Optimal decay of p-Sobolev extremals on Carnot groups

ANNUNZIATA LOIUDICE

Dipartimento di Matematica

Università degli Studi di Bari

Via Orabona, 4 - 70125 Bari (Italy)

annunziata.loiudice@uniba.it

Abstract

We determine the sharp asymptotic behavior at infinity of solutions to quasilinear critical problems involving the p-sublaplacian operator $\Delta_{p,\mathbb{G}}$ on a Carnot group $\mathbb{G}$, $1 < p < Q$. As a remarkable consequence, we obtain the exact rate of decay of the extremal functions for the subelliptic Sobolev inequality involving the L_p-norm of the horizontal gradient.

1 Introduction

In this paper we obtain the exact asymptotic behavior at infinity of finite energy solutions to quasilinear subelliptic problems with critical growth nonlinearity on Carnot groups. From this general result, we deduce the sharp rate of decay of the p-Sobolev extremals on such groups.

Let us introduce our problem. Let $\mathbb{G}$ be a Carnot group of arbitrary step $r \geq 1$, i.e. a connected simply connected nilpotent Lie group whose Lie algebra $\mathcal{G}$ admits a stratification $\mathcal{G} = V_1 \oplus V_2 \oplus \ldots \oplus V_r$, and it is generated via commutation by its first layer V_1.

Given a basis $\{X_j\}_{j=1}^m$ of V_1, the associated p-Laplacian operator $\Delta_{p,\mathbb{G}}$, where $1 < p < Q$ and Q denotes the homogeneous dimension of $\mathbb{G}$, is defined by

$$\Delta_{p,\mathbb{G}} := \sum_{i=1}^m X_i(|Xu|^{p-2}X_i u),$$

where Xu is the so-called *horizontal gradient* of u with length $|Xu| = (\sum_{i=1}^m |X_i u|^2)^{1/2}$. We are interested in subelliptic problems of the type

$$\begin{cases} -\Delta_{p,\mathbb{G}}u = f(\xi, u) & \text{in } \mathbb{G} \\ u \in \mathcal{D}^{1,p}(\mathbb{G}) \end{cases} \tag{1.1}$$

where $\mathcal{D}^{1,p}(\mathbb{G})$ is the completion of $C_0^\infty(\mathbb{G})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,p}(\mathbb{G})} := (\int_{\mathbb{G}} |Xu|^p \, \mathrm{d}\xi)^{1/p}$$

and $f : \mathbb{G} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(\xi, s)| \leq \Lambda |s|^{p^*-1}, \quad \text{for all } s \in \mathbb{R} \text{ and a.e. } \xi \in \mathbb{G}, \tag{1.2}$$

for some $\Lambda > 0$, where $p^* = \dfrac{pQ}{Q-p}$ is the critical Sobolev exponent in this context.

We shall deal with weak solutions of problem (1.1), i.e. functions $u \in \mathcal{D}^{1,p}(\mathbb{G})$ such that

$$\int_{\mathbb{G}} |Xu|^{p-2} <Xu, X\phi> \mathrm{d}\xi = \int_{\mathbb{G}} f(\xi, u) \, \phi \, \mathrm{d}\xi \qquad \forall \phi \in \mathcal{C}_0^\infty(\mathbb{G}).$$

In the ordinary Euclidean setting the problem of determining the sharp rate of decay of weak solutions to equations of the type (1.1) under critical growth assumptions on the nonlinearity f was firstly addressed by Egnell in [10]. He pointed out that the difficulty to treat the quasilinear case with respect to the semilinear one was due to the absence of the Kelvin transform in the quasilinear setting and suggested the necessity to find a more direct method to treat the p-Laplacian case.

The problem was firstly studied in the Euclidean elliptic setting by Vassilev [32], who obtained an almost-sharp decay estimate for the solutions in the case of critical singular nonlinearities, by mainly using their L^q-regularity.

Recently, by exploiting the sharp regularity of solutions in the framework of weak Lebesgue spaces, the exact rate of decay of solutions to problem (1.1) in $\mathbb{R}^n$ has been obtained by Vétois in [33]. We also quote the paper by Brasco-Mosconi-Squassina [4], where the optimal decay of the p-Sobolev minimizers for the fractional Sobolev inequality has been established, by using the sharp L^q-weak regularity of such extremals together with their radial symmetry. We also refer to Xiang [34], where asymptotic estimates are proved for the quasilinear problem with Hardy perturbation, without the use of weak Lebesgue norms.

In the present paper, we generalize to the subelliptic context of Carnot groups the general asymptotic results proved in [33].

Throughout the paper, d will indicate a fixed homogeneous norm on $\mathbb{G}$. Then, our main result can be stated as follows.

Theorem 1.1. *Let* $1 < p < Q$, $f : \mathbb{G} \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function satisfying condition (1.2) and let* $u \in \mathcal{D}^{1,p}(\mathbb{G})$ *be a solution of pb. (1.1). Then, there exists a constant* $C = C(Q, p, \Lambda, u)$ *such that*

$$|u(\xi)| \leq Cd(\xi)^{\frac{p-Q}{p-1}}, \quad \text{for } d(\xi) \text{ large.} \tag{1.3}$$

If, moreover, $u \geq 0$ *in* $\mathbb{G}$ *and* $\int_{\mathbb{G}} f(\xi, u) \, \mathrm{d}\xi > 0$, *then*

$$u(\xi) \geq C_1 d(\xi)^{\frac{p-Q}{p-1}}, \quad \text{for } d(\xi) \text{ large,} \tag{1.4}$$

for some constant $C_1 = C_1(Q, p, \lambda, \Lambda, u)$, *where* λ *is any fixed number such that* $0 < \lambda < \int_{\mathbb{G}} f(\xi, u) \, \mathrm{d}\xi$.

2

The above result extends to the quasilinear case analogous asymptotic estimates obtained for the case $p = 2$ by means of convolution representation techniques by Lanconelli-Uguzzoni [22], [23], Bonfiglioli-Uguzzoni [3] and the author in [25]; see also [27] for a first decay result in the case of sublaplacians with Hardy perturbation.

Our proof for the present quasilinear case relies on the sharp weak Lebesgue regularity of solutions, the scale invariance of the critical equations and the application of Moser-type estimates and Harnack inequality. In particular, a Moser-type estimate on annuli involving the sharp L^q-weak norm of the solutions (see Theorem 4.2 below) is our key ingredient to get the optimal decay result.

Our technique is mainly inspired to Vétois' Euclidean proof, but it differs from it, since it does not require any preliminary partial decay estimate on the solution of the type $|u(\xi)| \le C d(\xi)^{\frac{p-Q}{p}}$, which instead constitutes the most technical part of Vétois proof. In our approach, also taking some ideas from [34], we can avoid this step and simplify the proof, by restricting to suitable Moser-type estimates the functional tools needed to get the optimal estimate from above.

We remark that the functional analytic background on quasilinear subelliptic equations needed in the proof is based on the fundamental regularity results by Capogna-Danielli-Garofalo [5].

Now, we point out that, as a remarkable consequence of Theorem 1.1, we obtain the exact behavior at infinity of the extremal functions for the Sobolev inequality on Carnot groups due to Folland [11], which we here recall: there exists a positive constant $S_p = S_p(\mathbb{G}) > 0$ such that

$$\int_{\mathbb{G}} |Xu|^p \, \mathrm{d}\xi \ge S_p \left(\int_{\mathbb{G}} |u|^{p^*} \, \mathrm{d}\xi \right)^{p/p^*} \qquad \forall \, u \in C_0^\infty(\mathbb{G}). \tag{1.5}$$

We know that the best constant in (1.5) is achieved. Indeed, the existence of Sobolev minimizers was proved by concentration compactness arguments adapted to the Carnot setting by Garofalo and Vassilev [15]. However, the explicit form of the extremal functions is not known, except for the case when $p = 2$ and $\mathbb{G}$ is a group of Iwasawa type (see Jerison-Lee [20], but also Frank-Lieb [13] for the Heisenberg case, Ivanov-Minchev-Vassilev [17] and Christ-Liu-Zhang [7] for the remaining cases).

Now, since any extremal function U for inequality (1.5), up to multiplicative constant, is a nonnegative nontrivial entire solution of the equation

$$-\Delta_{p,\mathbb{G}} U = U^{p^*-1} \qquad \text{in } \mathbb{G},$$

by means of Theorem 1.1 we immediately obtain for U the following sharp decay result.

Theorem 1.2. *Let $1 < p < Q$ and let $U \in \mathcal{D}^{1,p}(\mathbb{G})$ be an extremal function for Sobolev inequality (1.5). Then, the following estimate holds*

$$U(\xi) \sim d(\xi)^{\frac{p-Q}{p-1}}, \quad \text{as } d(\xi) \to \infty.$$

We conclude by observing that the knowledge of the exact asymptotic behavior at infinity of Sobolev extremals turns out to be a crucial ingredient in order to obtain existence results for Brezis-Nirenberg type problems whenever the explicit form of Sobolev minimizers is not known, as shown by the author in [24] for the semilinear Carnot case $p = 2$ (see also [26], [18]; see, furthermore, [28] where different variational techniques not involving the knowledge of minimizers are used to obtain existence results).

The plan of the paper is the following: Section 2 is devoted to introduce the main notations and definitions about Carnot groups; in Section 3 we prove the sharp regularity of solutions in the scale of weak-Lebesgue spaces; finally, in Section 4 we establish their sharp asymptotic decay at infinity.

2 The functional setting

Let us briefly introduce the Carnot groups functional setting. For a complete treatment, we refer the reader to the monograph [2] and the classical papers [11], [12].

A *Carnot group* $(\mathbb{G}, \circ)$ is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, namely a decomposition $\mathfrak{g} = \bigoplus_{j=1}^{r} \mathfrak{G}_j$, such that $[\mathfrak{G}_1, \mathfrak{G}_j] = \mathfrak{G}_{j+1}$ for $1 \leq j < r$, and $[\mathfrak{G}_1, \mathfrak{G}_r] = \{0\}$. The number r is called the *step* of the group $\mathbb{G}$ and the integer $Q = \sum_{i=1}^{r} i \dim(\mathfrak{G}_i)$ is the *homogeneous dimension* of $\mathbb{G}$. We shall assume throughout that $Q \geq 3$. Note that, if $Q \leq 3$, then $\mathbb{G}$ is necessarily the ordinary Euclidean space $\mathbb{G} = (\mathbb{R}^N, +)$.

By means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map (which we shall assume throughout), it is not restrictive to suppose that $\mathbb{G}$ is a homogeneous Lie group on $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \ldots \times \mathbb{R}^{N_r}$, with $N_i = \dim(\mathfrak{G}_i)$, equipped with a family of group automorphisms (called *dilations*) δ_λ of the form

$$\delta_\lambda(\xi) = (\lambda \xi^{(1)}, \lambda^2 \xi^{(2)}, \cdots, \lambda^r \xi^{(r)}),$$

where $\xi^{(j)} \in \mathbb{R}^{N_j}$ for $j = 1, \ldots, r$. Let $m := N_1$ and let $X_1, \ldots, X_m$ be the set of left invariant vector fields of $\mathfrak{G}_1$ that coincide at the origin with the first m partial derivatives. We shall denote by

$$X = (X_1, \ldots, X_m)$$

such system of vector fields, which we shall refer to as the *horizontal* gradient. The differential operator

$$\Delta_{p,\mathbb{G}} := \sum_{i=1}^{m} X_i(|Xu|^{p-2} X_i u),$$

is called the canonical *p-sublaplacian* on $\mathbb{G}$. Note that for any $c > 0$ one has $\Delta_{p,\mathbb{G}}(cu) = c^{p-1} \Delta_{p,\mathbb{G}} u$ and furthermore, since the X_j's are homogeneous of degree one with respect to the dilations δ_λ, the operator $\Delta_{p,\mathbb{G}}$ is homogeneous of degree p with respect to δ_λ, namely

$$\Delta_{p,\mathbb{G}}(u \circ \delta_\lambda) = \lambda^p \Delta_{p,\mathbb{G}} u \circ \delta_\lambda. \tag{2.1}$$

By definition, a *homogeneous norm* on $\mathbb{G}$ is a continuous function $d : \mathbb{G} \to [0, +\infty)$, smooth away from the origin, such that $d(\delta_\lambda(\xi)) = \lambda d(\xi)$, for every $\lambda > 0$ and $\xi \in \mathbb{G}$,

$d(\xi^{-1}) = d(\xi)$ and $d(\xi) = 0$ iff $\xi = 0$. Moreover, if we define $d(\xi, \eta) := d(\eta^{-1} \circ \xi)$, then d is a pseudo-distance on $\mathbb{G}$. We recall that any two homogeneous norms on a Carnot group $\mathbb{G}$ are equivalent, as observed in [2, Prop. 5.1.4].

Throughout the paper, d will indicate a fixed homogeneous norm on $\mathbb{G}$. We shall denote by $B(\xi, r)$ the d-ball with center at ξ and radius r, i.e.

$$B(\xi, r) = \{\eta \in \mathbb{G} \,|\, d(\xi^{-1} \circ \eta) < r\}.$$

For the main regularity results we shall use in our proof, i.e. Moser-type estimates and Harnack-type inequality for quasilinear subelliptic equations, we refer to the seminal papers by Capogna-Danielli-Garofalo [5], [6], where the classical results by Moser [29] and Serrin [30] were generalized to quasilinear operators constructed by means of Hörmander vector fields. We also quote the papers by D'Ambrosio and Mitidieri [8], [9], where further functional tools related to quasilinear degenerate problems, such as Kato's inequality for subelliptic equations, have been obtained. Moreover, we indicate the paper [1] for an overview on the main aspects of nonlinear potential theory on Carnot groups. Concerning the variational formulation of nonlinear subelliptic problems and classical existence and non-existence results, we refer to [14], [15], [22], [23], [31].

3 L^q-weak regularity of solutions

In this section, we determine the sharp regularity of solutions to problem (1.1) in the framework of weak-Lebesgue spaces.

In the ordinary Euclidean space, the result for the case $p = 2$ goes back to Jannelli and Solimini [19], where they stated that any weak solution of a semilinear critical growth problem in $\mathbb{R}^n$, whose model example is given by the equation $-\Delta u = |u|^{2^*-2} u$ in $\mathbb{R}^n$, belongs to the space $L^{2^*/2, \infty}(\mathbb{R}^n)$. A generalization of this result to the polyharmonic Euclidean case can be found in [18]. The result of Jannelli and Solimini was then extended to the case of Stratified Lie groups by the author in [25].

Recently, this type of result has been extended to the quasilinear elliptic case by Vétois in [33, Lemma 2.2] (see also [4]) by a very direct approach; in what follows we adapt the proof in [33] to the present subelliptic setting. A preliminary step is to establish the global boundedness of the solutions, which we state in the following proposition.

Proposition 3.1. *Let $f : \mathbb{G} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying condition (1.2). Then, any solution $u \in \mathcal{D}^{1,p}(\mathbb{G})$ to (1.1) belongs to $L^\infty(\mathbb{G})$.*

Proof. The result can be proved by simply adapting the proof by Vassilev [31, Sect. 4], where the model case $f(\xi, u) = |u|^{p^*-2} u$ is considered. We omit the details. □

Let us, now, recall the definition of weak Lebesgue spaces. For any $s \in (0, \infty)$ and any open set $\Omega \subset \mathbb{G}$, we define the space $L^{s,\infty}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$[u]_{L^{s,\infty}(\Omega)} := \sup_{h>0} h \cdot \mu(\{|u| > h\})^{1/s} < \infty,$$

where $\mu(\{|u| > h\})$ denotes the Lebesgue measure of the set $\{\xi \in \Omega : |u(\xi)| > h\}$. The map $[u]_{L^{s,\infty}(\Omega)}$ is a quasi-norm on $L^{s,\infty}(\Omega)$. For a complete treatment of such spaces we refer to Grafakos [16].

The optimal regularity of solutions in the scale of weak Lebesgue spaces is the following:

Proposition 3.2. *Let $f : \mathbb{G} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth assumption (1.2). Then, any solution $u \in \mathcal{D}^{1,p}(\mathbb{G})$ to problem (1.1) belongs to $L^{q_0,\infty}(\mathbb{G})$, with $q_0 = \dfrac{p^*}{p'}$, p' being the conjugate exponent of p. Hence, by interpolation, any solution u of (1.1) belongs to $L^q(\mathbb{G})$, for any $q \in \left(\dfrac{p^*}{p'}, \infty \right]$.*

Proof. Let u be a nontrivial weak solution of pb. (1.1). In order to estimate the distribution function of u, we consider the test function

$$T_h(u) := sgn(u) \cdot \min(|u|, h), \quad h > 0.$$

By Sobolev inequality (1.5), we get that

$$h^{p^*} \mu(\{|u| > h\}) \leq \int_{\mathbb{G}} |T_h(u)|^{p^*} \, d\xi \leq S \left(\int_{|u| \leq h} |Xu|^p \, d\xi \right)^{\frac{Q}{Q-p}}, \tag{3.1}$$

where $S = S(Q, p)$. On the other hand, by testing equation (1.1) with $T_h(u)$ and using the growth assumption (1.2) on f we get

$$\int_{|u| \leq h} |Xu|^p d\xi = \int_{|u| \leq h} f(\xi, u) \cdot u \, d\xi + h \int_{|u| > h} f(\xi, u) \cdot sgn(u) \, d\xi$$
$$\leq \Lambda \left(\int_{|u| \leq h} |u|^{p^*} \, d\xi + h \int_{|u| > h} |u|^{p^* - 1} \, d\xi \right). \tag{3.2}$$

Let us estimate the terms in the right hand side of (3.2). We have that

$$\int_{|u| \leq h} |u|^{p^*} \, d\xi = \int_{\mathbb{G}} |T_h(u)|^{p^*} \, d\xi - h^{p^*} \mu(\{|u| > h\}) \tag{3.3}$$

and

$$\int_{|u| > h} |u|^{p^* - 1} \, d\xi = (p^* - 1) \int_0^\infty s^{p^* - 2} \mu(\{|u| > \max(s, h)\}) \, ds$$
$$= h^{p^* - 1} \mu(\{|u| > h\}) + (p^* - 1) \int_h^\infty s^{p^* - 2} \mu(\{|u| > s\}) \, ds. \tag{3.4}$$

Hence, from (3.2), (3.3) and (3.4), we get

$$\int_{|u| \leq h} |Xu|^p \, d\xi \leq \Lambda \left(\int_{\mathbb{G}} |T_h(u)|^{p^*} \, d\xi + (p^* - 1)h \int_h^\infty s^{p^* - 2} \mu(\{|u| > s\}) \, ds \right). \tag{3.5}$$

6

Then, by (3.1) and (3.5), and taking into account that $\int_{\mathbb{G}} |T_h(u)|^{p^*} \, d\xi = o(1)$ as $h \to 0$, we get the key estimate

$$h^{p^*} \mu(\{|u| > h\}) \leq C \left(h \int_h^\infty s^{p^*-2} \mu(\{|u| > s\}) \, ds \right)^{\frac{Q}{Q-p}}, \tag{3.6}$$

for small $h > 0$, for some constant $C = C(Q, p, \Lambda)$. Now, in order to estimate the r.h.s. of (3.6) and simplify notation, let us define

$$F(h) := \left(\int_h^\infty f(s) \, ds \right)^{-\frac{p}{Q-p}}, \quad \text{where} \quad f(s) := s^{p^*-2} \mu(\{|u| > s\}). \tag{3.7}$$

Taking into account definition (3.7) and the fact that $p^* - \frac{Q}{Q-p} = \frac{Q(p-1)}{Q-p} = \frac{p^*}{p'}$, the estimate (3.6) can be rewritten as follows

$$h^{p^*/p'} \mu(\{|u| > h\}) \leq C F(h)^{-Q/p}, \text{ for small } h > 0.$$

Now, it is not difficult to verify that F is a non-decreasing function and that $F(0) > 0$. So, we can conclude that

$$h^{p^*/p'} \mu(\{|u| > h\}) \leq C F(0)^{-Q/p}$$

for small h, which implies, together with Proposition 3.1, that $[u]_{L^{p^*/p',\infty}(\mathbb{G})} < \infty$. $\qquad \square$

4 Asymptotic behavior of solutions

This section is devoted to the proof of the main Theorem 1.1, which will provide the optimal decay estimates on the solutions to the critical problem (1.1). As an immediate consequence, we shall obtain the asymptotic behavior of the p-Sobolev minimizers.

The proof of Theorem 1.1 will be divided into two parts: the proof of the estimate from above (1.3) and that of the estimate from below (1.4). Each of the two parts will require some preliminary lemmas.

Let us begin with the upper bound estimate, which will follow as a direct consequence of Theorem 4.2 below. The first step of the proof is the following preliminary Moser-type estimate inspired to Xiang [34, Lemma 2.3].

In what follows, denoted by B_R the d-ball with center at 0 and radius R, we let

$$A_R = B_{5R} \setminus \overline{B}_{2R} \quad \text{and} \quad \widetilde{A}_R = B_{6R} \setminus \overline{B}_R, \quad R > 0. \tag{4.1}$$

The following uniform estimate with respect to R holds.

Lemma 4.1. *Let $V \in L^{Q/p}(\mathbb{G})$ and let $u \in \mathcal{D}^{1,p}(\mathbb{G})$ be a nonnegative solution to*

$$-\Delta_{p,\mathbb{G}} u \leq V u^{p-1} \quad in \ \mathbb{G}. \tag{4.2}$$

Let $t > p^*$. Then, there exists $R_0 > 0$ depending on t such that for any $R \geq R_0$, it holds

$$\left(\fint_{A_R} u^t \right)^{1/t} \leq C \left(\fint_{\widetilde{A}_R} u^{p^*} \right)^{1/p^*}, \tag{4.3}$$

where $\fint_{A_R} u^t = \frac{1}{|A_R|} \int_{A_R} u^t$ and C is a positive constant depending on t, but not on R.

Proof. Adapting the proof in [34, Lemma 2.3], for any $R > 0$ and $\xi \in \mathbb{G}$, we define

$$v(\xi) := u(\delta_R \xi).$$

By (4.2) and (2.1), v satisfies

$$-\Delta_{p,\mathbb{G}} v \le V_R v^{p-1} \quad \text{in } \mathbb{G}, \tag{4.4}$$

where $V_R(\xi) = R^p V(\delta_R \xi)$, for any $\xi \in \mathbb{G}$. We shall prove estimate (4.3) for v on $\widetilde{A}_1$.

Let $v_m = \min(v, m)$, for $m \ge 1$. For any $\eta \in C_0^\infty(\widetilde{A}_1)$, $\eta \ge 0$ and $s \ge 1$, the test function $\varphi = \eta^p v_m^{p(s-1)} v$ into (4.4) gives

$$\int_{\widetilde{A}_1} |Xv|^{p-2} Xv \cdot X\varphi \le \int_{\widetilde{A}_1} V_R v^{p-1} \varphi. \tag{4.5}$$

Concerning the l.h.s. of (4.5), it is easy to see that for any sufficiently small $\delta > 0$, there exists $C_\delta > 0$ such that

$$\int_{\widetilde{A}_1} |Xv|^{p-2} Xv \cdot X\varphi \ge (1-\delta) \frac{p(s-1)+1}{s^p} \int_{\widetilde{A}_1} |X(\eta v_m^{s-1} v)|^p - C_\delta \int_{\widetilde{A}_1} |X\eta|^p v_m^{p(s-1)} v^p. \tag{4.6}$$

So, by choosing $\delta = 1/2$ in (4.6) and using Sobolev inequality (1.5), we obtain

$$\int_{\widetilde{A}_1} |Xv|^{p-2} Xv \cdot X\varphi \ge C_1 \left(\int_{\widetilde{A}_1} |\eta v_m^{s-1} v|^{p\chi} \right)^{1/\chi} - C_2 \int_{\widetilde{A}_1} |X\eta|^p v_m^{p(s-1)} v^p, \tag{4.7}$$

for some constants $C_1, C_2 > 0$ depending on Q, p, s, where $\chi = p^*/p$. On the other hand, by Hölder's inequality

$$\int_{\widetilde{A}_1} V_R v^{p-1} \varphi \le \|V_R\|_{\frac{Q}{p}, \widetilde{A}_1} \left(\int_{\widetilde{A}_1} |\eta v_m^{s-1} v|^{p\chi} \right)^{1/\chi} = \|V\|_{\frac{Q}{p}, \widetilde{A}_R} \left(\int_{\widetilde{A}_1} |\eta v_m^{s-1} v|^{p\chi} \right)^{1/\chi}. \tag{4.8}$$

So, by (4.5), (4.7) and (4.8), we get

$$\left(\int_{\widetilde{A}_1} |\eta v_m^{s-1} v|^{p\chi} \right)^{1/\chi} \le C_3 \int_{\widetilde{A}_1} |X\eta|^p v_m^{p(s-1)} v^p + C_3 \|V\|_{\frac{Q}{p}, \widetilde{A}_R} \left(\int_{\widetilde{A}_1} |\eta v_m^{s-1} v|^{p\chi} \right)^{1/\chi} \tag{4.9}$$

for some constant $C_3 = C_3(Q, p, s) > 0$.

Now, fix $t > p^*$ and let $k \in \mathbb{N}$ such that $p\chi^k \le t \le p\chi^{k+1}$. Then, there exists a positive constant $C_3 = C_3(Q, p, t)$ such that (4.9) holds for all $1 \le s \le \chi^k$.

Since $V \in L^{Q/p}(\mathbb{G})$, there exists $R_0 > 0$ such that

$$C_3 \|V\|_{\frac{Q}{p}, \widetilde{A}_R} \le 1/2 \quad \text{for any } R \ge R_0. \tag{4.10}$$

Therefore, for all $R \ge R_0$, it holds

$$\left(\int_{\widetilde{A}_1} |\eta v_m^{s-1} v|^{p\chi} \right)^{1/\chi} \le C \int_{\widetilde{A}_1} |X\eta|^p v_m^{p(s-1)} v^p,$$

for all $1 \leq s \leq \chi^k$, where $C > 0$ depends only on Q, p, t.

Now, by choosing an appropriate cut-off function η and applying Moser's iteration technique, after finitely many iterations we can conclude that

$$\left(\fint_{A_1} v^t \right)^{1/t} \leq C \left(\fint_{\widetilde{A}_1} v^{p^*} \right)^{1/p^*} \tag{4.11}$$

for $R \geq R_0$, where we recall that $v(\xi) = u(\delta_R \xi)$ and C does not depend on R. By a simple change of variable, (4.3) follows from (4.11). $\qquad\square$

We are now able to state our main Theorem, which gives an estimate of the L^∞-norm of the solutions to (1.1) on annuli by means of the sharp L^{q_0}-weak norm on larger annuli, thus providing the sharp decay of solutions at ∞.

In what follows, we shall indicate by

$$D_R = B_{4R} \setminus \overline{B}_{3R}, \quad R > 0, \tag{4.12}$$

and A_R will denote, as before, the larger annulus $B_{5R} \setminus \overline{B}_{2R}$.

Theorem 4.2. *Let $u \in \mathcal{D}^{1,p}(\mathbb{G})$ be a solution to (1.1) under the assumption (1.2). Let $q_0 = \frac{p^*}{p'}$ be the sharp L^q-weak summability exponent found in Prop. 3.2. Then, there exist constants $R_0, C > 0$, such that for any $R \geq R_0$*

$$\sup_{D_R} |u| \leq \frac{C}{|A_R|^{\frac{1}{q_0}}} [u]_{L^{q_0,\infty}(A_R)} \tag{4.13}$$

where C does not depend on R.

Proof. Firstly, notice that, if $u \in \mathcal{D}^{1,p}(\mathbb{G})$ is a solution to (1.1) under the assumption (1.2), then by Kato's inequality [21] adapted to the Stratified context (see [9]), $|u|$ satisfies

$$-\Delta_{p,\mathbb{G}}|u| \leq |f(\xi, u)| \leq \Lambda |u|^{p^*-1} \quad \text{in } \mathbb{G},$$

from which

$$-\Delta_{p,\mathbb{G}}|u| \leq V|u|^{p-1}, \qquad \text{where } V := \Lambda |u|^{p^*-p}. \tag{4.14}$$

Obviously, $V \in L^q(\mathbb{G})$ for any $q \geq Q/p$, being $u \in L^t(\mathbb{G})$ for any $t \geq p^*$.

Let us set, as before,

$$v(\xi) := |u(\delta_R \xi)|, \quad R > 0, \xi \in \mathbb{G}.$$

Then, in particular, v weakly satisfies the inequality

$$-\Delta_{p,\mathbb{G}} v \leq V_R v^{p-1} \quad \text{in } A_1, \tag{4.15}$$

where $V_R(\xi) = R^p V(\delta_R \xi)$, with V as in (4.14).

Let $t > p^*$ be fixed. Thus, in particular, $V_R \in L^{t_0}(\mathbb{G})$, for $t_0 = \frac{t}{p^*-p} > \frac{Q}{p}$. So, by the subelliptic Moser-type estimates by Capogna et al. in [5] (see [5, Theorem 3.4 and Lemma 3.29]) applied to (4.15), we get that, for any $q > 0$ the following estimate holds

$$\sup_{B} v \leq C \left(\fint_{2B} v^q \right)^{1/q}, \tag{4.16}$$

for any ball $B = B(\xi, r)$ such that $2B = B(\xi, 2r) \subset A_1$, where $C = C(Q, q, \|V_R\|_{L^{t_0}(A_1)})$.

Reasoning as in [34, proof of Proposition 2.1], the crucial observation, here, is that the norm $\|V_R\|_{L^{t_0}(A_1)}$ is uniformly bounded with respect to R, for sufficiently large R. Precisely, if we choose $R_0 > 0$ so that (4.10) holds for $V = \Lambda|u|^{p^*-p}$, there exists a constant $C > 0$ depending on Q, p, t_0, Λ such that

$$\|V_R\|_{L^{t_0}(A_1)} \leq C\|u\|_{L^{p^*}(\mathbb{G})}^{p^*-p} \quad \forall R \geq R_0. \tag{4.17}$$

Indeed, by the definition of V_R and by Lemma 4.1 applied to (4.14), we get that, for any $R \geq R_0$

$$
\begin{aligned}
\|V_R\|_{L^{t_0}(A_1)} &= R^{p-\frac{Q}{t_0}} \|V\|_{L^{t_0}(A_R)} \\
&= \Lambda R^{p-\frac{Q}{t_0}} \|u\|_{L^t(A_R)}^{p^*-p} \\
&\leq C R^{p-\frac{Q}{t_0}-(\frac{Q}{p^*}-\frac{Q}{t})(p^*-p)} \|u\|_{L^{p^*}(\widetilde{A}_R)}^{p^*-p} \\
&\leq C \|u\|_{L^{p^*}(\mathbb{G})}^{p^*-p},
\end{aligned}
$$

with $C > 0$ not depending on R, where we have used that $p - \frac{Q}{t_0} - (\frac{Q}{p^*} - \frac{Q}{t})(p^*-p) = 0$. Therefore, the constant C in (4.16) does not depend on R, for $R \geq R_0$.

Finally, by a covering argument on the inner annulus $D_1 \subset\subset A_1$, we deduce from (4.16) that

$$\sup_{D_1} v \leq C \left(\fint_{A_1} v^q \right)^{1/q},$$

that is, by rescaling

$$\sup_{D_R} |u| \leq C \left(\fint_{A_R} |u|^q \right)^{1/q}, \tag{4.18}$$

for $R \geq R_0$, where C depends on q, but not on R.

Now, let us choose q in (4.18) so that $0 < q < q_0 = p^*/p'$. By Hölder's inequality for weak Lebesgue norms (see Grafakos [16], Ex. 1.1.11) we have

$$\left(\fint_{A_R} |u|^q \right)^{1/q} \leq C_{q,q_0} |A_R|^{1/q-1/q_0} [u]_{L^{q_0,\infty}(A_R)}. \tag{4.19}$$

Henceforth, by (4.18) and (4.19), estimate (4.13) follows. $\qquad\square$

We are now able to prove the estimate from above (1.3) of Theorem 1.1.

Proof of Theorem 1.1 - estimate (1.3). From Theorem 4.2, by taking into account that $|A_R| \sim R^Q$ and that, by Proposition 3.2, $u \in L^{q_0,\infty}(\mathbb{G})$, the asymptotic estimate (1.3) follows by letting $R = \frac{2}{7}d(\xi)$ in (4.13), for $d(\xi) \geq \frac{7}{2}R_0$. $\qquad\square$

To complete the proof of Theorem 1.1 with the estimate from below (1.4), we follow the outline in [33]. We shall need the following Lemmas, which will be proved by means of the regularity results in [5].

In what follows, we shall use the same notation for annuli as before (see (4.1) and (4.12)).

Lemma 4.3. *Let $f : \mathbb{G} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that (1.2) holds and let u be a nonnegative solution of (1.1). Then, there exists a constant $C > 0$, not depending on R, such that*

$$\sup_{A_R} u \leq C \inf_{A_R} u \qquad (4.20)$$

for sufficiently large R.

Proof. Reasoning as in the previous proofs, we prove estimate (4.20) for the linear transformation $v(\xi) = u(\delta_R \xi)$ on the annulus A_1. Consider the equation (4.15) for v in the larger annulus $\widetilde{A}_1$. Since $V_R \in L^{t_0}(\widetilde{A}_1)$, for any fixed $t_0 > Q/p$, by the subelliptic Harnack inequality in [5, Theorem 3.1], there exists a constant $C_1 = C_1(Q, p, \|V_R\|_{L^{t_0}(\widetilde{A}_1)}) > 0$ such that

$$\sup_{B(\eta,1/4)} v \leq C_1 \inf_{B(\eta,1/4)} v, \qquad (4.21)$$

for all points η in the annulus A_1. Reasoning as in the proof of Theorem 4.2, we can recognize that the constant C_1 in (4.21) can be made independent of R, for sufficiently large R. Moreover, since every two points in A_1 can be jointed by a finite number of connected balls of radius $1/4$ and center in A_1, by a covering argument we get from (4.21) that

$$\sup_{A_1} v \leq C \inf_{A_1} v,$$

where C does not depend on R, for R sufficiently large. Thus, estimate (4.20) holds. $\qquad\square$

Lemma 4.4. *Let $f : \mathbb{G} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that (1.2) holds and let u be a nonnegative solution of (1.1). Then, there exists a constant $C > 0$, not depending on R, such that*

$$\|Xu\|_{L^p(D_R)} \leq CR^{-1}\|u\|_{L^p(A_R)} \qquad (4.22)$$

for sufficiently large R.

Proof. First of all, let us notice that an estimate of the type (4.22) can be found in Serrin [30, Theorem 1]. Reasoning as in the preceding lemmas, we prove that, for sufficiently large R, the following estimate holds

$$\|Xv\|_{L^p(D_1)} \leq C\|v\|_{L^p(A_1)} \qquad (4.23)$$

11

for the linear transformation $v(\xi) = u(\delta_R \xi)$, where the constant C does not depend on R, that is equivalent to (4.22).

We give a sketch of the proof. Let us test the inequality $-\Delta_{p,\mathbb{G}} v \leq V_R v^{p-1}$ with $\varphi = \eta^p v$, where $\eta \in C_0^\infty(A_1)$ is a suitable nonnegative cut-off function to be specified later. We get

$$\int_{A_1} |Xv|^{p-2} Xv \cdot X(\eta^p v) \, \mathrm{d}\xi \leq \int_{A_1} V_R(\eta v)^p \, \mathrm{d}\xi,$$

which leads to

$$\int_{A_1} \eta^p |Xv|^p \, \mathrm{d}\xi \leq p \int_{A_1} |vX\eta| |\eta Xv|^{p-1} \, \mathrm{d}\xi + \int_{A_1} V_R(\eta v)^p \, \mathrm{d}\xi. \tag{4.24}$$

By estimating the terms in the r.h.s. of (4.24) as in [5, Theorem 3.4], after some standard computations we obtain

$$\|\eta Xv\|_{L^p(A_1)} \leq C \left(\|\eta v\|_{L^p(A_1)} + \|vX\eta\|_{L^p(A_1)} \right) \tag{4.25}$$

(see [5], formula (3.12)) where C only depends on Q, p and $\|V_R\|_{L^{t_0}(A_1)}$, which is uniformly bounded with respect to R for sufficiently large R, due to (4.17).

So, by choosing $\eta \in C_0^\infty(A_1)$ so that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on the inner annulus $D_1 \subset\subset A_1$, from (4.25) we easily deduce (4.23). $\qquad\square$

Proof of Theorem 1.1 - estimate (1.4). Let u be a nonnegative solution of (1.1) such that $\int_{\mathbb{G}} f(\xi, u) \, \mathrm{d}\xi > 0$. Then, by taking into account (1.2), it follows that $u \not\equiv 0$. By virtue of Lemma 4.3, in order to prove estimate (1.4), it is sufficient to prove a lower bound for $\|u\|_{L^\infty(A_R)}$ in terms of $R^{\frac{p-Q}{p-1}}$, for large R. First of all, by Lemma 4.4 and Hölder's inequality, we get that

$$\|Xu\|_{L^p(D_R)} \leq CR^{\frac{Q-p}{p}} \|u\|_{L^\infty(A_R)}. \tag{4.26}$$

On the other hand, if $\int_{\mathbb{G}} f(\xi, u) \, \mathrm{d}\xi > \lambda$ for some $\lambda > 0$, we claim that

$$C'R^{\frac{p-Q}{p(p-1)}} \leq \|Xu\|_{L^p(D_R)} \tag{4.27}$$

for large R, for some constant C' depending on Q, p, λ. Indeed, if we test equation (1.1) with a cut-off function $\eta_R(\xi) = \eta(d(\xi)/R)$, where $\eta \in C^\infty(0, \infty)$ is such that $\eta \equiv 1$ on $[0, 3]$, $\eta \equiv 0$ on $[4, \infty)$, $0 \leq \eta \leq 1$ on $(3, 4)$, we have that

$$\begin{aligned}
\int_{\mathbb{G}} f(\xi, u) \, \eta_R \mathrm{d}\xi &= \int_{\mathbb{G}} |Xu|^{p-2} Xu \cdot X\eta_R \, \mathrm{d}\xi \\
&\leq \|Xu\|_{L^p(supp(X\eta_R))}^{p-1} \cdot \|X\eta_R\|_{L^p(supp(X\eta_R))} \\
&\leq CR^{\frac{Q-p}{p}} \|Xu\|_{L^p(D_R)}^{p-1},
\end{aligned} \tag{4.28}$$

where we have used that

$$\begin{aligned}
\int_{D_R} |X\eta_R|^p \, \mathrm{d}\xi &= \frac{1}{R^p} \int_{D_R} |X(\eta \circ d)(\delta_{\frac{1}{R}} \xi)|^p \, \mathrm{d}\xi = \frac{1}{R^p} \int_{D_R} |\eta'(d(\delta_{\frac{1}{R}} \xi))|^p |Xd(\delta_{\frac{1}{R}} \xi)|^p \, \mathrm{d}\xi \\
&\leq \frac{C}{R^p} |D_R| = CR^{Q-p}.
\end{aligned}$$

Hence, by choosing R sufficiently large so that $\int_{\mathbb{G}} f(\xi, u) \, \eta_R \mathrm{d}\xi > \lambda$, estimate (4.27) follows by (4.28) with $C' = (\lambda/C)^{\frac{1}{p-1}}$. Finally, by (4.20), (4.26) and (4.27), we get

$$CR^{\frac{p-Q}{p-1}} \leq \inf_{A_R} u \qquad \text{for large } R,$$

from which estimate (1.4) follows. □

References

[1] Z. BALOGH, J. T. TYSON, *Potential theory on Carnot groups*, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 1527, Contemp. Math., 320, Amer. Math. Soc., Providence, RI, 2003.

[2] A. BONFIGLIOLI, E. LANCONELLI, F. UGUZZONI, Stratified Lie groups and potential theory for their Sub-Laplacians, Springer Monographs in Mathematics. Springer, Berlin, 2007.

[3] A. BONFIGLIOLI, F. UGUZZONI, *Nonlinear Liouville theorems for some critical problems on H-type groups*, J. Funct. Anal. 207 (2004), 161-215.

[4] L. BRASCO, S. MOSCONI, M. SQUASSINA, *Optimal decay of extremals for the fractional Sobolev inequality*, Calc. Var. Partial Differential Equations 55(2015), 55-23.

[5] L. CAPOGNA, D. DANIELLI, N. GAROFALO, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. Partial Differential Equations 18 (1993), no. 9-10, 1765-1794.

[6] L. CAPOGNA, D. DANIELLI, N. GAROFALO, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations*, Amer. J. Math. 118 (1996), no. 6, 1153-1196.

[7] M. CHRIST, H. LIU, A. ZHANG, *Sharp Hardy-Littlewood-Sobolev inequalities on the octonionic Heisenberg group*, Calc. Var. Partial Differential Equations 55 (2016), no. 1, Art. 11, 18 pp.

[8] L. D'AMBROSIO, E. MITIDIERI, *A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities*, Adv. Math. 224 (2010), no. 3, 967-1020.

[9] L. D'AMBROSIO, E. MITIDIERI, *A priori estimates and reduction principles for quasilinear elliptic problems and applications*, Advances in Differential Equations, 17 (2012), 935-1000.

[10] H. EGNELL, *Asymptotic results for finite energy solutions of semilinear elliptic equations*, J. Differential Equations 98 (1992), no. 1, 34-56.

[11] G.B. FOLLAND, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. 13 (1975), 161-207.

[12] G.B. FOLLAND, E. STEIN, Hardy spaces on homogeneous groups, Mathematical Notes, 28, University Press, Princeton, N.J., 1982.

[13] R.L. FRANK, E.H. LIEB, *Sharp constants in several inequalities on the Heisenberg group*, Ann. of Math. (2) 176, no. 1 (2012), 349-381.

[14] N. GAROFALO, E. LANCONELLI, *Existence and nonexistence results for semilinear equations on the Heisenberg group*, Indiana Univ. Math. J. 41(1992), 71-98.

[15] N. GAROFALO, D. VASSILEV, *Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot Groups*, Math. Ann. 318 (2000), 453-516.

[16] L. GRAFAKOS, Classical Fourier Analysis. 2nd Edition, Graduate Text in Mathematics 249, Springer, New York, 2008.

[17] S. IVANOV, I. MINCHEV, D. VASSILEV, *The optimal constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 11 (2012), no. 3, 635-652.

[18] E. JANNELLI, A. LOIUDICE, *Critical polyharmonic problems with singular nonlinearities*, Nonlinear Anal. 110 (2014), 77-96.

[19] E. JANNELLI, S. SOLIMINI, *Concentration estimates for critical problems*, Ricerche Mat. 48 (1999), Suppl., 233-257.

[20] D. JERISON, J. LEE, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. 1 (1988), 1-13.

[21] T. KATO, *Schrödinger operators with singular potentials*, Israel J. Math. 13 (1972), 135-148.

[22] E. LANCONELLI, F. UGUZZONI, *Asymptotic behavior and non existence theorems for semilinear Dirichlet problems involving critical exponent on unbounded domains of the Heisenberg group*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), 139-168.

[23] E. LANCONELLI, F. UGUZZONI, *Non-existence results for semilinear Kohn-Laplace equations in unbounded domains*, Comm. Partial Differential Equations 25 (2000), 1703-1739.

[24] A. LOIUDICE, *Semilinear subelliptic problems with critical growth on Carnot groups*, Manuscripta Math. 124 (2007), 247-259.

[25] A. LOIUDICE, L^p-weak regularity and asymptotic behavior of solutions for critical equations with singular potentials on Carnot groups, NoDEA Nonlinear Differential Equations and Applications 17 (2010), 575-589.

[26] A. LOIUDICE, Critical growth problems with singular nonlinearities on Carnot groups, Nonlinear Anal. 126 (2015), 415-436.

[27] A. LOIUDICE, Local behavior of solutions to subelliptic problems with Hardy potential on Carnot groups, Mediterr. J. Math. 15 (2018), no. 3, Art. 81, 20 pp.

[28] G. MOLICA BISCI, D. REPOVŠ, Yamabe-type equations on Carnot groups, Potential Anal. 46 (2017), no. 2, 369-383.

[29] J. MOSER, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961), 577-591.

[30] J. SERRIN, Local behavior of solutions of quasi-linear equations, Acta Math. 111 (1964), n.1, 247-302.

[31] D. VASSILEV, Existence of solutions and regularity near the characteristic boundary for sub-laplacian equations on Carnot Groups, Pacific J. Math. 227(2) (2006), 361-397.

[32] D. VASSILEV, L^p-estimates and asymptotic behavior for finite energy solutions of extremals to Hardy-Sobolev inequalities, Trans. Amer. Math. Soc. 363 (2011), no. 1, 37-62.

[33] J. VÉTOIS, A priori estimates and application to the symmetry of solutions for critical p-Laplace equations, J. Differential Equations 260 (2016), no. 1, 149-161.

[34] C.L. XIANG, Asymptotic behaviors of solutions to quasilinear elliptic equations with critical Sobolev growth and Hardy potential, J. Differential Equations 259 (2015), 3929-3954.