Quantifying high-order interdependencies in entangled quantum states

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We leverage recent advances in information theory to develop a method to characterize the dominant character of the high-order dependencies of quantum systems. To this end, we introduce the *Q-information*: an information-theoretic measure capable of distinguishing quantum states dominated by synergy or redundancy. We illustrate the measure by investigating the properties of paradigmatic entangled qubit states and find that—in contrast to classical systems—quantum systems need at least four variables to exhibit high-order properties. Furthermore, our results reveal that unitary evolution can radically affect the internal information organization in a way that strongly depends on the corresponding Hamiltonian. Overall, the Q-information sheds light on aspects of the internal organization of quantum systems and their time evolution, opening different avenues for studying several quantum phenomena and related technologies.

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I. INTRODUCTION

Entropy connects the physics of a system with the information it carries, being a fundamental quantity for describing the relationships among the degrees of freedom in a system. However, a fundamental open question is how to best characterize the emergence of various collective modes of interdependency among several degrees of freedom and, more broadly, which are the laws describing the informational architecture of complex systems. While physical systems and processes can be effectively described (either from first principles or as a modeling strategy) via pairwise interactions, many-body interactions [1–4] are gathering the attention of several communities—even beyond physics, including neuroscience, economics, and many others [5–9]. Crucially, it has been recently noticed that the link between the relationship of the order (e.g., pairwise vs higher order) of mechanisms modeled via Hamiltonians and the resulting activity patterns can be highly nontrivial [10]. Therefore, a principled understanding of the informational structure of a physical system cannot be based solely on determining its effective Hamiltonian but requires additional tools to investigate the resulting phenomena that arise from it.

Information theory and statistical mechanics provide an ideal toolbox to prove the informational relationships within complex systems [11], benefiting from a solid mathematical formalism for describing multivariate information and high-order interactions. For instance, the partial information decomposition (PID) framework [12–14] and its extension to time-series analysis [15] provide an encompassing and thorough approach to investigate the different information modes—e.g., synergy and redundancy—within a system, which can yield important scientific insights. .For instance, recent investigations on brain dynamics uncovered that synergy and redundancy [16,17] relate to the interplay between brain segregation and integration [18], namely, redundancy speaks to the robustness of input/output (I/O) in sensory areas, whilst synergy dominates high-level networks and ensures information integration.

Given the relevance of synergies and redundancies, capturing their balance is a key summary marker of the informational architecture of a system. Interestingly, while the computation of the full PID of a system may be sometimes challenging, it was shown that the calculation of the overall balance between synergy and redundancy can be done in a relatively straightforward way via a quantity named *O-information* [19]. Despite its recent inception, the O-information has found numerous applications [20–22] and has various theoretical extensions including pointwise [23], spectral [24], and harmonic decompositions [25].

Motivated by the success of this measure and building on the growing interest in studying quantum systems via the lens of partial information decomposition [26,27], and in quantum information [28], here, we investigate the feasibility of extending the O-information to quantum systems. To reach this goal, we map the O-information to its quantum

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counterpart and perform a series of measures on relevant qubit states, concentrating on four-qubit systems. As shown later, this choice leads us to identify notable properties of the quantum O-information and the way it can be changed using suitable time evolution operators. The remainder of the manuscript is organized as follows. In Sec. II, we provide a general introduction to the O-information. Then, in Sec. III we map the classical O-information to its quantum counterpart, i.e., the Q-information, and measure it on various n-qubit states. In Sec. IV, we focus on four-qubit states, and analyze the Q-information on these states undergoing time evolution in Sec. V. Finally, in Sec. VI, we end the manuscript by summarizing the main findings and discussing potential developments.

II. O-INFORMATION

Shannon's mutual information provides a reliable metric of interdependence between two (groups of) variables, capturing both linear and nonlinear components. Despite providing a foundation, the mutual information does not allow assessing triple- or higher-order interdependencies, which can play important roles even in systems governed by pairwise mechanisms [10]. On the other hand, well-known nonnegative multivariate extensions of the mutual information, namely, the total correlation (TC) [29],

$$\mathrm{TC}(\boldsymbol{X}) := \sum_{i=1}^{N} H(X_i) - H(\boldsymbol{X}), \tag{1}$$

and the dual total correlation (DTC) [30],

$$\mathrm{DTC}(\boldsymbol{X}) := H(\boldsymbol{X}) - \sum_{i=1}^{N} H(X_i \mid \boldsymbol{X}_{-i}), \tag{2}$$

allow assessing high-order interdependencies. Here, $H(Y) = -\sum_y p_Y(y)\log_2 p_Y(y)$ is Shannon's entropy of the random variable (or vector) Y with distribution $p_Y(y)$, $H(Y|Z) = H(Y,Z) - H(Z)$ is the conditional Shannon entropy, and $\boldsymbol{X}_{-i} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_N)$. Importantly, both TC and DTC are zero if and only if all variables $X_1, \dots, X_N$ are jointly statistically independent—i.e., if their joint distribution can be factorized as $p_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{N} p_{X_i}(x_i)$.

Unfortunately, TC and DTC provide alternative metrics for high-order interdependencies that are difficult to analyze together. A recent approach, introduced in Ref. [19], proposes to consider the sum and the difference of those two metrics to obtain the O-information and the S-information, which have more intuitive interpretations. Specifically, given N random variables $\boldsymbol{X} = (X_1, \dots, X_N)$, their O-information is defined as

$$\Omega(\boldsymbol{X}) := \mathrm{TC}(\boldsymbol{X}) - \mathrm{DTC}(\boldsymbol{X}), \tag{3}$$

which reads

$$\Omega(\boldsymbol{X}) = (N-2)H(\boldsymbol{X}) + \sum_{i=1}^{N} [H(X_i) - H(\boldsymbol{X}_{-i})]. \tag{4}$$

The O-information is a signed metric capturing the balance between high- and low-order statistical constraints. Low-order constraints impose strict restrictions on the system and allow little shared information between variables, whereas high-order constraints impose collective restrictions that enable large amounts of shared randomness [19]. Crucially, high-order constraints can generate global interdependencies that do not impose corresponding pairwise dependencies, as observed, for instance, in the XOR logic gates.

Consequently, $\Omega(\boldsymbol{X}) < 0$ implies a predominance of high-order constraints within the system, a condition usually referred to as *statistical synergy*. Conversely, $\Omega(\boldsymbol{X}) > 0$ indicates that low-order constraints dominate the system and implies *redundancy* of information. The adopted terminology finds support in many properties of the O-information. Firstly, the O-information is maximized by redundant distributions where the same information is copied in multiple variables and is minimized by synergistic ("XOR-like") distributions: e.g., for binary variables, Ω is maximized by the "N-bit copy" where X_1 is a Bernoulli random variable (r.v.) with parameter $p = 1/2$ and $X_1 = \cdots = X_N$, and is minimized when $X_1, \dots, X_{N-1}$ are independent and identically distributed (i.i.d). fair coins and $X_N = \sum_{j=1}^{N-1} X_j \pmod 2$. Additional relevant properties are, in order:

(1) It captures genuine high-order effects, as it is zero for systems with only pairwise interdependencies: if the joint distribution of $\boldsymbol{X}$ (with an even number N of components) can be factorized as $p_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{k=1}^{N/2} p_{X_{2k-1},X_{2k}}(x_{2k-1}, x_{2k})$, then $\Omega(\boldsymbol{X}) = 0$.

(2) The O-information characterizes the dominant tendency, being additive over noninteractive subsystems: if the system can be factorized as $p_{\boldsymbol{X}}(\boldsymbol{x}) = p_{X_1,\dots,X_m}(x_1, \dots, x_m) \times p_{X_{m+1},\dots,X_N}(x_{m+1}, \dots, x_N)$, then $\Omega(\boldsymbol{X}) = \Omega(X_1, \dots, X_m) + \Omega(X_{m+1}, X_N)$.

It is worth mentioning that for $N = 3$, Ω coincides with the interaction information [31], a classical multivariate information-theoretic measure [32] with interesting topological properties [33]. While the interaction information represents the balance of synergy-vs-redundancy only for three variables [34], the O-information has this capability for any system size—see Ref. [19] for additional insights, properties, and mathematical proofs related to the O-information.

Summarizing, while the TC and DTC provide alternative representations to the same construct, Ω qualitatively characterizes their dominant nature. Eventually, notice that the O-information finds an additional interpretation as a revision of the classic measure of neural complexity proposed by Tononi *et al.* in Ref. [35], which provides a mathematical formulation with properties that are closer to their original desiderata [19].

III. QUANTUM O-INFORMATION

Before delving into the quantum realm, it is important to note that Eq. (4) relies solely on a linear combination of Shannon entropies computed on various collections of variables, while it does not require any conditional entropy terms. By leveraging this, one can identify a quantum counterpart to the O-information by substituting the Shannon entropy H with the von Neumann entropy $S = -\mathrm{Tr}[\rho \log_2 \rho]$, where ρ denotes the density matrix of the quantum system under consideration. Accordingly, we define the *quantum O-information*—or, more succinctly, "Q-information"—of a quantum system composed of N parties as follows:

$$\Omega_Q(\rho) := (N-2)S(\rho) + \sum_{i=1}^{N}[S(\rho_i) - S(\rho_{-i})], \tag{5}$$

where ρ_i is the density matrix of the ith component of the system, obtained by tracing out all the other degrees of freedom, whereas ρ_{-i} denotes the density matrix obtained by tracing out the ith component from ρ and corresponds to the rest of the system. Note that (5) vanishes if $N = 2$, hence three is the minimum number of parties that can yield a nonvanishing Q-information.

Observe that, according to the definition (5), the Q-information of pure states $\rho = |\psi\rangle\langle\psi|$ is identically zero for any N. Indeed, the von Neumann entropy of pure states is zero and, from the Schmidt decomposition, each term in the summation in (5) is zero, since $S(\rho_i) = S(\rho_{-i})$ for all i. We can nonetheless associate nonzero Q-information values to pure states by making use of their reduced density matrices [36], namely, by decomposing them into subsystems.

Therefore, we define the Q-information of an N-party pure vector state $|\psi\rangle$ as the average Q-information of its $(N-1)$-party reduced density matrices, namely,

$$\overline{\Omega}_Q(|\Psi\rangle) := \frac{1}{N}\sum_{j=1}^{N}\Omega_Q(\rho_{-j}), \tag{6}$$

with $\rho = |\Psi\rangle\langle\Psi|$. Now, since ρ is a pure state, we get $S(\rho_{-j}) = S(\rho_j)$ and $S(\rho_{-j,-i}) = S(\rho_{ji})$, whence

$$\begin{aligned}\Omega_Q(\rho_{-j}) &= (N-3)S(\rho_j) + \sum_{\substack{i=1\\ i\neq j}}^{N}[S(\rho_i) - S(\rho_{ij})]\\ &= (N-4)S(\rho_j) + \sum_{i=1}^{N}S(\rho_i) - \sum_{\substack{i=1\\ i\neq j}}^{N}S(\rho_{ij}).\end{aligned} \tag{7}$$

Therefore, the Q-information for an N-party pure quantum state (6) reads

$$\overline{\Omega}_Q(|\psi\rangle) = \left(2 - \frac{4}{N}\right)\sum_j S(\rho_j) - \frac{1}{N}\sum_{i\neq j}S(\rho_{ij}). \tag{8}$$

By abusing notation, we will remove the bar in the definition (6) and refer henceforth to $\Omega_Q(|\psi\rangle)$ also for pure states, whenever confusion cannot arise.

The definition of the Q-information Ω_Q, both for pure and mixed states, is central in this work, and a few comments are in order. First of all, notice that in a general (pure or mixed) state, the von Neumann entropy depends on the jth component one traces out. It is for this reason that average procedures are necessary. This seemingly trivial comment will be useful in the following. Second, in light of the above comment, an alternative definition for pure states might be

$$\tilde{\Omega}_Q(|\Psi\rangle) := \Omega_Q(\overline{\rho}_{\text{red}}) \tag{9}$$

with

$$\overline{\rho}_{\text{red}} = \frac{1}{N}\sum_{j=1}^{N}\rho_{-j}, \tag{10}$$

but this expression is more difficult to analyze.

Third, it follows by direct calculation that a pure state of $N = 3$ components yields a vanishing Q-information. Therefore, $N = 4$ is the minimum number of components that may lead to a nonvanishing Q-information, as one component has to be traced out, leaving a mixed state of a tripartite system. Finally, from the above observation, a remarkable difference emerges between the classical O-information and its quantum counterpart, namely, the minimum number of components yielding a nonvanishing O-information is $N = 3$ in classical systems and $N = 4$ in pure-state quantum systems.

Given this premise, to develop further insight about the Q-information, let us apply it to some well-known qubit states: the Greenberger–Horne–Zeilinger (GHZ) state [37], the W state, and the maximally multipartite entangled states (MMES) [38,39]. For each of these states, various sizes will be considered. Additionally, we shall also evaluate states with a fixed number of qubits: the Yeo-Chua (YC) state [40], the "hyperdeterminant" (HD) state [41], and the Higuchi-Sudbery (HS) state [42]. Interestingly, calculations on these (pure) states suggest that the resulting Q-information does not rely on the choice of the traced-out qubit—additional details on this issue will be discussed later. For the sake of clarity, let us recall the definition of the considered states. The GHZ state, in its more general form (i.e., with N qubits), reads

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}). \tag{11}$$

Similarly, the W state, which constitutes a specific case of Dicke states [43], can be written in a generic form as

$$|W\rangle = \frac{1}{\sqrt{N}}(|10\ldots0\rangle + |01\ldots0\rangle + \cdots + |00\ldots1\rangle). \tag{12}$$

Considering $N = 4$, the states YC, HD, and HS read

$$\begin{aligned}|\text{YC}\rangle &= \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle\\ &\quad + |1001\rangle + |1010\rangle + |1100\rangle) + |1111\rangle),\\ |\text{HD}\rangle &= \frac{1}{\sqrt{6}}(|1000\rangle + |0100\rangle + |0001\rangle + \sqrt{2}|1111\rangle),\\ |\text{HS}\rangle &= \frac{1}{\sqrt{6}}[|0011\rangle + |1100\rangle + e^{\frac{2i\pi}{3}}(|0101\rangle + |1010\rangle)\\ &\quad + e^{\frac{4i\pi}{3}}(|0110\rangle + |1001\rangle)].\end{aligned} \tag{13}$$

Finally, uniform MMES states are defined as

$$|\text{MMES}\rangle = \frac{1}{N}\sum_{k\in\mathbb{Z}_2^N} e^{i\phi_k}|k\rangle, \tag{14}$$

where the ϕ_k are suitable phases. For instance, a five-qubit MMES state can be obtained by using the following entries for the term ϕ_k in (14):

$$\begin{aligned}\boldsymbol{\phi} = (&0, 0, 0, 0, 0, \pi, \pi, 0, 0, \pi, \pi, 0, 0, 0, 0, 0,\\ &0, 0, \pi, \pi, 0, \pi, 0, \pi, \pi, 0, \pi, 0, \pi, \pi, 0, 0);\end{aligned} \tag{15}$$

see Ref. [38] for additional details. The analysis of the Q-information of these states shows that subsystems of the GHZ and the W states are redundancy dominated and the value of the Q-information scales with N, i.e., the number of qubits.

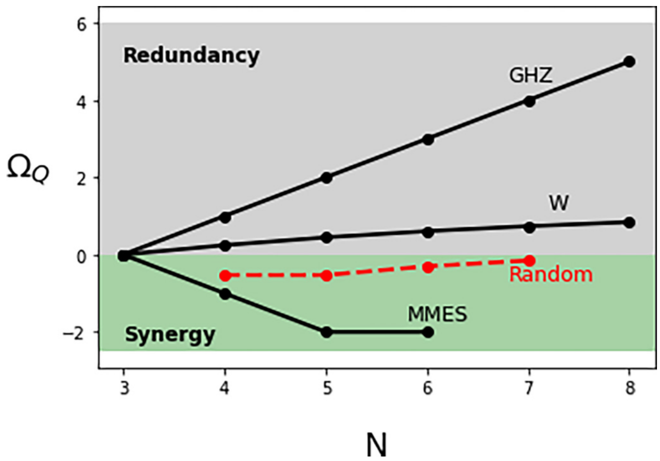

FIG. 1. Q-informationΩ_Q vs number of qubits N for different pure quantum states. The calculation of Ω_Q is performed differently for random states, see text.

In contrast, MMES are synergy dominated, with the absolute value of the Q-information also increasing with the number of qubits — see Fig. 1. Additionally, results show that the HS state is redundancy dominated, while the HD and the YC states are synergy dominated.

We note in passing that, in the states above considered, the contributions $\Omega_Q(\rho_{-j})$ in (6) do not rely on the jth traced out qubit. Yet, such independence comes from the symmetries of the analyzed states and is certainly *not* a general feature of the Q-information.

To build intuition on these results, let us recall that the GHZ state is the ground state of the ferromagnetic quantum Ising model described by the Hamiltonian $H = -J\sum \sigma_j^z\sigma_{j+1}^z$, with interaction strength $J > 0$ and σ^z being the third Pauli matrix. Hence, the fact that GHZ is redundancy dominated is consistent with previous works in classical spin systems [23,44], where the connection between synergy and frustration (i.e., when $J < 0$) has been stressed. Finally, all the above-described properties suggest that Ω_Q departs from separableness, with a redundant character ($\Omega_Q > 0$) for entanglement typical of the GHZ-like states, and a synergistic character ($\Omega_Q < 0$) for MMES-like states.

IV. FOUR-QUBIT PURE STATES

Let us remark that a nonvanishing Q-information in pure states can be obtained considering at least four qubits. Interestingly, that connects with the concept of interaction information [31] of classical variables. Also, we remark that many open problems concern four-qubit states [45], such as their classification [46–48], and are crucial for the efficiency of quantum protocols. Therefore, here, we concentrate on four-qubit pure states $|\Psi\rangle$, whose decomposition leads to identifying four components (subsystems), say, A, B, C, and D.

Tracing out A from $\rho_{ABCD} = |\Psi\rangle\langle\Psi|$, we obtain the reduced density matrix ρ_{BCD}. The Q-information derived through the reduced density matrix ρ_{BCD} reads

$$\Omega_Q(\rho_{BCD}) = S_{BCD} - S_{BC} - S_{BD} - S_{CD} + S_B + S_C + S_D. \tag{16}$$

Notice that Eq. (16) corresponds to the well-known topological entanglement entropy [49]. It is easy to prove that $\Omega_Q(\rho_{BCD})$ does not depend on the choice of the traced-out subsystem A (the same property does not generally hold for $N > 4$, however, as previously reported, it does hold for the states considered in the previous section). In this respect, it can be shown that, exploiting Schmidt's decomposition, $\Omega_Q(\rho_{BCD})$ can be written in the symmetrical form

$$\Omega_Q(\rho_{BCD}) = \sum_X S_X - \frac{1}{4}\sum_{X\neq Y} S_{XY}, \tag{17}$$

where X and Y take value in $\{A, B, C, D\}$.

This coincides with expression (8). Therefore, we can unambiguously associate the quantity

$$\overline{\Omega}_Q(|\Psi\rangle) = \Omega_Q(\rho_{BCD}) \tag{18}$$

for all four-party pure states (and in particular for four qubits), measuring the high-order informational character of $|\Psi\rangle$.

We also remark that, for $N = 4$, there is only one way to obtain four subsystems.

The following bounds can be proven by using subadditivity [50] and strong subadditivity inequalities [51]:

$$-2\min_X S_X \leqslant \Omega_Q(\rho_{BCD}) \leqslant 2\min_X S_X, \tag{19}$$

where X is one of the four subsystems A, B, C, and D. To prove the upper bound, note that strong subadditivity inequality implies that $S_{BCD} + S_B - S_{BC} - S_{BD} \leqslant 0$, and substituting in Eq. (16) yields $\Omega_Q(\rho_{BCD}) \leqslant S_C + S_D - S_{CD}$; now, assuming that $S_C \geqslant S_D$, subadditivity implies that $S_C - S_D - S_{CD} \leqslant 0$, hence $\Omega_Q(\rho_{BCD}) \leqslant 2\,S_D$. Since D is arbitrary, the upper bound follows. Similarly, to prove the lower bound, first substitute $S_B + S_C - S_{BC} \geqslant 0$ in Eq. (16), obtaining $\Omega_Q(\rho_{BCD}) \geqslant S_{BCD} + S_D - S_{CD} - S_{BD}$; using twice the subadditivity property implies $\Omega_Q(\rho_{BCD}) \geqslant -2\,S_C$. Since C is arbitrary, the lower bound follows. Note that, if the qubit state is factorizable with respect to a single qubit X, the two bounds of Eq. (19) vanish. This is a general property; if a pure state of N qubits (with arbitrary N) is factorizable, with respect to a single qubit, then after tracing out that qubit, the resulting state remains pure, therefore the Ω_Q associated to that state is zero. Finally, this observation shows that the property that a pure state of $N = 3$ components yields a vanishing Q-information, discussed after Eq. (10), is, in fact, a consequence of the fact that in a four-qubit system $\Omega_Q(\rho_{BCD})$ does not depend on the choice of the subsystem A that is traced out. Eventually, Eq. (19) entails $-2 \leqslant \overline{\Omega}_Q \leqslant 2$.

We now turn to consider states that are "random" in some respect. First, we look at all pure states of four qubits that are obtained by assigning 1 or 0 to the coefficients of the 16 basis states: the distribution of the Ω_Q of these states, displayed in Fig. 2, shows that negative values are very numerous, signaling synergy.

On the other hand, in Fig. 3 we consider the distribution of the Q-information Ω_Q on *mixed* states of three qubits (8×8 density matrices), obtained as reduced states of uniformly distributed random pure states of four, five, six, and seven qubits. This is shown in Fig. 3. In all cases, the distribution of Ω_Q peaks at negative values, i.e., we typically observe synergy. Observe that, by virtue of their typicality, for these

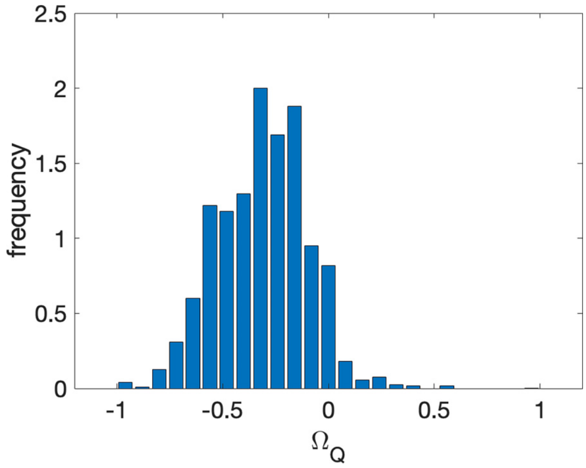

FIG. 2. Distribution of the Q-information Ω_Q evaluated on all 65 536 pure states that can be built by assigning 1 or 0 to the coefficients of the 16 basis states. Ω_Q ranges in $[-1, 1]$.

states Ω_Q shows (almost) no dependence on the particular subsystem that is traced out, but only on its dimension.

Thus, we conclude that the random generation of pure states shows that it is easier to produce synergistic (qubit) configurations than redundant ones. Moreover, the case $N = 4$ of Fig. 3 suggests that the bounds (19) are too loose. As discussed later, finding better bounds represents a valuable goal, requiring additional work.

V. HAMILTONIAN TIME EVOLUTION

Now we use the Q-information to address a fundamental question: how does the structure of high-order interdependencies change due to the unitary time evolution of a closed quantum system? As the unitary time evolution alters the state vector, the quantum system can experience variations in the internal organization of information as captured by Eq. (5).

The time evolution operator $U = e^{-itH}$ for a time t is generated by the Hamiltonian H of the system of interest. The specific form of the Hamiltonian H depends on how the

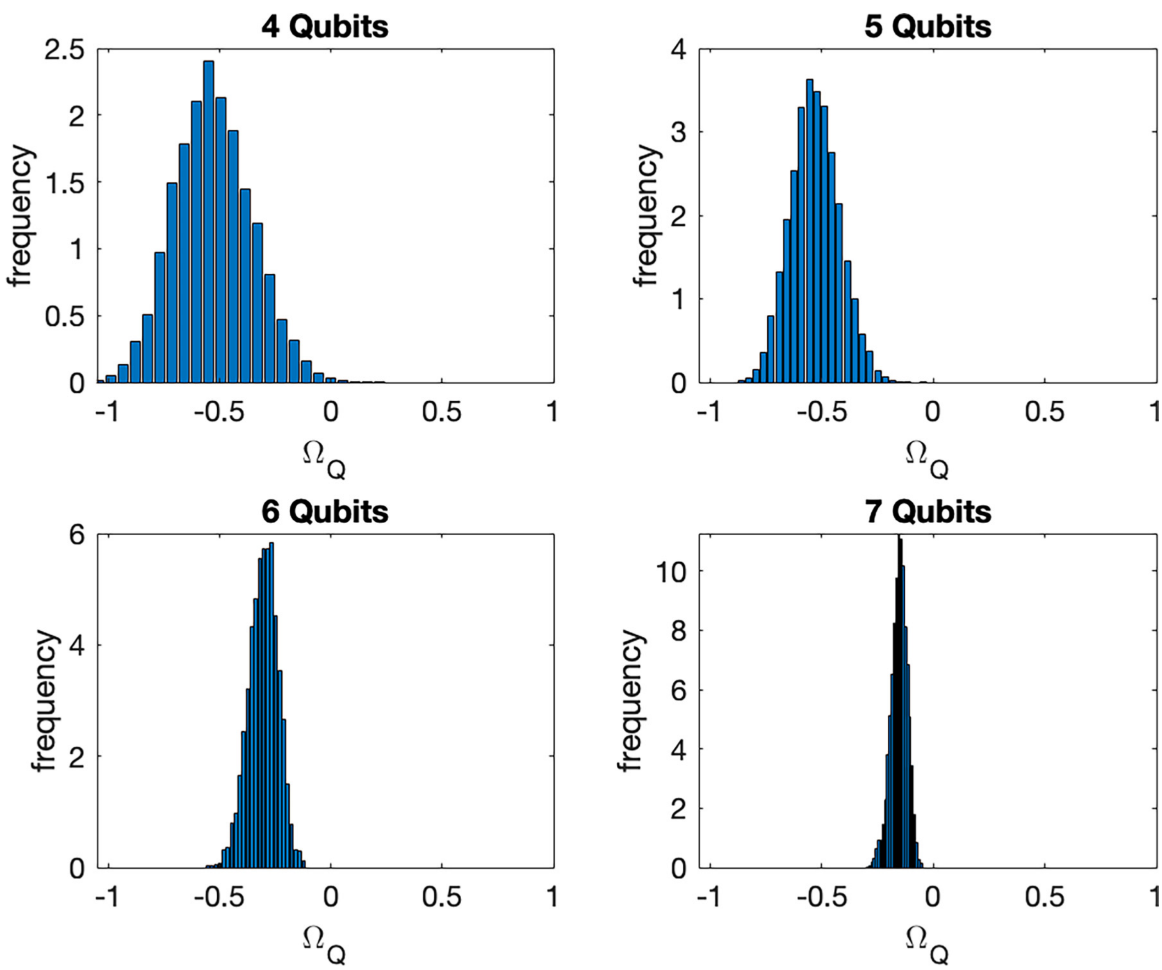

FIG. 3. Distribution of the Q-information Ω_Q, evaluated on three-qubit mixed states (8×8 density matrices) randomly sampled as follows: 10 000 uniformly distributed random pure states of n qubits ($n = 4$,5,6, and 7) are sampled by normalizing a vector of n i.i.d. complex Gaussian random variables. Then, the 8×8 density matrices are obtained by tracing out $n - 3$ qubits. In the first case ($n = 4$), Ω_Q ranges in $[-1.1, 033]$; as n increases, the measure of the reduced three-qubit density matrices becomes more and concentrated, see Ref. [36] for a similar concentration phenomenon in induced measure on mixed states.

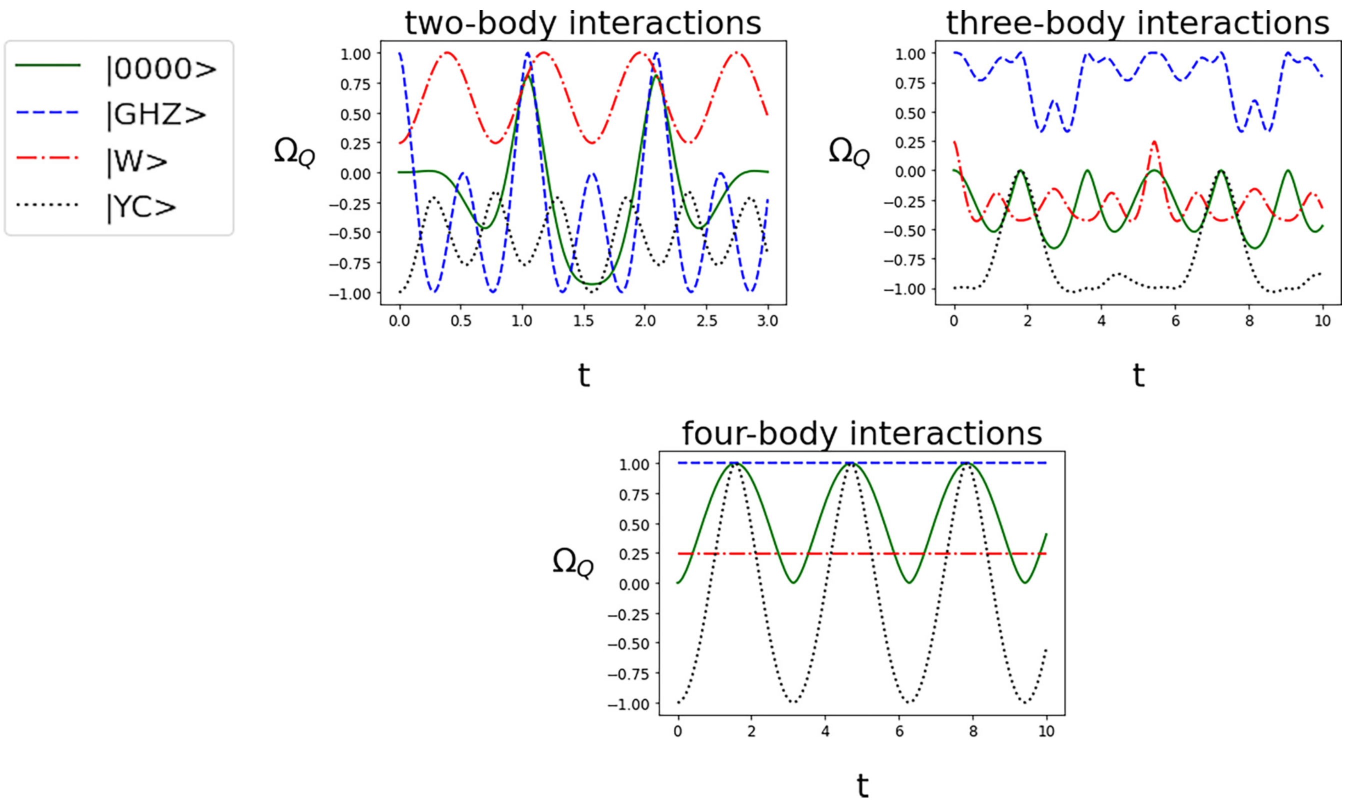

FIG. 4. Time evolution of the Q-information for the Hamiltonians defined in the main text, made of two-body interactions H_2 (top left), three-body interactions H_3 (top right), and four-body interaction H_4 (bottom). The initial states are YC (black dotted line), GHZ (blue dashed line), W (red dash-dotted line), and $|0000\rangle$ (green solid line).

system is organized. For instance, by considering a quantum register with the qubits in a chain, a relevant choice is given by the Heisenberg model [52]. According to this model, we consider three local Hamiltonians corresponding to systems with different interactions. Starting with two-body interactions, the Hamiltonian reads

$$H_2 = -\frac{1}{2}\sum_{i=1}^{4}\left(J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y + J_z\sigma_i^z\sigma_{i+1}^z\right), \quad (20)$$

where $\sigma_{i+1=\sigma_1}$ if $i = 4$ to implement periodic boundary conditions. Then, for three- and four-body interactions we have

$$H_3 = -\frac{1}{3}\sum_{i=1}^{3}\left(J_x\sigma_i^x\sigma_{i+1}^x\sigma_{i+2}^x + J_y\sigma_i^y\sigma_{i+1}^y\sigma_{i+2}^y + J_z\sigma_i^z\sigma_{i+1}^z\sigma_{i+2}^z\right), \quad (21)$$

$$H_4 = -\tfrac{1}{4}\left(J_x\sigma_1^x\sigma_2^x\sigma_3^x\sigma_4^x + J_y\sigma_1^y\sigma_2^y\sigma_3^y\sigma_4^y + J_z\sigma_1^z\sigma_2^z\sigma_3^z\sigma_4^z\right), \quad (22)$$

where, in H_3, $\sigma_{i+2=\sigma_1}$ if $i = 3$ for the boundary conditions above mentioned.

We used the above Hamiltonians to investigate what types of interactions allow dynamical reconfigurations of high-order phenomena—notice that the relationship between the order of the interaction and the order of the resulting patterns can be highly nontrivial [10]. In particular, many-body interactions are necessary for the Hamiltonian to have multipartite entangled eigenstates [37,53]. Also, multipartite high-order correlations have been shown to detect quantum phase transitions [54]

For this purpose, we measured the Q-information of $U(t)|\Psi\rangle$ as a function of time t, considering U arising from Hamiltonians of different orders. In the plots in Fig. 4 we set $J_x = J_y = J_z = 1$, and considered the initial states $|0000\rangle$, $|\mathrm{GHZ}\rangle$, $|W\rangle$, and $|\mathrm{YC}\rangle$.

The time evolution induced by the two-body Hamiltonian H_2 [Eq. (20)] leads to significant reorganization of high-order interdependencies [see Fig. 4(b)]. Remarkably, under suitable Hamiltonians, all states can evolve by alternating synergy- with redundancy-dominated configurations and vice versa. Similar observations apply for the three-body Hamiltonian H_3 as well as for H_4; note that GHZ and W states are eigenvectors of H_4, so their action leaves their internal organization unaltered.

Alternations observed through these Hamiltonians show an oscillatory behavior of Q-information, which can be limited to a small range or even turn a synergistic configuration into a redundant one (and vice versa). In summary, results show how unitary time evolution can deeply affect the internal organization of information in entangled qubit states. In particular, two-body interactions are shown to be enough to completely rearrange the overall information organization.

VI. CONCLUSION

We have extended the concept of O-information, successfully applied so far in a variety of complex macroscopic systems (e.g., see Refs. [20,21,23,55]), to the quantum case, showing its usefulness in grasping the informational character of the internal dependencies among the components of pure quantum states. The GHZ states represent the prototype of redundant dependency, while the MMES represent the synergistic prototypes. After analyzing the four-qubit case, we have found that random pure states are typically synergistic. Furthermore, we have observed that unitary time evolution generated by suitable local Hamiltonians affects the internal organization of entangled states. In other words, we have observed that two-body interactions, beyond increasing the quantum complexity of a state (see Refs. [56–59]), can drastically change the internal information organization of quantum systems.

To conclude, the Q-information deserves many developments as we deem may shed light on relevant quantum phenomena. For instance, to cite a few, future works could aim to understand the relationship between the quantum complexity and the Q-information, the effects of the time evolution on this quantity, and potential connections with the entanglement monogamy [60]. Also, it is relevant to generalize the properties of Q-information to collections with $N > 4$ qubits. Eventually, because the Q-information is a combination of quantum entropies, reaching a deeper comprehension of its operational meaning is far from trivial, yet it represents a fundamental aim to develop in future investigations.

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