with payoff $(S_1 - X)^+$, X being the strike price. Assume also that the transaction costs for unit of stock are

given by c_β. Our aim is to replicate the investment portfolio through the call option, i.e., we want to compute the initial price $\widetilde{c}_0$ of this derivative.
Let us start with initial wealth X_0 at time 0, buy Δ_0 shares of stock S_0 and invest the remaining part at a risk-free rate r. The future value at time 1 of our portfolio of stock and money market account is given by

$$X_1 = \Delta_0 \frac{S_1}{\beta^2} + (1+r)(X_0 - \Delta_0 \frac{S_0}{\beta}) = (1+r)X_0 + \Delta_0 \left(\frac{S_1}{\beta^2} - (1+r)\frac{S_0}{\beta} \right).$$

Observe that β^2 involves both the costs of buying and (possible) selling of the stock S_1, which we assume to be both equal to c_β. The replication procedure requires that the system in the unknowns X_0 and Δ_0

$$\begin{cases} \frac{1}{1+r}\, c_1^u = X_0 + \Delta_0 \left(\frac{S_1^u}{\beta^2(1+r)} - \frac{S_0}{\beta} \right), & c_1^u := (S_1^u - X)^+, \\ \frac{1}{1+r}\, c_1^d = X_0 + \Delta_0 \left(\frac{S_1^d}{\beta^2(1+r)} - \frac{S_0}{\beta} \right), & c_1^d := (S_1^d - X)^+. \end{cases} \tag{19}$$

should be fulfilled.
If we multiply the first equation of (19) by a number $p^{\overline{\mathbb{Q}}}$ and the second by $(1 - p^{\overline{\mathbb{Q}}})$ and then add them together, we get

$$X_0 + \Delta_0 \left(\frac{1}{1+r} \left(p^{\overline{\mathbb{Q}}} \frac{S_1^u}{\beta^2} + (1 - p^{\overline{\mathbb{Q}}}) \frac{S_1^d}{\beta^2} \right) - \frac{S_0}{\beta} \right) = \frac{1}{1+r} \big(p^{\overline{\mathbb{Q}}} c_1^u + (1 - p^{\overline{\mathbb{Q}}}) c_1^d \big).$$

In particular, choosing $p^{\overline{\mathbb{Q}}}$ so that

$$\frac{1}{1+r} \left(p^{\overline{\mathbb{Q}}} \frac{S_1^u}{\beta^2} + (1 - p^{\overline{\mathbb{Q}}}) \frac{S_1^d}{\beta^2} \right) = \frac{S_0}{\beta},$$

namely

$$p^{\overline{\mathbb{Q}}} = \frac{\beta(1+r) - d}{u - d}, \tag{20}$$

soon entails

$$\begin{cases} X_0 = \widetilde{c}_0 = \frac{1}{1+r} \big(p^{\overline{\mathbb{Q}}} c_1^u + (1 - p^{\overline{\mathbb{Q}}}) c_1^d \big), \\ \Delta_0 = \beta^2 \big(\frac{c_1^u - c_1^d}{S_1^u - S_1^d} \big). \end{cases} \tag{21}$$

In the interesting case $dS_0 < X < uS_0$, in which $c_1^u = uS_0 - X$ and $c_1^d = 0$, we have

$$\widetilde{c}_0 = \frac{1}{1+r} \big(uS_0 p^{\overline{\mathbb{Q}}} - X p^{\overline{\mathbb{Q}}} \big).$$

Obviously, in absence of transaction costs, i.e., if $\beta = 1$, the probability $p^{\overline{\mathbb{Q}}}$ reduces to the risk-neutral measure $p^{\mathbb{Q}}$.
The next proposition shows that really $p^{\overline{\mathbb{Q}}}$ coincides with the probability $p^{\widetilde{\mathbb{Q}}}$ (see (5)), as we might expect.

Proposition 3.3. *The probability $p^{\overline{\mathbb{Q}}}$ defined in* (20) *may be written as*

$$p^{\overline{\mathbb{Q}}} = \frac{u}{1+r} p^{\mathbb{Q}},$$

i.e., $p^{\overline{\mathbb{Q}}} = p^{\widetilde{\mathbb{Q}}}$, whence $\widetilde{c}_0 > c_0$ (compare (2) *and the first equation of* (21)*).*

Proof. First of all note that, on account of i)-Proposition 3.2, $p^{\overline{\mathbb{Q}}}$ is really a probability; by using the definition (17) of β and the risk-neutral relation $up^{\mathbb{Q}} + d(1 - p^{\mathbb{Q}}) = 1 + r$, it is easy to check that

$$p^{\overline{\mathbb{Q}}} = \frac{\frac{u^2 p^{\mathbb{Q}} + d^2(1-p^{\mathbb{Q}})}{1+r} - d}{u - d} = \frac{p^{\mathbb{Q}}(u^2 - d^2) + d^2 - udp^{\mathbb{Q}} - d^2(1 - p^{\mathbb{Q}})}{(1+r)(u-d)} = \frac{u}{1+r} p^{\mathbb{Q}} = p^{\widetilde{\mathbb{Q}}},$$

as claimed. Since $p^{\widetilde{\mathbb{Q}}} > p^{\mathbb{Q}}$, we also have $\widetilde{c}_0 > c_0$ and the proof is now complete. $\square$

Our next goal is to extend the procedure to the multi-period case with $N \geq 2$, where at time n, $1 \leq n \leq N$ (fixed one and for all) the stock price process is described by (6). As in the one-period case, we want to replicate a risky portfolio by means of some derivative security, for instance a call option. Let us assume to be at time $n-1$ starting at time 0, with current wealth X_{n-1}; next buy Δ_{n-1} shares of stock, pay c_β costs for unit of stock and invest the remaining part at a risk-free rate r. The value of our portfolio of stock and money market account at time n is given by

$$X_n = (1+r)X_{n-1} + \Delta_{n-1}\left(\frac{S_n}{\beta^{n+1}} - (1+r)\frac{S_{n-1}}{\beta^n}\right). \tag{22}$$

Arguing as in the one-period case, we can compute the price $\widetilde{c}_{n-1}$ recursively, exactly as in [1, formula (30)], achieving

$$\widetilde{c}_{n-1}^{\rho(i,(n-1)-i)(u,d)} = \frac{1}{1+r}(\widetilde{c}_n^{\rho(i+1,n-(i+1))(u,d)} p^{\widetilde{\mathbb{Q}}} + \widetilde{c}_n^{\rho(i,n-i)(u,d)}(1 - p^{\widetilde{\mathbb{Q}}})) \tag{23}$$

for any $i \in \{0, ..., n-1\}$. Henceforth we may thereby recover some formulas stated in [1, Section 3] simply replacing $p^{\mathbb{Q}}$ by $p^{\widetilde{\mathbb{Q}}}$ and $p^{\widetilde{\mathbb{Q}}}$ by $\widetilde{p^{\widetilde{\mathbb{Q}}}} := \frac{u}{1+r} p^{\widetilde{\mathbb{Q}}} = \frac{u^2}{(1+r)^2} p^{\mathbb{Q}}$

everywhere. Specifically, by means of an induction argument it is easy to see that the price $\widetilde{c}_0$ is given by

$$\widetilde{c}_0 = \frac{1}{(1+r)^n}\left[\sum_{i=0}^{n}\binom{n}{i}\widetilde{c}_n^{\rho^{(i,n-i)}(u,d)}(p^{\widetilde{\mathbb{Q}}})^i(1-p^{\widetilde{\mathbb{Q}}})^{n-i}\right]. \tag{24}$$

Moreover, if m_0 is the minimum number of upward moves necessary for the stock to end in the money at time n, (24) may be rewritten as

$$\widetilde{c}_0 = \frac{1}{(1+r)^n}\left[\sum_{i=m_0}^{n}\binom{n}{i}(u^i d^{n-i}S_0 - X)(p^{\widetilde{\mathbb{Q}}})^i(1-p^{\widetilde{\mathbb{Q}}})^{n-i}\right] =$$

$$\frac{1}{(1+r)^n}\left[\sum_{i=m_0}^{n}\binom{n}{i}(up^{\widetilde{\mathbb{Q}}})^i(d(1-p^{\widetilde{\mathbb{Q}}}))^{n-i}S_0 - X\sum_{i=m_0}^{n}\binom{n}{i}(p^{\widetilde{\mathbb{Q}}})^i(1-p^{\widetilde{\mathbb{Q}}})^{n-i}\right] \tag{25}$$

(compare with [1, formulas (31) and (32)]). More generally, starting from a fixed $k \leq n$, if m_k is defined as in (9), as a potential price of the call at time k one has

$$\widetilde{c}_k = \frac{1}{(1+r)^{n-k}}\left[\sum_{j=m_k}^{n-k}\binom{n-k}{j}\Big(S_n - X\Big)(p^{\widetilde{\mathbb{Q}}})^j(1-p^{\widetilde{\mathbb{Q}}})^{(n-k)-j}\right] =$$

$$S_k\sum_{j=m_k}^{n-k}\binom{n-k}{j}(up^{\widetilde{\mathbb{Q}}})^j(d(1-p^{\widetilde{\mathbb{Q}}}))^{(n-k)-j} - \frac{X}{(1+r)^{n-k}}\sum_{j=m_k}^{n-k}\binom{n-k}{j}(p^{\widetilde{\mathbb{Q}}})^j(1-p^{\widetilde{\mathbb{Q}}})^{(n-k)-j}.$$

In particular the realizations of the random variable $\widetilde{c}_k$ may be expressed as

$$\widetilde{c}_k^{\rho^{(i,k-i)}(u,d)} = \frac{1}{(1+r)^{n-k}}\left[\sum_{j=m_k^i}^{n-k}\binom{n-k}{j}\Big(S_n^{\rho^{(i+j,n-(i+j))}(u,d)} - X\Big)(p^{\widetilde{\mathbb{Q}}})^j(1-p^{\widetilde{\mathbb{Q}}})^{(n-k)-j}\right] =$$

$$\frac{1}{(1+r)^{n-k}}\sum_{j=m_k^i}^{n-k}\binom{n-k}{j}S_k^{\rho^{(i,k-i)}(u,d)}(up^{\widetilde{\mathbb{Q}}})^j(d(1-p^{\widetilde{\mathbb{Q}}}))^{(n-k)-j} -$$

$$\frac{X}{(1+r)^{n-k}}\sum_{j=m_k^i}^{n-k}\binom{n-k}{j}(p^{\widetilde{\mathbb{Q}}})^j(1-p^{\widetilde{\mathbb{Q}}})^{(n-k)-j} \qquad \text{for any } i \in \{0,1,...,k\}, \tag{26}$$

where m_k^i is, in turn, a realization of the random variable m_k. It seems convenient to write down the general option pricing formula (26) in the following compact form

$$\widetilde{c}_k^{\rho^{(i,k-i)}(u,d)} = S_k^{\rho^{(i,k-i)}(u,d)}\Phi(\widetilde{p^{\widetilde{\mathbb{Q}}}}, n-k, m_k^i) - \frac{X}{(1+r)^{n-k}}\Phi(p^{\widetilde{\mathbb{Q}}}, n-k, m_k^i), \tag{27}$$

Φ being the complementary binomial distribution with parameter

$$p^{\widetilde{\widetilde{\mathbb{Q}}}} = \frac{u}{1+r} p^{\widetilde{\mathbb{Q}}}, \tag{28}$$

or, equivalently, in the form

$$\widetilde{c}_k = S_k \, \widetilde{\widetilde{\mathbb{Q}}}\{S_n > X | \mathcal{F}_k\} - \frac{X}{(1+r)^{n-k}} \, \widetilde{\mathbb{Q}}\{S_n > X | \mathcal{F}_k\}, \tag{29}$$

in perfect analogy with (8) and (7), respectively.
It may be easily seen that, if $k < N$, then $\widetilde{c}_k > c_k$ (hence $\widetilde{c}_k^{\rho^{(i,k-i)}(u,d)} > c_k^{\rho^{(i,k-i)}(u,d)}$ for all $i \leq k$), i.e., at time $k < N$, in presence of transaction costs, the call is nodewise more valuable. If $k = N$, then $\widetilde{c}_N = c_N$.
It is worthwhile remarking that, if we know the real transaction cost coefficient $c_\gamma = 1 - \frac{1}{\gamma}$ coming from financial institutions, we can restate the above analysis about the option pricing formulas, obtaing similar results. However, in this case, Proposition 3.3 is not true anymore and the probability to be used is given by

$$p^{\overline{\mathbb{Q}}} = \frac{\gamma(1+r) - d}{u - d}, \tag{30}$$

which is in general different from $p^{\widetilde{\mathbb{Q}}}$; in addition, in order to get a probability, it must be $1 < \gamma < \frac{u}{1+r}$ as already foretold when discussing about γ at page 10.
Summing up, if we do know γ (and hence c_γ), we may use the probability in (30), otherwise we should be content with making a prediction and referring to β, c_β (see (17), (18)) and to the corresponding probability defined in (20).
We conclude this Section with the following theorem which should be compared with [20, Theorem 1.2.2]. The proof is similar with some slight changes somewhere and is therefore omitted for the sake of brevity.

Theorem 3.4. *(Portfolio replication in the binomial model with transaction costs) Consider the N-period binomial model with $N \geq 1$, $d < 1+r < u$, and denote by $c_\gamma = 1 - \frac{1}{\gamma}$ the (real) transaction cost coefficient. As in (22), for any $n \in \{0, 1, ..., N-1\}$ let X_n be the hedging portfolio with the corresponding Δ_n; moreover let V_n be the value of a derivative security expiring at time N and defined (backward in time) by*

$$V_n = \frac{1}{1+r}(V_{n+1}^u p^{\overline{\mathbb{Q}}} + V_{n+1}^d (1 - p^{\overline{\mathbb{Q}}})) \qquad (0 \leq n \leq N-1), \tag{31}$$

where $p^{\overline{\mathbb{Q}}}$ is defined in (30) and the superscripts "u" and "d" record an "up" or "down" move of the stock price. Finally, for any $n \in \{0, 1, ..., N-1\}$, let us set

$$\Delta_n := \gamma^{n+2} \cdot \frac{V_{n+1}^u - V_{n+1}^d}{S_{n+1}^u - S_{n+1}^d}. \tag{32}$$

Then, if we assume $X_0 = V_0$, we have $X_N = V_N$.
Moreover, dealing with the expected transaction cost coefficient c_β given by (18) *yields the same result with $p^{\overline{\mathbb{Q}}}$ replaced by $p^{\widetilde{\mathbb{Q}}}$ everywhere.*

Summing up, the measure $\widehat{\mathbb{Q}}$ introduced in Section 2 provides a $sub-replication$[2] of our risky investment portfolio, since the discounted stock price process is a $\widehat{\mathbb{Q}}$-submartingale according to i)-Proposition 2.1. However, in presence of transaction costs, the market becomes incomplete and the discounted stock price process net of transaction costs is a $\widehat{\mathbb{Q}}$-martingale. In this sense we are implicitly adopting the one-step probability $p^{\widetilde{\mathbb{Q}}}$ (see (20) and Proposition 3.3) and the endogenous factor β. However, if the real transaction cost factor γ should be, in some way, available to us, we might choose the one-step probability $p^{\overline{\mathbb{Q}}}$ given in (30) and the corresponding probability measure which may be denoted by $\overline{\mathbb{Q}}$. Also in this case the discounted stock price process net of transaction costs is a $\overline{\mathbb{Q}}$-martingale. Of course, if $\gamma = \beta$, then $\overline{\mathbb{Q}} = \widetilde{\mathbb{Q}}$, whereas $\gamma = 1$ corresponds to no transaction cost which implies that the market is complete and $\overline{\mathbb{Q}} = \widetilde{\mathbb{Q}} = \mathbb{Q}$.

Remark 3.5. As a matter of fact, all the analysis developed so far is essentially based upon the choice of the probability measure $\widehat{\mathbb{Q}}$ among all the possible (infinitely many) equivalent martingale measures as appearing in an incomplete market.
Of course, one might make a different choice and, accordingly, reach different conclusions.

4. The Black-Scholes equation in presence of transaction costs

In this Section we present the modified Black-Scholes equation in the case of transaction costs by reproducing the usual portfolio hedging strategy (see, e.g., [21, Section 5.2]); in particular we shall obtain a linear PDE whose solution may be expressed explicitly.
In analogy with the discrete case described in Section 3, the first step is supplied by the following crucial result.

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[2] e.g., if $N = 1$, we have $\widehat{\mathbb{Q}} = \widetilde{\mathbb{Q}}$ and

$$c_0 = \frac{1}{1+r}\left(p^{\widetilde{\mathbb{Q}}}c_1^u + (1-p^{\widetilde{\mathbb{Q}}})c_1^d\right) = X_0 + \Delta_0\left(\frac{1}{1+r}\left(p^{\widetilde{\mathbb{Q}}}S_1^u + (1-p^{\widetilde{\mathbb{Q}}})S_1^d\right) - S_0\right) =$$

$$X_0 + \Delta_0\left(\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\frac{S_1}{1+r}\middle|S_0\right] - S_0\right) \geq X_0.$$

Theorem 4.1. *Let $\widetilde{\mathbb{Q}}$ be the probability measure defined in* (13) *and set*

$$\beta := e^{\sigma^2}. \tag{33}$$

Then the process $\left(\frac{e^{-rt}S_t}{\beta^t}\right)_{0\leq t\leq T}$ *is a $\widetilde{\mathbb{Q}}$-martingale.*

Proof. By virtue of i)-Proposition 2.2, $(e^{-rt}S_t)_{0\leq t\leq T}$ is a $\widetilde{\mathbb{Q}}$-submartingale; in particular, for any $0 \leq s \leq t$, one has

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[e^{-rt}S_t|\mathcal{F}_s] = e^{\sigma^2(t-s)}e^{-rs}S_s$$

(see [1, proof of i)-Proposition 2.1]) and therefore we may conclude that

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\frac{e^{-rt}S_t}{\beta^t}\middle|\mathcal{F}_s\right] = \frac{e^{-rs}S_s}{\beta^s}.$$

□

The role of the istantaneous factor β may be inferred from the next proposition, which is the continuous version of Proposition 3.2.

Proposition 4.2. *The following properties hold true:*

i) $\beta \geq 1$;

ii) $\beta = \frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t|\mathcal{F}_{t-1}]}{\mathbb{E}^{\mathbb{Q}}[S_t|\mathcal{F}_{t-1}]} \qquad (1 \leq t \leq T)$;

iii) $\beta^t = \frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t]}{\mathbb{E}^{\mathbb{Q}}[S_t]} \qquad (0 \leq t \leq T)$.

Proof. i) The proof is trivial.

ii) By using the Radon-Nikodym process $(Z_t)_{0\leq t\leq T}$ introduced in Section 2, it is easy to check out ii) for a fixed t, since explicitly

$$\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t|\mathcal{F}_{t-1}]}{\mathbb{E}^{\mathbb{Q}}[S_t|\mathcal{F}_{t-1}]} = \frac{\mathbb{E}^{\mathbb{Q}}[S_tZ_t|\mathcal{F}_{t-1}]}{e^rS_{t-1}Z_{t-1}} = \frac{e^{-r}S_{t-1}^2}{e^rS_{t-1}^2\sqrt{2\pi}}\int_{-\infty}^{+\infty} exp\left(2\left(r-\frac{\sigma^2}{2}\right)-2\sigma y\right)e^{-\frac{y^2}{2}}\,dy =$$

$$\frac{e^{\sigma^2}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(y+2\sigma)^2}{2}}\,dy = e^{\sigma^2} = \beta.$$

iii) Taking into account iv)-Proposition 2.2, it is already known that for $t \in [0,T]$ $\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t] = \frac{\mathbb{E}^{\mathbb{Q}}[S_t^2]}{\mathbb{E}^{\mathbb{Q}}[S_t]}$. Moreover, if we let $Y := -\frac{W_t^{\mathbb{Q}}}{\sqrt{t}} \sim N(0,1)$, computing the expectation yields

$$\mathbb{E}^{\mathbb{Q}}[S_t^2] = \mathbb{E}^{\mathbb{Q}}\left[S_0^2\, exp\Big(2\big(r - \frac{\sigma^2}{2}\big)t - 2\sigma\sqrt{t}Y\Big)\right] =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} S_0^2\, exp\Big(2\big(r - \frac{\sigma^2}{2}\big)t - 2\sigma\sqrt{t}y\Big)e^{-\frac{y^2}{2}}\, dy =$$

$$e^{(\sigma^2+2r)t}S_0^2 \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(y+2\sigma\sqrt{t})^2}{2}}\, dy = e^{(\sigma^2+2r)t}S_0^2.$$

Since $\mathbb{E}^{\mathbb{Q}}[S_t] = e^{rt}S_0$, it is immediate to conclude that

$$\frac{\mathbb{E}^{\widetilde{\mathbb{Q}}}[S_t]}{\mathbb{E}^{\mathbb{Q}}[S_t]} = \frac{\mathbb{E}^{\mathbb{Q}}[S_t^2]}{(\mathbb{E}^{\mathbb{Q}}[S_t])^2} = \frac{e^{(\sigma^2+2r)t}S_0^2}{e^{2rt}S_0^2} = e^{\sigma^2 t} = \beta^t,$$

whence our assertion □

Remark 4.3. In analogy with (17) and by virtue of iii) we might define the istantaneous factor β in the continuous case also as

$$\beta = \frac{\mathbb{E}^{\mathbb{Q}}[S_1^2]}{(\mathbb{E}^{\mathbb{Q}}[S_1])^2},$$

or, equivalently, as

$$\beta = \frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{t+1}^2}{S_t^2}\right]}{\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{t+1}}{S_t}\right]\right)^2} \qquad (0 \leq t \leq T-1)$$

(compare with iv)-Proposition 3.2).

In order to derive the modified Black-Scholes equation, we start with substituting the stock price process S_t with the discounted process $\frac{S_t}{\beta^t} = e^{-t\,ln\beta}S_t$. Assume that, at each time $t \in [0,T[$, we buy Δ_t shares of stock and invest the remaining part in the money market account at risk-free rate r. Applying the Ito-Doeblin formula, the evolution of our portfolio, denoted by X_t, is described by the dynamics (see [20, Sections 4.5.1-4.5.3])

$$dX_t = \Delta_t d\big(e^{-t\,ln\beta}S_t\big) + r(X_t - \Delta_t e^{-t\,ln\beta}S_t)dt;$$

since $dS_t = \mu S_t dt + \sigma S_t dW_t$ and, on the other hand,

$$d\big(e^{-t\, ln\beta} S_t\big) = e^{-t\, ln\beta}(dS_t - ln\,\beta S_t dt),$$

we finally get

$$dX_t = \Delta_t(\mu - ln\beta)e^{-t\, ln\beta} S_t dt + \Delta_t \sigma e^{-t\, ln\beta} S_t dW_t + rX_t dt - \Delta_t r e^{-t\, ln\beta} S_t dt.$$

Therefore the evolutions of the discounted portfolio and of the discounted value of a derivative we want to replicate, e.g., a call option, are given respectively by

$$d(e^{-rt}X_t) = -re^{-rt}X_t dt + e^{-rt}dX_t =$$

$$\Delta_t\big(-re^{-r't}S_t dt - ln\,\beta e^{-r't}S_t dt + \sigma e^{-r't}S_t dW_t + \mu e^{-r't}S_t dt\big) = \Delta_t d\big(e^{-r't}S_t\big)$$

and by

$$d(e^{-rt}\widetilde{c}_t) = e^{-rt}\left(\frac{\partial \widetilde{c}_t}{\partial t} + \mu S_t \frac{\partial \widetilde{c}_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \widetilde{c}_t}{\partial S_t^2} - r\widetilde{c}_t\right)dt + \sigma e^{-rt} S_t \frac{\partial \widetilde{c}_t}{\partial S_t} dW_t,$$

where, according to our notation, we have set $\widetilde{c}_t = \widetilde{c}(t, S_t)$ as the price of the call and

$$r' := r + ln\beta = r + \sigma^2. \tag{34}$$

Letting $d(e^{-rt}X_t) = d(e^{-rt}\widetilde{c}_t)$ and rearranging the terms lead to

$$\begin{cases} \Delta_t = e^{t\, ln\beta} \frac{\partial \widetilde{c}_t}{\partial S_t}, \\ \frac{\partial \widetilde{c}_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \widetilde{c}_t}{\partial S_t^2} + r' S_t \frac{\partial \widetilde{c}_t}{\partial S_t} - r\widetilde{c}_t = 0. \end{cases} \tag{35}$$

The first equation represents the $delta - hedging\ rule$, while the second one may be referred to as the $Black - Scholes\ partial\ differential\ equation\ in\ presence\ of\ transaction\ costs$. Like in the classical case, the resulting PDE is of parabolic-backward type; in order to find a solution, first of all let us reverse the time through the change of variable $t \mapsto T - t$ and then pass to consider the general forward PDE

$$\frac{\partial \widetilde{c}_t}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \widetilde{c}_t}{\partial x^2} + r' x \frac{\partial \widetilde{c}_t}{\partial x} - r\widetilde{c}_t \qquad (x \geq 0,\, t \in [0, T[), \tag{36}$$

coupled with the initial condition $\widetilde{c}_0 = (x - X)^+$ $(x \geq 0)$, where now $\widetilde{c}_t$ has to be meant as $\widetilde{c}(T - t, x)$ for any $x \geq 0$. In other words, we are dealing with a forward Cauchy problem whose solution may be expressly determined, as shown in the next theorem.

Theorem 4.4. *The solution of the forward Cauchy problem stated above is given by*

$$\widetilde{c}_t = e^{\tau\, ln\beta} x N(\widetilde{d}_1) - e^{-r\tau} X N(\widetilde{d}_2) \qquad (x \geq 0,\, t \in [0,T[), \tag{37}$$

where $\tau = T - t$ *and*

$$\widetilde{d}_1 := \frac{1}{\sigma\sqrt{\tau}}\Big(ln\frac{x}{X} + \big(r' + \frac{\sigma^2}{2}\big)\tau \Big), \qquad \widetilde{d}_2 = \widetilde{d}_1 - \sigma\sqrt{\tau}. \tag{38}$$

Proof. Following [12, Section 3], the solution of the general (abstract) Cauchy problem

$$\begin{cases} \frac{\partial u_t}{\partial t} = ax^2 \frac{\partial^2 u_t}{\partial x^2} + bx\frac{\partial u_t}{\partial x} + cu_t, \qquad t \geq 0,\, x \in \mathbb{R}_+,\, (a > 0,\, b, c \in \mathbb{R}), \\ u_0 = f(x), \qquad x \in \mathbb{R}_+, \end{cases}$$

is given by

$$u_t = \frac{e^{c\tau}}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} f\Big(x\, exp\Big((b-a)\tau - \sqrt{2a\tau} y \Big)\Big) e^{-\frac{y^2}{2}}\, dy \tag{39}$$

for any $t \geq 0, x \geq 0$ and for any nonnegative function $f : [0, +\infty[\to \mathbb{R}$, provided the integral on the right-hand side is convergent; here again $\tau = T - t$ and x is the usual dummy variable. Hence, in our case ($a = \frac{\sigma^2}{2}, b = r', c = -r$), we soon obtain

$$\widetilde{c}_t = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Big(x\, exp\Big(\big(r' - \frac{\sigma^2}{2}\big)\tau - \sigma\sqrt{\tau} y \Big) - X \Big)^+ e^{-\frac{y^2}{2}}\, dy.$$

The above integrand is positive if and only if

$$y < \widetilde{d}_2 := \frac{1}{\sigma\sqrt{\tau}}\Big(ln\frac{x}{X} + \big(r' - \frac{\sigma^2}{2}\big)\tau \Big)$$

and therefore

$$\widetilde{c}_t = \frac{e^{\tau\, ln\beta} x}{\sqrt{2\pi}} \int_{-\infty}^{\widetilde{d}_2} exp\Big(-\frac{\sigma^2}{2}\tau - \sigma\sqrt{\tau} y - \frac{y^2}{2} \Big)\, dy - \frac{e^{-r\tau} X}{\sqrt{2\pi}} \int_{-\infty}^{\widetilde{d}_2} e^{-\frac{y^2}{2}}\, dy =$$

$$\frac{e^{\tau\, ln\beta} x}{\sqrt{2\pi}} \int_{-\infty}^{\widetilde{d}_1} e^{-\frac{z^2}{2}}\, dz - \frac{e^{-r\tau} X}{\sqrt{2\pi}} \int_{-\infty}^{\widetilde{d}_2} e^{-\frac{y^2}{2}}\, dy = e^{\tau\, ln\beta} x N(\widetilde{d}_1) - e^{-r\tau} X N(\widetilde{d}_2),$$

where $z := y + \sigma\sqrt{\tau}$ and $\widetilde{d}_1 = \widetilde{d}_2 + \sigma\sqrt{\tau}$. □

In particular, since $\beta = e^{\sigma^2}$, letting $x = S_t$, we get the $Black-Scholes\ formula\ in\ presence\ of\ transaction\ costs$, namely

$$\widetilde{c}_t = e^{\sigma^2\tau} S_t N(\widetilde{d}_1) - e^{-r\tau} X N(\widetilde{d}_2) \qquad (0 \le t < T), \tag{40}$$

where

$$\widetilde{d}_1 = \frac{1}{\sigma\sqrt{\tau}}\left(ln\frac{S_t}{X} + \big(r + \frac{3}{2}\sigma^2\big)\tau\right),\ \widetilde{d}_2 = \widetilde{d}_1 - \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}\left(ln\frac{S_t}{X} + \big(r + \frac{\sigma^2}{2}\big)\tau\right). \tag{41}$$

It goes without saying that, should the exogenous transaction cost factor γ be known in some way, we could recover the equation in (35) and the related solution (40) starting from $\frac{S_t}{\gamma^t} = e^{-t\ln\gamma} S_t$ and setting $r' = r + \ln\gamma$ in (34).
Explicitly, (40) would become

$$\widetilde{c}_t = e^{\tau\ln\gamma} S_t N(\widetilde{d}_1) - e^{-r\tau} X N(\widetilde{d}_2),$$

with $r' = r + \ln\gamma$ in the expression of $\widetilde{d}_1$ and $\widetilde{d}_2$.

Remark 4.5. According to Remark 3.5, formula (37) provides the risk-neutral price of an European call option corresponding to our selection of the probability measure $\widetilde{\mathbb{Q}}$ (see (33)) in the set of all (infinitely many) equivalent martingale measures in the incomplete setting. Of course, choosing a different measure would lead to a (in general) different risk-neutral price; for an optimal choice of the equivalent martingale measure in an incomplete framework, we refer the reader, for instance, to [16] and [4].

Remark 4.6. (Feynman-Kac formula for expected transaction costs)
It is possible to get the modified Black-Scholes formula (40) by using the Feynman-Kac formula. At this purpose let $\beta = e^{\sigma^2}$ be the expected transaction cost factor and $\widetilde{\mathbb{Q}}$ the measure defined in (13); passing from $\widetilde{\mathbb{Q}}$ to $\mathbb{Q}$ (for more details see [1, proof of i) of Proposition 2.1] and [21, Lemma 5.2.2]) yields for any $t \in [0, T[$

$$\widetilde{c}_t = e^{-r\tau} \cdot \mathbb{E}^{\widetilde{\mathbb{Q}}}[(S_T - X)^+|\mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{Q}}[S_T e^{-r(2T-t)}(S_T - X)^+|\mathcal{F}_t]}{e^{-rt}S_t} =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-2r\tau} exp\left(\big(r - \frac{\sigma^2}{2}\big)\tau - \sigma\sqrt{\tau}y\right)\left(S_t exp\left(\big(r - \frac{\sigma^2}{2}\big)\tau - \sigma\sqrt{\tau}y\right) - X\right)^+ e^{-\frac{y^2}{2}}\,dy,$$

where the above integrand is positive if and only if

$$y < d := \frac{1}{\sigma\sqrt{\tau}}\left(ln\frac{S_t}{X} + \big(r - \frac{\sigma^2}{2}\big)\tau\right);$$

therefore an easy computation leads to

$$\widetilde{c}_t = \frac{S_t}{\sqrt{2\pi}} \int_{-\infty}^{d} exp\Big(-\sigma^2\tau - 2\sigma\sqrt{\tau}y - \frac{y^2}{2}\Big)\, dy - \frac{Xe^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{d} exp\Big(-\frac{\sigma^2}{2}\tau - \sigma\sqrt{\tau}y - \frac{y^2}{2}\Big)\, dy =$$

$$\frac{e^{\sigma^2\tau}S_t}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{(y+2\sigma\sqrt{\tau})^2}{2}}\, dy - \frac{Xe^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}}\, dy = e^{\sigma^2\tau}S_t N(\widetilde{d}_1) - e^{-r\tau}XN(\widetilde{d}_2),$$

where

$$\widetilde{d}_1 = \frac{1}{\sigma\sqrt{\tau}}\Big(ln\frac{S_t}{X} + \big(r + \frac{3}{2}\sigma^2\big)\tau\Big), \qquad \widetilde{d}_2 = \widetilde{d}_1 - \sigma\sqrt{\tau},$$

as already obtained in a different way.
Alternatively, we might exploit the $\widetilde{\mathbb{Q}}$-dynamics of $(S_t)_{t\geq 0}$ indicated in (16), obtaining explicitly

$$S_T = S_t exp\Big(\big(r + \frac{\sigma^2}{2}\big)\tau + \sigma W_\tau^{\widetilde{\mathbb{Q}}}\Big) = S_t exp\Big(\big(r + \frac{\sigma^2}{2}\big)\tau - \sigma\sqrt{\tau}Y\Big) \qquad (0 \leq t < T),$$

where $Y := -\frac{W_T^{\widetilde{\mathbb{Q}}} - W_t^{\widetilde{\mathbb{Q}}}}{\sqrt{\tau}} \sim N(0,1)$; consequently, for any $t \in [0, T[$, the measurability of S_t with respect to the information $\mathcal{F}_t$ together with the $\mathcal{F}_t$-indipendence of Y enables us to obtain quickly

$$\widetilde{c}_t = e^{-r\tau}\mathbb{E}^{\widetilde{\mathbb{Q}}}[(S_T - X)^+ | \mathcal{F}_t] = e^{-r\tau}\mathbb{E}^{\widetilde{\mathbb{Q}}}\Big[\Big(S_t exp\Big(\big(r + \frac{\sigma^2}{2}\big)\tau - \sigma\sqrt{\tau}Y\Big) - X\Big)^+\Big] =$$

$$\frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\widetilde{d}_2} \Big(S_t exp\Big(\big(r + \frac{\sigma^2}{2}\big)\tau - \sigma\sqrt{\tau}y\Big) - X\Big) e^{-\frac{y^2}{2}}\, dy,$$

where $\widetilde{d}_2$, together with $\widetilde{d}_1$, is defined by (41).
Manipulating the last integral leads to

$$\widetilde{c}_t = \frac{e^{-r\tau}S_t}{\sqrt{2\pi}} \int_{-\infty}^{\widetilde{d}_2} exp\Big(\big(r + \frac{\sigma^2}{2}\big)\tau - \sigma\sqrt{\tau}y - \frac{y^2}{2}\Big)\, dy - \frac{e^{-r\tau}X}{\sqrt{2\pi}} \int_{-\infty}^{\widetilde{d}_2} e^{-\frac{y^2}{2}}\, dy =$$

$$\frac{e^{\sigma^2\tau}S_t}{\sqrt{2\pi}} \int_{-\infty}^{\widetilde{d}_2} e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}} - e^{-r\tau}XN(\widetilde{d}_2) = e^{\sigma^2\tau}S_t N(\widetilde{d}_1) - e^{-r\tau}XN(\widetilde{d}_2).$$

Incidentally observe that $\widetilde{d}_2 = d_1$ (see (12)) in the Black-Scholes formula. Below we state that, when transaction costs are taken into account, a call option is more valuable as one might easily expect.

Proposition 4.7. *For any $t \in [0, T]$ one has $c_t \leq \widetilde{c}_t$, where the equality holds true only for $t = T$.*

Proof. Let us choose $t \in [0, T[$ and recall that, according to [21, p. 220], one has

$$c_t = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[(S_T - X)^+|\mathcal{F}_t] = \frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{-\infty}^{d_2}\left(S_t exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau - \sigma\sqrt{\tau}y\right) - X\right)e^{-\frac{y^2}{2}}\,dy;$$

on the other hand, we have just shown above that

$$\widetilde{c}_t = e^{-r\tau}\mathbb{E}^{\widetilde{\mathbb{Q}}}[(S_T - X)^+|\mathcal{F}_t] = \frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{-\infty}^{d_1}\left(S_t exp\left(\left(r + \frac{\sigma^2}{2}\right)\tau - \sigma\sqrt{\tau}y\right) - X\right)e^{-\frac{y^2}{2}}\,dy,$$

because $\widetilde{d}_2 = d_1$. The two integrands are positive (in $] - \infty, d_2]$ and $] - \infty, d_1]$, respectively) and the former is strictly less that the second: this, together with $d_2 < d_1$, soon entails $c_t < \widetilde{c}_t$ as claimed. When $t = T$, then $\tau = 0$, $N(d_i) = N(\widetilde{d}_i)$ $(i = 1, 2)$ are both equal to 1 or 0 depending on the call ends in the money or out of the money, so that, accordingly, $c_T = \widetilde{c}_T = (S_T - X)^+$. The case "at the money" yields in turn $N(d_i) = N(\widetilde{d}_i) = \frac{1}{2}$ and again $c_T = \widetilde{c}_T = 0 = (S_T - X)^+$. The proof is now complete. □

Remark 4.8. At this stage a natural question arises about pricing an European put option whose value, consistently with the notations adopted so far, will be henceforth denoted by $\widetilde{p}_t$ $(0 \leq t \leq T)$.
As a matter of fact, also in the transaction cost framework, a *put − call parity* will be available to us, allowing to get $\widetilde{p}_t$ knowing $\widetilde{c}_t$.
Indeed, following [21, pp. 162-163], let $\widetilde{f}_t = \widetilde{f}(t, S_t)$ be the value of a forward contract at a fixed time t, with current asset value S_t, expiration date T and delivery price X; it is quite easy to realize that, in presence of transaction costs, one has

$$\widetilde{f}_t = e^{\sigma^2\tau}S_t - Xe^{-r\tau},$$

where the first term is affected by the continuously compouned transaction costs $e^{\sigma^2\tau}$ relative to the (future) time interval $]t, T]$. At expiration T one has $\tau = 0$ so that

$$\widetilde{f}_T = S_T - X = \widetilde{c}_T - \widetilde{p}_T,$$

which, by a standard no-arbitrage argument, soon implies

$$\widetilde{f}_t = \widetilde{c}_t - \widetilde{p}_t \;\text{ for any } t \in [0, T[,$$

or, more explicitly,

$$\widetilde{c}_t - \widetilde{p}_t = e^{\sigma^2 \tau} S_t - X e^{-r\tau}. \tag{42}$$

Once the above relationship has been established, plugging (25) and (40) into (42) gives immediately

$$\widetilde{p}_0 = \widetilde{c}_0 - \beta^n S_0 + \frac{X}{(1+r)^n}, \tag{43}$$

as well as

$$\widetilde{p}_t = X e^{-r\tau} N(-\widetilde{d}_2) - e^{-\tau \ln \beta} S_t N(-\widetilde{d}_1) \qquad (0 \leq t < T), \tag{44}$$

keeping the some meaning of the relevant terms; in particular in (43) the factor β is the one defined in (17).
It is easy to verify that Proposition 4.7 still holds true replacing c_t and $\widetilde{c}_t$ with p_t and $\widetilde{p}_t$, respectively; the same happens for the discrete case as already discussed for call options just after formula (29).

5. Numerical implementations

In this Section we implement the procedure described in Section 3. The advantage of formula (26) (or (25)) lies on the fact that it allows to compute the call price $\widetilde{c}_k^{\rho^{(i,k-i)}(u,d)}$ at any of the nodes i at time k in just one step, differently from other (classic) methods which realize a "backward" procedure from n to k (see, e.g., [5] and [20]). From a computational point of view, this is especially useful while dealing with binomial trees with many steps to gain high numerical accuracy and/or for pricing long expiring options. In our approach we need to know only the stock price $S_k^{\rho^{(i,k-i)}(u,d)}$ for fixed i and k, $0 \leq i \leq k$ (that is generally known) and the number of upwards m_k^i as defined in (9). Further, m_k^i is also given in a closed formulation if we know only $S_k^{\rho^{(i,k-i)}(u,d)}$; indeed we are interested to the scenario "in the money" corresponding to

$$S_k^{\rho^{(i,k-i)}(u,d)} u^j d^{(n-k)-j} > X;$$

this happens if and only if

$$j > \frac{\ln\left(\frac{X}{S_k^{\rho^{(i,k-i)}(u,d)} d^{n-k}}\right)}{\ln\left(\frac{u}{d}\right)}, \tag{45}$$

which leads to choose m_k^i equal to the ceiling of the term on the right-hand side.
In particular, we set the parameters $u, d, p^{\mathbb{Q}}$ according to the Cox-Ross-Rubinstein model [5], i.e.,

$$u = e^{\sigma\sqrt{\Delta}},\ d = \frac{1}{u},\ p^{\mathbb{Q}} = \frac{e^{r\Delta} - d}{u - d},$$

where Δ is the time step. Figure 1 shows the expected transaction cost coefficient c_β when the risk-free interest rate r and the volatility σ move in the set $[0, 0.4] \times [0, 1]$. We observe a remarkable agreement with the empirical results listed in [10, Table 2].

Coming back to our main issue, as a study case, we consider the dayly stock prices of Unicredit S.p.A. (UCG.MI)[3] observed from 8 October 2018 to 4 October 2019 and the call prices[4] written on these stocks with trade date October 2019 and maturity

[3]See Yahoo Finance `https://finance.yahoo.com/quote/UCG.MI/history?p=UCG.MI`.

[4]Data taken from Borsa Italiana `https://www.borsaitaliana.it/borsa/derivati/opzioni-su-azioni/lista.html?isin=IT0005239360&lang=it`. We consider the "1 month" maturity due to the largest dataset available on the site.

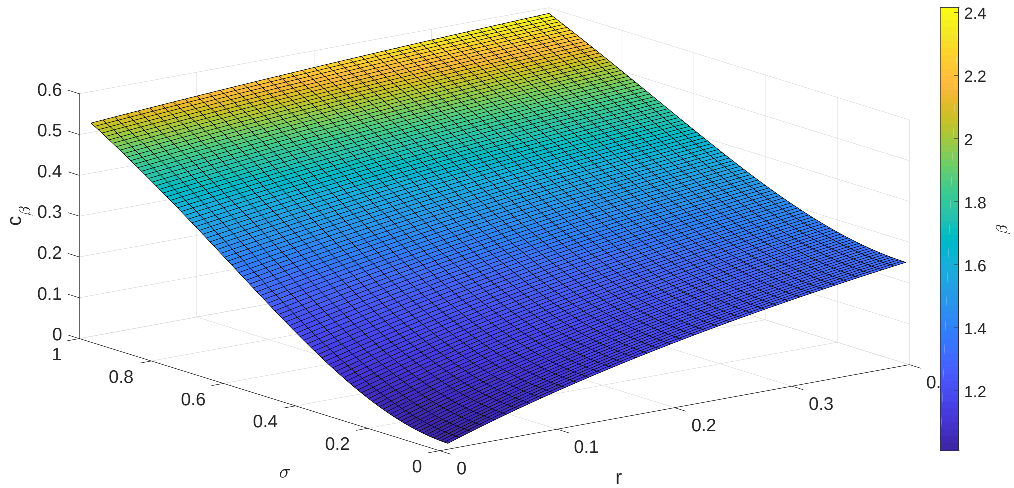

Figure 1: Expected transaction cost coefficient c_β for different risk-free rate r and volatility σ.

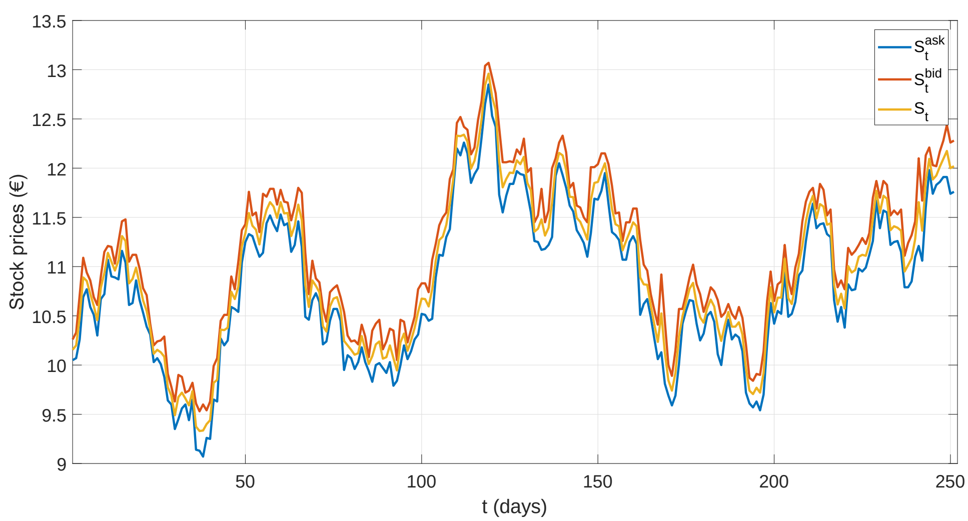

Figure 2: Ask and Bid Unicredit dayly stock prices observed from 8 October 2018 to 4 October 2019.

November 2019. Figure 2 graphically represents the ask/bid stock prices and their mean.
Table 1 displays the call and put ask/bid prices for different level of strike X and volatility σ so that we analyze both the "in the money" and "out of money" cases. The risk-free interest rate[5] and the dividend [6] are set to be $r = -0.47\%$ and $\delta = 2.3$ €, respectively. To include the dividend δ in formulas (26) or (25) we have only to replace r with $r - \delta$ under the assumption that the dividend is paid at maturity.
For each date t we compute the transaction cost coefficient as

$$c_\gamma(t) = \frac{S_t^{ask} - S_t^{bid}}{S_t},$$

where S_t is the mean of S_t^{ask} and S_t^{bid}. Comparing these real costs with the (expected) transaction cost coefficient c_β defined in (18), we conclude that c_β captures effectively the mean of c_γ with a (maximum) absolute error of order of 10^{-3}, as showed in Figure 3. Finally, referring to Table 1, we compute the (ask) call and put prices with the following parameters

$$S_0^{ask} = 10.05, \ |r| = 0.47\%, \ \delta = 0.27, \ \Delta = 1/30, \ T = 1\,(month).$$

As displayed in Figure 4 (right and left top), the prices provided by formulas (25) and (43) coincide with the true prices of call and put options quoted in Table 1. The same holds true for the continuous formulas (40) and (44) (see Figure 4, right and left down). In particular, both the expected and real transaction cost coefficients c_β and c_γ are involved in our simulation of prices and compared quite satisfactorily with the true prices.

[5] The risk-free interest rate is set to be equal to the 1 month euribor, see `https://it.global-rates.com/tassi-di-interesse/euribor/euribor.aspx`. As explained in [2] and [7], the current short interest-rate in the Eurozone is permanently negative in the last years, hence we adjust our model taking the absolute value of r.

[6] See `https://www.unicreditgroup.eu/it/investors/share-information/dividends.html`.

Bid call prices	Ask call prices	Strike	Bid put prices	Ask put prices	Volatility
2.571	3.071	7.4	0.0001	0.15	0.9500
2.376	2.876	7.6	0.0001	0.15	0.9330
2.185	2.685	7.8	0.0001	0.15	0.8670
2.002	2.502	8	0.0001	0.15	0.8050
1.865	2.165	8.2	0.0001	0.15	0.7450
1.6775	1.9775	8.4	0.044	0.0855	0.6850
1.582	1.7315	8.6	0.061	0.105	0.6260
1.4055	1.5525	8.8	0.09	0.1215	0.5740
1.2325	1.3805	9	0.1155	0.1505	0.5500
0.85	0.9745	9.5	0.2125	0.257	0.5280
0.553	0.604	10	0.379	0.43	0.5080
0.307	0.358	10.5	0.5275	0.808	0.4890
0.151	0.1945	11	0.935	1.063	0.4480
0.066	0.099	11.5	1.3355	1.4865	0.4150
0.0006	0.1465	12	1.7165	2.0165	0.3660
0.0001	0.15	12.5	2.085	2.585	0.3780
0.0001	0.15	13	2.591	3.091	0.3950
0.0001	0.15	13.5	3.0905	3.5905	0.4160
0.0001	0.15	14	3.5525	4.0525	0.4410
0.0001	0.15	14.5	4.09	4.59	0.4700
0.0001	0.15	15	4.558	5.058	0.5020
0.0001	0.15	15.5	5.00	6.00	0.5380
0.0001	0.15	16	5.288	6.288	0.5730

Table 1: Ask/Bid prices (€) of call and put options for different strike X and volatility σ, with maturity $T = 1$ month, risk-free rate $r = -0.47\%$, dividend $\delta = 0.27$€ and (initial) underlying asset prices $S_0^{ask} = 10.05$ € and $S_0^{bid} = 10.26$ €.

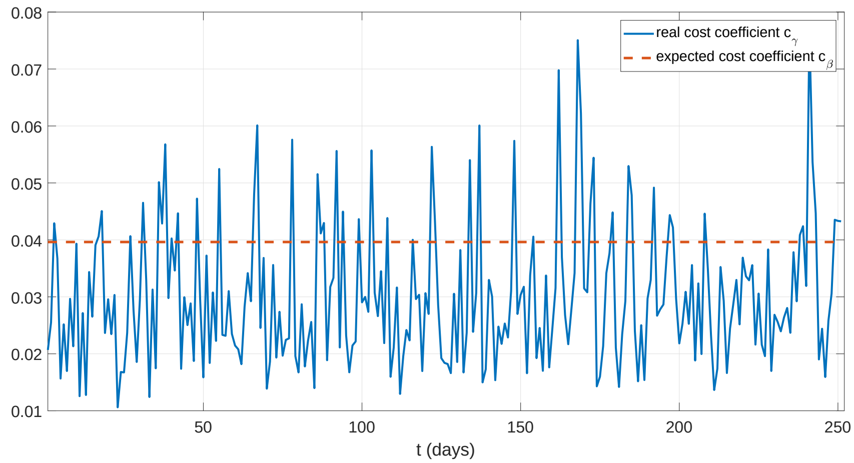

Figure 3: Real transaction cost coefficient c_γ versus the expected transaction cost coefficient c_β.

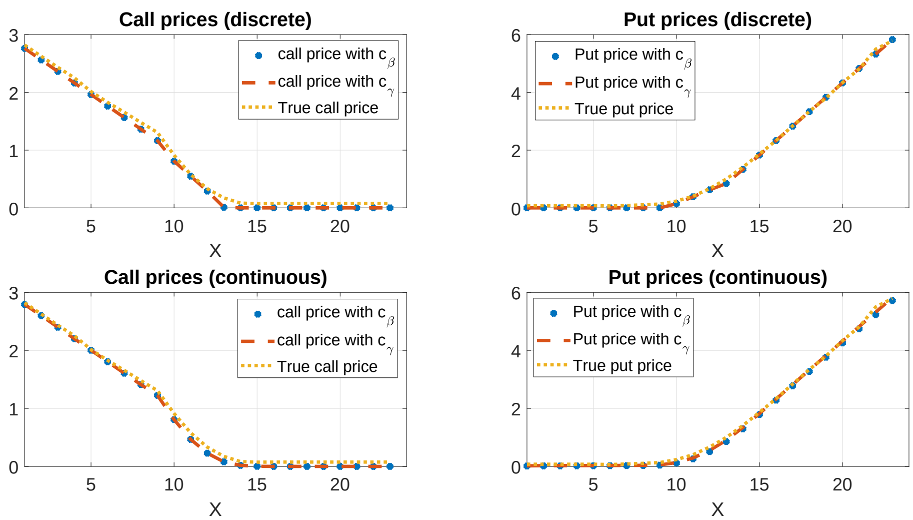

Figure 4: True call/put prices versus simulated call/put prices (with c_γ and c_β, respectively).

As a final step, we compare our results with the ask and bid option prices computed by using the Leland model [15]. As well-known, in the Leland model the call and put option prices are obtained like in the Black-Scholes model except that the market volatility σ is replaced by the following "adjusted" volatility

$$\sigma' = \sigma \cdot \sqrt{1 \pm \frac{c_\gamma}{\sigma}\sqrt{\frac{2}{\pi\Delta}}}, \tag{46}$$

where the signs $\pm$ refer to the ask/bid price, respectively (see [15, p. 1289, formula (13)]).

As a measure of the forecasting accuracy we use the root mean squared error (RMSE) that captures the closeness between the observed data and the simulated values. It is defined as

$$RMSE := \frac{1}{N}\sqrt{\sum_{h=1}^{N} e_h^2},$$

where e_h denotes the residuals between the observed data and their simulations over N times. Lower values (close to zero) represent a good result for the RMSE.

Table 2 reports the RMSE between the true ask/bid call and put prices and those computed through our approach (formulas (40) and (44)) and the Leland model, respectively.

	our model	Leland model
ask call price	12.52%	18.47%
bid call price	10.38%	13.40%
ask put price	20.46%	22.10%
bid put price	19.85%	22.23%

Table 2: RMSE between the true ask/bid call and put prices and simulated values.

6. Conclusions

In this paper we develop a technique for computing the price of European options in presence of transaction costs. The procedure is essentially based upon the choice, among all the (infinitely many) equivalent martingale measures, of new probability measures introduced in [1], under which the discounted stock price process, which is a submartingale in a complete market framework, becomes, on the countrary, a martingale if one considers transaction costs in stock trading. The results obtained highlight the financial meaning of such measures, which thereby turn out to be not a merely theoretical mathematical expedient, but rather a powerful applicative tool. Accordingly, we state explicit pricing formulas in the discrete and continuous case which preserve strong analogies with the classic results; however, as expected, in this new background options are more valuable than their no transaction cost counterparts.

The notion of expected transaction cost, as defined in (17) and (33) in a quite elementary way, plays a fundamental role, having no equivalent, as far as we know, in the existing literature apart some generic hints in few selected papers (see, e.g., [8], [10] and [19]); perhaps it is a new concept in Finance, which probably would deserve to be analyzed more deeply.

As a concrete test, we price European call and put options written on the Unicredit daily stocks observed from 8 October 2018 to 4 October 2019, finding many agreements between our expected transaction cost, market data and other models, like those developed in [8], [10] and [15]. Finally, we point out that our procedure is simply to apply and provides prices which fit well enough with real market values.

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