

# Critical fluctuations in renewal models of statistical mechanics

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## Abstract

We investigate the sharp asymptotic behavior at criticality of the large fluctuations of extensive observables in renewal models of statistical mechanics, such as the Poland-Scheraga model of DNA denaturation, the Fisher-Felderhof model of fluids, the Wako-Saitô-Muñoz-Eaton model of protein folding, and the Tokar-Dreyssé model of strained epitaxy. These models amount to Gibbs changes of measure of a classical renewal process and can be identified with a constrained pinning model of polymers. The extensive observables that enter the thermodynamic description turn out to be cumulative rewards corresponding to deterministic rewards that are uniquely determined by the waiting time and grow no faster than it. The probability decay with the system size of their fluctuations switches from exponential to subexponential at criticality, which is a regime corresponding to a discontinuous pinning-depinning phase transition. We describe such decay by proposing a precise large deviation principle under the assumption that the subexponential correction term to the waiting time distribution is regularly varying. This principle is in particular used to characterize the fluctuations of the number of renewals, which measures the DNA-bound monomers in the Poland-Scheraga model, the particles in the Fisher-Felderhof model and the Tokar-Dreyssé model, and the native peptide bonds in the Wako-Saitô-Muñoz-Eaton model.

Keywords: Renewal processes; Polymer pinning models; Critical phenomena; Renewal-reward processes; Precise large deviations; Regular varying tails

Mathematics Subject Classification 2020: Primary 60F10; 60K05; 60G50, Secondary 62E20; 60K35; 82B20; 82B23

## 1 Introduction

The Poland-Scheraga model of DNA denaturation [1,2], the Fisher-Felderhof model of fluids [3–7], the model of protein folding introduced independently by Wako and Saitô first [8,9] and Muñoz and Eaton later [10–12], and the Tokar-Dreyssé model of strained epitaxy [13–15] have been proved to share a common regenerative structure in Ref. [16], where they have been named *renewal models of statistical mechanics*. In fact, these models can be related to the pinning model of polymers [17,18], which amounts to a Gibbs change of measure of a classical discrete-time renewal process. Precisely, in Ref. [16] they have been mathematically mapped into a constrained pinning model obtained by the pinning model under the condition that one of the renewals occurs at a predetermined time corresponding to the system size. This constrained pinning model is the prototype of renewal models of statistical mechanics and introduces the language of renewal theory in the above lattice-gas models of equilibrium statistical physics that apparently have nothing to do with renewal theory.

The extensive variables that enter the thermodynamic description of renewal models of statistical mechanics have been shown in Ref. [16] to be cumulative rewards, supposing that each renewal involves a deterministic reward that is uniquely determined by the waiting time and grows no faster than it. Examples are the number of DNA-bound monomers in the

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Poland-Scheraga model and the total contact energy of a protein in the Wako-Saitô-Muñoz-Eaton model [16]. The fluctuations of extensive variables in small systems such as DNA molecules and proteins have become observable by the recent advent of micromanipulation techniques [19], motivating the development of a theoretical framework. A sharp large deviation principle for multivariate extensive observables in renewal models of statistical mechanics has been proposed in Ref. [16]. This principle comes from more general sharp large deviation principles established in Ref. [20] for cumulative rewards in constrained and non-constrained pinning models of polymers with broad-sense rewards taking values in a separable Banach space.

Large deviation principles represent a general tool for describing the exponential decay of probabilities of fluctuations in terms of a rate function. A thorough study of the rate functions associated with extensive observables in renewal models of statistical mechanics has been provided in Ref. [16]. There are however systems for which the decay of probabilities is slower than exponential and large deviation principles lose effectiveness due to a rate function that displays a wide region of zeros. Such systems generalize a polymer at the discontinuous pinning-depinning phase transition and have been identified and named *critical* in Ref. [16]. Critical renewal models are found in the theory of DNA and proteins since the DNA denaturation transition is generally believed to be discontinuous [21, 22] and the protein folding process is regarded as an “all-or-none” transition [23], which is a microscopic analog of a discontinuous phase transition. Phase transitions are investigated by changing a control parameter, such as the temperature or the denaturant concentration, and the hallmark of discontinuous phase transitions is a discontinuity in the graph of the expected value of some extensive observable as a function of the control parameter. The number of renewals, which counts for instance the pinned monomers in pinning models of polymers and the DNA-bound monomers in the Poland-Scheraga model, is the extensive observable commonly considered to characterize the phase transition in renewal models of statistical mechanics [16–18]. Criticality as defined in Ref. [16] however does not require to introduce and change a control parameter since the focus is on persistent fluctuations that lead to subexponential decay of probabilities, rather than on a comparison of the system in different conditions. Despite the large amount of work devoted to pinning models of polymers [17, 18], which among others has elucidated the sharp asymptotic behavior of partition functions by means of methods from renewal theory, to the best of our knowledge criticality in renewal models of statistical mechanics has never been investigated in terms of fluctuations.

While a large deviation principle for extensive observables in renewal models of statistical mechanics has been proposed in Ref. [16] with no assumption on the waiting time distribution, the study of critical fluctuations requires that some hypothesis is made. This paper aims to describe the sharp asymptotics of critical fluctuations, proposing a so-called precise large deviation principle, under the assumption that the subexponential correction term to the waiting time distribution is regularly varying. This case covers pinning models of polymers and the Poland-Scheraga model, for which the literature has mostly considered polynomial-tailed corrections [16–18]. The paper is organized as follows. Section 1 introduces pinning models and critical systems. Section 2 presents and discusses the main result of the paper and its application to the number of renewals. The proof of this result is reported in section 3, which resorts to three appendices for the most technical details.

## 1.1 Pinning models

Let on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  be given independent and identically distributed random variables  $S_1, S_2, \dots$  taking values in  $\{1, 2, \dots\} \cup \{\infty\}$ . The variable  $S_i$  can be regarded as the *waiting time* for the  $i$ th occurrence at the *renewal time*  $T_i := S_1 + \dots + S_i$  of some event that is continuously renewed over time. The pinning model considered in Refs. [16] and [20] makes use of this formalism to describe a polymer that is pinned by a substrate at the monomers  $T_1, T_2, \dots$ . The polymer is supposed to consist of  $t \geq 1$  monomers so that the monomer  $T_i$  contributes an energy  $-v(S_i)$  provided that  $T_i \leq t$ , the real function  $v$  defined on  $\{1, 2, \dots\} \cup \{\infty\}$  being called the *potential*. The state of the polymer is described by the

law  $\mathbb{P}_t$  on the measurable space  $(\Omega, \mathcal{F})$  given by the Gibbs change of measure

$$\frac{d\mathbb{P}_t}{d\mathbb{P}} := \frac{e^{H_t}}{Z_t},$$

where  $H_t := \sum_{i \geq 1} v(S_i) \mathbb{1}_{\{T_i \leq t\}}$  is the *Hamiltonian* and  $Z_t := \mathbb{E}[e^{H_t}]$  is the *partition function* ensuring normalization. The *pinning model* is the probabilistic model  $(\Omega, \mathcal{F}, \mathbb{P}_t)$  supplied with the hypotheses of aperiodicity and extensivity. We say that the *waiting time distribution*  $p := \mathbb{P}[S_1 = \cdot]$  is *aperiodic* if its support  $\mathcal{S} := \{s \geq 1 : p(s) > 0\}$  is nonempty and there does not exist an integer  $\tau > 1$  with the property that  $\mathcal{S}$  includes only some multiples of  $\tau$ . It is worth observing that aperiodicity of  $p$  can be obtained by simply changing the time unit whenever  $\mathbb{P}[S_1 < \infty] > 0$ .

**Assumption 1.1.** *The waiting time distribution  $p$  is aperiodic.*

We say that the potential  $v$  is *extensive* if  $\limsup_{s \uparrow \infty} (1/s) \ln e^{v(s)} p(s) < +\infty$ . Extensivity is necessary to make the thermodynamic limit, as  $t$  goes to infinity, of the pinning model meaningful since  $Z_t \geq \mathbb{E}[e^{H_t} \mathbb{1}_{\{S_1=t\}}] = e^{v(t)} p(t)$ .

**Assumption 1.2.** *The potential  $v$  is extensive.*

The *constrained pinning model* where the last monomer is forced to be always pinned by the substrate is the mathematical skeleton of the Poland-Scheraga model, the Fisher-Felderhof model, the Wako-Saitô-Muñoz-Eaton model, and the Tokar-Dreyssé model. According to Refs. [16] and [20], it corresponds to the law  $\mathbb{P}_t^c$  on the measurable space  $(\Omega, \mathcal{F})$  defined through the change of measure

$$\frac{d\mathbb{P}_t^c}{d\mathbb{P}} := \frac{U_t e^{H_t}}{Z_t^c},$$

where  $U_t := \sum_{i \geq 1} \mathbb{1}_{\{T_i=t\}}$  is the renewal indicator taking value 1 if and only if  $t$  is a renewal and  $Z_t^c := \mathbb{E}[U_t e^{H_t}]$  is the partition function. As explained in Ref. [16], the constraint that  $t$  is a renewal is needed to set the system size, that is the number of monomers per strand for the Poland-Scheraga model, the number of lattice sites for the Fisher-Felderhof model and the Tokar-Dreyssé model, and the number of peptide bonds for the Wako-Saitô-Muñoz-Eaton model, which turn out to be  $t$  (see [16], section 3). Aperiodicity of the waiting time distribution gives  $Z_t^c > 0$  for all sufficiently large  $t$  (see [20], section 1.1), thus ensuring that the constrained pinning model is well-defined at least for such  $t$ .

## 1.2 Deterministic rewards and critical systems

Let us suppose that the  $i$ th renewal involves a *deterministic reward*  $f(S_i)$ , where  $f$  is a function that maps  $\{1, 2, \dots\} \cup \{\infty\}$  in the Euclidean  $d$ -space  $\mathbb{R}^d$ . The extensive observables of renewal models of statistical mechanics are *cumulative reward* by the time  $t$  of the form  $W_t := \sum_{i \geq 1} f(S_i) \mathbb{1}_{\{T_i \leq t\}}$  with  $f(s)$  at most of the order of magnitude of  $s$  (see [16], section 3). The number  $N_t := \sum_{i \geq 1} \mathbb{1}_{\{T_i \leq t\}}$  of renewals by  $t$  is the cumulative reward associated with the function  $f$  identically equal to 1. We aim to characterize the large fluctuations of  $W_t$  under the following hypothesis inherited from Ref. [16].

**Assumption 1.3.** *If the support  $\mathcal{S}$  of the waiting time distribution is infinite, then  $f(s)/s$  has a limit  $r \in \mathbb{R}^d$  when  $s$  goes to infinity through  $\mathcal{S}$ .*

The study of the large fluctuations of  $W_t$  on the exponential scale has been carried out in Ref. [16]. This section collects those results of Ref. [16] that introduce the problem of critical fluctuations in renewal models of statistical mechanics and that will serve the proof of the main contribution of the present paper. In the thermodynamic limit, the scaled cumulative reward  $W_t/t$  converges in probability to a constant vector  $\rho \in \mathbb{R}^d$  under assumptions 1.1, 1.2, and 1.3, which are tacitly supposed to be satisfied in the sequel. In order to introduce  $\rho$ , let us set  $\ell := \limsup_{s \uparrow \infty} (1/s) \ln e^{v(s)} p(s)$ , which fulfills  $-\infty \leq \ell < +\infty$  by assumption

1.2, and if  $\ell > -\infty$ , then let us consider an “effective statistical weight”  $p_o$  for waiting times defined for all  $s \geq 1$  by

$$p_o(s) := e^{v(s)-\ell s} p(s). \quad (1.1)$$

If  $\ell = -\infty$  or  $\ell > -\infty$  and  $\sum_{s \geq 1} p_o(s) > 1$ , then let  $\zeta$  denote that unique real number larger than  $\ell$  that satisfies  $\sum_{s \geq 1} e^{v(s)-\zeta s} p(s) = 1$ . Bearing in mind that  $\mathcal{S}$  is necessarily infinite when  $\ell > -\infty$  and letting  $r$  be given by assumption 1.3, the vector  $\rho$  turns out to be

$$\rho := \begin{cases} \frac{\sum_{s \geq 1} f(s) e^{v(s)-\zeta s} p(s)}{\sum_{s \geq 1} s e^{v(s)-\zeta s} p(s)} & \text{if } \ell = -\infty \text{ or } \ell > -\infty \text{ and } \sum_{s \geq 1} p_o(s) > 1; \\ \frac{\sum_{s \geq 1} f(s) p_o(s)}{\sum_{s \geq 1} s p_o(s)} & \text{if } \ell > -\infty, \sum_{s \geq 1} p_o(s) = 1, \text{ and } \sum_{s \geq 1} s p_o(s) < +\infty; \\ r & \text{otherwise.} \end{cases}$$

We stress that  $\sum_{s \geq 1} s e^{v(s)-\zeta s} p(s)$  is finite and  $\sum_{s \geq 1} f(s) e^{v(s)-\zeta s} p(s)$  exists due to assumption 1.3 whenever  $\zeta$  is a real number larger than  $\ell$ . The following proposition states that  $W_t/t$  converges in probability to  $\rho$  as  $t$  is sent to infinity (see [16], theorem 4). It follows that  $W_t/t$  converges to  $\rho$  also in mean since  $W_t/t$  takes bounded values with probability 1. Indeed, assumption 1.3 implies that there exists a positive constant  $M < +\infty$  such that  $\|f(s)\| \leq Ms$  for all  $s \in \mathcal{S}$ , which gives  $\|W_t\| \leq M \sum_{i \geq 1} S_i \mathbb{1}_{\{T_i \leq t\}} \leq Mt$  with probability 1. Hereafter,  $u \cdot v$  denotes the usual dot product between  $u$  and  $v$  in  $\mathbb{R}^d$  and  $\|u\| := \sqrt{u \cdot u}$  is the Euclidean norm of  $u$ .

**Proposition 1.1.**  $\lim_{t \uparrow \infty} \mathbb{P}_t^c[\|W_t/t - \rho\| \geq \delta] = 0$  for any  $\delta > 0$ .

According to Ellis [24], we say that  $W_t/t$  converges exponentially to  $\rho$  if for any  $\delta > 0$  there exists a real number  $\lambda > 0$  such that  $\mathbb{P}_t^c[\|W_t/t - \rho\| \geq \delta] \leq e^{-\lambda t}$  for all sufficiently large  $t$ . The following result improves proposition 1.1 by identifying exponential convergence (see [16], theorem 4).

**Proposition 1.2.**  $W_t/t$  converges exponentially to  $\rho$  if and only if the conditions  $\ell > -\infty$ ,  $\sum_{s \geq 1} p_o(s) = 1$ ,  $\sum_{s \geq 1} s p_o(s) < +\infty$ , and  $\rho \neq r$  are not simultaneously satisfied.

Proposition 1.2 tells us that the convergence in probability to  $\rho$  of the scaled cumulative reward  $W_t/t$  is slower than exponential if  $\ell > -\infty$ ,  $\sum_{s \geq 1} p_o(s) = 1$ ,  $\sum_{s \geq 1} s p_o(s) < +\infty$ , and  $\rho \neq r$ . The facts that only the condition  $\rho \neq r$  involves the function  $f$  and that such condition is verified by most of  $f$  justify the following definition of critical model, which was originally proposed in Ref. [16]. We stress that the condition  $\sum_{s \geq 1} p_o(s) = 1$ , if fulfilled, promotes  $p_o$  to a probability distribution on  $\{1, 2, \dots\}$ , which we call *effective waiting time distribution*.

**Definition 1.1.** The constrained pinning model is critical if  $\ell > -\infty$ ,  $\sum_{s \geq 1} p_o(s) = 1$ , and  $\sum_{s \geq 1} s p_o(s) < +\infty$ .

We point out that this notion of criticality is borrowed from the literature on large deviation principles in statistical mechanics [25, 26], where the focus is on the breaking of exponential convergence. A different broader definition of critical system comes from the theory of random polymers [17, 18], where a control parameter  $\beta$  that plays the role of a binding energy can drive a pinning-depinning phase transition. We recover the standard framework of polymers by taking  $v(s) = \beta$  for every  $s$ , and in this framework our critical scenario corresponds to the conditions  $\ell = \limsup_{s \uparrow \infty} (1/s) \ln p(s) > -\infty$ ,  $\sum_{s \geq 1} s e^{-\ell s} p(s) < +\infty$ , and  $\beta = \beta_c$  with  $\beta_c := -\ln \sum_{s \geq 1} e^{-\ell s} p(s)$ . Clearly,  $\beta_c$  is finite if  $\sum_{s \geq 1} s e^{-\ell s} p(s) < +\infty$ . These conditions identify a discontinuous pinning-depinning phase transition (see [17], theorem 2.1, or [18], theorem 7.4), leaving out continuous phase transitions which preserve exponential convergence. In order to clarify the point and make contact with the literature [16–18] that looks at the expected value of the number  $N_t$  of renewals by  $t$  to characterize the phase transition, let us recall that  $N_t$  is  $W_t$  when the function  $f$  is identically equal to 1 and let us observe that this function satisfies assumption 1.3 with  $r = 0$ . Then,  $N_t/t$  converges in probability and in mean to  $\rho$  by proposition 1.1. One can verify (see [16], section 4.4.1) that

$\rho$  as a function of  $\beta$  is analytic on  $(-\infty, +\infty)$  if  $\ell = -\infty$  or  $\ell > -\infty$  and  $\beta_c = -\infty$ . If instead  $\ell > -\infty$  and  $\beta_c > -\infty$ , then  $\rho$  is positive and analytic on the open interval  $(\beta_c, +\infty)$ , continuous on the closed interval  $[\beta_c, +\infty)$ , and equal to 0 for all  $\beta < \beta_c$ . In this case there is a phase transition for  $\beta = \beta_c$ . The phase transition is continuous, namely  $\rho$  as a function of  $\beta$  is continuous at  $\beta_c$ , if  $\sum_{s \geq 1} s e^{-\ell s} p(s) = +\infty$ . On the contrary, if  $\sum_{s \geq 1} s e^{-\ell s} p(s) < +\infty$ , then  $\rho$  jumps from the value  $w_c := \frac{\sum_{s \geq 1} e^{-\ell s} p(s)}{\sum_{s \geq 1} s e^{-\ell s} p(s)} > 0$  at  $\beta_c$  to the value 0 when  $\beta$  is let to decrease and the phase transition is discontinuous.

Precise exponential rates for probability decays are provided by large deviation principles. The cumulative reward  $W_t$  satisfies a *large deviation principle* with *good rate function* according to the following theorem (see [16], theorem 1).

**Theorem 1.1.** *There exists a proper convex lower semicontinuous function  $I$  from  $\mathbb{R}^d$  to  $[0, \infty]$  with compact level sets such that*

- (a)  $\liminf_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P}_t^c \left[ \frac{W_t}{t} \in G \right] \geq -\inf_{w \in G} \{I(w)\}$  for each open set  $G \subseteq \mathbb{R}^d$ ;
- (b)  $\limsup_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P}_t^c \left[ \frac{W_t}{t} \in F \right] \leq -\inf_{w \in F} \{I(w)\}$  for each Borel convex set  $F \subseteq \mathbb{R}^d$  or closed set  $F \subseteq \mathbb{R}^d$ .

The function  $I$  is the rate function and the compactness of its level sets entails that  $\inf_{w \in F} \{I(w)\} > 0$  whenever  $F$  is a closed set that does not contain a zero of  $I$  (see [16], section 4.3). The explicit expression of  $I$  as the convex conjugate of the scaled cumulant generating function has been given in Ref. [16], where the zeros of  $I$  have in particular been determined (see [16], section 4.3) and the following proposition has been deduced,  $r$  being the vector of assumption 1.3.

**Proposition 1.3.** *Let  $\mathcal{Z}$  be the set of zeroes of  $I$ . Then*

- (a)  $\mathcal{Z} = \{\rho\}$  if the model is not critical;
- (b)  $\mathcal{Z} = \{(1 - \alpha)r + \alpha\rho : \alpha \in [0, 1]\}$  if the model is critical.

The closed line segment  $\mathcal{Z}$  is not a singleton at criticality if  $\rho \neq r$  and, according to the literature on large deviation principles in statistical mechanics [25, 26], we call it the *phase transition segment*. Thus, whatever  $f$  is, the scaled cumulative reward  $W_t/t$  converges exponentially to  $\rho$  in the non-critical scenario with exponential rate  $\inf_{w \in F} \{I(w)\} > 0$  for the probability of a fluctuation over a closed set  $F$  that does not contain  $\rho$ . On the contrary, convergence in probability of  $W_t/t$  to  $\rho$  is slower than exponential in the critical constrained pinning model provided that  $f$  obeys  $\rho \neq r$ . In this case, the above large deviation principle tells us that the probability that  $W_t/t$  fluctuates over a closed set  $F$  that does not intersect the phase transition segment  $\mathcal{Z}$  decays exponentially with rate  $\inf_{w \in F} \{I(w)\} > 0$ , whereas it says nothing about the fluctuations that reach  $\mathcal{Z}$  and that prevent exponential convergence. The situation  $\rho = r$  constitutes an exception because convergence is exponential even in the critical scenario.

## 2 Main results

The present paper describes, in a critical scenario with regularly varying corrections to the waiting time distribution, the fluctuations of the scaled cumulative reward  $W_t/t$  that reach the phase transition segment  $\mathcal{Z}$ . Let us suppose that the constrained pinning model is critical and that  $\rho \neq r$ . We tackle the issue by studying the fluctuations in the closed half-space  $\mathcal{H}_\alpha$  that contains the fraction  $\alpha \in [0, 1)$  starting from  $r$  of the segment  $\mathcal{Z}$  and that is delimited by the hyperplane normal to it:

$$\mathcal{H}_\alpha := \left\{ w \in \mathbb{R}^d : [\rho - r] \cdot [w - (1 - \alpha)r - \alpha\rho] \leq 0 \right\}.$$

At large  $t$ , the probability  $\mathbb{P}_t^c[W_t/t \in \mathcal{H}_\alpha]$  is dominated by the fluctuations of  $W_t/t$  over the phase transition segment that prevent exponential convergence because we know that  $\mathbb{P}_t^c[W_t/t \in \mathcal{H}_\alpha \cap F]$  decays exponentially fast for every closed set  $F$  that does not intersect  $\mathcal{Z}$ .

## 2.1 A precise large deviation principle

We determine the large- $t$  behavior of the probability  $\mathbb{P}_t^c[W_t/t \in \mathcal{H}_\alpha]$  under the assumption that there exist a real number  $\kappa \geq 1$  and a slowly varying function  $\mathcal{L}$  such that for each  $s \geq 1$

$$p_o(s) = \frac{\mathcal{L}(s)}{s^{\kappa+1}}, \quad (2.1)$$

$p_o$  being the effective waiting time distribution defined by (1.1). A measurable function  $\mathcal{L}$  on the positive semi-axis is *slowly varying* if it is positive on some neighborhood of infinity and satisfies the scale-invariance property  $\lim_{x \uparrow +\infty} \mathcal{L}(\gamma x)/\mathcal{L}(x) = 1$  for any number  $\gamma > 0$ . Trivially, a measurable function  $\mathcal{L}$  with a positive limit at infinity is slowly varying. The simplest non-trivial example is represented by the logarithm. Due to slow variation of  $\mathcal{L}$ , the sequence whose  $s$ th term is  $p_o(s)$  is *regularly varying with index  $-\kappa - 1$* . We refer to [27] for the theory of slow and regular variation. We point out that the constraint  $\kappa \geq 1$  reflects our focus on critical models, which fulfill  $\sum_{s \geq 1} s p_o(s) < +\infty$ . Indeed, since  $\mathcal{L}(x) \geq x^{-\delta}$  for any fixed  $\delta > 0$  and all sufficiently large  $x$  (see [27], proposition 1.3.6),  $\kappa < 1$  would entail  $\sum_{s \geq 1} s p_o(s) = +\infty$ . The following theorem is the main result of the paper.

**Theorem 2.1.** *Suppose that assumptions 1.1, 1.2, and 1.3 are satisfied, that the model is critical, and that  $\rho \neq r$ . If there exist a real number  $\kappa \geq 1$  and a slowly varying function  $\mathcal{L}$  such that (2.1) holds, then for every  $\alpha \in [0, 1]$*

$$\lim_{t \uparrow \infty} \frac{\mathbb{P}_t^c[W_t/t \in \mathcal{H}_\alpha]}{t^{1-\kappa} \mathcal{L}(t)} = \frac{1}{\sum_{s \geq 1} s p_o(s)} \begin{cases} \frac{\alpha}{1-\alpha} + \ln(1-\alpha) & \text{if } \kappa = 1; \\ \frac{1+(\alpha\kappa-1)(1-\alpha)^{-\kappa}}{\kappa(\kappa-1)} & \text{if } \kappa > 1. \end{cases}$$

Theorem 2.1 states a precise large deviation principle for the cumulative reward  $W_t$ . It completes the picture of the large fluctuations of extensive variables in renewal models of statistical mechanics by supplying the leading order of probabilities at criticality, where a large deviation principle loses effectiveness. It is worth noting that the function  $f$  of rewards enters only the shape of the half-space  $\mathcal{H}_\alpha$  through  $\rho$  and  $r$ , so that, basically, all extensive observables in critical models display the same fluctuations, whose probability decays polynomially in the system size under our assumption of regular variation.

The proof of theorem 2.1 is provided in section 3. We stress that the cumulative reward is a random sum of random variables, which can be written down explicitly by introducing the number  $N_t$  of renewals by  $t$  as  $W_t = \sum_{i=1}^{N_t} X_i$  with  $X_i := f(S_i)$  for any  $i$ . Precise large deviations for random sums of random variables with several types of subexponential distributions have been investigated by many researchers [28–34]. However, their work does not cover our case in two respects. First, the hypothesis they have made is that the random variables  $X_1, X_2, \dots$  are independent of the counting process  $N_t$ , but this hypothesis is not satisfied by our problem. Second, we had to implement the constraint that  $t$  is a renewal in order to set the system size of renewal models of statistical mechanics at  $t$ , whereas they did not have such a special need. For these reasons, theorem 2.1 is a new result that requires a new proof, which is guided by the idea that the only significant way in which a large fluctuation of  $W_t$  occurs at criticality is that one and only one of the waiting times takes a large value. This idea underlies many heavy-tailed problems and is known with the folklore name of *principle of a single big jump* [35], whose a detailed picture can be given for sums of independent and identically distributed random variables [36, 37].

## 2.2 Critical fluctuations of $N_t$

As shown in Ref. [16], renewal models of statistical mechanics can be mapped in a standard constrained pinning model where  $v(s) = \beta$  for any  $s$ ,  $\beta$  being a real number that acts as a control parameter. Precisely,  $\beta$  is the binding energy for the Poland-Scheraga model, the chemical potential for the Fisher-Felderhof model and the Tokar-Dreyssé model, and an entropic loss for the Wako-Saitô-Muñoz-Eaton model (see [16], section 3). From section 1.2 we know that criticality is achieved if  $\ell = \limsup_{s \uparrow \infty} (1/s) \ln p(s) > -\infty$ ,  $\sum_{s \geq 1} s e^{-\ell s} p(s) < +\infty$ , and  $\beta = \beta_c$  with  $\beta_c := -\ln \sum_{s \geq 1} e^{-\ell s} p(s)$ . Let us suppose here that  $\ell > -\infty$  and

that  $\sum_{s \geq 1} s e^{-\ell s} p(s) < +\infty$  and let us investigate the fluctuations of the number  $N_t$  of renewals by  $t$  for different values of  $\beta$ . The extensive observable  $N_t$  measures the DNA-bound monomers in the Poland-Scheraga model, the particles in the Fisher-Felderhof model and the Tokar-Dreyssé model, and the native peptide bonds in the Wako-Saitô-Muñoz-Eaton model (see [16], section 3). By proposition 1.1,  $N_t/t$  converges in probability and in mean to  $\rho = \frac{\sum_{s \geq 1} e^{-\zeta s} p(s)}{\sum_{s \geq 1} s e^{-\zeta s} p(s)}$  for  $\beta > \beta_c$  with  $\zeta > \ell$  satisfying  $\sum_{s \geq 1} e^{\beta - \zeta s} p(s) = 1$ , to  $\rho = w_c := \frac{\sum_{s \geq 1} e^{-\ell s} p(s)}{\sum_{s \geq 1} s e^{-\ell s} p(s)}$  for  $\beta = \beta_c$ , and to  $\rho = 0$  for  $\beta < \beta_c$ . By combining parts (a) and (b) of theorem 1.1 we find for any  $\beta$  and open interval  $G$

$$\lim_{t \uparrow \infty} \frac{1}{t} \ln \mathbb{P}_t^c \left[ \frac{N_t}{t} \in G \right] = - \inf_{w \in G} \{I(w)\}.$$

The rate function  $I$  has been determined in Ref. [16] and for each real  $w$  takes the value (see [16], section 4.4.2)

$$I(w) := \begin{cases} w(\beta_c - \beta) - \ell & \text{if } w \in [0, w_c]; \\ -w \ln \left[ \sum_{s \geq 1} e^{\beta - \eta s} p(s) \right] - \eta & \text{if } w \in (w_c, 1); \\ -\ln [e^\beta p(1)] & \text{if } w = 1; \\ +\infty & \text{otherwise} \end{cases} + \begin{cases} \zeta & \text{if } \beta > \beta_c; \\ \ell & \text{if } \beta \leq \beta_c, \end{cases}$$

where  $\eta$  is the unique number larger than  $\ell$  that fulfills  $\frac{\sum_{s \geq 1} e^{-\eta s} p(s)}{\sum_{s \geq 1} s e^{-\eta s} p(s)} = w$ . The rate function  $I$  has an affine stretch on  $[0, w_c]$ , is analytic on  $(w_c, 1)$ , and is continuously differentiable at  $w_c$  (see [16], section 4.4.2). If  $\beta \neq \beta_c$ , then the model is not critical and  $I(w) = 0$  only for  $w = \rho$ . If instead  $\beta = \beta_c$ , then the model is critical and  $I(w) = 0$  for all  $w \in [0, w_c]$ . The phase transition segment  $\mathcal{Z}$  is the closed interval  $[0, w_c]$ .

Theorem 2.1 completes the study of the large fluctuations of  $N_t$  initiated in Ref. [16] by resolving for every  $\alpha \in [0, 1)$  the subexponential decay at criticality of the probability  $\mathbb{P}_t^c[N_t \leq \alpha w_c t]$ . Indeed, under the assumption that there exist an index  $\kappa \geq 1$  and a slowly varying function  $\mathcal{L}$  such that  $e^{-\ell s} p(s) = s^{-\kappa-1} \mathcal{L}(s)$  for all  $s \geq 1$ , as  $\mathcal{H}_\alpha = (-\infty, \alpha w_c]$  theorem 2.1 shows that

$$\lim_{t \uparrow \infty} \frac{\mathbb{P}_t^c[N_t \leq \alpha w_c t]}{t^{1-\kappa} \mathcal{L}(t)} = \frac{1}{\sum_{s \geq 1} s e^{-\ell s} p(s)} \begin{cases} \frac{\alpha}{1-\alpha} + \ln(1-\alpha) & \text{if } \kappa = 1; \\ \frac{1 + (\alpha\kappa - 1)(1-\alpha)^{-\kappa}}{\kappa(\kappa-1)} & \text{if } \kappa > 1. \end{cases}$$

### 3 Proof of theorem 2.1

In this section we report the proof of theorem 2.1, postponing the most technical details in the appendices in order not to interrupt the flow of the presentation. The proof is facilitated by a natural change of measure. Regarding the effective waiting time distribution  $p_o$  as a new waiting time distribution, which is non-defective because  $\sum_{s \geq 1} p_o(s) = 1$  by the hypothesis of criticality, let us consider a new probability space  $(\Omega_o, \mathcal{F}_o, \mathbb{P}_o)$  where a sequence  $\{S_i\}_{i \geq 1}$  of independent waiting times distributed according to  $p_o$  is given. Denoting by  $\mathbb{E}_o$  the expectation with respect to  $\mathbb{P}_o$ , we have  $\mathbb{E}_o[S_1] = \sum_{s \geq 1} s p_o(s) < +\infty$  and  $\mathbb{E}_o[f(S_1)] = \rho \mathbb{E}_o[S_1]$  again by criticality. The tail probability  $\mathcal{Q}$  of  $S_1$  under the law  $\mathbb{P}_o$  is the measurable function that maps each real number  $x > 0$  in

$$\mathcal{Q}(x) := \mathbb{P}_o[S_1 > x] = \sum_{s \geq 1} \mathbf{1}_{\{s > x\}} p_o(s).$$

The hypothesis that  $p_o(s) = s^{-\kappa-1} \mathcal{L}(s)$  for all  $s \geq 1$  with  $\mathcal{L}$  a slowly varying function results in regular variation with index  $-\kappa$  for  $\mathcal{Q}$ , meaning that  $\lim_{x \uparrow +\infty} \mathcal{Q}(\gamma x) / \mathcal{Q}(x) = \gamma^{-\kappa}$  for any number  $\gamma > 0$ . Indeed, the following lemma holds as demonstrated in appendix A.

**Lemma 3.1.**  $\mathcal{Q}$  varies regularly with index  $-\kappa$  and  $\lim_{x \uparrow +\infty} x^\kappa \mathcal{Q}(x) / \mathcal{L}(x) = 1 / \kappa$ .

The transition to the new probabilistic model  $(\Omega_o, \mathcal{F}_o, \mathbb{P}_o)$  proceeds as follows. By observing that  $\prod_{i=1}^n e^{v(s_i)} p(s_i) = e^{\ell t} \prod_{i=1}^n p_o(s_i)$  whenever  $s_1 + \dots + s_n = t$ , for every integer  $t \geq 1$  and Borel set  $\mathcal{B} \subseteq \mathbb{R}^d$  we find

$$\begin{aligned}
Z_t^c \cdot \mathbb{P}_t^c \left[ \frac{W_t}{t} \in \mathcal{B} \right] &= \mathbb{E} \left[ \mathbb{1}_{\left\{ \frac{W_t}{t} \in \mathcal{B} \right\}} U_t e^{H_t} \right] \\
&= \sum_{n \geq 1} \mathbb{E} \left[ \mathbb{1}_{\left\{ \frac{1}{t} \sum_{i=1}^n f(S_i) \in \mathcal{B} \right\}} \mathbb{1}_{\{T_n=t\}} e^{\sum_{i=1}^n v(S_i)} \right] \\
&= \sum_{n \geq 1} \sum_{s_1 \geq 1} \dots \sum_{s_n \geq 1} \mathbb{1}_{\left\{ \frac{1}{t} \sum_{i=1}^n f(s_i) \in \mathcal{B} \right\}} \mathbb{1}_{\{s_1 + \dots + s_n = t\}} \prod_{i=1}^n e^{v(s_i)} p(s_i) \\
&= e^{\ell t} \sum_{n \geq 1} \sum_{s_1 \geq 1} \dots \sum_{s_n \geq 1} \mathbb{1}_{\left\{ \frac{1}{t} \sum_{i=1}^n f(s_i) \in \mathcal{B} \right\}} \mathbb{1}_{\{s_1 + \dots + s_n = t\}} \prod_{i=1}^n p_o(s_i) \\
&= e^{\ell t} \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \frac{W_t}{t} \in \mathcal{B} \right\}} U_t \right]. \tag{3.1}
\end{aligned}$$

This identity with  $\mathcal{B} = \mathbb{R}^d$  yields  $Z_t^c = e^{\ell t} \mathbb{E}_o[U_t]$ , which allows us to recast (3.1) as

$$\mathbb{E}_o[U_t] \cdot \mathbb{P}_t^c \left[ \frac{W_t}{t} \in \mathcal{B} \right] = \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \frac{W_t}{t} \in \mathcal{B} \right\}} U_t \right]. \tag{3.2}$$

Formula (3.2) is precisely the bridge between constrained pinning models with respect to  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $(\Omega_o, \mathcal{F}_o, \mathbb{P}_o)$ . We observe that  $\lim_{t \uparrow \infty} \mathbb{E}_o[U_t] = 1/\mathbb{E}_o[S_1]$  since  $p_o$  is non-defective. This limit is due to an application of the renewal theorem (see [38], theorem 1 in chapter XIII.10) to the renewal equation  $\mathbb{E}_o[U_t] = \sum_{s=1}^t p_o(s) \mathbb{E}_o[U_{t-s}]$  valid for every  $t \geq 1$ . The renewal equation is deduced by conditioning on  $T_1 = S_1$  and then by using the fact that a renewal process starts over at every renewal.

Fix  $\alpha \in [0, 1)$ . If  $t$  is a renewal, then  $\sum_{i \geq 1} S_i \mathbb{1}_{\{T_i \leq t\}} = t$  and the condition  $W_t/t \in \mathcal{H}_\alpha$ , namely  $[\rho - r] \cdot [W_t/t - (1 - \alpha)r - \alpha\rho] \leq 0$ , becomes  $\sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq t\}} \leq \alpha t$  with

$$g(s) := \frac{[\rho - r] \cdot [f(s) - rs]}{\|\rho - r\|^2}.$$

This way, by setting  $\mathcal{B} = \mathcal{H}_\alpha$  in (3.2) we find for each  $t \geq 1$

$$\mathbb{E}_o[U_t] \cdot \mathbb{P}_t^c \left[ \frac{W_t}{t} \in \mathcal{H}_\alpha \right] = \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq t\}} \leq \alpha t \right\}} U_t \right] =: \mathcal{E}(t).$$

Since  $\lim_{t \uparrow \infty} \mathbb{E}_o[U_t] = 1/\mathbb{E}_o[S_1]$  and  $\lim_{t \uparrow \infty} t^\kappa \mathcal{Q}(t)/\mathcal{L}(t) = 1/\kappa$  by lemma 3.1, it follows that in order to prove theorem 2.1 it suffices to demonstrate that

$$\begin{aligned}
\lim_{t \uparrow \infty} \frac{\mathcal{E}(t)}{t \mathcal{Q}(t)} &= \frac{1}{\mathbb{E}_o[S_1]^2} \begin{cases} \frac{\alpha}{1-\alpha} + \ln(1-\alpha) & \text{if } \kappa = 1; \\ \frac{1 + (\alpha\kappa - 1)(1-\alpha)^{-\kappa}}{\kappa - 1} & \text{if } \kappa > 1 \end{cases} \\
&= \frac{1}{\mathbb{E}_o[S_1]^2} \left[ \alpha(1-\alpha)^{-\kappa} - \int_{1-\alpha}^1 \frac{dx}{x^\kappa} \right]. \tag{3.3}
\end{aligned}$$

We shall demonstrate (3.3) by verifying a lower bound first and an upper bound later. The features of the function  $g$  we will use to this aim are the equality  $\mathbb{E}_o[g(S_1)] = \mathbb{E}_o[S_1]$  and the fact that  $g(s)/s$  goes to zero when  $s$  is sent to infinity through  $\mathcal{S}$ , which is the support of both  $p$  and  $p_o$ . Since the values  $g(s)$  when  $s \notin \mathcal{S}$  do not affect the problem, in order to simplify the notations we redefine the function  $g$  by setting  $g(s) := 0$  for all  $s \notin \mathcal{S}$  so that the limit  $\lim_{s \uparrow \infty} g(s)/s = 0$  is valid.

### 3.1 A lower bound

In order to prove (3.3) we show at first that

$$\liminf_{t \uparrow \infty} \frac{\mathcal{E}(t)}{t \mathcal{Q}(t)} \geq \frac{1}{\mathbb{E}_o[S_1]^2} \left[ \alpha(1-\alpha)^{-\kappa} - \int_{1-\alpha}^1 \frac{dx}{x^\kappa} \right]. \tag{3.4}$$



This bound is trivial if  $\alpha = 0$  since the l.h.s. of (3.4) is not negative. In the case  $\alpha > 0$ , it follows if we demonstrate that for all real numbers  $\gamma, \eta$ , and  $\epsilon$  such that  $1 - \alpha < \gamma < \eta < 1$  and  $\epsilon > 0$

$$\liminf_{t \uparrow \infty} \frac{\mathcal{E}(t)}{t \mathcal{Q}(t)} \geq \frac{1 - 2\epsilon}{\mathbb{E}_o[S_1]^2} \left[ (1 - \gamma)\gamma^{-\kappa} - (1 - \eta)\eta^{-\kappa} - \int_{\gamma}^{\eta} \frac{dx}{x^{\kappa}} \right]. \quad (3.5)$$

Indeed, (3.5) gives (3.4) when  $\gamma$  is sent to  $1 - \alpha$ ,  $\eta$  is sent to 1, and  $\epsilon$  is sent to 0.

Let us assume  $\alpha > 0$  and let us pick  $\gamma, \eta$ , and  $\epsilon$  such that  $1 - \alpha < \gamma < \eta < 1$  and  $\epsilon > 0$ . To get at a proof of (3.5) we observe that  $3y/2 - y^2/2 \leq 1$  for any integer  $y \in \mathbb{Z}$ , so that for each  $n \geq 2$  we find

$$1 \geq \frac{3}{2} \sum_{j=1}^n \mathbb{1}_{\{S_j > \gamma t\}} - \frac{1}{2} \left[ \sum_{j=1}^n \mathbb{1}_{\{S_j > \gamma t\}} \right]^2 = \sum_{j=1}^n \mathbb{1}_{\{S_j > \gamma t\}} - \sum_{j=1}^{n-1} \sum_{k=j+1}^n \mathbb{1}_{\{S_j > \gamma t\}} \mathbb{1}_{\{S_k > \gamma t\}}.$$

It follows that for all  $t \geq 2$

$$\begin{aligned} \mathcal{E}(t) &:= \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq t\}} \leq \alpha t \right\}} U_t \right] = \sum_{n=1}^t \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i=1}^n g(S_i) \leq \alpha t \right\}} \mathbb{1}_{\{T_n = t\}} \right] \\ &\geq \sum_{n=1}^t \sum_{j=1}^n \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i=1}^n g(S_i) \leq \alpha t \right\}} \mathbb{1}_{\{S_j > \gamma t\}} \mathbb{1}_{\{S_1 + \dots + S_n = t\}} \right] + \\ &\quad - \sum_{n=2}^t \sum_{j=1}^{n-1} \sum_{k=j+1}^n \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i=1}^n g(S_i) \leq \alpha t \right\}} \mathbb{1}_{\{S_j > \gamma t\}} \mathbb{1}_{\{S_k > \gamma t\}} \mathbb{1}_{\{S_1 + \dots + S_n = t\}} \right] \\ &\geq \sum_{n=1}^t n \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i=1}^n g(S_i) \leq \alpha t \right\}} \mathbb{1}_{\{S_n > \gamma t\}} \mathbb{1}_{\{T_n = t\}} \right] - t^2 \mathbb{E}_o \left[ \mathbb{1}_{\{S_1 > \gamma t\}} \mathbb{1}_{\{S_2 > \gamma t\}} U_t \right] \\ &\geq \sum_{n=1}^t n \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i=1}^n g(S_i) \leq \alpha t \right\}} \mathbb{1}_{\{S_n > \gamma t\}} \mathbb{1}_{\{T_n = t\}} \right] - [t \mathcal{Q}(\gamma t)]^2. \end{aligned} \quad (3.6)$$

Since  $\alpha + \gamma - 1 > 0$  by hypothesis, the limit  $\lim_{s \uparrow \infty} g(s)/s = 0$  entails that there exists a positive integer  $t_1$  such that  $g(s) \leq (\alpha + \gamma - 1)s$  whenever  $s > \gamma t_1$ . If  $t > t_1$ , then the conditions  $s > \gamma t$  and  $\sum_{i=1}^n g(S_i) < (\alpha + \gamma)(t - s)$  together imply

$$\sum_{i=1}^n g(S_i) + g(s) \leq (\alpha + \gamma)(t - s) + (\alpha + \gamma - 1)s = (\alpha + \gamma)t - s \leq \alpha t.$$

This way, from (3.6) we get for every  $t > t_1 \geq 1$

$$\begin{aligned} \mathcal{E}(t) &\geq \sum_{n=2}^t \sum_{s=1}^{t-n+1} n \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i=1}^{n-1} g(S_i) + g(s) \leq \alpha t \right\}} \mathbb{1}_{\{T_{n-1} = t-s\}} \right] \mathbb{1}_{\{s > \gamma t\}} p_o(s) - [t \mathcal{Q}(\gamma t)]^2 \\ &\geq \sum_{s=1}^{t-1} \sum_{n=1}^{t-s} n \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i=1}^n g(S_i) + g(s) \leq \alpha t \right\}} \mathbb{1}_{\{T_n = t-s\}} \right] \mathbb{1}_{\{s > \gamma t\}} p_o(s) - [t \mathcal{Q}(\gamma t)]^2 \\ &\geq \sum_{s=1}^{t-1} \sum_{n=1}^{t-s} n \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i=1}^n g(S_i) < (\alpha + \gamma)(t-s) \right\}} \mathbb{1}_{\{T_n = t-s\}} \right] \mathbb{1}_{\{s > \gamma t\}} p_o(s) - [t \mathcal{Q}(\gamma t)]^2 \\ &= \sum_{s=1}^{t-1} \mathbb{E}_o \left[ N_{t-s} \mathbb{1}_{\left\{ \sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq t-s\}} < (\alpha + \gamma)(t-s) \right\}} U_{t-s} \right] \mathbb{1}_{\{s > \gamma t\}} p_o(s) - [t \mathcal{Q}(\gamma t)]^2. \end{aligned}$$

By introducing the restriction  $s < \eta t$ , for any  $t > t_1$  we obtain the further lower bound

$$\begin{aligned} \mathcal{E}(t) &\geq \sum_{s \geq 1} \mathbb{E}_o \left[ N_{t-s} \mathbb{1}_{\left\{ \sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq t-s\}} < (\alpha + \gamma)(t-s) \right\}} U_{t-s} \right] \mathbb{1}_{\{\gamma t < s \leq \eta t\}} p_o(s) - [t \mathcal{Q}(\gamma t)]^2 \\ &\geq \sum_{s \geq 1} \mathbb{E}_o [N_{t-s} U_{t-s}] \mathbb{1}_{\{\gamma t < s \leq \eta t\}} p_o(s) - [t \mathcal{Q}(\gamma t)]^2 + \\ &\quad - \sum_{s \geq 1} (t-s) \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq t-s\}} \geq (\alpha + \gamma)(t-s) \right\}} U_{t-s} \right] \mathbb{1}_{\{\gamma t < s \leq \eta t\}} p_o(s), \end{aligned} \quad (3.7)$$

where the fact that  $N_{t-s} \leq t-s$  has been used in the last equality.

To continue estimation we need the following lemma, which is proved in appendix B.

**Lemma 3.2.**  $\lim_{t \uparrow \infty} \mathbb{E}_o[(N_t/t)U_t] = 1/\mathbb{E}_o[S_1]^2$ .

We also observe that  $\lim_{t \uparrow \infty} \mathbb{E}_o[\mathbb{1}_{\{\sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq \tau\}} \geq (\alpha + \gamma)\tau\}} U_\tau] = 0$  since formula (3.2) and proposition 1.1 with  $g$  in place of  $f$  yield respectively  $\mathbb{E}_o[\mathbb{1}_{\{\sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq \tau\}} \geq (\alpha + \gamma)\tau\}} U_\tau] \leq \mathbb{P}_\tau^c[\sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq \tau\}} \geq (\alpha + \gamma)\tau]$  and  $\lim_{\tau \uparrow \infty} \mathbb{P}_\tau^c[\sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq \tau\}} \geq (\alpha + \gamma)\tau] = 0$  as  $\mathbb{E}_o[g(S_1)]/\mathbb{E}_o[S_1] = 1$  and  $\alpha + \gamma > 1$ . Then, by these arguments we deduce that there exists  $t_2 \geq t_1$  such that  $\tau > (1 - \eta)t_2$  implies both  $\mathbb{E}_o[N_\tau U_\tau] \geq (1 - \epsilon)\tau/\mathbb{E}_o[S_1]^2$  and  $\mathbb{E}_o[\mathbb{1}_{\{\sum_{i \geq 1} g(S_i) \mathbb{1}_{\{T_i \leq \tau\}} \geq (\alpha + \gamma)\tau\}} U_\tau] \leq \epsilon/\mathbb{E}_o[S_1]^2$ . If  $t > t_2$ , then the condition  $s \leq \eta t$  giving  $t-s > (1 - \eta)t_2$  allows us to replace (3.7) with

$$\begin{aligned} \mathcal{E}(t) &\geq \frac{1-2\epsilon}{\mathbb{E}_o[S_1]^2} \sum_{s \geq 1} (t-s) \mathbb{1}_{\{\gamma t < s \leq \eta t\}} p_o(s) - [t \mathcal{Q}(\gamma t)]^2 \\ &= \frac{1-2\epsilon}{\mathbb{E}_o[S_1]^2} \left[ (1-\gamma)t \mathcal{Q}(\gamma t) - (1-\eta)t \mathcal{Q}(\eta t) - t \int_\gamma^\eta \mathcal{Q}(x) dx \right] - [t \mathcal{Q}(\gamma t)]^2. \end{aligned}$$

From here, we obtain (3.5) by dividing by  $t \mathcal{Q}(t)$  first and by sending  $t$  to infinity later because on the one hand the limit  $\lim_{t \uparrow \infty} \mathcal{Q}(x t)/\mathcal{Q}(t) = x^{-\kappa}$  is uniform with respect to  $x$  in the compact interval  $[\gamma, \eta]$  (see [27], theorem 1.5.2), and on the other hand  $\lim_{t \uparrow \infty} t \mathcal{Q}(t) = 0$  as  $t \mathcal{Q}(t) \leq \mathbb{E}_o[S_1 \mathbb{1}_{\{S_1 > t\}}]$  and  $\mathbb{E}_o[S_1] < +\infty$ .

### 3.2 An upper bound

Now we show that for all real numbers  $\gamma, \eta$ , and  $\epsilon$  such that  $0 < \gamma < 1 - \alpha$ ,  $\gamma < \eta < 1$ , and  $\epsilon > 0$

$$\limsup_{t \uparrow \infty} \frac{\mathcal{E}(t)}{t \mathcal{Q}(t)} \leq \frac{1+\epsilon}{\mathbb{E}_o[S_1]^2} \left[ (1-\gamma)\gamma^{-\kappa} - (1-\eta)\eta^{-\kappa} - \int_\gamma^\eta \frac{dx}{x^\kappa} \right] + \eta^{-\kappa} - 1. \quad (3.8)$$

In the light of (3.4), this bound leads us to prove (3.3) because sending  $\gamma$  to  $1 - \alpha$ ,  $\eta$  to 1, and  $\epsilon$  to 0 we find

$$\limsup_{t \uparrow \infty} \frac{\mathcal{E}(t)}{t \mathcal{Q}(t)} \leq \frac{1}{\mathbb{E}_o[S_1]^2} \left[ \alpha(1-\alpha)^{-\kappa} - \int_{1-\alpha}^1 \frac{dx}{x^\kappa} \right].$$

Pick  $\gamma, \eta$ , and  $\epsilon$  such that  $0 < \gamma < 1 - \alpha$ ,  $\gamma < \eta < 1$ , and  $\epsilon > 0$ . Obviously, for every  $n \geq 1$  and  $t \geq 1$  we have

$$1 \leq \prod_{j=1}^n \mathbb{1}_{\{S_j \leq \gamma t\}} + \sum_{j=1}^n \mathbb{1}_{\{S_j > \gamma t\}}$$

since either  $S_j \leq \gamma t$  for all  $j \leq n$  or  $S_j > \gamma t$  for at least one  $j \leq n$ . It follows that for any  $t \geq 1$

$$\mathcal{E}(t) = \sum_{n=1}^t \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \sum_{i=1}^n g(S_i) \leq \alpha t \right\}} \mathbb{1}_{\{T_n = t\}} \right] \leq \mathcal{E}_1(t) + \mathcal{E}_2(t) \quad (3.9)$$

with

$$\mathcal{E}_1(t) := \sum_{n=1}^t \mathbb{E}_o \left[ \mathbf{1}_{\{\sum_{i=1}^n g(S_i) \leq \alpha t\}} \prod_{j=1}^n \mathbf{1}_{\{S_j \leq \gamma t\}} \mathbf{1}_{\{T_n = t\}} \right]$$

and

$$\mathcal{E}_2(t) := \sum_{n=1}^t n \mathbb{E}_o \left[ \mathbf{1}_{\{S_n > \gamma t\}} \mathbf{1}_{\{T_n = t\}} \right].$$

The term  $\mathcal{E}_1(t)$  can be estimated by means of the following lemma which is proved in appendix C. Set  $\xi_t := -\ln t \mathcal{Q}(\gamma t)$  for all  $t$  for brevity and notice that  $\lim_{t \uparrow \infty} \xi_t = +\infty$  since  $x \mathcal{Q}(x) \leq \mathbb{E}_o[S_1 \mathbf{1}_{\{S_1 > x\}}]$  and  $\mathbb{E}_o[S_1] < +\infty$ .

**Lemma 3.3.** *For all sufficiently large  $t$  there exist two positive real numbers  $z_t$  and  $\lambda_t$  such that  $\mathbb{E}_o[\mathbf{1}_{\{S_1 \leq \gamma t\}} e^{z_t S_1 - \lambda_t g(S_1)}] \leq 1$  and  $(z_t - \alpha \lambda_t)t \geq \xi_t + \ln |\xi_t|$ .*

Let  $z_t$  and  $\lambda_t$  be the numbers introduced by lemma 3.3, so that the inequalities  $0 < \mathbb{E}_o[\mathbf{1}_{\{S_1 \leq \gamma t\}} e^{z_t S_1 - \lambda_t g(S_1)}] \leq 1$  and  $(z_t - \alpha \lambda_t)t \geq \xi_t + \ln |\xi_t|$  hold for all  $t > t_1$  with some  $t_1 > 0$ . The fact that  $\lambda_t \geq 0$  allows us to invoke the inequality  $\mathbf{1}_{\{\sum_{i=1}^n g(S_i) \leq \alpha t\}} \leq e^{\alpha \lambda_t t - \lambda_t \sum_{i=1}^n g(S_i)}$  to obtain

$$\begin{aligned} \mathcal{E}_1(t) &\leq e^{\alpha \lambda_t t} \sum_{n=1}^t \mathbb{E}_o \left[ \prod_{j=1}^n \mathbf{1}_{\{S_j \leq \gamma t\}} e^{-\lambda_t g(S_j)} \mathbf{1}_{\{T_n = t\}} \right] \\ &= e^{(\alpha \lambda_t - z_t)t} \sum_{n=1}^t \mathbb{E}_o \left[ \prod_{j=1}^n \mathbf{1}_{\{S_j \leq \gamma t\}} e^{z_t S_j - \lambda_t g(S_j)} \mathbf{1}_{\{T_n = t\}} \right] \\ &= e^{(\alpha \lambda_t - z_t)t} \sum_{n=1}^t \frac{\mathbb{E}_o \left[ \prod_{j=1}^n \mathbf{1}_{\{S_j \leq \gamma t\}} e^{z_t S_j - \lambda_t g(S_j)} \mathbf{1}_{\{T_n = t\}} \right]}{\mathbb{E}_o \left[ \mathbf{1}_{\{S_1 \leq \gamma t\}} e^{z_t S_1 - \lambda_t g(S_1)} \right]^{t-n}}. \end{aligned}$$

At this point, we make use of  $\mathbb{E}_o[\mathbf{1}_{\{S_1 \leq \gamma t\}} e^{z_t S_1 - \lambda_t g(S_1)}] \leq 1$  and  $(z_t - \alpha \lambda_t)t \geq \xi_t + \ln |\xi_t|$  to get for any  $t > t_1$  at the upper bound

$$\begin{aligned} \mathcal{E}_1(t) &\leq e^{-\xi_t - \ln |\xi_t|} \sum_{n=1}^t \frac{\mathbb{E}_o \left[ \prod_{j=1}^n \mathbf{1}_{\{S_j \leq \gamma t\}} e^{z_t S_j - \lambda_t g(S_j)} \mathbf{1}_{\{T_n = t\}} \right]}{\mathbb{E}_o \left[ \mathbf{1}_{\{S_1 \leq \gamma t\}} e^{z_t S_1 - \lambda_t g(S_1)} \right]^t} \\ &= \frac{t \mathcal{Q}(\gamma t)}{|\xi_t|} \frac{\mathbb{E}_o \left[ \prod_{j=1}^t \mathbf{1}_{\{S_j \leq \gamma t\}} e^{z_t S_j - \lambda_t g(S_j)} U_t \right]}{\mathbb{E}_o \left[ \mathbf{1}_{\{S_1 \leq \gamma t\}} e^{z_t S_1 - \lambda_t g(S_1)} \right]^t} \\ &\leq \frac{t \mathcal{Q}(\gamma t)}{|\xi_t|} \frac{\mathbb{E}_o \left[ \prod_{j=1}^t \mathbf{1}_{\{S_j \leq \gamma t\}} e^{z_t S_j - \lambda_t g(S_j)} \right]}{\mathbb{E}_o \left[ \mathbf{1}_{\{S_1 \leq \gamma t\}} e^{z_t S_1 - \lambda_t g(S_1)} \right]^t} = \frac{t \mathcal{Q}(\gamma t)}{|\xi_t|}. \end{aligned} \quad (3.10)$$

As far as  $\mathcal{E}_2(t)$  is concerned, we let  $\eta$  come into play by writing for all  $t \geq 2$

$$\begin{aligned} \mathcal{E}_2(t) &= \sum_{n=1}^t n \mathbb{E}_o \left[ \mathbf{1}_{\{\gamma t < S_n \leq \eta t\}} \mathbf{1}_{\{T_n = t\}} \right] + \sum_{n=1}^t n \mathbb{E}_o \left[ \mathbf{1}_{\{S_n > \eta t\}} \mathbf{1}_{\{T_n = t\}} \right] \\ &= \sum_{n=2}^t (n-1) \mathbb{E}_o \left[ \mathbf{1}_{\{\gamma t < S_n \leq \eta t\}} \mathbf{1}_{\{T_{n-1} + S_n = t\}} \right] + \mathbb{E}_o \left[ \mathbf{1}_{\{\gamma t < S_1 \leq \eta t\}} U_t \right] + \mathbb{E}_o \left[ \mathbf{1}_{\{S_1 > \eta t\}} N_t U_t \right] \\ &= \sum_{s=1}^{t-1} \sum_{n=1}^{t-s} n \mathbb{E}_o \left[ \mathbf{1}_{\{T_n = t-s\}} \right] \mathbf{1}_{\{\gamma t < s \leq \eta t\}} p_o(s) + \mathbb{E}_o \left[ \mathbf{1}_{\{\gamma t < S_1 \leq \eta t\}} U_t \right] + \mathbb{E}_o \left[ \mathbf{1}_{\{S_1 > \eta t\}} N_t U_t \right] \\ &= \sum_{s \geq 1} \mathbb{E}_o \left[ N_{t-s} U_{t-s} \right] \mathbf{1}_{\{\gamma t < s \leq \eta t\}} p_o(s) + \mathbb{E}_o \left[ \mathbf{1}_{\{\gamma t < S_1 \leq \eta t\}} U_t \right] + \mathbb{E}_o \left[ \mathbf{1}_{\{S_1 > \eta t\}} N_t U_t \right]. \end{aligned}$$

We notice that  $\mathbb{E}_o[\mathbf{1}_{\{\gamma t < S_1 \leq \eta t\}} U_t] \leq \mathbb{E}_o[\mathbf{1}_{\{S_1 > \gamma t\}}] = \mathcal{Q}(\gamma t)$ . We also observe that the condition  $U_t = 1$  implies  $S_1 \leq t$ , so that  $\mathbb{E}_o[\mathbf{1}_{\{S_1 > \eta t\}} N_t U_t] = \mathbb{E}_o[\mathbf{1}_{\{\eta t < S_1 \leq t\}} N_t U_t]$  and hence  $\mathbb{E}_o[\mathbf{1}_{\{S_1 > \eta t\}} N_t U_t] \leq t \mathbb{E}_o[\mathbf{1}_{\{\eta t < S_1 \leq t\}}] = t \mathcal{Q}(\eta t) - t \mathcal{Q}(t)$ . Finally, we recall that

$\lim_{\tau \uparrow \infty} \mathbb{E}_o[(N_\tau/\tau)U_\tau] = 1/\mathbb{E}_o[S_1]^2$  by lemma 3.2, which entails that there exists  $t_2 \geq t_1$  with the property that  $\mathbb{E}_o[N_\tau U_\tau] \leq (1 + \epsilon)\tau/\mathbb{E}_o[S_1]^2$  for all  $\tau > (1 - \eta)t_2$ . This way, since  $t - s \geq (1 - \eta)t$  when  $s \leq \eta t$ , for every  $t > t_2$  we get the bound

$$\begin{aligned} \mathcal{E}_2(t) &\leq \frac{1 + \epsilon}{\mathbb{E}_o[S_1]^2} \sum_{s \geq 1} (t - s) \mathbb{1}_{\{\gamma t < s \leq \eta t\}} p_o(s) + \mathcal{Q}(\gamma t) + t \mathcal{Q}(\eta t) - t \mathcal{Q}(t) \\ &= \frac{1 + \epsilon}{\mathbb{E}_o[S_1]^2} \left[ (1 - \gamma) t \mathcal{Q}(\gamma t) - (1 - \eta) t \mathcal{Q}(\eta t) - t \int_\gamma^\eta \mathcal{Q}(xt) dx \right] \\ &\quad + \mathcal{Q}(\gamma t) + t \mathcal{Q}(\eta t) - t \mathcal{Q}(t). \end{aligned} \tag{3.11}$$

In conclusion, by combining (3.9) with (3.10) and (3.11) we obtain

$$\begin{aligned} \mathcal{E}(t) &\leq \frac{1 + \epsilon}{\mathbb{E}_o[S_1]^2} \left[ (1 - \gamma) t \mathcal{Q}(\gamma t) - (1 - \eta) t \mathcal{Q}(\eta t) - t \int_\gamma^\eta \mathcal{Q}(xt) dx \right] \\ &\quad + \mathcal{Q}(\gamma t) + \frac{t \mathcal{Q}(\gamma t)}{|\xi_t|} + t \mathcal{Q}(\eta t) - t \mathcal{Q}(t) \end{aligned}$$

for all  $t > t_2$ . Recalling that the limit  $\lim_{t \uparrow \infty} \mathcal{Q}(xt)/\mathcal{Q}(t) = x^{-\kappa}$  is uniform with respect to  $x$  in the compact interval  $[\gamma, \eta]$  and that  $\lim_{t \uparrow \infty} \xi_t = +\infty$ , (3.8) follows from here by dividing by  $t \mathcal{Q}(t)$  first and by sending  $t$  to infinity later.

## Acknowledgments

The author is grateful to Giambattista Giacomin for a critical reading of the manuscript and valuable comments.

## A Proof of lemma 3.1

If  $\mathcal{L}$  varies slowly and

$$\lim_{x \uparrow +\infty} \frac{\mathcal{Q}(x)}{x^{-\kappa} \mathcal{L}(x)} = \frac{1}{\kappa}, \tag{A.1}$$

then  $\mathcal{Q}$  varies regularly with index  $-\kappa$ .

Let us demonstrate the limit (A.1). Pick two real numbers  $\delta \in (0, \kappa)$  and  $K > 1$  and observe that there exists  $x_o > 1$  such that  $(x/y)^\delta \mathcal{L}(x)/K \leq \mathcal{L}(y) \leq K(x/y)^{-\delta} \mathcal{L}(x)$  when  $x_o \leq x \leq y$  (see [27], theorem 1.5.6). Recalling that  $p_o(s) = s^{-\kappa-1} \mathcal{L}(s)$ , it follows that  $x^\delta \mathcal{L}(x) s^{-\kappa-\delta-1}/K \leq p_o(s) \leq K x^{-\delta} \mathcal{L}(x) s^{-\kappa+\delta-1}$  if  $x_o \leq x < s$ . The upper bound gives for  $x \geq x_o > 1$

$$\mathcal{Q}(x) = \sum_{s \geq 1} \mathbb{1}_{\{s > x\}} p_o(s) \leq \sum_{s \geq 1} \mathbb{1}_{\{s > x\}} K x^{-\delta} \mathcal{L}(x) s^{-\kappa+\delta-1} \leq \frac{K}{\kappa - \delta} \frac{x^{-\delta}}{(x-1)^{\kappa-\delta}} \mathcal{L}(x),$$

where the inequality  $(\kappa - \delta) s^{-\kappa+\delta-1} \leq (s-1)^{-\kappa+\delta} - s^{-\kappa+\delta}$  valid for all  $s > 1$  has been used to change the second series with a telescoping series. Similarly, the lower bound and the inequality  $s^{-\kappa-\delta} - (s+1)^{-\kappa-\delta} \leq (\kappa + \delta) s^{-\kappa-\delta-1}$  valid for all  $s \geq 1$  yield for  $x \geq x_o$

$$\mathcal{Q}(x) \geq \sum_{s \geq 1} \mathbb{1}_{\{s > x\}} x^\delta \mathcal{L}(x) s^{-\kappa-\delta-1}/K \geq \frac{1}{K(\kappa + \delta)} x^{-\kappa} \mathcal{L}(x).$$

This way, we find

$$\frac{1}{K(\kappa + \delta)} \leq \liminf_{x \uparrow +\infty} \frac{\mathcal{Q}(x)}{x^{-\kappa} \mathcal{L}(x)} \leq \limsup_{x \uparrow +\infty} \frac{\mathcal{Q}(x)}{x^{-\kappa} \mathcal{L}(x)} \leq \frac{K}{\kappa - \delta}$$

and (A.1) follows from here by sending  $\delta$  to 0 and  $K$  to 1.

## B Proof of lemma 3.2

Recall that  $N_t$  is the cumulative reward corresponding to a function  $f$  identically equal to 1. Since  $0 \leq N_t/t \leq 1$  and  $0 < 1/\mathbb{E}_o[S_1] < 1$ , for any  $t \geq 1$  and  $\delta > 0$  we can write

$$\begin{aligned} \left| \mathbb{E}_o \left[ \left( \frac{N_t}{t} - \frac{1}{\mathbb{E}_o[S_1]} \right) U_t \right] \right| &\leq \delta \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \left| \frac{N_t}{t} - \frac{1}{\mathbb{E}_o[S_1]} \right| < \delta \right\}} U_t \right] + 2 \mathbb{E}_o \left[ \mathbb{1}_{\left\{ \left| \frac{N_t}{t} - \frac{1}{\mathbb{E}_o[S_1]} \right| \geq \delta \right\}} U_t \right] \\ &\leq \delta + 2 \mathbb{P}_t^c \left[ \left| \frac{N_t}{t} - \frac{1}{\mathbb{E}_o[S_1]} \right| \geq \delta \right], \end{aligned}$$

where the second bound is obtained by applying formula (3.2) to  $N_t$ . From here, we get at the proof of the lemma by combining the limit  $\lim_{t \uparrow \infty} \mathbb{E}_o[U_t] = 1/\mathbb{E}_o[S_1]$  with the convergence in probability stated by proposition 1.1.

## C Proof of lemma 3.3

Recall that  $\xi_t := -\ln t \mathcal{Q}(\gamma t)$  and for every  $t \geq 1$  set

$$\lambda_t := \frac{(1-\gamma)\xi_t - 4\kappa \ln |\xi_t|}{\gamma(1-\gamma)t}$$

and

$$z_t := \alpha \lambda_t + \frac{\xi_t + \ln |\xi_t|}{t}.$$

It is manifest that  $(z_t - \alpha \lambda_t)t \geq \xi_t + \ln |\xi_t|$  for each  $t$  and, since  $\xi_t$  goes to infinity when  $t$  is sent to infinity, there exists a positive integer  $\tau_1$  such that  $\xi_t > 1$  and the real numbers  $\lambda_t$  and  $z_t$  are positive for any  $t > \tau_1$ . We prove the lemma by showing that for all sufficiently large  $t > \tau_1$

$$\sum_{s \geq 1} \mathbb{1}_{\{s \leq \gamma t / \xi_t^2\}} e^{z_t s - \lambda_t g(s)} p_o(s) \leq 1 - \frac{2}{t} \quad (\text{C.1})$$

and

$$\sum_{s \geq 1} \mathbb{1}_{\{\gamma t / \xi_t^2 < s \leq \gamma t\}} e^{z_t s - \lambda_t g(s)} p_o(s) \leq \frac{2}{t}. \quad (\text{C.2})$$

We point out that  $\lim_{t \uparrow \infty} \gamma t / \xi_t^2 = +\infty$  since  $\mathcal{Q}(x) \geq x^{-\kappa-\delta}$  for any  $\delta > 0$  and all sufficiently large  $x$  as  $\mathcal{Q}$  varies regularly with index  $-\kappa$  (see [27], proposition 1.3.6).

Let us verify (C.1) at first. Let  $M < +\infty$  be a positive constant such that  $|g(s)| \leq Ms$  for all  $s$ , which certainly exists because  $\lim_{s \uparrow \infty} g(s)/s = 0$ . Then, let  $K < +\infty$  be a positive constant such that  $\gamma(tz_t + Mt\lambda_t)^2 \xi_t^{-2} e^{\gamma(tz_t + Mt\lambda_t)\xi_t^{-2}} \leq K$  for all  $t > \tau_1$ , which exists because  $\lim_{t \uparrow \infty} \gamma(tz_t + Mt\lambda_t)^2 \xi_t^{-2} e^{\gamma(tz_t + Mt\lambda_t)\xi_t^{-2}} = \gamma^{-1}(\alpha + \gamma + M)^2$ . Finally, let  $\tau_2 \geq \tau_1$  be an integer such that  $z_t s - \lambda_t g(s) \geq 0$  for all  $t > \tau_2$  and  $s > \gamma t / \xi_t^2$ , which exists because  $\lim_{t \uparrow \infty} z_t / \lambda_t = \alpha + \gamma > 0$ ,  $\lim_{s \uparrow \infty} g(s)/s = 0$ , and  $\lim_{t \uparrow \infty} \gamma t / \xi_t^2 = +\infty$ . Pick  $t > \tau_2$ . The bound  $e^y \leq 1 + y + y^2 e^{|y|}$  valid for all  $y \in \mathbb{R}$  yields for any  $s \leq \gamma t / \xi_t^2$

$$\begin{aligned} e^{z_t s - \lambda_t g(s)} &\leq 1 + z_t s - \lambda_t g(s) + (z_t + M\lambda_t)^2 s^2 e^{(z_t + M\lambda_t)s} \\ &\leq 1 + z_t s - \lambda_t g(s) + \frac{\gamma}{t} \left( \frac{tz_t + Mt\lambda_t}{\xi_t} \right)^2 s e^{\gamma \frac{tz_t + Mt\lambda_t}{\xi_t^2}} \\ &\leq 1 + z_t s - \lambda_t g(s) + \frac{K}{t} s. \end{aligned}$$

This bound, in combination with the facts that  $z_t s - \lambda_t g(s) \geq 0$  for  $s > \gamma t / \xi_t^2$  and that  $\mathbb{E}_o[g(S_1)] = \mathbb{E}_o[S_1]$ , gives

$$\begin{aligned} \sum_{s \geq 1} \mathbb{1}_{\{s \leq \gamma t / \xi_t^2\}} e^{z_t s - \lambda_t g(s)} p_o(s) &\leq 1 + \sum_{s \geq 1} \mathbb{1}_{\{s \leq \gamma t / \xi_t^2\}} [z_t s - \lambda_t g(s)] p_o(s) + \frac{K}{t} \mathbb{E}_o[S_1] \\ &\leq 1 + \mathbb{E}_o[S_1] \left\{ \frac{tz_t - t\lambda_t}{\xi_t} + \frac{K}{\xi_t} + \frac{2}{\mathbb{E}_o[S_1]\xi_t} \right\} \frac{\xi_t}{t} - \frac{2}{t}. \end{aligned}$$

The term between braces goes to  $-(1 - \alpha - \gamma)/\gamma < 0$  when  $t$  is sent to infinity, so that it is non-positive for all  $t$  larger than some  $\tau_3 \geq \tau_2$ . It follows that (C.1) holds for all  $t > \tau_3$ .

Let us now prove (C.2). To begin with, we observe that for any  $\delta > 0$  and  $K > 1$  there exists  $x_o > 0$  such that  $y^{\kappa+\delta} \mathcal{Q}(y) \leq K x^{\kappa+\delta} \mathcal{Q}(x)$  if  $x_o \leq y \leq x$  (see [27], theorem 1.5.6). Thus, taking  $\delta = \kappa - \gamma/2$  and  $K = 2$ ,  $\delta$  being positive because  $\kappa \geq 1$  and  $\gamma < 1 - \alpha$  by hypothesis, the fact that  $\lim_{t \uparrow \infty} \gamma t / \xi_t^2 = +\infty$  implies that an integer  $\tau_4 \geq \tau_3$  can be found in such a way that  $\mathcal{Q}(\gamma t / \xi_t^2) \leq 2 \xi_t^{4\kappa - \gamma} \mathcal{Q}(\gamma t)$  for all  $t > \tau_4$ . Then, since  $1 - \alpha - \gamma > 0$ , the limit  $\lim_{s \uparrow \infty} g(s)/s = 0$  ensures us that there exists an integer  $\tau_5 \geq \tau_4$  with the property that  $g(s) \geq -(1 - \alpha - \gamma)s$  for all  $s > \gamma t / \xi_t^2$  whenever  $t > \tau_5$ . This way, for every  $t > \tau_5$  we find the bound

$$\begin{aligned} \sum_{s \geq 1} \mathbb{1}_{\{\gamma t / \xi_t^2 < s \leq \gamma t\}} e^{z_t s - \lambda_t g(s)} p_o(s) &\leq e^{[z_t + (1 - \alpha - \gamma)\lambda_t] \gamma t} \mathcal{Q}(\gamma t / \xi_t^2) \\ &= \xi_t^{\gamma - 4\kappa} e^{\xi_t} \mathcal{Q}(\gamma t / \xi_t^2) = \frac{\xi_t^{\gamma - 4\kappa}}{t} \frac{\mathcal{Q}(\gamma t / \xi_t^2)}{\mathcal{Q}(\gamma t)} \leq \frac{2}{t}. \end{aligned}$$

In conclusion, we have that both (C.1) and (C.2) are satisfied if  $t > \tau_5$ .

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