

CONES OF LINES HAVING HIGH CONTACT WITH GENERAL HYPERSURFACES AND APPLICATIONS

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ABSTRACT. Given a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2$, we study the cones $V_p^h \subset \mathbb{P}^{n+1}$ swept out by lines having contact order $h \geq 2$ at a point $p \in X$. In particular, we prove that if X is general, then for any $p \in X$ and $2 \leq h \leq \min\{n+1, d\}$, the cone V_p^h has dimension exactly $n+2-h$. Moreover, when X is a very general hypersurface of degree $d \geq 2n+2$, we describe the relation between the cones V_p^h and the degree of irrationality of k -dimensional subvarieties of X passing through a general point of X . As an application, we give some bounds on the least degree of irrationality of k -dimensional subvarieties of X passing through a general point of X , and we prove that the connecting gonality of X satisfies $d - \left\lfloor \frac{\sqrt{16n+25}-3}{2} \right\rfloor \leq \text{conn. gon}(X) \leq d - \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor$.

1. INTRODUCTION

Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface of degree $d \geq 2$. Given a point $p \in X$ and an integer $h \geq 2$, we consider the cone $V_p^h \subset \mathbb{P}^{n+1}$ swept out by lines having intersection multiplicity at least h with X at p . These cones reflect the geometry of hypersurfaces and occur in various issues, concerning both the local geometry of hypersurfaces (see e.g. [11, 13, 14]) and the study of their global properties, such as unirationality ([6]) and their covering gonality, i.e. the least gonality of curves passing through a general point of X ([4]).

In this paper, we study the cones V_p^h of general hypersurfaces $X \subset \mathbb{P}^{n+1}$, and we apply our results to achieve some bounds concerning the degree of irrationality of k -dimensional subvarieties of X passing through general points of X , where we recall that the degree of irrationality of an irreducible variety Y of dimension k is the least degree of a dominant rational map $Y \dashrightarrow \mathbb{P}^k$.

Given a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2$, a point $p \in X$ and an integer $2 \leq h \leq d$, the cone $V_p^h \subset \mathbb{P}^{n+1}$ is defined by the vanishing of $h-1$ polynomials of degree $1, 2, \dots, h-1$, respectively, where the linear polynomial defines the tangent hyperplane of X at p (cf. Section 2). When $p \in X$ is a general point, then V_p^h is a complete intersection defined by those polynomials, i.e. $\dim V_p^h = n+2-h$ (cf. [13]). However, it may happen that for some special point of X , the cone V_p^h fails to be a complete intersection of multi-degree $(1, 2, \dots, h-1)$ and its dimension is larger than expected. We prove that when $X \subset \mathbb{P}^{n+1}$ is a general hypersurface of degree $d \geq 2$, this is not the case.

Theorem 1.1. *Let $n \geq 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geq 2$. Then, for any point $p \in X$ and for any integer $2 \leq h \leq \min\{n+1, d\}$, the cone V_p^h has dimension*

$$\dim(V_p^h) = n + 2 - h.$$

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In recent years there has been a great deal of interest concerning measures of irrationality of projective varieties, that is birational invariants which somehow measure the failure of a given variety to be rational (see e.g. [3, 5, 15, 8, 10, 16, 17]), and several interesting results have been obtained in this direction for very general hypersurfaces of large degree (cf. [2, 3, 4, 18]).

Given an irreducible complex projective variety X of dimension n and an integer k such that $1 \leq k \leq n$, we are interested in the following birational invariants. According to [4, Section 5.3], we define the k -irrationality degree of X as the integer

$$\text{irr}_k(X) := \min \left\{ c \in \mathbb{N} \left| \begin{array}{l} \text{Given a general point } p \in X, \exists \text{ an irreducible subvariety } Z \subseteq X \\ \text{of dimension } k \text{ such that } p \in Z \text{ and there is a dominant rational} \\ \text{map } Z \dashrightarrow \mathbb{P}^k \text{ of degree } c \end{array} \right. \right\}$$

and, in line with [3], we define the *connecting gonality* of X as the integer

$$\text{conn. gon}(X) := \min \left\{ c \in \mathbb{N} \left| \begin{array}{l} \text{Given two general points } q, q' \in X, \exists \text{ an irreducible} \\ \text{curve } C \subset X \text{ such that } q, q' \in C \text{ and } \text{gon}(C) = c \end{array} \right. \right\}.$$

Therefore, $\text{conn. gon}(X)$ can be thought as a measure of the failure of X to be rationally connected, whereas $\text{irr}_k(X)$ measures how X is far from being covered by k -dimensional rational varieties. We note further that $\text{irr}(X) := \text{irr}_n(X)$ is the *degree of irrationality* of X and $\text{cov. gon}(X) := \text{irr}_1(X)$ is the *covering gonality* of X . Moreover, these invariants satisfy the obvious inequalities

$$\text{cov. gon}(X) \leq \text{conn. gon}(X) \leq \text{irr}(X) \quad \text{and} \quad \text{irr}_1(X) \leq \text{irr}_2(X) \leq \dots \leq \text{irr}_n(X). \quad (1.1)$$

In [8], it has been proved that $\text{irr}_k(A) \geq k + \frac{1}{2} \dim A$, provided that A is a very general abelian variety of dimension at least 3 and $1 \leq k \leq \dim A$. Apart from this result and the cases $k = 1, n$, very little is known about the k -irrationality degrees and the connecting gonality of projective varieties. When $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d \geq 2n + 2$, it follows from [3, Theorem C] and [4, Theorem 1.1] that

$$\text{irr}_n(X) = d - 1 \quad \text{and} \quad d - \left\lfloor \frac{\sqrt{16n+9} - 1}{2} \right\rfloor \leq \text{irr}_1(X) \leq d - \left\lfloor \frac{\sqrt{16n+1} - 1}{2} \right\rfloor, \quad (1.2)$$

where the latter relation is often a chain of equalities. Under the same assumption on $X \subset \mathbb{P}^{n+1}$, we are concerned with its connecting gonality and the k -irrationality degree, with $2 \leq k \leq n - 1$.

In this direction, we extend [4, Proposition 2.12], which relates curves of low gonality covering X to the cones V_p^h of lines having high contact with X . In particular, we prove that if $Z \subset X$ is a k -dimensional subvariety passing through a general point, endowed with a dominant rational map $\varphi: Z \dashrightarrow \mathbb{P}^k$ of degree $c \leq d - 3$, then $Z \subset V_p^{d-c}$ for some $p \in X$, and the map φ is the projection from p (cf. Proposition 4.1). Combining Theorem 1.1 and the latter result, we achieve bounds on the k -irrationality degrees of X .

Theorem 1.2. *Let $n \geq 3$ and let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$. Then*

$$\text{irr}_k(X) \geq d - 1 - n + k \quad \text{for } 1 \leq k \leq n. \quad (1.3)$$

Moreover, equality holds for $n - 2 \leq k \leq n$, that is

$$\text{irr}_{n-2}(X) = d - 3, \quad \text{irr}_{n-1}(X) = d - 2 \quad \text{and} \quad \text{irr}_n(X) = d - 1.$$

We point out that the assertion for $k = n$ is given by [3, Theorem C]. Furthermore, the larger the value of k is, the more significant Theorem 1.2 becomes, since for small values of k the bound (1.3) is superseded by (1.2) and (1.1).

In order to discuss the connecting gonality of X , we prove further that for any pair of general points $q, q' \in X$ and for any $2 \leq h \leq \frac{n}{2} + 1$, there exists a general point $p \in X$ such that $q, q' \in V_p^h$ (cf. Lemma 3.2). Then, using the fact that for $h = \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor$ the locus V_p^h is a cone over a rationally connected variety, we bound from above the connecting gonality of a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of large degree.

Theorem 1.3. *Let $n \geq 4$ and let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$. Then*

$$\text{conn. gon}(X) \leq d - \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor. \quad (1.4)$$

Finally, using our results and the Grassmannian techniques introduced in [20], we also obtain a lower bound on the connecting gonality of very general hypersurfaces, which slightly improves the bound descending from (1.2) and (1.1).

Theorem 1.4. *Let $n \geq 4$ and let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$. Then*

$$\text{conn. gon}(X) \geq d - \left\lfloor \frac{\sqrt{16n+25}-3}{2} \right\rfloor. \quad (1.5)$$

In particular,

$$\text{conn. gon}(X) > \text{cov. gon}(X)$$

$$\forall n \in \mathbb{Z}_{\geq 4} \setminus \{4a^2 + 3a, 4a^2 + 5a, 4a^2 + 5a + 1, 4a^2 + 7a + 2, 4a^2 + 9a + 4, 4a^2 + 11a + 6 \mid a \in \mathbb{N}\}.$$

We note that the second part of the statement is obtained from (1.2) by determining the values of n such that $\left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor \neq \left\lfloor \frac{\sqrt{16n+25}-3}{2} \right\rfloor$. We believe that the bound (1.5) is far from being sharp. However, for any $4 \leq n \leq 16$ with $n \neq 9, 13, 14$, the right-hand sides of (1.4) and (1.5) do coincide, hence Theorems 1.3 and 1.4 compute the connecting gonality of X in these cases (cf. Example 4.5).

The paper is organized as follows. In Section 2, we recall some basic facts on the cones of lines of high contact with a general hypersurface $X \subset \mathbb{P}^{n+1}$ and we prove Theorem 1.1.

In Section 3, we consider polar hypersurfaces of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, in order to discuss when any pair $q, q' \in X$ lies on a cone V_p^h for some $p \in X$.

Finally, Section 4 is concerned with the applications to measures of irrationality. In particular, after describing the relation between the cones V_p^h and the degree of irrationality of k -dimensional subvarieties of X passing through a general point, we prove Theorems 1.2, 1.3 and 1.4, and we discuss the behavior of the connecting gonality for small values of $n = \dim X$.

Notation. We work throughout over the field \mathbb{C} of complex numbers. By *variety* we mean a complete reduced algebraic variety X , and by *curve* we mean a variety of dimension 1. We say that a property holds for a *general* (resp. *very general*) point $x \in X$ if it holds on a Zariski open nonempty subset of X (resp. on the complement of the countable union of proper subvarieties of X).

2. DIMENSION OF CONES OF LINES HAVING HIGH CONTACT

Let $n \geq 2$ be an integer, and let $X = V(F) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface defined by the vanishing of a homogeneous polynomial $F \in \mathbb{C}[x_0, \dots, x_{n+1}]$ of degree $d \geq 2$. Given a point $p \in X$ and an integer $2 \leq h \leq d$, we define the *cone* $V_p^h = V_{p,X}^h \subset \mathbb{P}^{n+1}$ of *tangent lines of order h at p* as the Zariski closure of the locus swept out by lines $\ell \subset \mathbb{P}^{n+1}$ such that either $\ell \subset X$ or $\ell \cdot X \geq hp$. Therefore V_p^h is a cone with vertex at p , defined by the vanishing of the following $h-1$ polynomials occurring in the Taylor expansion of F at p

$$\begin{aligned} G_k(x_0, \dots, x_{n+1}) &:= \left(x_0 \frac{\partial}{\partial x_0} + \dots + x_{n+1} \frac{\partial}{\partial x_{n+1}} \right)^{(k)} F(p) \\ &= \sum_{i_0 + \dots + i_{n+1} = k} \frac{k!}{i_0! \dots i_{n+1}!} x_0^{i_0} \dots x_{n+1}^{i_{n+1}} \frac{\partial^k F}{\partial x_0^{i_0} \dots \partial x_{n+1}^{i_{n+1}}}(p) \end{aligned} \quad (2.1)$$

where $(-)^{(k)}$ denotes the usual symbolic power, $1 \leq k \leq h-1$ and $\deg G_k = k$ (cf. [6, p. 186]). In particular, the cone V_p^2 coincides with the (projective) tangent hyperplane $T_p X \subset \mathbb{P}^{n+1}$ to X at p . When instead $h \geq 3$, we denote by $\Lambda_p^h = \Lambda_{p,X}^h$ the intersection of V_p^h with a general hyperplane $H \subset \mathbb{P}^{n+1}$ not containing p , so that Λ_p^h is defined in $H \cap T_p X \cong \mathbb{P}^{n-1}$ by $h-2$ polynomial equations of degree $2, 3, \dots, h-1$, respectively, and V_p^h is the cone over Λ_p^h with vertex at p .

Setting $2 \leq h \leq \min\{n+1, d\}$, it follows from this description of V_p^h that

$$\dim V_p^h \geq n+2-h. \quad (2.2)$$

When X is a general hypersurface of degree $d \geq 2$ and $p \in X$ is a general point, [4, Lemma 2.2] guarantees that $V_p^h \subset \mathbb{P}^{n+1}$ is a general complete intersection of multi-degree $(1, 2, \dots, h-1)$ and, in particular, (2.2) is an equality.

According to the assertion of Theorem 1.1, we want to prove that equality holds in (2.2) for *any* point $p \in X$. The case $h=2$ is trivial, since X is smooth and hence $V_p^2 = T_p X \cong \mathbb{P}^n$. Besides, the assertion for $h=3$ is implied by the following result.

Lemma 2.1. *Let $n \geq 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geq 2$. Then, for any point $p \in X$, the intersection of X with the tangent hyperplane $T_p X$ has a point of multiplicity exactly 2 at p .*

Proof. Let \mathcal{L} be the linear system of all hypersurfaces of degree d in \mathbb{P}^{n+1} , which has dimension

$$\dim \mathcal{L} = \binom{d+n+1}{n+1} - 1.$$

Consider the variety \mathcal{V} consisting of all triples (p, Π, Y) where $\Pi \subset \mathbb{P}^{n+1}$ is a hyperplane, $p \in \Pi$ and $Y \subset \Pi$ is a hypersurface of degree d with a point of multiplicity at least 3 at p . Then we define the variety

$$\mathcal{Z} := \{(p, \Pi, Y, X) \in \mathcal{V} \times \mathcal{L} \mid Y \subset X\}.$$

endowed with the projection $\pi_1: \mathcal{Z} \rightarrow \mathcal{V}$, whose fibers are all isomorphic to linear systems of hypersurfaces of degree d of the same dimension $\binom{d+n}{n+1}$. Looking at the map $\mathcal{V} \rightarrow \mathbb{P}^{n+1} \times (\mathbb{P}^{n+1})^*$ given by $(p, \Pi, Y) \mapsto (p, \Pi)$, it is easy to see that \mathcal{V} is irreducible of dimension

$$\dim \mathcal{V} = 2n + \binom{d+n}{n} - \binom{n+2}{2}.$$

Hence \mathcal{Z} is irreducible too, with dimension

$$\dim \mathcal{Z} = 2n + \binom{d+n}{n} - \binom{n+2}{2} + \binom{d+n}{n+1} = 2n + \binom{d+n+1}{n+1} - \binom{n+2}{2}.$$

Consider now the projection $\pi_2: \mathcal{Z} \rightarrow \mathcal{L}$, whose image is the locus \mathcal{T} of all hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d having a point $p \in X$ and a hyperplane $\Pi \subset \mathbb{P}^{n+1}$ such that the intersection scheme $X \cap \Pi$ has a point of multiplicity at least 3 at p . Hence

$$\dim \mathcal{T} \leq \dim \mathcal{Z} = \dim \mathcal{L} + 2n + 1 - \binom{n+2}{2}$$

and, since $\binom{n+2}{2} > 2n + 1$ as soon as $n \geq 2$, we conclude that \mathcal{T} is a proper closed subset of \mathcal{L} , as wanted. \square

Now we prove Theorem 1.1.

Proof of Theorem 1.1. We already discussed the trivial case $h = 2$. Besides, Lemma 2.1 ensures that for any $p \in X$, a general line $\ell \subset T_p X$ tangent to X at p intersects X at p with multiplicity exactly 2. Hence V_p^3 is a proper subvariety of $T_p X$, and (2.2) is an equality. Thus we assume hereafter $h \geq 4$.

Let $[x_0 : \dots : x_{n+1}]$ be homogeneous coordinates in \mathbb{P}^{n+1} and, for any positive integer k , we set

$$S_k := \mathbb{C}[x_0, \dots, x_{n+1}]_k \quad \text{and} \quad S_k^* := \mathbb{C}[x_0, \dots, x_{n+1}]_k \setminus \{0\}. \quad (2.3)$$

For any $F \in S_d^*$, we denote by $V(F) \subset \mathbb{P}^{n+1}$ the hypersurface defined by the vanishing of F , and for any $\mathbf{G} := (G_1, \dots, G_{h-1}) \in \prod_{k=1}^{h-1} S_k$, we denote by $V(\mathbf{G})$ the intersection scheme of the hypersurfaces $V(G_k)$ for $1 \leq k \leq h-1$.

For $F \in S_d^*$ and $p \in V(F) \subset \mathbb{P}^{n+1}$, let $G_k = G_{p,k}(F)(x_0, \dots, x_{n+1})$ be the homogeneous polynomial of degree k defined in (2.1), where $1 \leq k \leq h-1$, and let

$$\mathbf{G} = \mathbf{G}_p(F) := (G_{p,1}(F), \dots, G_{p,h-1}(F)) \in \prod_{k=1}^{h-1} S_k.$$

Therefore V_p^h is the cone $V(\mathbf{G}_p(F))$ with vertex at p , and (2.2) fails to be an equality if and only if $\mathbf{G}_p(F)$ is not a *regular sequence*. In order to prove the assertion, we then show that if $F \in S_d^*$ is general, then the sequence $\mathbf{G}_p(F)$ is regular for all points $p \in V(F)$.

To this aim, let $U_d \subset S_d^*$ be the open dense subset parametrizing those $F \in S_d^*$ such that $V(F)$ is smooth and let

$$\mathcal{J} := \left\{ (p, F, \mathbf{G}) \in \mathbb{P}^{n+1} \times U_d \times \prod_{k=1}^{h-1} S_k \mid p \in V(F) \text{ and } \mathbf{G} = \mathbf{G}_p(F) \right\},$$

which is endowed with the two projections $\pi_1: \mathcal{J} \rightarrow U_d$ and $\pi_2: \mathcal{J} \rightarrow \mathbb{P}^{n+1} \times \prod_{k=1}^{h-1} S_k$. The map π_1 is surjective, and for any $F \in U_d$, the fiber $\pi_1^{-1}(F)$ is isomorphic to $V(F)$, which is irreducible of dimension n . Thus \mathcal{J} is irreducible of dimension

$$\dim \mathcal{J} = \dim(U_d) + n = \binom{n+1+d}{d} + n. \quad (2.4)$$

Let us define $\mathcal{W} := \pi_2(\mathcal{J}) \subset \mathbb{P}^{n+1} \times \prod_{k=1}^{h-1} S_k$, which is irreducible too.

Claim 2.2. *All fibers of $\pi_2: \mathcal{J} \rightarrow \mathcal{W}$ have dimension*

$$f = \binom{n+h}{h} + \binom{n+h+1}{h+1} + \cdots + \binom{n+d}{d} = \binom{n+d+1}{d} - \binom{n+h}{h-1}.$$

Proof of Claim 2.2. Let $(p, \mathbf{G}) = \pi_2(p, F, \mathbf{G}) \in \mathcal{W}$, with $(p, F, \mathbf{G}) \in \mathcal{J}$, $\mathbf{G} = (G_1, \dots, G_{h-1})$ and $G_k = G_{p,k}(F)$ for $1 \leq k \leq h-1$. Up to projective transformations, we may assume $p = [1 : 0 : \dots : 0]$ and $T_p V(F) = V(x_{n+1})$. Then F is of the form

$$F(x_0, \dots, x_{n+1}) = cx_{n+1}x_0^{d-1} + F_2(x_1, \dots, x_{n+1})x_0^{d-2} + \cdots + F_d(x_1, \dots, x_{n+1}), \quad (2.5)$$

where c is a non-zero constant and each $F_i \in \mathbb{C}[x_1, \dots, x_{n+1}]$ is homogeneous of degree i . Easy computations show that

$$\begin{aligned} G_1 &= cx_{n+1} \\ G_2 &= 2(d-1)cx_{n+1}x_0 + 2F_2 \\ G_3 &= 3(d-1)(d-2)cx_{n+1}x_0^2 + 6(d-2)x_0F_2 + 6F_3 \\ &\dots \end{aligned} \quad (2.6)$$

$$G_{h-1} = (h-1) \frac{(d-1)!}{(d-h+1)!} cx_{n+1}x_0^{h-2} + \sum_{i=2}^{h-1} \frac{(h-1)!}{(h-i-1)!} \frac{(d-i)!}{(d-h+1)!} x_0^{h-i-1} F_i.$$

To determine the fiber of π_2 over (p, \mathbf{G}) , we have to find all forms $F' \in U_d$ such that $(p, F', \mathbf{G}) \in \mathcal{J}$. As in (2.5), we have $F' = c'x_{n+1}x_0^{d-1} + F'_2x_0^{d-2} + \cdots + F'_d$, which satisfies the corresponding equations in (2.6). Therefore, equations (2.6) imply $c = c'$ and $F'_k = F_k$ for any $2 \leq k \leq h-1$. Thus F' may differ from F by the terms F'_i for $h \leq i \leq d$, which can be chosen arbitrarily in $\mathbb{C}[x_1, \dots, x_{n+1}]_i$. \square

We deduce from the claim and (2.4) that

$$\dim \mathcal{W} = \dim \mathcal{J} - f = \binom{n+h}{h-1} + n.$$

Then we set

$$\mathcal{W}_0 := \{(p, \mathbf{G}) \in \mathcal{W} \mid \mathbf{G} \text{ is not a regular sequence}\} \subset \mathcal{W},$$

so that proving the assertion is equivalent to showing that $\pi_1(\pi_2^{-1}(\mathcal{W}_0))$ is a proper closed subset of U_d . In particular, it suffices to prove that

$$\text{codim}_{\mathcal{W}} \mathcal{W}_0 > n, \quad (2.7)$$

because in this case Claim 2.2 and (2.4) yield

$$\begin{aligned} \dim \pi_1(\pi_2^{-1}(\mathcal{W}_0)) &\leq \dim \pi_2^{-1}(\mathcal{W}_0) = \dim \mathcal{W}_0 + f = \dim \mathcal{W}_0 + \dim \mathcal{J} - \dim \mathcal{W} = \\ &= \dim \mathcal{J} - \text{codim}_{\mathcal{W}} \mathcal{W}_0 < \dim \mathcal{J} - n = \dim U_d, \end{aligned}$$

so that the assertion holds. Hence we focus on proving (2.7).

To this aim, let \mathcal{Z} be an irreducible component of \mathcal{W}_0 and let $(p, \mathbf{G}) \in \mathcal{Z}$ be a general point, with $\mathbf{G} = (G_1, \dots, G_{h-1})$. Then \mathbf{G} is not a regular sequence, and we may define $\alpha = \alpha_{\mathcal{Z}}$ to be the greatest integer such that (G_1, \dots, G_{α}) is a regular sequence. Since we already showed that $V_p^3 = V(G_1, G_2)$ has dimension $n-1$, we have $2 \leq \alpha < h-1$. By maximality of α , there exists an irreducible component Y of $V(G_1, G_2, \dots, G_{\alpha})$ of dimension $m := \dim Y = n - \alpha + 1$ and contained in $V(G_{\alpha+1})$. Therefore, in order to prove (2.7), it is enough to show that the *Hilbert function* $h_Y: \mathbb{N} \rightarrow \mathbb{N}$ of Y satisfies $h_Y(\alpha+1) > n$.

For any $\ell = 0, \dots, m$, let Y_ℓ denote a general linear section of Y of dimension ℓ . In particular, $Y_m = Y$ and Y_{m-1} is a general hyperplane section of Y . Then, for any positive integer t , we have

$$h_Y(t) - h_Y(t-1) \geq h_{Y_{m-1}}(t)$$

(see [12, Lemma 3.1]). Analogously, for any $1 \leq \ell \leq m$ and for any integer $t > 0$, we obtain

$$h_{Y_\ell}(t) \geq h_{Y_\ell}(t-1) + h_{Y_{\ell-1}}(t), \quad (2.8)$$

and we deduce by iteration that for any integer $t > 0$,

$$h_Y(t) \geq \sum_{\ell=1}^m h_{Y_\ell}(t-1) + h_{Y_0}(t) \geq \sum_{\ell=1}^m h_{Y_\ell}(t-1) + 1. \quad (2.9)$$

Claim 2.3. For any $1 \leq \ell \leq m$ and for any integer $t > 0$,

$$h_{Y_\ell}(t) \geq \binom{\ell+t}{t}.$$

Proof of Claim 2.3. We recall that Y_ℓ is a general linear section of $Y \subset V(G_1) \cong \mathbb{P}^n$ of dimension ℓ , so that Y_ℓ is irreducible and sits in a projective space $\Lambda_\ell \cong \mathbb{P}^{n+\ell-m}$, with $1 \leq \ell \leq m$. If $t = 1$, the claim is true as the linear span of $Y_\ell \subset \Lambda_\ell$ has dimension at least $\ell = \dim Y_\ell$, i.e. Y_ℓ contains at least $\ell + 1$ independent points of Λ_ℓ . Then we argue by induction on t , and using inequality (2.9) applied to Y_ℓ , we obtain

$$h_{Y_\ell}(t) \geq \sum_{j=1}^{\ell} h_{Y_j}(t-1) + 1 \geq \sum_{j=1}^{\ell} \binom{j+t-1}{t-1} + 1 = \sum_{j=0}^{\ell} \binom{j+t-1}{t-1} = \binom{\ell+t}{t}$$

as desired. \square

Finally, setting $t = \alpha + 1$ and $\ell = m = n - \alpha + 1$, Claim 2.3 ensures that

$$h_Y(\alpha + 1) \geq \binom{m + \alpha + 1}{\alpha + 1} = \binom{n + 2}{\alpha + 1}.$$

By assumption, we have $\alpha < h - 1 \leq n$, and hence $\binom{n+2}{\alpha+1} \geq n + 1$. Thus $h_Y(\alpha + 1) > n$, which concludes the proof of Theorem 1.1. \square

3. POLAR HYPERSURFACES AND CONES OF LINES HAVING HIGH CONTACT

Let $n \geq 2$ be an integer, and let $X := V(F) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$. Given a point $q = [q_0 : \dots : q_{n+1}] \in \mathbb{P}^{n+1}$ and an integer $0 \leq s \leq d$, we introduce the s -th polar hypersurface of X with respect to q as the hypersurface $\Delta_q^s = \Delta_q^s(X) \subset \mathbb{P}^{n+1}$ defined by the vanishing of the polynomial of degree $d - s$

$$\text{Pol}_q^s(F)(x_0, \dots, x_{n+1}) := \left(q_0 \frac{\partial}{\partial x_0} + \dots + q_{n+1} \frac{\partial}{\partial x_{n+1}} \right)^{(s)} F(x_0, \dots, x_{n+1}), \quad (3.1)$$

where $(-)^{(s)}$ denotes the usual symbolic power and $\text{Pol}_q^0(F) = F$, that is $\Delta_q^0 = X$ for any $q \in \mathbb{P}^{n+1}$. Furthermore, we define the intersection scheme

$$\Delta_{q,h}(X) := \bigcap_{s=0}^{h-1} \Delta_q^s. \quad (3.2)$$

In this section, we use polar hypersurfaces of X and Theorem 1.1 to show that for general $q, q' \in X$ and for any $2 \leq h \leq \frac{n}{2} + 1$, there exists a general point $p \in X$ such that $q, q' \in V_p^h$. In particular, this fact is crucial in order to prove Theorem 1.3. To start, we prove the following.

Lemma 3.1. *Let $n \geq 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$. For any integer $2 \leq h \leq \frac{n}{2} + 1$ and for any $q, q' \in X$, there exists a point $p \in X$ such that $q, q' \in V_p^h$.*

Proof. We point out that for $q \in \mathbb{P}^{n+1}$, the intersection $X \cap \Delta_q^1$ consists of the points $p \in X$ such that $q \in T_p X$, i.e. the line $\langle q, p \rangle$ intersects X with multiplicity at least 2 at p , provided that $p \neq q$. Similarly, given two points $q \in \mathbb{P}^{n+1}$ and $p \in X$ with $p \neq q$, the line $\langle q, p \rangle$ intersects X with multiplicity at least h at p —that is $q \in V_p^h$ —if and only if p belongs to $\Delta_{q,h}(X)$ defined in (3.2).

Therefore, proving the statement is equivalent to showing that for any $q, q' \in X$, the intersection of $\Delta_{q,h}(X)$ and $\Delta_{q',h}(X)$ is not empty. Since both $\Delta_{q,h}(X)$ and $\Delta_{q',h}(X)$ lie on $X = \Delta_q^0 = \Delta_{q'}^0$, then $\Delta_{q,h}(X) \cap \Delta_{q',h}(X)$ is the intersection of $2h - 1$ hypersurfaces of \mathbb{P}^{n+1} , which is non-empty because of the assumption $h \leq \frac{n}{2} + 1$. \square

In addition, when both $X \subset \mathbb{P}^{n+1}$ and the points $q, q' \in X$ are assumed to be general, the following holds.

Lemma 3.2. *Let $n \geq 2$ be an integer and let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geq 2$. Then, for any integer $2 \leq h \leq \frac{n}{2} + 1$ and for general $q, q' \in X$, there exists a general point $p \in X$ such that $q, q' \in V_p^h$.*

Proof. Set $2 \leq h \leq \frac{n}{2} + 1$ and consider the variety

$$\mathcal{P}_h := \overline{\{(q, q', p) \in X \times X \times X \mid q \neq q' \text{ and } p \in \Delta_{q,h} \cap \Delta_{q',h}\}}$$

endowed with the projections

$$X \times X \xleftarrow{\pi_{12}} \mathcal{P}_h \xrightarrow{\pi_3} X.$$

Thanks to Lemma 3.1, the map π_{12} is surjective. Let $\mathcal{Z} \subset \mathcal{P}_h$ be an irreducible component dominating $X \times X$. Therefore, the proof of Lemma 3.1 gives that for any $(q, q') \in X \times X$,

$$\dim(\Delta_{q,h}(X) \cap \Delta_{q',h}(X)) \geq n + 2 - 2h \quad \text{and hence} \quad \dim \mathcal{Z} \geq 3n + 2 - 2h. \quad (3.3)$$

Moreover, given any point $p \in \pi_3(\mathcal{Z})$ and setting $Z_p := V_p^h \cap X$, we have

$$(\pi_{3|\mathcal{Z}})^{-1}(p) \subseteq \pi_3^{-1}(p) \cong Z_p \times Z_p. \quad (3.4)$$

In order to prove that for general $q, q' \in X$, there exists a general point $p \in X$ such that $q, q' \in V_p^h$, it is enough to prove that \mathcal{Z} dominates X via π_3 . We assume by contradiction that $\pi_{3|\mathcal{Z}}$ is not dominant, i.e. $\dim \pi_3(\mathcal{Z}) < n = \dim X$. If $p \in \pi_3(\mathcal{Z})$ is a general point, then (3.3) and (3.4) give

$$\dim(Z_p \times Z_p) \geq \dim\left((\pi_{3|\mathcal{Z}})^{-1}(p)\right) = \dim \mathcal{Z} - \dim \pi_3(\mathcal{Z}) > 2n + 2 - 2h. \quad (3.5)$$

It follows that $\dim Z_p > n + 1 - h$ and hence $\dim V_p^h > n + 2 - h$. Since the latter inequality contradicts Theorem 1.1, we conclude that $\pi_{3|\mathcal{Z}}$ is dominant. \square

4. k -IRRATIONALITY DEGREE AND CONNECTING GONALITY OF GENERAL HYPERSURFACES

In this section we apply the previous results in order to bound the k -irrationality degree and the covering gonality of a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2n + 2$.

According to Section 2, we recall that if $V_p^h \subset \mathbb{P}^{n+1}$ is the cone of tangent lines of order h at $p \in X$, we denote by Λ_p^h a general hyperplane section of V_p^h . The link between the cones $V_p^h \subset \mathbb{P}^{n+1}$ and the invariants $\text{irr}_k(X)$ and $\text{conn. gon}(X)$ is expressed by the following result, which extends [4, Proposition 2.12] to higher dimensional subvarieties of X .

Proposition 4.1. *Let $n \geq 3$ and $1 \leq k \leq n - 1$ be integers. Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$. Suppose that for a general point $q \in X$, there exist a k -dimensional irreducible subvariety $Z \subset X$ containing q and a dominant rational map $\varphi: Z \dashrightarrow \mathbb{P}^k$ of degree $c \leq d - 3$. Then:*

- (i) *there exists a point $p \in Z$ such that $Z \subset V_p^{d-c} \cap X$;*
- (ii) *the map $\varphi: Z \dashrightarrow \mathbb{P}^k$ of degree c is the projection from p .*

In particular, the image of Z under φ is a k -dimensional rational variety $R \subset \Lambda_p^{d-c}$.

Proof. The case $k = 1$ is covered by [4, Proposition 2.12], so we assume $2 \leq k \leq n - 1$. Since $d \geq 2n + 2$, we have that $c \leq d - 3 < 2d - 2n - 1$. Let $z \in Z$ be a general point and let $\ell \subset \mathbb{P}^k$ be a general line passing through $\varphi(z)$. Consider the curve $C_\ell \subset \varphi^{-1}(\ell)$ which is the union of all irreducible components of curves in Z which dominate ℓ via φ . We claim that C_ℓ is irreducible. Indeed, if C' and C were two irreducible components, then $\text{gon}(C_{\text{red}}) + \text{gon}(C'_{\text{red}}) \leq \text{deg}(\varphi|_{C_\ell})$. Being $\varphi|_{C_\ell}: C_\ell \dashrightarrow \ell \cong \mathbb{P}^1$ a map of degree c , either C_{red} or C'_{red} would have gonality at most $\frac{c}{2} < d - n - \frac{1}{2}$. By varying $\ell \subset \mathbb{P}^k$, $z \in Z$ and $q \in X$, we conclude that X is covered by curves of gonality smaller than $d - n$, which is impossible (cf. [3, Theorem A]). The same argument shows that C_ℓ is reduced.

Therefore, we may define a family $\mathcal{C} \xrightarrow{\pi} U \subset \mathbb{G}(1, k)$ of curves with a map of degree c to \mathbb{P}^1 , where $U \cong \mathbb{P}^{k-1}$ parametrizes lines through $\varphi(z)$, and for any $[\ell] \in U$, the corresponding curve is C_ℓ . As we vary $q \in X$ (and hence Z), we may define a family of curves covering X , each endowed with a c -gonal map. Thus [4, Proposition 2.12] ensures that for general $[\ell] \in U$, there exists a point $x_\ell \in C_\ell$ such that $C_\ell \subset X \cap V_{x_\ell}^{d-c}$ and the degree c map $\varphi|_{C_\ell}: C_\ell \dashrightarrow \ell \cong \mathbb{P}^1$ is the projection from x_ℓ .

Next we need to show that all the points x_ℓ coincide with some fixed point $p \in Z$. For this we consider the map $\psi: U \dashrightarrow Z \subset X$ sending $[\ell] \in U$ to the corresponding point x_ℓ . Since $U \cong \mathbb{P}^{k-1}$, the image of ψ is unirational. As X does not contain rational curves (see e.g. [9]), we conclude that $\psi(U)$ is a point $p \in Z$. Thus Z is covered by the curves $C_\ell \subset V_p^{d-c}$ for general $[\ell] \in U$, and the degree c map $\varphi|_{C_\ell}: C_\ell \dashrightarrow \ell \cong \mathbb{P}^1$ is the projection from p . The assertion follows. \square

Remark 4.2. Let X be an irreducible projective variety of dimension n and let $\mathcal{Z} \xrightarrow{\pi} T$ be a family of k -dimensional subvarieties of X . If $\mathcal{Z} \xrightarrow{\pi} T$ is a *covering family* (i.e. for general $q \in X$, there exists $t \in T$ such that $q \in Z_t = \pi^{-1}(t)$), then $\dim(T) \geq n - k$. Indeed, the map $f: \mathcal{Z} \rightarrow X$ must be dominant and hence $\dim(\mathcal{Z}) = \dim(T) + k \geq n$.

If in addition $\mathcal{Z} \xrightarrow{\pi} T$ is a *connecting family* (i.e. for general $q, q' \in X$, there exists $t \in T$ such that $q, q' \in Z_t$), then $\dim(T) \geq 2n - 2k$. Indeed the map $\mathcal{Z} \times_T \mathcal{Z} \rightarrow X \times X$ induced by $f: \mathcal{Z} \rightarrow X$ must be dominant, hence $\dim(\mathcal{Z} \times_T \mathcal{Z}) = 2k + \dim(T) \geq 2n$.

Remark 4.3. If $X \subset \mathbb{P}^{n+1}$ and $Y \subset \mathbb{P}^{m+1}$ are very general hypersurfaces of degree d , with $n \leq m$, then

$$\text{irr}_k(Y) \leq \text{irr}_k(X) \text{ for any } 1 \leq k \leq n \text{ and } \text{conn. gon}(Y) \leq \text{conn. gon}(X).$$

Indeed, the section of Y by a general $(n+1)$ -plane of \mathbb{P}^{m+1} is a very general hypersurface of \mathbb{P}^{n+1} .

We can now prove Theorem 1.2.

Proof of Theorem 1.2. When $k = n$, the assertion is covered by [3, Theorem C] and $\text{irr}_n(X) = d - 1$.

If $k = n - 1$, we claim that $\text{irr}_{n-1}(X) \leq d - 2$. Indeed, tangent hyperplane sections $Z = X \cap T_p X$ of X are $(n - 1)$ -dimensional varieties of degree d having a double point at p (see Lemma 2.1), so that the projection from p is a dominant rational map $Z \dashrightarrow \mathbb{P}^{n-1}$ of degree $d - 2$. On the other hand, suppose by contradiction that $\text{irr}_{n-1}(X) = c \leq d - 3$. Proposition 4.1 ensures that any $(n - 1)$ -dimensional subvariety $Z \subset X$ computing $\text{irr}_{n-1}(X)$ is contained in $X \cap V_p^{d-c}$ for some $p \in X$. Thanks to (1.1) and [3, Theorem A], we have $c \geq \text{irr}_1(X) \geq d - n$, so that $3 \leq d - c \leq n$. Then Theorem 1.1 gives that $\dim V_p^{d-c} = n + 2 - (d - c)$. In order to cover X by $(n - 1)$ -dimensional varieties cut out by the cones V_p^{d-c} , we must have that $\dim(X \cap V_p^{d-c}) = n + 1 - (d - c) \geq n - 1$ and hence $c \geq d - 2$, a contradiction. Thus $\text{irr}_{n-1}(X) = d - 2$.

If $1 \leq k \leq n - 2$, we claim that $\text{irr}_k(X) \leq d - 3$. To see this, we note that for any $p \in X$, Theorem 1.1 ensures that V_p^3 is a cone in $T_p X \cong \mathbb{P}^n$ over a quadric $\Lambda_p^3 \subset \mathbb{P}^{n-1}$ (cf. Section 2). Then the variety $Z = X \cap V_p^3$ has dimension $n - 2$ and the projection from p is a dominant map $Z \dashrightarrow \Lambda_p^3$ of degree $d - 3$ to a rational variety. Thus $\text{irr}_1(X) \leq \dots \leq \text{irr}_{n-2}(X) \leq d - 3$.

Finally, any k -dimensional subvariety $Z \subset X$ computing $c := \text{irr}_k(X)$ is contained in some $X \cap V_p^{d-c}$ by Proposition 4.1. As above, we deduce $d - c \leq n$ and for any $p \in X$, Theorem 1.1 gives $\dim(X \cap V_p^{d-c}) = n + 1 - (d - c)$. Thus, in order to cover X by k -dimensional varieties in $X \cap V_p^{d-c}$, we must have that $n + 1 - (d - c) \geq k$, that is $c \geq d - 1 - n + k$.

For $k = n - 2$, the latter inequality gives $\text{irr}_{n-2}(X) \geq d - 3$, so the assertion follows. \square

Now, we prove Theorem 1.3.

Proof of Theorem 1.3. By [4, Lemma 2.2], if $p \in X$ is a general point and $3 \leq h \leq \min\{n + 1, d\}$, then Λ_p^h is a general complete intersection of type $(2, 3, \dots, h - 1)$ in \mathbb{P}^{n-1} . If Λ_p^h is a Fano variety, then it is rationally connected (see [19]). The canonical bundle of $\Lambda_p^h \subset \mathbb{P}^{n-1}$ is $\mathcal{O}_{\Lambda_p^h}(\sum_{i=2}^{h-1} i - n)$. Therefore, Λ_p^h is a Fano variety if and only if

$$\sum_{i=2}^{h-1} i \leq n - 1 \iff \frac{h(h-1)}{2} - 1 \leq n - 1 \iff h \leq \left\lfloor \frac{\sqrt{8n+1} + 1}{2} \right\rfloor. \quad (4.1)$$

We note that $d > n + 1 \geq \left\lfloor \frac{\sqrt{8n+1} + 1}{2} \right\rfloor \geq 3$ and we assume hereafter $h := \left\lfloor \frac{\sqrt{8n+1} + 1}{2} \right\rfloor$, so that the general Λ_p^h is a smooth, rationally connected, complete intersection, whose dimension is $n + 1 - h$.

Setting $Z_p := V_p^h \cap X$, the projection $\varphi: Z_p \dashrightarrow \Lambda_p^h$ from p has degree $d - h \leq d - 3$. Given two general points $q, q' \in Z_p$, let $D \subset \Lambda_p^h$ be a rational curve connecting $\varphi(q)$ and $\varphi(q')$, and let $C := \varphi^{-1}(D)$. By arguing as for the curves C_ℓ in the proof of Proposition 4.1, we deduce that C is integral. Then C is an irreducible curve passing through two general points $q, q' \in Z_p$ endowed with a map $\varphi|_C: C \dashrightarrow D$ of degree $d - h$. Thus $\text{conn. gon}(Z_p) \leq d - h = d - \left\lfloor \frac{\sqrt{8n+1} + 1}{2} \right\rfloor$.

Since $n \geq 4$, we have $h = \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \leq \frac{n}{2} + 1$. So Lemma 3.2 ensures that for general $q, q' \in X$, there exists a general point $p \in X$ such that $q, q' \in Z_p$, i.e. the varieties Z_p produce a connecting family. Thus $\text{conn. gon}(X) \leq \text{conn. gon}(Z_p) \leq d - \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor$. \square

Let us consider integers $n, d \geq 2$ and $2 \leq h \leq \min\{n+1, d\}$. Before proving Theorem 1.4, we aim at introducing a suitable parameter space \square_h for 4-tuples (p, ℓ_1, ℓ_2, X) , where $X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree d , $p \in X$ and $\ell_1, \ell_2 \subset V_{p,X}^h$ are lines having intersection multiplicity at least h with X at p .

To this aim, we define $S_d := \mathbb{C}[x_0, \dots, x_{n+1}]_d$ and $S_d^* := S_d \setminus \{0\}$ as in (2.3), and we set

$$\mathbb{P} := \mathbb{P}^{n+1}, \quad \mathbb{G} := \mathbb{G}(1, n+1) \quad \text{and} \quad N+1 := \dim_{\mathbb{C}}(S_d) = \binom{d+n+1}{d}.$$

Let $\mathcal{P} \subset \mathbb{P} \times \mathbb{G}$ be the universal family of lines over \mathbb{G} , endowed with the projections $\mathbb{P} \xleftarrow{\pi_1} \mathcal{P} \xrightarrow{\pi_2} \mathbb{G}$. The morphism π_1 makes \mathcal{P} a \mathbb{P}^n -bundle over \mathbb{P} , whereas π_2 makes \mathcal{P} a \mathbb{P}^1 -bundle over \mathbb{G} , so that $\dim(\mathcal{P}) = 2n+1$. Consider the fibred product

$$\mathcal{P} \times_{\mathbb{P}} \mathcal{P} := \{(p, [\ell_1], [\ell_2]) \mid p \in \ell_1 \cap \ell_2\} \subset \mathbb{P} \times \mathbb{G} \times \mathbb{G}$$

and its *diagonal locus*

$$\Delta := \{(p, [\ell], [\ell]) \in \mathcal{P} \times_{\mathbb{P}} \mathcal{P} \mid p \in \ell\} \cong \mathcal{P}.$$

Let \mathcal{P}^{\square} denote the blow-up of $\mathcal{P} \times_{\mathbb{P}} \mathcal{P}$ along Δ and let Δ^{\square} be the exceptional divisor. Thus,

$$\dim(\mathcal{P}^{\square}) = 3n+1 = \dim(\Delta^{\square}) + 1.$$

Moreover, as a set, we have

$$\Delta^{\square} = \{(p, [\ell], [\Pi]) \mid p \in \ell \subset \Pi\} \subset \mathbb{P} \times \mathbb{G} \times \mathbb{G}(2, n+1).$$

Given a polynomial $F \in S_d^*$, let $X_F := V(F) \subset \mathbb{P}$ denote its vanishing locus. Then we define the variety $\square_h \subset \mathcal{P}^{\square} \times S_d^*$ as

$$\square_h = \square_h^{n+1} := \overline{\left\{ (p, [\ell_1], [\ell_2], F) \in \mathcal{P}^{\square} \times S_d^* \mid \begin{array}{l} \ell_1 \neq \ell_2 \text{ and for } 1 \leq i \leq 2, \\ \text{either } \ell_i \subset X_F \text{ or } X_F \cdot \ell_i \geq hp \end{array} \right\}}.$$

Lemma 4.4. *For any $2 \leq h \leq \min\{n+1, d\}$, \square_h is smooth, irreducible, of dimension $3n+3+N-2h$, dominating both \mathcal{P}^{\square} and S_d^* via the projection maps*

$$\mathcal{P}^{\square} \xleftarrow{\Psi} \square_h \xrightarrow{\Phi} S_d^*.$$

Proof. Let $(p, [\ell_1], [\ell_2]) \in \mathcal{P}^{\square} \setminus \Delta^{\square}$ and $F \in S_d^*$. Requiring that $(p, [\ell_1], [\ell_2], F) \in \square_h$ amounts to impose $2h-1$ independent linear conditions to F , corresponding to the conditions $\ell_1 \cdot X_F \geq hp$ and $\ell_2 \cdot X_F \geq hp$.

We claim that the same happens at $(p, [\ell], [\Pi]) \in \Delta^{\square}$, when we require $(p, [\ell], [\Pi], F) \in \square_h$. In fact, choose affine coordinates (η, ζ) on Π such that $p = (0, 0)$ and $\ell = V(\eta)$. Write $F|_{\Pi} = F_0 + F_1 + \dots + F_d$, where $F_i = \sum_{0 \leq j \leq i} a_{i,j} \eta^{i-j} \zeta^j$ is a homogeneous polynomial of degree i . Imposing the condition $X_F \cdot \ell \geq hp$ gives $a_{i,i} = 0$, for any $0 \leq i \leq h-1$. Now, consider a general line $\ell' := V(\eta - t\zeta)$ in π through $p = (0, 0)$. Imposing the condition $X \cdot \ell' \geq hp$ and letting ℓ' approach ℓ , i.e. letting t approach zero, gives $a_{i,i-1} = 0$ for any $1 \leq i \leq h-1$. Therefore, there are again $2h-1$ independent conditions for the coefficients of F in order to have $(p, [\ell], [\Pi], F) \in \square_h$.

Hence the projection $\Psi: \square_h \rightarrow \mathcal{P}^\square$ is onto, and its fibers are parameterized by $(2h-1)$ -codimensional linear subspaces of S_d . Therefore, \square_h is smooth, irreducible, of dimension

$$\dim(\square_h) = \dim(\mathcal{P}^\square) + \dim(S_d) - (2h-1) = 3n+2+N-2h+1.$$

The surjectivity of $\Phi: \square_h \rightarrow S_d^*$ is clear; indeed, for any $F \in S_d^*$ and any $p \in X_F$, we have that $\dim(V_{p,X_F}^h) \geq n+2-h > 0$, as $h \leq n+1$ by assumption. \square

For a general polynomial $F \in S_d^*$, we set

$$\square_{h,F} = \square_{h,F}^{n+1} := \Phi^{-1}(F), \quad (4.2)$$

which is smooth, equidimensional, of dimension $3n+2-2h$.

Now, we argue as in [20, Proof of Theorem 3.3] and we prove Theorem 1.4.

Proof of Theorem 1.4. Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n+2$ and let $h \in \mathbb{N}$ such that $\text{conn. gon}(X) = d-h$. It follows from Theorem 1.3 that $h \geq \left\lfloor \frac{1+\sqrt{8n+1}}{2} \right\rfloor$. We note that if $h = \left\lfloor \frac{1+\sqrt{8n+1}}{2} \right\rfloor$, we are done because $\left\lfloor \frac{1+\sqrt{8n+1}}{2} \right\rfloor \leq \left\lfloor \frac{\sqrt{16n+25}-3}{2} \right\rfloor$ for any $n \geq 4$. Hence we assume hereafter

$$h > \left\lfloor \frac{1+\sqrt{8n+1}}{2} \right\rfloor. \quad (4.3)$$

Given two general points $x_1, x_2 \in X$, there exists an irreducible curve $C \subset X$ containing x_1 and x_2 such that $\text{gon}(C) = \text{conn. gon}(X) = d-h$. Since $h \geq 3$, by Proposition 4.1 there exists a point $p \in X$ such that $C \subset V_p^h$, and the projection $\pi_p: V_p^h \dashrightarrow \Lambda_p^h$ from p maps C to a rational curve $D \subset \Lambda_p^h$. In particular, if $\ell_1, \ell_2 \subset V_p^h$ denote the lines connecting the vertex p to the points $x_1, x_2 \in X$, respectively, then D passes through the corresponding points $\ell_1 \cap \Lambda_p^h$ and $\ell_2 \cap \Lambda_p^h$. So, we are interested in the locus $\mathcal{R}_{n+1} \subset \square_h$ defined as

$$\mathcal{R}_{n+1} := \overline{\left\{ (p, [\ell_1], [\ell_2], F) \in \square_h \left| \begin{array}{l} \ell_1 \neq \ell_2 \text{ and } \exists \text{ a rational curve } D \subset \Lambda_{p,X_F}^h \text{ passing} \\ \text{through the points } \ell_1 \cap \Lambda_{p,X_F}^h \text{ and } \ell_2 \cap \Lambda_{p,X_F}^h \end{array} \right. \right\}}. \quad (4.4)$$

It follows from [4, Lemma 2.2] that for $F \in S_d^*$ general and $p \in X_F$ general, the variety $\Lambda_p^h = \Lambda_{p,X_F}^h$ is a general complete intersection of type $(1, 1, 2, \dots, h-1)$ in \mathbb{P}^{n+1} . Thus its canonical bundle is isomorphic to $\mathcal{O}_{\Lambda_p^h}(\sum_{i=2}^{h-1} i - n)$, which is effective by (4.1) and (4.3). In particular, Λ_p^h is not covered by rational curves, so that \mathcal{R}_{n+1} consists of (at most) countably many proper closed subsets of \square_h .

Let $F \in S_d^*$ be the polynomial defining the very general hypersurface $X \subset \mathbb{P}^{n+1}$. According to the discussion above, for general $x_1, x_2 \in X$, we may find (at least) one 4-tuple $(p, [\ell_1], [\ell_2], F) \in \mathcal{R}_{n+1}$, where $\ell_i = \langle p, x_i \rangle$ for $i = 1, 2$. Since $\ell_i \cap X$ consists of finitely many points, as we vary the pair $(x_1, x_2) \in X \times X$, the corresponding 4-tuples $(p, [\ell_1], [\ell_2], F)$ describe a subset of $\mathcal{R}_{n+1} \cap \square_{h,F}^{n+1}$ having dimension at least $2n$. In particular, $\dim(\mathcal{R}_{n+1} \cap \square_{h,F}^{n+1}) \geq 2n$ and, being $F \in S_d^*$ very general, we deduce that $\dim \mathcal{R}_{n+1} \geq 2n + N + 1$. Therefore,

$$\text{codim}_{\square_h} \mathcal{R}_{n+1} \leq n+2-2h. \quad (4.5)$$

We point out that for any subfamily $\mathcal{F} \subset \square_h$ such that $(p, [\ell_1], [\ell_2], F) \in \mathcal{F}$, we have $\text{codim}_{\mathcal{F}}(\mathcal{R}_{n+1} \cap \mathcal{F}) \leq \text{codim}_{\square_h} \mathcal{R}_{n+1}$. Hence (4.5) gives

$$\text{codim}_{\mathcal{F}}(\mathcal{R}_{n+1} \cap \mathcal{F}) \leq n+2-2h. \quad (4.6)$$

We construct a subfamily $\mathcal{F} \subset \square_h$ with $(p, [\ell_1], [\ell_2], F) \in \mathcal{F}$, as follows. Let

$$m := \frac{h(h-1)}{2} - 2$$

and let $(p', [\ell'_1], [\ell'_2], F') \in \square_h^{m+2}$ be a 4-tuple such that $Y' := V(F') \subset \mathbb{P}^{m+2}$ is a very general hypersurface of degree d , $p' \in Y'$ is a very general point, ℓ'_1 is very general among lines in $V_{p', Y'}^h$, passing through p' , and $\ell'_2 \neq \ell'_1$. Moreover, we deduce from (4.3) that $m+2 \geq n+1$.

Let $M \geq m+2 \geq n+1$ and let $(p'', [\ell''_1], [\ell''_2], F'') \in \square_h^M$, where $Y'' := V(F'') \subset \mathbb{P}^M$ is a hypersurface of degree d such that X is a $(n+1)$ -plane section and Y' is a $(m+2)$ -plane section, with $p = p' = p''$, $\ell_1 = \ell'_1 = \ell''_1$ and $\ell_2 = \ell'_2 = \ell''_2$.

Now, let $Z_r \subset \text{Hom}(\mathbb{P}^r, \mathbb{P}^M)$ be the set of parameterized r -planes in \mathbb{P}^M containing the plane $\langle \ell_1, \ell_2 \rangle$, and let $Z'_r \subset Z_r$ be the subset of parameterized r -planes $\Lambda \subset \mathbb{P}^M$ such that $(p, [\ell_1], [\ell_2], F_\Lambda) \in \mathcal{R}_r$, where F_Λ is the polynomial defining the section of Y'' by Λ .

We point out that for any $\Lambda \in Z_r$, we have $(p, [\ell_1], [\ell_2], F_\Lambda) \in \square_h^r$. To see this fact, consider the hypersurface $Y := V(F_\Lambda) = \Lambda \cap Y''$ of degree d in $\Lambda \cong \mathbb{P}^r$. For $i = 1, 2$, we have $\ell_i \subset \Lambda$, so the intersection schemes $\ell_i \cdot Y$ and $\ell_i \cdot Y''$ are supported on the same 0-cycle of degree d , i.e. $\text{mult}_q(\ell_i \cdot Y) = \text{mult}_q(\ell_i \cdot Y'')$ for any $q \in Y \cap \ell_i$. In particular, $Y \cdot \ell_i \geq hp$ for $i = 1, 2$, so that $(p, [\ell_1], [\ell_2], F_\Lambda) \in \square_h^r$.

As in [20], let \mathcal{F} be the image of Z_{n+1} in \square_h^{n+1} under the map sending a $(n+1)$ -plane $\Lambda \in Z_{n+1}$ to the point $(p, [\ell_1], [\ell_2], F_\Lambda) \in \square_h^{n+1}$. Thus $\mathcal{R}_{n+1} \cap \mathcal{F}$ is the image of Z'_{n+1} . According to (4.6), we have

$$\text{codim}_{Z_{n+1}} Z'_{n+1} \leq n+2-2h. \quad (4.7)$$

Let $\varepsilon_r := \text{codim}_{Z_r} Z'_r$. Since $Y' \subset \mathbb{P}^{m+2}$ and $p \in Y'$ are very general, then $\Lambda_{p, Y'}^h$ is a smooth complete intersection of type $(1, 1, 2, \dots, h-1)$ in \mathbb{P}^{m+2} by [4, Lemma 2.2]. Hence its canonical bundle is $\mathcal{O}_{\Lambda_{p, Y'}^h}(\sum_{i=2}^{h-1} i - m - 1)$, which is trivial by the choice of m . In particular, $\Lambda_{p, Y'}^h$ is not covered by rational curves and, being $\ell_1 \subset V_{p, Y'}^h$, a very general line through p , there are no rational curves of $\Lambda_{p, Y'}^h$ passing through the point $\ell_1 \cap \Lambda_{p, Y'}^h$. Thus $(p, [\ell_1], [\ell_2], F') \notin \mathcal{R}_{m+2}$ and $\varepsilon_{m+2} \geq 1$.

Applying [20, Proposition 3.5], we obtain

$$\varepsilon_{m+1} = \text{codim}_{Z_{m+1}} Z'_{m+1} \geq \varepsilon_{m+2} + 1 \geq 2,$$

and by recursion

$$\varepsilon_{n+1} = \text{codim}_{Z_{n+1}} Z'_{n+1} \geq m - n + 2.$$

By (4.7), we must have $m - n + 2 \leq n + 2 - 2h$, and being $m := \frac{h(h-1)}{2} - 2$, we deduce

$$\frac{h(h-1)}{2} - n \leq n + 2 - 2h, \quad \text{so that} \quad h \leq \frac{-3 + \sqrt{16n+25}}{2}.$$

Thus the connecting gonality of X satisfies $\text{conn. gon}(X) \geq d - \left\lfloor \frac{-3 + \sqrt{16n+25}}{2} \right\rfloor$.

The final part of the statement is achieved by using (1.2) and noting that $\left\lfloor \frac{-1 + \sqrt{16n+1}}{2} \right\rfloor = \left\lfloor \frac{-3 + \sqrt{16n+25}}{2} \right\rfloor$ if and only if n belongs to the set

$$\{4a^2 + 3a, 4a^2 + 5a, 4a^2 + 5a + 1, 4a^2 + 7a + 2, 4a^2 + 9a + 4, 4a^2 + 11a + 6 \mid a \in \mathbb{N}\}.$$

□

Finally, we discuss the values of $\text{conn. gon}(X)$, when the hypersurface X has small dimension.

Example 4.5. Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n+2$, with $1 \leq n \leq 16$ and $n \neq 9, 13, 14$.

Case $n=1$. When X is a plane curve, $\text{conn. gon}(X)$ equals the gonality of X , which is $\text{gon}(X) = d-1$ (cf. [7, Teorema 3.14]).

Case $n=2$. The connecting gonality of very general surfaces $X \subset \mathbb{P}^3$ of degree $d \geq 5$ is computed by tangent hyperplane sections $X \cap T_p X$, so that $\text{conn. gon}(X) = d-2$ (see e.g. [1]).

Case $n=3$. When $n=3$, we have $\text{conn. gon}(X) = d-2$. To see this fact, notice that $\text{conn. gon}(X) \leq d-2$ by Remark 4.3 and case $n=2$ above. On the other hand, $\text{conn. gon}(X) \geq \text{cov. gon}(X) = d-3$ by (1.1) and (1.2). Suppose by contradiction that there exists a connecting family $\mathcal{C} \xrightarrow{\pi} T$ of $(d-3)$ -gonal curves. Then Proposition 4.1 ensures that the general curve $C_t := \pi^{-1}(t)$ lies on $X \cap V_p^3$ for some $p \in X$. By Theorem 1.1, the varieties $Z_p := X \cap V_p^3$ are curves and, as we vary $p \in X$, we obtain a 3-dimensional family. However, according to Remark 4.2, the family $\mathcal{C} \xrightarrow{\pi} T$ should have dimension at least 4, a contradiction.

Cases $4 \leq n \leq 16$ with $n \neq 9, 13, 14$. For all these values of n , we may apply Theorems 1.3 and 1.4, and the bounds included therein coincide. Thus

$$\text{conn. gon}(X) = \begin{cases} d-3 & \text{if } n = 4, 5 \\ d-4 & \text{if } n = 6, 7, 8 \\ d-5 & \text{if } n = 10, 11, 12 \\ d-6 & \text{if } n = 15, 16. \end{cases}$$

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