Research Article

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Nontrivial solutions for resonance quasilinear elliptic systems

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Abstract: We establish an Amann-Zehnder-type result for resonance systems of quasilinear elliptic equations with homogeneous Dirichlet boundary conditions, involving nonlinearities growing asymptotically (p, q)-linear at infinity. The proof relies on a cohomological linking in a product Banach space where the properties of cones of the sublevels are missing, differently from the single quasilinear equation. We also perform critical group computations of the energy functional at the origin, in spite of the lack of its C^2 regularity, to exclude that the found mini-max solution is trivial. Finally, we furnish a local condition which guarantees that the found solution is not semi-trivial.

Keywords: quasilinear elliptic systems, Fadell Rabinowitz index, asymptotically (p, q) linear, Morse index, resonance, critical groups

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1 Introduction

The interaction between a nonlinearity g and the spectrum of $-\Delta$ at 0 and infinity has been used in the celebrated article [2], under the assumption of nonresonance at infinity, namely, $\bar{\lambda}$ is not an eigenvalue of $-\Delta$. Precisely, Amann and Zehnder [2] proved the existence of a nontrivial solution for the nonresonance asymptotically linear elliptic equation:

$$\begin{cases} -\Delta u = g(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary, $g \in C^1(\mathbb{R}, \mathbb{R})$ such that $g(0) = 0$, and there exists $\bar{\lambda} \in \mathbb{R}$ such that $\lim_{|s|\to\infty} g'(s) = \bar{\lambda}$. Using Morse theory for manifolds with boundary, the same result has been found by Chang [7]. Successively, Lazer and Solimini [35] recognized that such nontrivial solution can be detected, combining mini-max characterization of the critical point and Morse index estimates (see also [41]).

Recently, the quasilinear counterpart of the Amann-Zehnder existence result has been obtained in [12] (see also [10,26]) for a class of quasilinear elliptic equations, involving as principal part either the p-Laplace operator or the operator related to the p-area functional, and a nonlinearity with p-linear growth at infinity, exploiting new techniques of Morse theory in Banach spaces (see [13–15,46]), and introducing a Saddle theorem where linear subspaces are substituted by symmetric cones (see Theorem 7.1 in [12]).

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In this article, we are interested to derive Amann-Zehnder-type results for a class of quasilinear elliptic systems, having as principal parts (p, q)-Laplace operators or (p, q)-area-type operators, and nonlinearities growing (p, q)-linearly at infinity.

Precisely, we will seek for nontrivial solutions for the following autonomous quasilinear system:

$$\begin{cases} -\mathrm{div}\left((\alpha + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u\right) = G_s(u, v), & x \in \Omega, \\ -\mathrm{div}\left((\alpha + |\nabla v|^2)^{\frac{q-2}{2}}\nabla v\right) = G_t(u, v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $2 \le p < N$, $2 \le q < N$, $\alpha \ge 0$, Ω is a smooth bounded domain of $\mathbb{R}^N$, and $G \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies the following conditions:

(a1) $\nabla G(0, 0) = (0, 0)$ and there exists $\bar{\lambda} \in \mathbb{R}$ such that

$$\lim_{|(s,t)|\to\infty} \frac{G_s(s, t) - \bar{\lambda}F_s(s, t)}{|s|^{p-1} + |t|^{q\frac{p-1}{p}}} = \lim_{|(s,t)|\to\infty} \frac{G_t(s, t) - \bar{\lambda}F_t(s, t)}{|s|^{p\frac{q-1}{q}} + |t|^{q-1}} = 0,$$

where $F \in C^1(\mathbb{R}^2, \mathbb{R})$ is defined by

$$F(s, t) = \frac{1}{p}|s|^p + \frac{1}{q}|t|^q + \frac{1}{(\beta + 1)(\gamma + 1)}|s|^\beta|t|^\gamma st,$$

with $\beta, \gamma > 0$ such that $(\beta + 1)/p + (\gamma + 1)/q = 1$;

(a2) there is a suitable neighborhood U of $(0, 0)$ such that $G \in C^2(U, \mathbb{R})$.

Let X be the product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ endowed with the norm:

$$||z|| = ||u||_{1,p} + ||v||_{1,q}, \quad z = (u, v) \in X,$$

where $||\cdot||_s$ and $||\cdot||_{1,s}$ denote the usual norms in $L^s(\Omega)$ and $W_0^{1,s}(\Omega)$, respectively.

Weak solutions of Problem (1.2) correspond to critical points of the Euler functional J_α on X defined by setting

$$\begin{aligned} J_\alpha(z) = J_\alpha(u, v) = & \frac{1}{p}\int_\Omega(\alpha + |\nabla u(x)|^2)^{\frac{p}{2}}\mathrm{d}x + \frac{1}{q}\int_\Omega(\alpha + |\nabla v(x)|^2)^{\frac{q}{2}}\mathrm{d}x \\ & - \int_\Omega G(u(x), v(x))\mathrm{d}x, \quad z = (u, v) \in X. \end{aligned} \tag{1.3}$$

By (a_1), the functional J_α is C^1 on X and for any $z_0 = (u_0, v_0) \in X$, and $z = (u, v) \in X$, it results

$$\begin{aligned} \langle J_\alpha'(z_0), z\rangle = & \int_\Omega(\alpha + |\nabla u_0|^2)^{\frac{p-2}{2}}\nabla u_0\nabla u\mathrm{d}x + \int_\Omega(\alpha + |\nabla v_0|^2)^{\frac{q-2}{2}}\nabla v_0\nabla v\mathrm{d}x \\ & - \int_\Omega(G_s(u_0, v_0)u + G_t(u_0, v_0)v)\mathrm{d}x. \end{aligned}$$

These systems model some phenomena in non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology (see [24,31,38,40,43]). Existence, nonexistence, and regularity results for such quasilinear elliptic systems are obtained by various authors (see, for instance, [3–5,18,20,25,37,39,48]).

In this article, we address the study of the interaction between the spectrum of the operators involved in System (1.2) and nonlinearities that grow (p, q)-linearly at infinity, in both the asymptotic resonant and nonresonant case.

To this aim, we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda\, |u|^{p-2}u + \dfrac{\lambda}{\gamma+1}\, |u|^{\beta}|v|^{\gamma}v, & x \in \Omega, \\ -\Delta_q v = \lambda\, |v|^{q-2}v + \dfrac{\lambda}{\beta+1}\, |u|^{\beta}|v|^{\gamma}u, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $2 \le p < N$, $2 \le q < N$, $\beta > 0$, and $\gamma > 0$ are the real numbers satisfying $(\beta+1)/p + (\gamma+1)/q = 1$.

The real number λ is called an eigenvalue of (1.4) if there exists a nontrivial solution (u, v) of (1.4).

In [42], the existence of an unbounded sequence of mini-max eigenvalues was proved using $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [28,29].

Precisely, let $\Phi : X \to \mathbb{R}$, $\Psi : X \to \mathbb{R}$ be the following functionals:

$$\Phi(u, v) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p \mathrm{d}x + \frac{1}{q}\int_\Omega |\nabla v(x)|^q \mathrm{d}x,$$

$$\Psi(u, v) = \int_\Omega F(u, v)\mathrm{d}x.$$

By Young's inequality, we infer that

$$\frac{1}{p}\frac{\gamma}{\gamma+1}\,|s|^p + \frac{1}{q}\frac{\beta}{\beta+1}\,|t|^q \le F(s,t) \le \frac{1}{p}\frac{\gamma+2}{\gamma+1}\,|s|^p + \frac{1}{q}\frac{\beta+2}{\beta+1}\,|t|^q. \tag{1.5}$$

Note that Φ and Ψ are (p, q)-homogeneous, i.e.,

$$\Phi\left(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v\right) = t\Phi(u, v) \quad \text{and} \quad \Psi\left(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v\right) = t\Psi(u, v), \tag{1.6}$$

for all $t > 0$ and $(u, v) \in X$.

Moreover, λ is an eigenvalue iff there is $(u, v) \in X\backslash\{(0, 0)\}$ such that

$$\Phi'(u, v) = \lambda\Psi'(u, v).$$

Ou and Tang [42] proved that (1.4) has a nondecreasing and unbounded sequence of eigenvalues with the variational characterization:

$$\lambda_k = \inf_{A\in\Sigma_k}\sup_{(u,v)\in A} \Phi(u, v), \tag{1.7}$$

where Σ is the C^1 manifold

$$\Sigma := \{(u, v) \in X | \Psi(u, v) = 1\}$$

and

$$\Sigma_k := \{A \subset \Sigma | A \text{ is symmetric, compact and } i(A) \ge k\},$$

where $i(A)$ denotes the $\mathbb{Z}_2$-cohomological index of A, introduced by Fadell and Rabinowitz [28,29]. For a matter of convenience, we also put $\lambda_0 = -\infty$.

In [45], it is shown that the first eigenvalue λ_1 is simple and isolated with a first strictly positive eigenfunction $\psi = (\psi_1, \psi_2)$, i.e., $\psi_1 > 0$, $\psi_2 > 0$ in Ω (see also [3,19,20,27]). However, it is not clear if the set of the eigenvalues described by (1.7) contains all the eigenvalues of Problem (1.4). Really, also for the single scalar eigenvalue problem $-\Delta_p u = \lambda\, |u|^{p-2}u$, the spectral properties of $-\Delta_p$ are not yet well understood. Denoted $\sigma(-\Delta_p)$ the set of such eigenvalues λ, one can define in at least three different ways a diverging sequence (λ_m) of eigenvalues of $-\Delta_p$ [10,26], but it is not known if they agree for $m \ge 3$ and if the whole set $\sigma(-\Delta_p)$ is covered.

Furthermore, we remark that in the context of quasilinear systems, with $p \neq q$, the level set $E_{\lambda_k} = \{(u, v) \in X | \Phi(u, v) = \lambda_k \Psi(u, v)\}$ is not a linear space or a cone, but, by (1.6), we have $\left(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v\right) \in E_{\lambda_k}$ for any $t \geq 0$ and $(u, v) \in E_{\lambda_k}$. This makes delicate to recognize a saddle-type geometry for the energy functional associated with (1.2).

In this article, we derive a quite general abstract linking theorem (see Theorem 3.3), that extends Theorem 7.1 in [12], which deals with symmetric cones. This abstract result can be successfully applied in the framework of systems and allows us to detect the existence of a mini-max critical point for J_a with a suitable nontrivial critical group. Finally, in order to prove that the solution we found is not trivial, we need to compute the critical groups at the origin in terms of differential notions. However, the applicability of Morse arguments in Banach spaces presents severe difficulties since classical Morse lemma and generalized Morse lemma [32] are so far known, also due to the lack of Fredholm properties of the second derivative of the functionals. Moreover, the energy functional J_a is not C^2, so that we cannot define the classical notion of Morse index. In order to overcome this lack of regularity, the first idea is to consider the quadratic form $Q : X \to \mathbb{R}$ defined as

$$\begin{aligned} Q(z) = Q(u, v) &= a^{\frac{p-2}{2}}\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right) + a^{\frac{q-2}{2}}\left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right) \\ &\quad - \int_\Omega (G_{ss}(0, 0)u^2 + G_{tt}(0, 0)v^2 + 2G_{st}(0, 0)uv)\mathrm{d}x \\ &= a^{\frac{p-2}{2}}\|u\|_{1,2}^2 + a^{\frac{q-2}{2}}\|v\|_{1,2}^2 - \int_\Omega H_G(0, 0)[z]^2 \mathrm{d}x, \end{aligned} \tag{1.8}$$

where $H_G(0, 0)$ is the Hessian matrix of G at $(0, 0)$ and consider m_0 and m_0^* defined as the supremum of the dimensions of the subspaces of X, where Q is negative definite and negative semidefinite, respectively. These objects will play the role of Morse index and large Morse index, respectively.

The second problem becomes to relate such differential notions m_0 and m_0^* to the behavior of the functional J_a near the origin. This delicate issue will be obtained, by means of a penalized functional of J_a, which is C^2 just in a ball centered at the origin. We stress that even if $p \geq 2$ and $q \geq 2$, the functional F is just C^1 on X, differently from the case of a single equation. Roughly speaking, we say that the eigenvalue Problem (1.4) is intrinsically less regular than the eigenvalue problem for the single p-Laplace equation, due to the presence of the right-hand side coupled terms.

Our main results are the following:

Theorem 1.1. *Suppose that* $2 \leq p < N$, $2 \leq q < N$, $a \geq 0$, *and* $(a_1) - (a_2)$ *hold. Assume that* $\bar{\lambda}$ *is not an eigenvalue for System* (1.4) *and denote by* m_∞ *the integer such that*

$$\lambda_{m_\infty} < \bar{\lambda} < \lambda_{m_\infty+1}.$$

If

$$m_\infty \notin [m_0, m_0^*],$$

then there exists a nontrivial weak solution $z = (u, v)$ *of* (1.2).

We remark that in Theorem 1.1, the parameter $\bar{\lambda}$ is not an eigenvalue of (1.4) and $\lambda_m < \bar{\lambda} < \lambda_{m+1}$ for some $m \in \mathbb{N}$. This condition seems to be possible, for instance, if $m = 1$ taking into account that λ_1 is isolated and it is the only eigenvalue of (1.4) to which corresponds a componentwise positive eigenfunction [45]. However, such nonresonant restriction would be quite severe, and it becomes relevant to face with systems at resonance. In the following theorems, the value $\bar{\lambda}$ is allowed to be an eigenvalue of (1.4), under an additional condition of G at infinity.

Theorem 1.2. *Suppose that* $2 \leq p < N$, $2 \leq q < N$, $a \geq 0$, *and* (a_1)–(a_2) *hold. Denote by* m_∞ *the integer such that*

$$\lambda_{m_\infty} < \bar{\lambda} \leq \lambda_{m_\infty+1}.$$

If

$$m_\infty \notin [m_0, m_0^*]$$

and

$$(b_-) \quad \lim_{|(s,t)|\to\infty}\left[G(s,t) - \frac{1}{p}G_s(s,t)s - \frac{1}{q}G_t(s,t)t\right] = -\infty,$$

then there exists a nontrivial weak solution $z = (u, v)$ *of* (1.2).

Theorem 1.3. *Suppose that* $2 \le p < N$, $2 \le q < N$, $\alpha = 0$, *and* $(a_1) - (a_2)$ *hold. Denote by* m_∞ *the integer such that*

$$\lambda_{m_\infty} \le \bar{\lambda} < \lambda_{m_\infty+1}.$$

If

$$m_\infty \notin [m_0, m_0^*]$$

and

$$(b_+) \quad \lim_{|(s,t)|\to\infty}\left[G(s,t) - \frac{1}{p}G_s(s,t)s - \frac{1}{q}G_t(s,t)t\right] = +\infty,$$

then there exists a nontrivial weak solution $z = (u, v)$ *of* (1.2).

We remark that the nontrivial solutions of (1.2), including those found through Theorems 1.1–1.3, cannot be semitrivial if we assume that $G_t(s, 0) \neq 0$ for any $s \in (-\varepsilon, \varepsilon)\backslash\{0\}$ and $G_s(0, t) \neq 0$ for any $t \in (-\varepsilon, \varepsilon)\backslash\{0\}$, for an arbitrary $\varepsilon > 0$ (see Proposition 5.2 and also [9, Lemma 3.1]).

Finally, we note that if $\gamma = \beta = 0$, then $p = q = 2$. In this case, mini-max arguments and Morse theory are applied in [30] for deriving the existence of nontrivial solutions of a semilinear elliptic system with a C^2 nonlinear function G.

2 Palais-Smale (PS) and Cerami-Palais-Smale (CPS) conditions

We begin to recall a classical definition in a reflexive Banach space.

Definition 2.1. Let X be a reflexive Banach space and $D \subset X$. A map $H : D \to X'$ is said to be of class $(S)_+$, if, for every sequence u_k in D weakly convergent to u in X with

$$\limsup_{k\to\infty}\langle H(u_k), u_k - u\rangle \le 0,$$

we have $||u_k - u|| \to 0$.

Lemma 2.2. *For any* $\alpha \ge 0$ *and* $r \ge 2$, *the map* $H_{\alpha,r} : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ *defined by*

$$H_{\alpha,r}(u) = -\operatorname{div}\left((\alpha + |\nabla u|^2)^{\frac{r-2}{2}}\nabla u\right)$$

is of class $(S)_+$.

Proof. Let us denote by $a_{\alpha,r} : \mathbb{R}^n \to \mathbb{R}^n$ the map

$$a_{\alpha,r}(\xi) = (\alpha + |\xi|^2)^{\frac{r-2}{2}}\xi.$$

We can see that for any $\alpha \ge 0$, there is $C > 0$ such that

$$|a_{\alpha,r}(\xi)| \le C(1 + |\xi|^{r-1}), \tag{2.1}$$

$$a_{\alpha,r}(\xi)\xi \ge |\xi|^r, \tag{2.2}$$

$$(a_{\alpha,r}(\xi) - a_{\alpha,r}(\eta))(\xi - \eta) > 0, \tag{2.3}$$

for any $\xi \in \mathbb{R}^n$ and $\eta \ne \xi \in \mathbb{R}^n$.

Let us consider $\eta \ne \xi \in \mathbb{R}^N$. If $|\xi| = |\eta|$, then

$$(a_{\alpha,r}(\xi) - a_{\alpha,r}(\eta))\cdot(\xi - \eta) = (\alpha + |\eta|^2)^{\frac{r-2}{2}}|\xi - \eta|^2 > 0.$$

Otherwise, if $|\xi| \ne |\eta|$, by the monotonicity of the real function $t \in \mathbb{R} \mapsto t(\alpha + t^2)^{\frac{r-2}{2}}$, we obtain

$$|\xi||\eta|\left((\alpha + |\xi|^2)^{\frac{r-2}{2}} + (\alpha + |\eta|^2)^{\frac{r-2}{2}}\right) < |\xi|^2(\alpha + |\xi|^2)^{\frac{r-2}{2}} + |\eta|^2(\alpha + |\eta|^2)^{\frac{r-2}{2}},$$

which gives (2.3). As (2.1) and (2.2) are trivial and $H_{\alpha,r}(u) = -\mathrm{div}(a_{\alpha,r}(\nabla u))$, applying [1, Theorem 3.5], we complete the proof. □

Proposition 2.3. *If z_n is a bounded sequence in X such that $\|J_\alpha'(z_n)\| \to 0$, then z_n has a convergent subsequence in X.*

Proof. Up to a subsequence, $z_n = (u_n, v_n)$ converges to some $(\bar{u}, \bar{v})$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$; hence, $\int_\Omega G_s(z_n(x))(u_n(x) - \bar{u}(x))\mathrm{d}x \to 0$.

Moreover, $\langle J_\alpha'(z_n), (u_n - \bar{u}, 0)\rangle \to 0$ and, by Lemma 2.2, $H_{\alpha,p}$ is of class $(S)_+$, so that $u_n \to \bar{u}$ strongly in $W_0^{1,p}(\Omega)$.

In the same way, we obtain that also, $v_n \to \bar{v}$ strongly in $W_0^{1,q}(\Omega)$. □

Proposition 2.4. *If $\bar{\lambda}$ is not an eigenvalue of* (1.4)*, then J_α satisfies the (PS) condition at any level $c \in \mathbb{R}$, namely, if $z_n = (u_n, v_n)$ is a sequence in X such that $J_\alpha(z_n) \to c$ and $\|J_\alpha'(z_n)\| \to 0$, then z_n has a convergent subsequence in X.*

Proof. Let $z_n = (u_n, v_n)$ be a sequence in X such that $J_\alpha(z_n)$ is bounded and $\|J_\alpha'(z_n)\| \to 0$. Due to the previous proposition, it is sufficient to show that

$$\|z_n\| \quad \text{is bounded}. \tag{2.4}$$

By contradiction, suppose $\|(u_n, v_n)\| \to \infty$ and, denoting by $r_n = \|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q$, set $u_n' = u_n/r_n^{1/p}$ and $v_n' = v_n/r_n^{1/q}$.

It is immediate that $\|u_n'\|_{1,p}, \|v_n'\|_{1,q} \le 1$; hence, up to a subsequence, (u_n', v_n') converges to some $z' = (u', v')$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$.

In particular,

$$\frac{1}{r_n^{\frac{p-1}{p}}}\langle J_\alpha'(z_n), (u_n' - u', 0)\rangle \to 0 \tag{2.5}$$

and

$$\frac{1}{r_n^{\frac{q-1}{q}}}\langle J_\alpha'(z_n), (0, v_n' - v')\rangle \to 0. \tag{2.6}$$

Considering that $\frac{\beta}{p-1} + \frac{(\gamma+1)p}{q(p-1)} = 1$, by assumption (a_1) and Young's inequality, there is $c > 0$ such that

$$|G_s(s,t)| \le c\left(1 + |s|^{p-1} + |t|^{\frac{q(p-1)}{p}}\right)$$

so that

$$\frac{1}{r_n^{\frac{p-1}{p}}}\left|\int_\Omega G_s(z_n)(u_n'-u')\mathrm{d}x\right| \le c\left(\int_\Omega |u_n'-u'|\mathrm{d}x + \int_\Omega |u_n'|^{p-1}|u_n'-u'|\mathrm{d}x + \int_\Omega |v_n'|^{\frac{q(p-1)}{p}}|u_n'-u'|\mathrm{d}x\right).$$

Hence, by Hölder inequality, we obtain

$$\lim_{n\to\infty}\frac{1}{r_n^{\frac{p-1}{p}}}\int_\Omega G_s(z_n)(u_n'-u')\mathrm{d}x = 0,$$

which, combined with (2.5), gives

$$\int_\Omega\left(\frac{\alpha}{r_n^{2/p}} + |\nabla u_n'|^2\right)^{\frac{p-2}{2}}\nabla u_n'\nabla(u_n'-u')\mathrm{d}x \to 0.$$

Therefore, using the convexity of the function $\xi\in\mathbb{R}^n \mapsto \left(\frac{\alpha}{r_n^{2/p}} + |\xi|^2\right)^{\frac{p}{2}}$, we see that

$$\begin{aligned}\|u'\|_{1,p}^p &\le \liminf_{n\to\infty}\|u_n'\|_{1,p}^p \le \limsup_{n\to\infty}\|u_n'\|_{1,p}^p \le \limsup_{n\to\infty}\int_\Omega\left(\frac{\alpha}{r_n^{2/p}} + |\nabla u_n'|^2\right)^{\frac{p}{2}}\\ &\le \limsup_{n\to\infty}\left[\int_\Omega\left(\frac{\alpha}{r_n^{2/p}} + |\nabla u'|^2\right)^{\frac{p}{2}} + p\int_\Omega\left(\frac{\alpha}{r_n^{2/p}} + |\nabla u_n'|^2\right)^{\frac{p-2}{2}}\nabla u_n'\nabla(u_n'-u')\right]\\ &\le \|u'\|_{1,p}^p.\end{aligned}$$

Hence $u_n'\to u'$ also strongly in $W_0^{1,p}(\Omega)$. Analogously, by (2.6), we obtain that $v_n'\to v'$ strongly in $W_0^{1,q}(\Omega)$.

In particular,

$$\|u'\|_{1,p}^p + \|v'\|_{1,q}^q = \lim_{n\to\infty}\|u_n'\|_{1,p}^p + \|v_n'\|_{1,q}^q = \lim_{n\to\infty}\frac{\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q}{r_n} = 1$$

so that $(u', v') \neq (0,0)$.

By assumption (a_1),

$$G_s(s,t) = \bar{\lambda}\left(|s|^{p-2}s + \frac{1}{\gamma+1}|s|^\beta|t|^\gamma t\right) + r_1(s,t),$$

where

$$\lim_{|(s,t)|\to\infty}\frac{r_1(s,t)}{|s|^{p-1}+|t|^{q\frac{p-1}{p}}} = 0. \tag{2.7}$$

Hence, there is $c>0$ such that

$$|r_1(s,t)| \le c\left(|s|^{p-1} + |t|^{q\frac{p-1}{p}} + 1\right), \tag{2.8}$$

for any $(s,t)\in\mathbb{R}^2$. Let us fix $\tilde{u}\in W_0^{1,p}(\Omega)$, we want to prove that

$$\lim_{n\to\infty}\frac{1}{r_n^{(p-1)/p}}\int_\Omega |r_1(z_n(x))||\tilde{u}(x)|\mathrm{d}x = 0. \tag{2.9}$$

Recalling that $\|u_n'\|_{1,p}, \|v_n'\|_{1,q}\le 1$, let $\bar{c}>0$ be such that

$$\|u_n'\|_p^{p-1} + \|v_n'\|_q^{q\frac{p-1}{p}} \le \bar{c}, \quad \text{for any } n\in\mathbb{N}. \tag{2.10}$$

Let us fix $\varepsilon>0$. By (2.7), there is $\delta_\varepsilon>0$ such that

$$|(s,t)| > \delta_\varepsilon \Rightarrow |r_1(s,t)| < \frac{\varepsilon}{2\bar{c}}\left(|s|^{p-1} + |t|^{q\frac{p-1}{p}}\right). \tag{2.11}$$

Denoting by $\Omega^-_{n,\varepsilon} = \{x \in \Omega : |z_n(x)| \le \delta_\varepsilon\}$ and $\Omega^+_{n,\varepsilon} = \Omega \setminus \Omega^-_{n,\varepsilon}$,

$$x \in \Omega^-_{n,\varepsilon} \Rightarrow |u_n(x)|, \quad |v_n(x)| \le \delta_\varepsilon$$

so, by (2.8),

$$\frac{1}{r_n^{(p-1)/p}} \int\limits_{\Omega^-_{n,\varepsilon}} |r_1(z_n(x))||\tilde{u}(x)|\mathrm{d}x \le \frac{c}{r_n^{(p-1)/p}}\left(|\delta_\varepsilon|^{p-1} + |\delta_\varepsilon|^{q\frac{p-1}{p}} + 1\right)||\tilde{u}||_1$$

and, choosing n is big enough,

$$\frac{1}{r_n^{(p-1)/p}} \int\limits_{\Omega^-_{n,\varepsilon}} |r_1(z_n(x))| \quad |\tilde{u}(x)|\mathrm{d}x < \frac{\varepsilon}{2}\ ||\tilde{u}||_p. \tag{2.12}$$

On the other hand, by (2.11), Hölder inequality, and (2.10),

$$\begin{aligned}
&\frac{1}{r_n^{(p-1)/p}} \int\limits_{\Omega^+_{n,\varepsilon}} |r_1(z_n(x))| \quad |\tilde{u}(x)|\mathrm{d}x \\
&\le \frac{1}{r_n^{(p-1)/p}} \frac{\varepsilon}{2\bar{c}} \int\limits_{\Omega^+_{n,\varepsilon}} \left(|u_n(x)|^{p-1} + |v_n(x)|^{q\frac{p-1}{p}}\right)|\tilde{u}(x)|\mathrm{d}x \\
&\le \frac{\varepsilon}{2\bar{c}} \int\limits_{\Omega} \left(|u_n'(x)|^{p-1} + |v_n'(x)|^{q\frac{p-1}{p}}\right)|\tilde{u}(x)|\mathrm{d}x \\
&\le \frac{\varepsilon}{2}||\tilde{u}||_p,
\end{aligned}$$

which, together with (2.12), proves (2.9). As $u_n' \to u'$ in $W_0^{1,p}(\Omega)$, we infer

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{r_n^{(p-1)/p}} \int\limits_{\Omega} (\alpha + |\nabla u_n(x)|^2)^{\frac{p-2}{2}} \nabla u_n(x) \nabla \tilde{u}(x)\mathrm{d}x \\
&= \lim_{n\to\infty} \int\limits_{\Omega} \left(\frac{\alpha}{r_n^{2/p}} + |\nabla u_n'(x)|^2\right)^{\frac{p-2}{2}} \nabla u_n'(x)\nabla\tilde{u}(x)\mathrm{d}x \\
&= \int\limits_{\Omega} |\nabla u'(x)|^{p-2}\nabla u'(x)\nabla\tilde{u}(x)\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{r_n^{(p-1)/p}} \int\limits_{\Omega} \left(|u_n(x)|^{p-2}u_n(x) + \frac{1}{\gamma+1}\ |u_n(x)|^{\beta}|v_n(x)|^{\gamma}v_n(x)\right)\tilde{u}(x)\mathrm{d}x \\
&= \int\limits_{\Omega} \left(|u'(x)|^{p-2}u'(x) + \frac{1}{\gamma+1}\ |u'(x)|^{\beta}|v'(x)|^{\gamma}v'(x)\right)\tilde{u}(x)\mathrm{d}x,
\end{aligned}$$

which, together with (2.9), gives

$$\begin{aligned}
&\int\limits_{\Omega} |\nabla u'|^{p-2}\nabla u'\nabla\tilde{u}\mathrm{d}x - \bar{\lambda}\int\limits_{\Omega}\left(|u'|^{p-2}u' + \frac{1}{\gamma+1}\ |u'|^{\beta}|v'|^{\gamma}v'\right)\tilde{u}\mathrm{d}x \\
&= \lim_{n\to\infty}\frac{1}{r_n^{(p-1)/p}}\ \langle J_\alpha'(z_n), (\tilde{u}, 0)\rangle = 0.
\end{aligned}$$

In the same manner, we can see that, for any $\tilde{v} \in W_0^{1,q}(\Omega)$,

$$\int_\Omega |\nabla v'|^{q-2}\nabla v'\nabla \tilde{v}\mathrm{d}x - \bar{\lambda}\int_\Omega \left(|v'|^{q-2}v' + \frac{1}{\beta+1}|u'|^\beta |v'|^\gamma u'\right)\tilde{v}\mathrm{d}x = \lim_{n\to\infty}\frac{1}{r_n^{(q-1)/q}}\langle J'_\alpha(z_n),(0,\tilde{v})\rangle = 0.$$

In other words, $(u', v') \neq (0,0)$ solves (1.4) with $\lambda = \bar{\lambda}$, so the contradiction proves (2.4). □

Proposition 2.5. *If $\bar{\lambda}$ is an eigenvalue of* (1.4) *and* (b_-) *holds, so that*

$$\lim_{|(s,t)|\to\infty}\left[G(s,t) - \frac{1}{p}G_s(s,t)s - \frac{1}{q}G_t(s,t)t\right] = -\infty,$$

then J_α satisfies the CPS condition at any level $c \in \mathbb{R}$, namely, any sequence $z_n = (u_n, v_n)$ in X such that $J_\alpha(z_n) \to c$ and $\|J'_\alpha(z_n)\|(1 + \|z_n\|) \to 0$ has a convergent subsequence in X.

Moreover, if $\bar{\lambda}$ is an eigenvalue of (1.4) *and* (b_+) *holds, so that*

$$\lim_{|(s,t)|\to\infty}\left[G(s,t) - \frac{1}{p}G_s(s,t)s - \frac{1}{q}G_t(s,t)t\right] = +\infty,$$

then J_0 satisfies the CPS condition at any level $c \in \mathbb{R}$.

Proof. Let us show that $z_n = (u_n, v_n)$ is bounded in X when (b_-) holds and $\alpha \geq 0$.

By contradiction, assume that $\|z_n\| \to \infty$ and set $r_n = \|u_n\|^p_{1,p} + \|v_n\|^q_{1,q}$, $u'_n = u_n/r_n^{1/p}$, $v'_n = v_n/r_n^{1/q}$. Reasoning as in the proof of Proposition 2.4, we infer that, up to a subsequence, (u'_n, v'_n) converges to some $z' = (u', v') \neq (0,0)$ strongly in X.

Denoting by $K(s,t) = G(s,t) - \frac{1}{p}G_s(s,t)s - \frac{1}{q}G_t(s,t)t$, CPS condition gives that, for a suitable $C \in \mathbb{R}$,

$$\begin{aligned} C &\geq J_\alpha(z_n) - \frac{1}{p}\langle J'_\alpha(z_n),(u_n,0)\rangle - \frac{1}{q}\langle J'_\alpha(z_n),(0,v_n)\rangle \\ &= \int_\Omega \frac{\alpha}{p}(\alpha + |\nabla u_n(x)|^2)^{(p-2)/2} + \frac{\alpha}{q}(\alpha + |\nabla v_n(x)|^2)^{(q-2)/2}\mathrm{d}x - \int_\Omega K(z_n(x))\mathrm{d}x \\ &\geq -\int_\Omega K(z_n(x))\mathrm{d}x. \end{aligned}$$

From (b_-), there is $c_1 \in \mathbb{R}$ such that

$$K(s,t) \leq c_1 \quad \forall (s,t) \in \mathbb{R}^2.$$

Hence, by Fatou's lemma,

$$-C \leq \int_\Omega \limsup_{n\to\infty} K(z_n(x))\mathrm{d}x. \tag{2.13}$$

As we are assuming $r_n \to +\infty$, by (b_-), we obtain that, for almost every $x \in \Omega$,

$$z'(x) \neq (0,0) \Rightarrow |z_n(x)| = |(r_n^{1/p}u'_n(x), r_n^{1/q}v'_n(x))| \to \infty \Rightarrow K(z_n(x)) \to -\infty.$$

So, from (2.13), we infer that $z'(x) = (0,0)$ almost everywhere in Ω, which contradicts $z' \neq (0,0)$. Consequently, $z_n = (u_n, v_n)$ is bounded and, by Proposition 2.3, has a convergent subsequence in X.

If instead (b_+) holds, $J_0(z_n) \to c \in \mathbb{R}$ and $\|J'(z_n)\|(1 + \|z_n\|) \to 0$, then

$$\lim_{n\to\infty}\int_\Omega K(z_n(x))\mathrm{d}x = -c.$$

Then, reasoning as earlier, we infer again that z_n has a convergent subsequence in X. □

3 Cohomological linking

Throughout this section, Y denotes a Banach space and $f: Y \to \mathbb{R}$ a C^1 function. We also denote by H^* the Alexander-Spanier cohomology [44] with coefficients in $\mathbb{Z}_2 = \{-1, 1\}$.

Let us recall next definitions (see [6,8,21,22]).

Definition 3.1. Let D, S, and A be three subsets of Y and m a nonnegative integer.

We say that (D, S) *links A cohomologically in dimension* m (over $\mathbb{Z}_2$), if $S \subseteq D$, $S \cap A = \emptyset$ and the restriction homomorphism $H^m(Y, Y\setminus A) \to H^m(D, S)$ is not identically zero.

If (D, S) links A cohomologically in some dimension, then (D, S) links A (c.f. [10, Definition 5.1]). Moreover, if (D, S) links A, then $D \cap A \neq \emptyset$.

Definition 3.2. Let G be an abelian group. Let the mth critical group of f at $z \in Y$ with coefficient in G be defined by

$$C_m(f;\, z) = H^m(f^c, f^c\setminus\{z\}),$$

where $c = f(z)$, $f^c = \{u \in Y | f(u) \leq c\}$.

In general, it can happen that $C_m(f, z)$ is not finitely generated for some m and that $C_m(f, z) \neq 0$ for infinitely many m's.

Now, we assume that f is Gauteax differentiable and we denote U an open subset of X. If z is an isolated critical point of f and $f' : U \to X'$ is a demicontinuous function (namely, it is continuous from the strong topology to the weak topology) and of class $(S)_+$ in a neighborhood of u, then $C_*(f, z)$ is of finite type (see [11, Theorem 1.1] and [1, Theorem 3.4].

Let us introduce a general result concerning a C^1 functional in a Banach space that extends Theorem 3.2 in [10] to the case in which f satisfies only the CPS condition.

Theorem 3.3. *Let Y be a Banach space, $f \in C^1(Y, \mathbb{R})$, D, S, and A be three subsets of Y, $m \in \mathbb{N}$.*

Assume that

(D, S) links A cohomologically in dimension m over $\mathbb{Z}_2$,

$$\sup_S f < \inf_A f = a \quad \textit{and} \quad b = \sup_D f < +\infty,$$

f verifies the (CPS) condition at any level $c \in [a, b]$ and $f^{-1}([a, b])$ contains a finite number of critical points.

Then, there exists a critical point z of f with $a \leq f(z) \leq b$ and $C_m(f;\, z) \neq \{0\}$.

Proof. Since $D \cap A \neq \emptyset$, $a \leq b$. We aim to apply [21, Theorem 5.2]. If (E, d) is a metric space, $f: E \to \mathbb{R}$ is a continuous function and $z \in E$, let us consider the notion of weak slope $|df|(z)$ as defined in [23, Definition 2.1] (see also [33, Definition 5.1]). Consequently, we say that

- z is a critical point if $|df|(z) = 0$
- f satisfies the PS condition at a level $c \in \mathbb{R}$ if any sequence z_n in E such that $f(z_n) \to c$ and $|df|(z_n) \to 0$ has a convergent subsequence in E.

In particular, if E is also a Banach space and f is C^1, we have

$$|df|(z) = \|f'(z)\|, \quad \forall z \in E.$$

Due to [16, Theorem 4.1, Remark 4.4], there exists a metric $\tilde{d}$ on Y, topologically equivalent to the metric induced by $\|\ \|_Y$, such that $(Y, \tilde{d})$ is complete and, denoting by $|\tilde{d}f|$ the weak slope of f with respect to the metric $\tilde{d}$,

$$|\tilde{d}f|(z) = \|f'(z)\|_{Y'}(1 + \|z\|_Y), \quad \forall z \in Y.$$

Therefore, as f satisfies the *CPS* condition at any level $c \in [a, b]$, then f satisfies the *PS* condition at any level $c \in [a, b]$, with respect to $\tilde{d}$.

By [21, Theorem 7.5], the assumptions of [21, Theorem 5.2] are satisfied. Then, the assertion follows, taking into account [21, Proposition 7.3 and Remark 5.3]. □

Denoting by $\mathcal{A}$ the class of symmetric subsets of Y, Fadell and Rabinowitz (see [28,29]) constructed an index theory $i : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ with the following properties:

(i) Definiteness: $i(A) \geq 0, \quad i(A) = 0 \Leftrightarrow A = \emptyset$;

(ii) Monotonicity: If there is an odd continuous map $A \to A'$, then

$$i(A) \leq i(A').$$

In particular, equality holds if A and A' are homeomorphic;

(iii) Subadditivity:

$$i(A \cup A') \leq i(A) + i(A');$$

(iv) Continuity: If A is closed, there is a neighborhood $U \in \mathcal{A}$ of A such that

$$i(U) = i(A);$$

(v) Neighborhood of zero: If U is a bounded symmetric neighborhood of 0 in Y,

$$i(\partial U) = \dim Y;$$

(vi) Stability: If A is closed and $A^*\mathbb{Z}_2$ is the join of A with $\mathbb{Z}_2$, realized in $Y \oplus \mathbb{R}$,

$$i(A^*\mathbb{Z}_2) = i(A) + 1,$$

where $A^*\mathbb{Z}_2$ is the union of all line segments in $Y \oplus \mathbb{R}$, joining $\{1\}$ and $\{-1\}$ to points of A;

(vii) Piercing property: Assume that A, A_0, and A_1 are closed and

$$\varphi : A \times [0, 1] \to A_0 \cup A_1$$

is an odd continuous map such that $\varphi(A \times [0, 1])$ is closed, and

$$\varphi(A \times \{0\}) \subset A_0, \quad \varphi(A \times \{1\}) \subset A_1.$$

Then,

$$i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A).$$

We can state the next theorem whose proof follows from [10, Theorem 3.6] and [22, Theorem 2.7]. For reader's convenience, we give a sketch of the proof.

Theorem 3.4. *Let S and A be two symmetric subsets of Y with $S \cap A = \emptyset$. Assume that $0 \in A$ and $i(S) = i(Y \backslash A) = m < \infty$.*

Then, (Y, S) links A cohomologically in dimension m over $\mathbb{Z}_2$.

Proof. We want to prove that $H^m(Y, Y \backslash A) \to H^m(Y, S)$ is not identically zero. So, considering the exact sequence

$$H^m(Y, Y \backslash A) \to H^m(Y, S) \to H^m(Y \backslash A, S),$$

it is sufficient to show that

$$H^m(Y, S) \to H^m(Y \backslash A, S) \quad \text{is not injective.} \tag{3.1}$$

If $m = 0$, then $S = \emptyset$ and $A = Y$, so $H^0(Y, S) \neq \{0\}$, while $H^0(Y \backslash A, S) = \{0\}$, so (3.1) is proved.

Let us denote by $j : H^{m-1}(Y\backslash A) \to H^{m-1}(S)$ and $i : H^{m-1}(Y) \to H^{m-1}(S)$ the homomorphisms induced by inclusions. We recall that $H^q(Y) = \{0\}$ if $q \neq 0$, while the dimension of $H^0(Y)$ is 1.

By Lemma 3.4 in [10], we infer that

(i) if $m = 1$, the dimension of $im(j)$ is at least 2, while the dimension of $im(i)$ is at most 1.
(ii) if $m \geq 2$, the dimension of $im(j)$ is at least 1, while $im(i) = \{0\}$.

Hence, $im(i)$ is a proper subset of $im(j)$, for any $m \geq 1$.

Consider now the commutative diagram

$$\begin{array}{ccccc} H^{m-1}(Y) & \xrightarrow{i} & H^{m-1}(S) & \xrightarrow{\delta} & H^m(Y,S) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow \\ H^{m-1}(Y\setminus A) & \xrightarrow{j} & H^{m-1}(S) & \xrightarrow{\bar{\delta}} & H^m(Y\setminus A, S) \end{array}$$

where the horizontal rows are exact sequences and the vertical rows are induced by inclusions.

Let $\omega \in im(j)\backslash im(i)$, then $\delta\omega \neq 0$ in $H^m(Y,S)$, while $\bar{\delta}\omega = (\delta\omega)_{|(Y\backslash A,S)} = 0$.

Hence, (3.1) is proved also for any $m \geq 1$, as $H^m(Y,S) \to H^m(Y\backslash A, S)$ is not injective. □

Now, we consider the Banach space $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We recall that the sequence (λ_m) is defined by (1.7), i.e.,

$$\lambda_m = \inf_{A\in\Sigma_m} \sup_{(u,v)\in A} \Phi(u,v),$$

where $\Sigma_m = \{A \subset \Sigma | A$ is symmetric, compact, and $i(A) \geq m\}$. Let us recall the following lemma, proved in [22].

Lemma 3.5. *For every symmetric and open subset A of Σ*

$$i(A) = \sup\{i(K) | K \text{ is compact and symmetric with } K \subseteq A\}.$$

Remark 3.6. Denoting by $\mathcal{B}_m = \{A \subset \Sigma | A$ is symmetric and $i(A) \geq m\}$, the previous lemma assures that

$$\lambda_m = \inf_{A\in\mathcal{B}_m} \sup_{(u,v)\in A} \Phi(u,v).$$

Proposition 3.7. *Let $\alpha \in \mathbb{R}$ and $r > 0$ and set*

$$X^\alpha = \{z \in X | \Phi(z) \leq \alpha\Psi(z)\}, \quad \Sigma^\alpha = \Sigma \cap X^\alpha,$$

$$S_r^\alpha = \{z \in X^\alpha | \Phi(z) = r\},$$

$$\mathring{X}^\alpha = \{z \in X | \Phi(z) < \alpha\Psi(z)\}, \quad \mathring{\Sigma}^\alpha = \Sigma \cap \mathring{X}^\alpha.$$

Then,

(*a*) $i(\Sigma^\alpha) \geq m$, *for any* $\alpha \geq \lambda_m$;
(*b*) $\lambda_m < \lambda_{m+1} \Rightarrow i(\Sigma^\alpha) = m$, *for any* $\alpha \in [\lambda_m, \lambda_{m+1})$;
(*c*) $\lambda_m < \lambda_{m+1} \Rightarrow i(\mathring{\Sigma}^{\lambda_{m+1}}) = m$;
(*d*) Σ^α *is an odd strong deformation retract of* $X^\alpha\backslash\{0\}$;
(*e*) $\mathring{\Sigma}^\alpha$ *is an odd strong deformation retract of* $\mathring{X}^\alpha\backslash\{0\}$;
(*f*) S_r^α *is an odd strong deformation retract of* $X^\alpha\backslash\{0\}$.

Proof. Let us consider $\alpha > \lambda_1$; otherwise, the proof is even simpler.

By contradiction, assume that $i(\Sigma^{\lambda_m}) \leq m - 1$. Continuity assures that there is a closed neighborhood N of Σ^{λ_m} such that $i(N) = i(\Sigma^{\lambda_m})$. As N is also a neighborhood of the critical set $\{z | \Phi_{|\Sigma}(z) = \lambda_m\}$, by the equivariant deformation theorem, there exist $\varepsilon > 0$ and an odd continuous map $\eta : \Sigma^{\lambda_m+\varepsilon} \to \Sigma^{\lambda_m-\varepsilon} \cup N = N$. So, due to monotonicity, $i(\Sigma^{\lambda_m+\varepsilon}) \le i(N) = i(\Sigma^{\lambda_m}) \le m-1$. By definition of λ_m, there is $\bar{A} \in \Sigma_m$ such that $\sup\Phi(\bar{A}) < \lambda_m + \varepsilon$. Hence, $\bar{A} \subset \Sigma^{\lambda_m+\varepsilon}$, and we have the contradiction $m \le i(\bar{A}) \le i(\Sigma^{\lambda_m+\varepsilon}) \le m-1$, which proves (a).

Now, if $\lambda_m < \lambda_{m+1}$ and $\alpha \in [\lambda_m, \lambda_{m+1})$, by (a) we know that $i(\Sigma^\alpha) \ge m$. By contradiction, assume that $i(\Sigma^\alpha) \ge m+1$. Recalling Remark 3.6, $\Sigma^\alpha \in \mathcal{B}_{m+1}$ and

$$\lambda_{m+1} = \inf_{A\in\mathcal{B}_{m+1}} \sup_A \Phi \le \sup_{\Sigma^\alpha} \Phi = \alpha < \lambda_{m+1},$$

so we conclude that $i(\Sigma^\alpha) = m$.

Moreover, still due to (a), $i(\mathring{\Sigma}^{\lambda_{m+1}}) \ge m$. Let K be a symmetric and compact subset of $\mathring{\Sigma}^{\lambda_{m+1}}$, so that $\max\Phi(K) \le \alpha$ for a suitable $\alpha \in [\lambda_m, \lambda_{m+1})$. (b) gives that $i(K) \le i(\Sigma^\alpha) = m$; hence, (c) comes from Lemma 3.5.

For any $s \in [0,1]$ and $z = (u,v) \in X\setminus\{0\}$, we define $\gamma_{s,z} = 1 - s + \frac{s}{\Psi(z)}$ and $\eta(s,(u,v)) = (\gamma_{s,z}^{1/p}u, \gamma_{s,z}^{1/q}v)$.

η is clearly continuous and odd. In addition, due to the (p,q)-homogeneity of Φ and Ψ, shown in (1.6),

$$\Phi(\eta(s,z)) = \gamma_{s,z}\Phi(z) \quad \text{and} \quad \Psi(\eta(s,z)) = \gamma_{s,z}\Psi(z).$$

Hence, (d) and (e) are proved as

$$z \in X^\alpha\setminus\{0\} \quad \Rightarrow \quad \eta(0,z) = z, \quad \eta(1,z) \in \Sigma^\alpha \quad \text{and} \quad \eta(s,z) \in X^\alpha\setminus\{0\} \quad \forall s \in [0,1]$$

$z \in \Sigma \quad \Rightarrow \quad \eta(s,z) = z$, for any $s \in [0,1]$.

Analogously, we obtain (f) setting
$\delta_{s,z} = 1 - s + s\frac{r}{\Phi(z)}$ and $\bar{\eta}(s,(u,v)) = (\delta_{s,z}^{1/p}u, \delta_{s,z}^{1/q}v)$. □

Let $m \in \mathbb{N}$ be such that $\lambda_m < \lambda_{m+1}$.
If $m \ge 1$, we set

$$X_-^m := \{z = (u,v) \in X | \Phi(u,v) \le \lambda_m \Psi(u,v)\},$$

$$X_+^m := \{z = (u,v) \in X | \Phi(u,v) \ge \lambda_{m+1} \Psi(u,v)\},$$

and if $m = 0$

$$X_-^0 := \{0\} \quad X_+^0 := X.$$

We underline that the sets X_-^m and X_+^m are symmetric and (p,q)-homogeneous, i.e.,

$$(u,v) \in X_\pm^m \Rightarrow \left(\delta^{\frac{1}{p}}u, \delta^{\frac{1}{q}}v\right) \in X_\pm^m, \quad \text{for any } \delta > 0.$$

In particular, if $p \ne q$ and $m \ge 1$, the sets X_+^m and X_-^m are not cones.

Theorem 3.8. *Setting*

$$D_r = \{z \in X_-^m | \Phi(z) \le r\} \quad \textit{and} \quad S_r = \{z \in X_-^m | \Phi(z) = r\},$$

(D_r, S_r) links X_+^m cohomologically in dimension m over $\mathbb{Z}_2$, for any $r > 0$.

Proof. From Proposition 3.7, we obtain that

$$i(S_r) = i(X\setminus X_+^m) = m.$$

As S_r and X_+^m are symmetric, $0 \in X_+^m$ and $S_r \cap X_+^m = \emptyset$, Theorem 3.4 gives that

$$(X, S_r) \text{ links } X_+^m \text{ cohomologically in dimension } m \text{ over } \mathbb{Z}_2. \tag{3.2}$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} H^{m-1}(X) & \to & H^{m-1}(X^m_-\setminus\{0\}) & \to & H^m(X, X^m_-\setminus\{0\}) & \to & H^m(X) & \to & H^m(X^m_-\setminus\{0\}) \\ \downarrow \gamma_1 & & \downarrow \gamma_2 & & \downarrow \gamma_3 & & \downarrow \gamma_4 & & \downarrow \gamma_5 \\ H^{m-1}(D_r) & \longrightarrow & H^{m-1}(S_r) & \longrightarrow & H^m(D_r, S_r) & \longrightarrow & H^m(D_r) & \longrightarrow & H^m(S_r), \end{array}$$

where the horizontal rows are exact sequences and the vertical rows are induced by inclusions.

In particular, γ_1, γ_2, γ_4, and γ_5 are isomorphisms, as D_r is homotopic to X and S_r is homotopic to $X^m_-\setminus\{0\}$. Hence, due to the five lemma [44], also γ_3 is an isomorphism, which, combined with (3.2), completes the proof. □

4 Critical group estimates at zero

From now on in this section, we assume that $p, q \in [2, N)$, $\alpha \geq 0$, $H \in C^1(\mathbb{R}^2, \mathbb{R})$ and there exist $C > 0$, $p' \in (p, p^*)$, and $q' \in (q, q^*)$ such that

$$\begin{aligned} |H_s(s,t)| &\leq C\left(|s|^{p'-1} + |t|^{q'\frac{p'-1}{p'}} + 1\right), \\ |H_t(s,t)| &\leq C\left(|s|^{p'\frac{q'-1}{q'}} + |t|^{q'-1} + 1\right). \end{aligned} \tag{4.1}$$

We denote by $I_{\alpha,H}$ the C^1-functional defined by

$$I_{\alpha,H}(z) = \frac{1}{p}\int_\Omega (\alpha + |\nabla u(x)|^2)^{\frac{p}{2}}\mathrm{d}x + \frac{1}{q}\int_\Omega (\alpha + |\nabla v(x)|^2)^{\frac{q}{2}}\mathrm{d}x - \int_\Omega H(u(x), v(x))\mathrm{d}x, \tag{4.2}$$

for any $z = (u, v) \in X$.

Theorem 4.1. *Let $p, q \in [2, N)$, $\alpha \geq 0$. Assume that $H \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies* (4.1) *and $\nabla H(0,0) = (0,0)$.*

If $\mathbf{0} = (0,0)$ is an isolated critical point for $I_{\alpha,H}$, there are $\overline{H} \in C^1(\mathbb{R}^2, \mathbb{R})$ and $\bar\eta > 0$ such that:
- *$\overline{H}(s,t) = H(s,t)$ when $|s|, |t| \leq \bar\eta$.*
- *$\mathbf{0}$ is an isolated critical point for $I_{\alpha,\overline{H}}$ and $C_m(I_{\alpha,H}, \mathbf{0}) = C_m(I_{\alpha,\overline{H}}, \mathbf{0})$, for any $m \in \mathbb{N}$.*
- *$\overline{H}(\mathbb{R}^2) = \overline{H}([-2\bar\eta, 2\bar\eta]^2) \subset H([-2\bar\eta, 2\bar\eta]^2)$.*

In particular, if $(s,t) \notin [-2\bar\eta, 2\bar\eta]^2$, then

$$\overline{H}(s,t) = \begin{cases} H(0,0), & \text{if } |s| \geq 2\bar\eta, \quad |t| \geq 2\bar\eta, \\ \overline{H}(0,t), & \text{if } |s| \geq 2\bar\eta, \quad |t| \leq 2\bar\eta, \\ \overline{H}(s,0), & \text{if } |s| \leq 2\bar\eta, \quad |t| \geq 2\bar\eta. \end{cases}$$

In addition, if H is C^2 at least in an open set containing $[-2\bar\eta, 2\bar\eta]^2$, then $\overline{H} \in C^2(\mathbb{R}^2, \mathbb{R})$.

Proof. Consider a C^∞-function $\theta : \mathbb{R} \to [0,1]$ such that $\theta(\tau) = 1$ for $|\tau| \leq 1$ and $\theta(\tau) = 0$ for $|\tau| \geq 2$.

For every $\delta \in [0,1]$, we define $H^\delta(s,t) = H(\theta(\delta s)s, \theta(\delta t)t)$, so that

$$\begin{aligned} H^\delta_s(s,t) &= H_s(\theta(\delta s)s, \theta(\delta t)t)(\theta'(\delta s)\delta s + \theta(\delta s)), \\ H^\delta_t(s,t) &= H_s(\theta(\delta s)s, \theta(\delta t)t)(\theta'(\delta t)\delta t + \theta(\delta t)). \end{aligned} \tag{4.3}$$

As the function $\tau \mapsto |\theta'(\tau)\tau + \theta(\tau)|$ is bounded in $\mathbb{R}$, by (4.1) and (4.3), there is a constant $C_0 > 0$, independent from δ, such that

$$\begin{aligned} |H^\delta_s(s,t)| &\leq C_0\left(|s|^{p'-1} + |t|^{q'\frac{p'-1}{p'}} + 1\right), \\ |H^\delta_t(s,t)| &\leq C_0\left(|s|^{p'\frac{q'-1}{q'}} + |t|^{q'-1} + 1\right). \end{aligned} \tag{4.4}$$

Hence, the functional $I_{a,H^\delta}: X \to \mathbb{R}$ is C^1 for any $a \geq 0$ and $\delta \in [0,1]$ and for any $z_0 = (u_0, v_0) \in X$, $z = (u, v) \in X$, it results in

$$\langle I'_{a,H^\delta}(z_0), z\rangle = \int_\Omega (a + |\nabla u_0|^2)^{\frac{p-2}{2}} \nabla u_0 \nabla u \mathrm{d}x + \int_\Omega (a + |\nabla v_0|^2)^{\frac{q-2}{2}} \nabla v_0 \nabla v \mathrm{d}x - \int_\Omega (H_s^\delta(u_0, v_0)u + H_t^\delta(u_0, v_0)v)\mathrm{d}x.$$

Let $r > 0$ be such that $\mathbf{0}$ is the unique critical point of $I_{a,H^0} = I_{a,H}$ in

$$D_r = \{z = (u, v) \in X : \|\nabla u\|_p + \|\nabla v\|_q \leq r\}.$$

The map $\delta \mapsto I_{a,H^\delta}$ is continuous from $[0,1]$ to $C_b^1(D_r, \mathbb{R})$.

Moreover, reasoning as in the proof of Proposition 2.3, we see that I_{a,H^δ} satisfies the Palais-Smale condition in D_r, for any $a \geq 0$ and $\delta \in [0,1]$.

We claim that there exists $\bar{\delta} \in (0,1]$ such that $\mathbf{0}$ is the unique critical point of I_{a,H^δ} in D_r, for any $\delta \in [0, \bar{\delta}]$. Assume, by contradiction, that $\delta_j \to 0$ and $z_j = (u_j, v_j) \in D_r \setminus \{\mathbf{0}\}$ is a critical point of $I_{a,H^{\delta_j}}$.

Taking account of (4.4), by Theorem 1.1 in [47], we infer that every u_j and v_j are in $L^\infty(\Omega)$ and there is $C_1 > 0$, depending on Ω, p, q, p', q', and r but independent from j, such that

$$\|u_j\|_\infty, \quad \|v_j\|_\infty \leq C_1.$$

Thus, $|\delta_j u_j(x)| < 1$ and $|\delta_j v_j(x)| < 1$ in Ω, when j is big enough.

Therefore, by (4.3), $z_j = (u_j, v_j)$ is a critical point of $I_{a,H}$ and a contradiction follows.

Setting $\overline{H} = H_{\bar{\delta}}$, from [17, Theorem 5.2], we deduce that $C_m(I_{a,H}, \mathbf{0}) = C_m\big(I_{a,\overline{H}}, \mathbf{0}\big)$ (for related results, see also [8, Theorem I.5.6]) and the assertion follows. □

Assume that $H \in C^2(\mathbb{R}^2, \mathbb{R})$, and there exist $C > 0$, $p' \in (p, p^*)$, $q' \in (q, q^*)$ such that

$$\begin{aligned} c|H_{ss}(s,t)| &\leq C\left(|s|^{p'-2} + |t|^{q'\frac{p'-2}{p'}} + 1\right), \\ |H_{st}(s,t)|, |H_{ts}(s,t)| &\leq C\left(|s|^{p'-1-\frac{p'}{q'}} + |t|^{q'-1-\frac{q'}{p'}} + 1\right), \\ |H_{tt}(s,t)| &\leq C\left(|s|^{p'\frac{q'-2}{q'}} + |t|^{q'-2} + 1\right). \end{aligned} \tag{4.5}$$

Then, $I_{a,H}$, defined through (4.2), is a C^2-functional, and for any $z_0 = (u_0, v_0)$, $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2) \in X$, we have

$$\begin{aligned} \langle I''_{a,H}(z_0)z_1, z_2\rangle = &\int_\Omega ((a + |\nabla u_0|^2)^{(p-2)/2}(\nabla u_1|\nabla u_2) + (p-2)(a + |\nabla u_0|^2)^{(p-4)/2}(\nabla u_0|\nabla u_1)(\nabla u_0|\nabla u_2))\mathrm{d}x \\ &+ \int_\Omega ((a + |\nabla v_0|^2)^{(q-2)/2}(\nabla v_1|\nabla v_2) + (q-2)(a + |\nabla v_0|^2)^{(q-4)/2}(\nabla v_0|\nabla v_1)(\nabla v_0|\nabla v_2))\mathrm{d}x \\ &- \int_\Omega (H_{ss}(u_0, v_0)u_1u_2 + H_{tt}(u_0, v_0)v_1v_2 + H_{st}(u_0, v_0)u_1v_2 + H_{ts}(u_0, v_0)u_2v_1)\mathrm{d}x. \end{aligned}$$

We recall the following notion.

Definition 4.2. If Y is a Banach space, $I \in C^2(Y, \mathbb{R})$ and z_0 a critical point of I, and the Morse index $m(I, z_0)$ of I at z_0 is the supremum of the dimensions of the subspaces of Y, where $I''(z_0)$ is negative definite. The large Morse index $m^*(I, z_0)$ is the sum of $m(I, z_0)$ and the dimension of the kernel of $I''(z_0)$.

Arguing as in [6, Theorem 1.4], we can establish the following critical group estimates for the C^2 functionals I_a associated with systems of (p, q)-area equations. Precisely, we have

Theorem 4.3. *Let $\alpha > 0$ and $H \in C^2(\mathbb{R}^2, \mathbb{R})$ satisfying assumptions* (4.5). *If z_0 is a critical point of the functional $I_{\alpha,H}$, then $m(I_{\alpha,H}, z_0)$ and $m^*(I_{\alpha,H}, z_0)$ are finite and*

$$C_m(I_{\alpha,H}, z_0) = \{0\},$$

whenever $m < m(I_{\alpha,H}, z_0)$ or $m > m^*(I_{\alpha,H}, z_0)$.

In the following theorem, we refer to m_0 and m_0^*, which have been defined through (1.8).

Theorem 4.4. *Let $\alpha \geq 0$ and $p, q \in [2, N)$. Let $G \in C^1(\mathbb{R}^2, \mathbb{R})$ a function that satisfies the conditions (a_1) and (a_2). If $\mathbf{0}$ is an isolated critical point of the functional J_α, defined as in* (1.3), *then*

$$C_m(J_\alpha, \mathbf{0}) = \{0\}, \quad \textit{whenever } m < m_0 \textit{ or } m > m_0^*.$$

Proof. From (4.2), $J_\alpha = I_{\alpha,G}$. Due to assumption (a_1) and Theorem 4.1, there is $\overline{G} \in C^1(\mathbb{R}^2, \mathbb{R})$ and $\overline{\eta} > 0$ such that

$$|s|, |t| \leq \overline{\eta} \quad \Rightarrow \quad \overline{G}(s, t) = G(s, t), \tag{4.6}$$

$$|s|, |t| \geq 2\overline{\eta} \quad \Rightarrow \quad \overline{G}(s, t) = G(0, 0), \tag{4.7}$$

$$C_m(J_\alpha, \mathbf{0}) = C_m(I_{\alpha,\overline{G}}, \mathbf{0}), \quad \text{for any} \quad m \in \mathbb{N}. \tag{4.8}$$

Taking account of assumption (a_2), where U is an open set containing $[-2\overline{\eta}, 2\overline{\eta}] \times [-2\overline{\eta}, 2\overline{\eta}]$, Theorem 4.1 assures that $\overline{G} \in C^2(\mathbb{R}^2, \mathbb{R})$. Furthermore, by (4.7), $\overline{G}$ satisfies (4.5), so that $I_{\alpha,\overline{G}} \in C^2(X, \mathbb{R})$.

Moreover, (4.6) and (1.8) give that $\langle I''_{\alpha,\overline{G}}(\mathbf{0})z, z\rangle = Q(z)$ for any z in X; thus, $m(I_{\alpha,\overline{G}}, \mathbf{0}) = m_0$ and $m^*(I_{\alpha,\overline{G}}, \mathbf{0}) = m_0^*$.

If 0, then Theorem 4.3 holds; thus, $C_m(I_{\alpha,\overline{G}}, \mathbf{0}) = 0$ whenever $m < m(I_{\alpha,\overline{G}}, \mathbf{0})$ or $m > m^*(I_{\alpha,\overline{G}}, \mathbf{0})$, so by (4.8), the assertion follows.

If instead $\alpha = 0$, we have to make a further distinction.

If $H_G(0, 0)$ is null or negative semidefinite, we have $m_0 = 0$, $m_0^* = +\infty$ and the assertion is obvious.

If $H_G(0, 0)$ is negative definite, we have $m_0 = m_0^* = 0$ and there is $\mu > 0$ such that

$$H_G(0, 0)[\xi]^2 \leq -\mu\, |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^2. \tag{4.9}$$

Moreover, due to assumption (a_1), there are $p' > p, q' > q$, and $C > 0$ such that $W_0^{1,p}(\Omega) \subset L^{p'}(\Omega)$, $W_0^{1,q}(\Omega) \subset L^{q'}(\Omega)$, and

$$|G(s, t)| \leq C(|s|^{p'} + |t|^{q'} + 1).$$

Combining with (4.9) and redefining $C > 0$, we obtain

$$-G(s, t) \geq -G(0, 0) - C(|s|^{p'} + |t|^{q'}), \quad \text{for any } (s, t) \in \mathbb{R}^2,$$

Thus,

$$J_0(u, v) - J_0(\mathbf{0}) \geq \frac{1}{p}||u||_p{}^p + \frac{1}{q}\,||v||_q{}^q - C(|u|_{p'}{}^{p'} + |v|_{q'}{}^{q'})$$

$$\geq ||u||_p{}^p\left(\frac{1}{p} - C||u||_p{}^{p'-p}\right) + ||v||_q{}^q\left(\frac{1}{q} - C||v||_q{}^{q'-q}\right) \geq 0,$$

if $||(u, v)||$ is small enough.

Hence, $\mathbf{0}$ is a local minimum for J_0, and, by the excision property, we have

$$C_m(J_0, \mathbf{0}) \approx H^m(\{0\}, \emptyset),$$

so the assertion follows.

If $H_G(0, 0)$ is positive semidefinite or indefinite, we have $m_0 = m_0^* = +\infty$.

As $I_{\alpha,\overline{G}} \in C^2(X, \mathbb{R})$, from [34, Theorem 3.1], we infer that $C_m(I_{\alpha,\overline{G}}, \mathbf{0}) = \{0\}$ for any m, so by (4.8), the assertion follows. □

5 Geometry of J_a

Let us recall that $F:\mathbb{R}^2\to\mathbb{R}$ is the C^1-function defined by

$$F(s,t)=\frac{1}{p}\,|s|^p+\frac{1}{q}\,|t|^q+\frac{1}{(\beta+1)(\gamma+1)}\,|s|^\beta|t|^\gamma st.$$

Assuming that $G(0,0)=0$, from assumption (a_1), we see that

$$G(s,t)=\bar{\lambda}F(s,t)+R(s,t),$$

where $R\in C^1(\mathbb{R}^2,\mathbb{R})$, $R(0,0)=0$, and

$$\lim_{|(s,t)|\to\infty}\frac{R_s(s,t)}{|s|^{p-1}+|t|^{q\frac{p-1}{p}}}=0\qquad\lim_{|(s,t)|\to\infty}\frac{R_t(s,t)}{|s|^{p\frac{q-1}{q}}+|t|^{q-1}}=0.\tag{5.1}$$

Theorem 5.1. *Following the previous notations,*

$$\lim_{|(s,t)|\to\infty}\frac{R(s,t)}{F(s,t)}=0.$$

Proof. From Young's inequality and (1.5), there exists $c>0$ be such that

$$\frac{1}{p}\,|s|^p+|s||t|^{q\frac{p-1}{p}}+\frac{1}{q}\,|t|^q\le cF(s,t),\quad\forall(s,t)\in\mathbb{R}^2.$$

By (5.1), for every $\varepsilon>0$, there are $\eta_\varepsilon,\ c_\varepsilon,\ \delta_\varepsilon>0$ such that

$$\begin{aligned}|(s,t)|>\eta_\varepsilon\Rightarrow|R_s(s,t)|&\le\frac{\varepsilon}{2c}\left(|s|^{p-1}+|t|^{q\frac{p-1}{p}}\right),\\ |R_t(s,t)|&\le\frac{\varepsilon}{2c}\left(|s|^{p\frac{q-1}{q}}+|t|^{q-1}\right);\\ |(s,t)|\le\eta_\varepsilon\Rightarrow|R_s(s,t)|&\le c_\varepsilon,\quad|R_t(s,t)|\le c_\varepsilon;\\ |(s,t)|>\delta_\varepsilon\Rightarrow F(s,t)&>\frac{4c_\varepsilon\eta_\varepsilon}{\varepsilon}.\end{aligned}$$

For any $r\in\mathbb{R}$, let us denote by $r^+=\max\{0,r\}$ and $r^-=\min\{0,r\}$. As

$$R(s,t)=\int_0^s R_s(\sigma,t)\mathrm{d}\sigma+\int_0^t R_t(0,\tau)\mathrm{d}\tau,$$

we infer that

$$\begin{aligned}|R(s,t)|&\le\int_{s^-}^{s^+}|R_s(\sigma,t)|\mathrm{d}\sigma+\int_{t^-}^{t^+}|R_t(0,\tau)|\mathrm{d}\tau\\ &=\int_{\{\sigma\in[s^-,s^+]:|(\sigma,t)|\le\eta_\varepsilon\}}|R_s(\sigma,t)|\mathrm{d}\sigma+\int_{\{\sigma\in[s^-,s^+]:|(\sigma,t)|>\eta_\varepsilon\}}|R_s(\sigma,t)|\mathrm{d}\sigma\\ &\quad+\int_{\{\tau\in[t^-,t^+]:|\tau|\le\eta_\varepsilon\}}|R_t(0,\tau)|\mathrm{d}\tau+\int_{\{\tau\in[t^-,t^+]\ :\ |\tau|>\eta_\varepsilon\}}|R_t(0,\tau)|\mathrm{d}\tau\\ &\le 2c_\varepsilon\eta_\varepsilon+\frac{\varepsilon}{2}F(s,t).\end{aligned}$$

Therefore,

$$|(s,t)|>\delta_\varepsilon\quad\Rightarrow\quad\frac{|R(s,t)|}{F(s,t)}<\varepsilon.$$

□

Proposition 5.2. *If* (a_1) *holds and* (u, v) *is a weak solution to Problem* (1.2), *then*

(i) $(u, v) \in (C^{1,\eta}(\overline{\Omega}))^2$, *for some* $\eta \in (0, 1)$;

(ii) *for every bounded set* $A \subset X$, *there exists* $\eta \in (0, 1)$ *and* $M > 0$ *such that* $\|u\|_{C^{1,\eta}(\overline{\Omega})} \leq M$ *and* $\|v\|_{C^{1,\eta}(\overline{\Omega})} \leq M$ *for every* $(u, v) \in A$ *solving* (1.2);

(iii) *if there is* $\varepsilon > 0$ *such that*

$$\begin{aligned} &G_s(0, t) \neq 0, \quad \forall t \in (-\varepsilon, \varepsilon)\setminus\{0\}, \\ &G_t(s, 0) \neq 0, \quad \forall s \in (-\varepsilon, \varepsilon)\setminus\{0\}, \end{aligned} \tag{*}$$

then every nontrivial solution (u, v) *of* (1.2) *is also not semitrivial, i.e.,* $u \neq 0$ *and* $v \neq 0$.

Proof. If (a_1) holds and A is a bounded subset of X, by Theorem 1.1 in [47], all the weak solutions $(u, v) \in A$ are in $(L^\infty(\Omega))^2$ and there is $M_0 > 0$ such that $\|u\|_\infty \leq M_0$ and $\|v\|_\infty \leq M_0$. Hence, due to [36], we infer (i) and (ii).

Let us prove (iii) arguing by contradiction. Suppose that (a_1) and (*) hold and that there is an $(u, 0) \neq (0, 0)$ solving (1.2). By (i), we obtain that $u \in C(\overline{\Omega})\setminus\{0\}$, $M = \max_{\overline{\Omega}}|u| > 0$, and there is $\bar{x} \in \Omega$ such that $0 < |u(\bar{x})| < \min\{\varepsilon, M\}$. Hence, by (*), $G_t(u(\bar{x}), 0) \neq 0$, in contradiction with (1.2). □

Lemma 5.3. *If* $r \geq 2$, $\alpha \geq 0$, *and* $\varepsilon > 0$, *there is* $c(r, \alpha, \varepsilon) > 0$ *such that*

$$\frac{1}{r}(\alpha + t^2)^{\frac{r}{2}} \leq \frac{1}{r}(1 + \varepsilon)|t|^r + c(r, \alpha, \varepsilon),$$

for any $t \in \mathbb{R}$.

Proof. The assertion is trivial if $r = 2$, so let us consider the case $r > 2$.

Setting $f_\alpha(t) = \frac{1}{r}(\alpha + t^2)^{\frac{r}{2}} - \frac{1}{r}(1 + \varepsilon)|t|^r$, we see that $\lim_{|t|\to+\infty} f_\alpha(t) = -\infty$.

Moreover, $f'_\alpha(t) = t\,|t|^{r-2}\left(\left(\frac{\alpha}{t^2} + 1\right)^{\frac{r-2}{2}} - 1 - \varepsilon\right)$ and, denoting by $t_\alpha = \left(\frac{\alpha}{(1+\varepsilon)^{\frac{2}{r-2}} - 1}\right)^{1/2}$, we obtain

$$\max_{t\in\mathbb{R}} f_\alpha(t) = f_\alpha(\pm t_\alpha) = \frac{\alpha^{r/2}}{r}\frac{1 + \varepsilon}{\left((1 + \varepsilon)^{\frac{2}{r-2}} - 1\right)^{\frac{r-2}{2}}}.$$
□

Setting

$$L(s, t) = \frac{1}{p}\,|s|^p + \frac{1}{q}\,|t|^q \quad \text{and} \quad L_\alpha(s, t) = \frac{1}{p}(\alpha + s^2)^{p/2} + \frac{1}{q}(\alpha + t^2)^{q/2},$$

and recalling Theorem 5.1, we obtain the following result.

Corollary 5.4. *For any* $\varepsilon > 0$, *there is* $c_\varepsilon > 0$ *such that*

$$L(s, t) \leq L_\alpha(s, t) \leq (1 + \varepsilon)L(s, t) + c_\varepsilon \tag{5.2}$$

$$-(\bar{\lambda} + \varepsilon)F(s, t) - c_\varepsilon \leq -G(s, t) \leq -(\bar{\lambda} - \varepsilon)F(s, t) + c_\varepsilon, \tag{5.3}$$

for any $(s, t) \in \mathbb{R}^2$.

Proof of Theorem 1.1. As $\bar{\lambda}$ is not an eigenvalue for System (1.4), let $m_\infty \in \mathbb{N}$ be such that $\lambda_{m_\infty} < \bar{\lambda} < \lambda_{m_\infty+1}$ and assume $m_\infty \notin [m_0, m_0^*]$.

Setting

$$\begin{aligned} &X_- = \{z \in X | \Phi(z) \leq \lambda_{m_\infty}\Psi(z)\}, \\ &X_+ = \{z \in X | \Phi(z) \geq \lambda_{m_\infty+1}\Psi(z)\}, \\ &D_r = \{z \in X_- \quad | \quad \Phi(z) \leq r\} \quad S_r = \{z \in X_- | \Phi(z) = r\}. \end{aligned}$$

Theorem 3.8 assures that (D_r, S_r) links X_+ cohomologically in dimension m_∞ over $\mathbb{Z}_2$, for any $r > 0$.

We take into consideration only the case $m_\infty \geq 1$, as when $m_\infty = 0$, the proof is similar and simpler. Let $\alpha', \alpha'', \beta'$ be such that

$$\lambda_{m_\infty} < \alpha' < \alpha'' < \bar{\lambda} < \beta' < \lambda_{m_\infty+1}.$$

By Corollary 5.4, there is $C > 0$ such that

$$-\beta' F(s,t) - C \leq -G(s,t) \leq -\alpha'' F(s,t) + C$$
$$L(s,t) \leq L_a(s,t) \leq \frac{\alpha'}{\lambda_{m_\infty}} L(s,t) + C.$$

Therefore, for any $(u, v) \in X_+$,

$$\begin{aligned} J_a(u,v) &= \int_\Omega L_a(|\nabla u(x)|, |\nabla v(x)|)\mathrm{d}x - \int_\Omega G(u(x), v(x))\mathrm{d}x \\ &\geq \int_\Omega L(|\nabla u(x)|, |\nabla v(x)|)\mathrm{d}x - \int_\Omega (\beta' F(u(x), v(x)) + C)\mathrm{d}x \\ &\geq (\lambda_{m_\infty+1} - \beta') \int_\Omega F(u(x), v(x))\mathrm{d}x - C|\Omega| \\ &\geq -C|\Omega|, \end{aligned}$$

so that

$$\inf_{z \in X_+} J_a(z) = a > -\infty.$$

On the other hand, for any $(u, v) \in X_-$,

$$\begin{aligned} J_a(u,v) &= \int_\Omega L_a(|\nabla u(x)|, |\nabla v(x)|)\mathrm{d}x - \int_\Omega G(u(x), v(x))\mathrm{d}x \\ &\leq \frac{\alpha'}{\lambda_{m_\infty}} \int_\Omega L(|\nabla u(x)|, |\nabla v(x)|)\mathrm{d}x + C|\Omega| - \alpha'' \int_\Omega F(u(x), v(x))\mathrm{d}x + C|\Omega| \\ &\leq 2C|\Omega| + \left(\frac{\alpha' - \alpha''}{\lambda_{m_\infty}}\right) \int_\Omega L(|\nabla u(x)|, |\nabla v(x)|)\mathrm{d}x, \end{aligned}$$

so that

$$\lim_{\substack{\|z\| \to \infty \\ z \in X_-}} J_a(z) = -\infty \quad \Rightarrow \quad \exists r > 0 : \sup_{S_r} J_a < a = \inf_{X_+} J_a.$$

Moreover, as D_r is compact, $b = \max_{D_r} J_a < +\infty$.

If $J_a^{-1}([a, b])$ contains an infinite number of critical points, the theorem is obviously proved. Otherwise, applying Theorem 3.3 combined with Proposition 2.4, there exists a critical point z_0 of J_a with $a \leq J_a(z_0) \leq b$ and

$$C_{m_\infty}(J_a, z_0) \neq \{0\}.$$

As $m_\infty \notin [m_0, m_0^*]$, Theorem 4.4 gives that

$$C_{m_\infty}(J_a, \mathbf{0}) = \{0\}.$$

Hence $z_0 \neq \mathbf{0}$. □

Lemma 5.5. *Let $\Gamma : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function such that*

$$\lim_{|(s,t)| \to \infty} \frac{\Gamma(s,t)}{F(s,t)} = 0 \tag{5.4}$$

and

$$\lim_{|(s,t)|\to\infty} \Gamma(s,t) - \frac{1}{p}\Gamma_s(s,t)s - \frac{1}{q}\Gamma_t(s,t)t = -\infty, \tag{5.5}$$

then

$$\lim_{|(s,t)|\to\infty} \Gamma(s,t) = -\infty. \tag{5.6}$$

In the same way, if Γ *satisfies Condition* (5.4) *and*

$$\lim_{|(s,t)|\to\infty} \Gamma(s,t) - \frac{1}{p}\Gamma_s(s,t)s - \frac{1}{q}\Gamma_t(s,t)t = +\infty, \tag{5.7}$$

then

$$\lim_{|(s,t)|\to\infty} \Gamma(s,t) = +\infty. \tag{5.8}$$

Proof. Let us observe $F(x^{1/p}s, x^{1/q}t) = xF(s,t)$, for any $x \geq 0$ and $(s,t) \in \mathbb{R}^2$.

Hence, if $(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}$, then, by (5.4),

$$\lim_{x\to+\infty} \frac{\Gamma(x^{1/p}s, x^{1/q}t)}{x} = 0. \tag{5.9}$$

Let us fix $M > 0$. By (5.5), there exists $\delta > 0$ such that

$\Gamma(s,t) - \frac{1}{p}\Gamma_s(s,t)s - \frac{1}{q}\Gamma_t(s,t)t < -M$ when $|(s,t)| > \delta$.

If $(s',t') \in \mathbb{R}^2$ is such that $|(s',t')| > \delta$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\Gamma(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t') + M}{x} = \frac{\frac{1}{p}\Gamma_s(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t')x^{\frac{1}{p}}s' + \frac{1}{q}\Gamma_t(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t')x^{\frac{1}{q}}t' - \Gamma(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t') - M}{x^2} > 0, \quad \text{if } x \geq 1,$$

so that

$$\frac{\Gamma(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t') + M}{x} \leq \frac{\Gamma(\bar{x}^{\frac{1}{p}}s', \bar{x}^{\frac{1}{q}}t') + M}{\bar{x}}, \quad \text{whenever } 1 \leq x \leq \bar{x}.$$

By (5.9), passing to the limit for $\bar{x} \to +\infty$, we obtain

$$\frac{\Gamma(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t') + M}{x} \leq 0 \quad \forall x \geq 1.$$

Hence, $\Gamma(s',t') \leq -M$, which proves (5.6).

Finally, (5.8) can be proved applying (5.6) to the function $\bar{\Gamma} = -\Gamma$. □

Proof of Theorem 1.2. As

$$\lim_{|(s,t)|\to\infty} \left[G(s,t) - \frac{1}{p}G_s(s,t)s - \frac{1}{q}G_t(s,t)t\right] = -\infty$$

and $F(s,t) - \frac{1}{p}F_s(s,t)s - \frac{1}{q}F_t(s,t)t = 0$ for any $(s,t) \in \mathbb{R}^2$, by Theorem 5.1 and Lemma 5.5, we obtain

$$\lim_{|(s,t)|\to\infty} G(s,t) - \bar{\lambda}F(s,t) = -\infty.$$

Hence, $G(s,t) - \bar{\lambda}F(s,t)$ is upperly bounded.

Therefore, recalling Corollary 5.4 and letting α', α'' be such that

$$\lambda_{m_\infty} < \alpha' < \alpha'' < \bar{\lambda} \leq \lambda_{m_\infty+1},$$

there is $C > 0$ such that

$$-\bar{\lambda}F(s,t) - C \leq -G(s,t) \leq -\alpha'' F(s,t) + CL(s,t) \leq L_\alpha(s,t) \leq \frac{\alpha'}{\lambda_{m_\infty}} L(s,t) + C.$$

So we can conclude as in the proof of Theorem 1.1, defining X_- and X_+ in the same way. □

Proof of Theorem 1.3. First, reasoning as in the previous proof, by assumption (b_+), we obtain

$$\lim_{|(s,t)|\to\infty} G(s,t) - \bar{\lambda}F(s,t) = +\infty. \tag{5.10}$$

Let $\beta' \in (\bar{\lambda}, \lambda_{m_\infty+1})$. By Corollary 5.4, there is $C > 0$ such that

$$-\beta' F(s,t) - C \leq -G(s,t).$$

We define X_- and X_+ as in the proof of Theorem 1.1 and, reasoning in the same way, we see that

$$\inf_{z\in X_+} J_0(z) = a > -\infty.$$

So it is sufficient to prove that, for any $M > 0$, we can choose a suitably big $r > 0$ such that

$$\sup_{z\in S_r} J_0(z) < -M.$$

By contradiction, there is $M > 0$ and a sequence $z_n = (u_n, v_n)$ in X_- such that $\Phi(z_n) = n$ and

$$J_0(z_n) \geq -M. \tag{5.11}$$

As $z_n \in X_-$, we see that

$$J_0(z_n) = \Phi(z_n) - \int_\Omega G(z_n(x))\mathrm{d}x \leq \int_\Omega \bar{\lambda}F(z_n(x)) - G(z_n(x))\mathrm{d}x. \tag{5.12}$$

Let $\delta_n = 1/\Phi(z_n)$, $u_n' = \delta_n^{1/p} u_n$, and $v_n' = \delta_n^{1/q} v_n$, $z_n' = (u_n', v_n')$.

As $\Phi(z_n') = 1$, z_n' converges to some $z' = (u', v')$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$, up to a subsequence.

It is easy to see that every $z_n' \in X_-$, as well as $z' \in X_-$.

Moreover,

$$\Psi(z') = \lim_{n\to\infty} \Psi(z_n') \geq \lim_{n\to\infty} \frac{1}{\lambda_{m_\infty}} \Phi(z_n') = \frac{1}{\lambda_{m_\infty}} > 0.$$

Hence,

$$z' \neq 0. \tag{5.13}$$

Since

$$|u_n(x)|^p + |v_n(x)|^q = \Phi(z_n)(|\delta_n^{1/p} u_n(x)|^p + |\delta_n^{1/q} v_n(x)|^q) \geq n(|u_n'(x)|^p + |v_n'(x)|^q)$$

and

$$|u_n'(x)|^p + |v_n'(x)|^q \longrightarrow |u'(x)|^p + |v'(x)|^q, \quad \text{a.e. in } \Omega,$$

taking (5.10), (5.12), and (5.13) into account, we infer that, up to a subsequence,

$$\lim_{n\to\infty} J_0(z_n) = -\infty,$$

which contradicts (5.11). □

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