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MINIMAL $*$-VARIETIES AND SUPERINVOLUTIONS

BY

Onofrio M. Di Vincenzo

Dipartimento di Matematica, Informatica ed Economia, Università della Basilicata
viale dell'Ateneo Lucano, 85100 Potenza, Italia
e-mail: onofrio.divincenzo@unibas.it

AND

Vincenzo C. Nardozza*

Dipartimento di Matematica, Università degli Studi di Bari
via Orabona 4, 70125 Bari, Italia
e-mail: vincenzo.nardozza@uniba.it

ABSTRACT

We introduce a class of finite-dimensional superalgebras over an algebraically closed field of characteristic zero, whose Grassmann envelopes generate all $*$-minimal varieties. Moreover, we prove that any affine minimal variety of superalgebras with superinvolution is generated by a suitable element in this selected class.

1. Introduction

The present paper is devoted to the description of the $*$-minimal varieties of associative algebras with involution over a field of characteristic zero. More specifically, we present a class of finite-dimensional algebras giving rise to all such varieties in a precise sense: any $*$-minimal variety is generated by the Grassmann envelope of a suitable element in the class. These algebras have a neat and essentially easy structure, with some kind of canonical features: they

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* Corresponding author.

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are block-triangular graded subalgebras of a full matrix algebra of even size, endowed with an elementary $\mathbb{Z}_2$-grading and a fixed type of superinvolution.

The problem of classifying the $*$-minimal varieties has been the subject of intensive investigations lasting over two decades, and takes its steps in the kin problem of classifying the varieties of ordinary (without involution) algebras on the scale provided by their PI-exponent. By the way, while this task has been completely accomplished soon after the 1998–99 results [Gi&Za1, Gi&Za2] on the existence of the exponent, precisely in the 2003 paper [Gi&Za4], the characterization of the $*$-minimal varieties took a much longer and winding route.

We briefly recall that a $*$-variety is a class $\mathscr{V}$ of algebras with involution satisfying a given set of $*$-polynomial identities. The size of $\mathscr{V}$ can be in some sense measured by a non-negative integer $\exp^*(\mathscr{V})$, called the $*$-PI-exponent of $\mathscr{V}$: if $\mathscr{U},\mathscr{V}$ are $*$-varieties, then $\mathscr{U}\subseteq\mathscr{V}$ implies $\exp^*(\mathscr{U})\leqslant\exp^*(\mathscr{V})$. A $*$-variety $\mathscr{V}$ is $*$-minimal if the $*$-exponent of any proper $*$-subvariety is strictly less than $\exp^*(\mathscr{V})$. From this point of view, $*$-minimal varieties are edge varieties with respect to the exponent, in a perfect analogy with the ordinary case.

The first difficulty to be confronted with is about the existence of the $*$-exponent: this has been obtained for finite-dimensional algebras in the 1999 as well [Gi&Za3], but it took almost fifteen years to prove the existence for finitely generated algebras [Sv]. Essentially, the troubles in passing to the general case ($*$-algebras without any finiteness condition) were due to the lack of a bridge between this general situation and a finite-dimensional-controlled one, in the spirit of the renowned Kemer's theorem stating that any (ordinary) PI-algebra is PI-equivalent to the Grassmann envelope of a finite-dimensional superalgebra. This issue has been overcome in 2017 [Al&Gi&Ka], so in the same year the most definitive statement about the existence of the $*$-exponent was finally proved [Gi&PM&Va].

As a consequence of the previous picture, the description of the $*$-minimal varieties developed in a parallel way: at first the $*$-varieties of polynomial growth and of almost polynomial growth have been characterized, by presenting the only two $*$-minimal varieties with $*$-exponent 2 [Gi&Mi, Mi&Va]. Then, the spotlight moved to $*$-minimal varieties generated by finite-dimensional $*$-algebras [DV&LS, DV&Sp]. A few years later, Sviridova proved that actually any affine $*$-variety can be generated by a finite-dimensional $*$-algebra [Sv].

Hence the results in the former papers, exhibiting certain finite-dimensional $*$-algebras $UT^*(A_1, \dots, A_m)$ built on finite-dimensional $*$-simple algebras $A_1, \dots, A_m$, actually hold in the affine case as well, and provide a precise description of affine $*$-minimal varieties: any affine $*$-variety $\mathscr{V}$ is $*$-minimal if and only if it is generated by a suitable $UT^*(A_1, \dots, A_m)$. This description also underlines a second difficulty in dealing with $*$-algebras instead of ordinary ones: while for ordinary varieties the T-ideal of a minimal variety is a product of T-ideals, if $\mathscr{V}$ is generated by $UT^*(A_1, \dots, A_m)$ it holds that

$$\mathrm{Id}^*(\mathscr{V}) = \mathrm{Id}^*(A_1) \cdots \mathrm{Id}^*(A_m) \cap \mathrm{Id}^*(A_m) \cdots \mathrm{Id}^*(A_1),$$

and in general the two products are different T^*-ideals (the easiest and only completely described instance was studied in [DV&N]).

The bridge across arbitrary and finite-dimensional algebras discovered in [Al&Gi&Ka], and the related result on the existence of the $*$-exponent for any $*$-variety in [Gi&PM&Va] allow us to provide a description of the general $*$-minimal varieties, without restricting to the affine ones, by means of a particular class of finite-dimensional algebras and their Grassmann envelopes, in some sense extending the previous description given in [DV&LS, DV&Sp]. By the way, a common and decisive ingredient in these recent results is the intervention of superalgebras with superinvolution.

Superalgebras with superinvolutions arise naturally, and play a fundamental role, in the study of Lie and Jordan superalgebras [Ka], [Ra&Ze]: the skew-symmetric elements of a superalgebra with a superinvolution constitute a Lie superalgebra with respect to the graded bracket

$$[a, b] = ab - (-1)^{|a||b|} ba,$$

while the symmetric ones form a Jordan superalgebra with respect to the signed Jordan product

$$a \circ b = ab + (-1)^{|a||b|} ba.$$

Actually, as anticipated it turns out that they play a significant role also in the theory of algebras with involutions satisfying polynomial identities: any $*$-variety can be generated by the Grassmann envelope of a finite-dimensional superalgebra with superinvolution [Al&Gi&Ka]; that is, in the framework of PI-algebras with involution, superalgebras with superinvolution play the role superalgebras play in the framework of PI-algebras.

When dealing with superalgebras with superinvolution, called $\overline{*}$-algebras for short throughout the present paper, the similar notion of $\overline{*}$-exponent of a variety can be considered, and actually it has been proved that, at least for finitely generated $\overline{*}$-algebras, it does exist and is a non-negative integer, as well [Io]. We will also describe the $\overline{*}$-minimal varieties by means of the same algebras involved in the general situation.

The paper is sectioned in the following way: Section 2 is devoted to notations and the basic facts about $\overline{*}$-algebras and their $\overline{*}$-polynomial identities; Section 3 deals with the classification of the finite-dimensional $\overline{*}$-simple algebras, and brings up a new presentation for them, bypassing the notational difficulties of the classical one. Their embedding into a full matrix algebra of even size, turned into a $\overline{*}$-algebra by an elementary grading and a superinvolution based on the reflection along the secondary diagonal is the main idea the paper is rooted on. The $\overline{*}$-algebras $UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)$, built on a sequence of finite-dimensional $\overline{*}$-simple algebras $(A_1, \dots, A_m)$ and a translation vector $\mathbf{g} \in \mathbb{Z}_2^m$, constitute the class of $\overline{*}$-algebras at the heart of the classification of $*$-minimal varieties, and are presented here. Section 4 develops the notion of affine $\overline{*}$-minimal varieties, and provides their description by means of the algebras $UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)$, in Theorem 4.6. This result leads immediately to the main result of this section, Theorem 4.7, stating that any $*$-minimal variety can be generated by the Grassmann envelope of a suitable $\overline{*}$-algebra $UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)$. Finally, Section 5 describes the T^*-ideal of the $*$-varieties obtained in this way (Theorem 5.8).

2. (Super)involutions and polynomial identities

Throughout the paper, let F denote an algebraically closed field of characteristic zero, and let the word algebra mean an associative F-algebra. An algebra A is a superalgebra, or a $\mathbb{Z}_2$-graded algebra, if it comes with a vector space decomposition $A = A_0 \oplus A_1$ satisfying $A_i A_j \subseteq A_{i+j}$ $(i, j \in \mathbb{Z}_2)$. If $a \in A_i$ we write $|a| = i$, and say that a is homogeneous of degree i. Given superalgebras A, B, the algebra homomorphisms preserving the gradings are called graded homomorphisms. Any algebra can be endowed with the trivial grading, where $A_0 = A$ and $A_1 = 0$, so in some sense the notion of superalgebra embraces the notion of algebra. The free superalgebra is denoted by $F\langle Y, Z\rangle$, and is the free algebra generated by the union of two disjoint sets Y, Z of homogeneous indeterminates, with assigned degrees $|y| = 0$ and $|z| = 1$; its elements are called graded polynomials. The superalgebra $F\langle Y, Z\rangle$ fulfills the following universal property: for any superalgebra A, any set-theoretic map $\varphi : Y \cup Z \to A$ such that $\varphi(Y) \subseteq A_0$ and $\varphi(Z) \subseteq A_1$ can be uniquely extended to a graded homomorphism $F\langle Y, Z\rangle \to A$. This leads to the notion of graded polynomial identities: an element $f \in F\langle Y, Z\rangle$ is a graded polynomial identity of A if $\varphi(f) = 0$ for all graded homomorphisms $\varphi : F\langle Y, Z\rangle \to A$. The set of all such polynomials is a graded ideal of the free superalgebra, stable under all graded endomorphisms, denoted $\mathrm{Id}_{\mathbb{Z}_2}(A)$.

When A is a superalgebra, a graded involutory element $\circ$ in $GL_F(A)$ is called a **superinvolution** if for all homogeneous elements $a, b \in A$ we have

$$(ab)^\circ = (-1)^{|a||b|} b^\circ a^\circ,$$

and a **graded involution** if $(ab)^\circ = b^\circ a^\circ$. We will usually denote $\overline{*}$ a superinvolution, and $*$ a graded involution; we will talk of **$\overline{*}$-algebra** or **$*$-superalgebra**, accordingly.

A couple of examples, which are also of relevance to our aims, are the following:

Example 2.1: If A is any superalgebra, its **super-opposite** A^{sop} consists of the same graded vector space $A = A_0 \oplus A_1$ endowed with the super-opposite product $\overline{\odot}$, defined on homogeneous elements by

$$a_i \overline{\odot} b_j = (-1)^{ij} b_j a_i,$$

for $a_i \in A_i$ and $b_j \in A_j$. The direct sum $A \oplus A^{\mathrm{sop}}$ is then a superalgebra, with i-th homogeneous component $A_i \oplus A_i$, and the exchange map $(a, b) \overset{\mathrm{ex}}{\to} (b, a)$ is a superinvolution for $A \oplus A^{\mathrm{sop}}$. We will usually not mention explicitly ex, and just talk of the $\overline{*}$-algebra $A \oplus A^{\mathrm{sop}}$.

Example 2.2: Let E be the Grassmann algebra of an infinite-dimensional vector space, endowed with the usual $\mathbb{Z}_2$-grading. Both $(a_0 + a_1)^{\overline{*}} = a_0 + a_1$ and $(a_0 + a_1)^{\overline{*}} = a_0 - a_1$ turn E into a $\overline{*}$-algebra. This last superinvolution has been denoted $\sharp$ in [Al&Gi&Ka]. We will reserve the symbol $\sharp$ for this superinvolution on E throughout the paper.

Remark 2.3: As a general fact, if A, B are $\overline{*}$-algebras with superinvolutions α, β respectively, their graded tensor product

$$A \hat{\otimes} B = A_0 \otimes B_0 + A_1 \otimes B_1$$

is a $*$-superalgebra with respect to the graded involution $\alpha \otimes \beta$, so in particular is an algebra with involution, as noted in [DV&Ko]. This applies in particular to the Grassmann envelope $\mathscr{G}(A) = E\hat{\otimes}A$ of a superalgebra A with superinvolution $\overline{*}$, turning $\mathscr{G}(A)$ into a (graded) $*$-algebra with involution $\star = \sharp \otimes \overline{*}$.

This easy observation actually marks a key point in bridging algebras with involution and $\overline{*}$-algebras: among its consequences, any PI-algebra with involution can be realized as the Grassmann envelope of a finite-dimensional $\overline{*}$-algebra ([Al&Gi&Ka, Thm. 4]), its T^*-ideal is finitely generated as a T^*-ideal (Thm. 5 of the same paper), and guarantees the existence of its $*$-exponent ([Gi&PM&Va, Thm. 4]).

Thus, in order to describe $*$-varieties, one needs to delve deeper into the study of $\overline{*}$-algebras.

The free superalgebra $F\langle Y, Z\rangle$ can be turned into a $\overline{*}$-algebra in a natural way, which we will denote $F\langle Y, Z|\overline{*}\rangle$; sometimes, setting $X := Y \cup Z$, we will denote it simply by $F\langle X|\overline{*}\rangle$. It has the property that any graded assignment of the indeterminates y_i, z_i to homogeneous elements of a $\overline{*}$-algebra A uniquely defines a $\overline{*}$-homomorphism $\varphi : F\langle Y, Z \mid \overline{*}\rangle \to A$. As expected, its elements are called $\overline{*}$-polynomials, and those among them lying in the kernel of all $\overline{*}$-homomorphisms from $F\langle Y, Z|\overline{*}\rangle$ to A are called $\overline{*}$-polynomial identities of A. They form a $\mathbb{Z}_2$-graded ideal of $F\langle Y, Z|\overline{*}\rangle$, $\overline{*}$-invariant as well as invariant under all $\overline{*}$-endomorphisms of the free $\overline{*}$-algebra (in short, a $T^{\overline{*}}$-ideal), denoted $\mathrm{Id}^{\overline{*}}(A)$.

A peculiar role in the formation of $\mathrm{Id}^{\overline{*}}(A)$ is played by the $\overline{*}$-multilinear polynomial identities, that is the $\overline{*}$-polynomial identities belonging to the vector space $P_n(\overline{*})$ spanned by the $2^{2n}n!$ monomials $x^{\delta_1}_{\pi(1)} \cdots x^{\delta_n}_{\pi(n)}$, for $1 \leqslant n \in \mathbb{N}$, $x_i \in \{y_i, z_i\}$, $\delta_i \in \{1, \overline{*}\}$ and $\pi \in S_n$. In fact, standard Vandermonde arguments and linearization process prove that the whole ideal $\mathrm{Id}^{\overline{*}}(A)$ is generated by the multilinear polynomials it contains, as in the ordinary case.

If A is a $\overline{*}$-algebra, the class of $\overline{*}$-algebras B such that $\mathrm{Id}^{\overline{*}}(A) \subseteq \mathrm{Id}^{\overline{*}}(B)$, called the **$\overline{*}$-variety generated by** A, will be denoted by $\mathrm{Var}^{\overline{*}}(A)$. Also, given a set $\mathscr{S}$ of $\overline{*}$-polynomials, the class $\mathscr{V}$ of $\overline{*}$-algebras A such that $\mathscr{S} \subseteq \mathrm{Id}^{\overline{*}}(A)$ is a $\overline{*}$-variety (defined by $\mathscr{S}$), and $\mathrm{Id}^{\overline{*}}(\mathscr{V})$ is by definition the intersection of the $T^{\overline{*}}$-ideals $\mathrm{Id}^{\overline{*}}(A)$'s as A ranges in $\mathscr{V}$. If for some $A \in \mathscr{V}$ it holds that $\mathrm{Id}^{\overline{*}}(A) = \mathrm{Id}^{\overline{*}}(\mathscr{V})$, we say that $\mathscr{V}$ is generated by A.

For an assigned $T^{\overline{*}}$-ideal I, $P_n(\overline{*}) \cap I$ is a vector subspace of $P_n(\overline{*})$, and the dimension $c_n^{\overline{*}}(I)$ of the factor space

$$P_n(I, \overline{*}) := P_n(\overline{*})/(P_n(\overline{*}) \cap I)$$

is called the **n-th $\overline{*}$-codimension of** I. The notations $c_n^{\overline{*}}(A)$ or $c_n^{\overline{*}}(\mathscr{V})$ mean the same number, in case $I = \mathrm{Id}^{\overline{*}}(A) = \mathrm{Id}^{\overline{*}}(\mathscr{V})$. The sequence $(c_n^{\overline{*}}(I))_{n\in\mathbb{N}}$ provides a measure on how big the $T^{\overline{*}}$-ideal is; in particular, if $\mathscr{U} \subseteq \mathscr{V}$ are $\overline{*}$-varieties, then $c_n^{\overline{*}}(\mathscr{U}) \leqslant c_n^{\overline{*}}(\mathscr{V})$ for all $n \in \mathbb{N}$, so the $\overline{*}$-codimension sequence provides also a measure of how big a $\overline{*}$-supervariety is. An important fact is that if A is an affine $\overline{*}$-algebra satisfying a non-trivial ordinary polynomial identity, then the limit of the numerical sequence $\sqrt[n]{c_n^{\overline{*}}(\mathscr{V})}$ exists and is a non-negative integer, called the **$\overline{*}$-exponent of** $\mathscr{V}$ and denoted $\exp^{\overline{*}}(\mathscr{V})$ ([Io, Theorem 3.2]). Of course, if $\mathscr{U} \subseteq \mathscr{V}$ then $\exp^{\overline{*}}(\mathscr{U}) \leqslant \exp^{\overline{*}}(\mathscr{V})$, but it may well happen that $\mathscr{U} \subsetneqq \mathscr{V}$ and still $\exp^{\overline{*}}(\mathscr{U}) = \exp^{\overline{*}}(\mathscr{V})$. When $\mathscr{V}$ fulfills the requirement that $\mathscr{U} \subsetneqq \mathscr{V}$ forces

$$\exp^{\overline{*}}(\mathscr{U}) \lneqq \exp^{\overline{*}}(\mathscr{V}),$$

the variety $\mathscr{V}$ is called a **$\overline{*}$-minimal variety**.

3. The antitranspose involution and related superinvolutions

Let A be a $\overline{*}$-algebra. A **$\overline{*}$-ideal** of A is a graded (two-sided) ideal of A which is $\overline{*}$-invariant; A is a **$\overline{*}$-simple algebra** if $A^2 \neq 0$ and its only $\overline{*}$-ideals are the trivial ones. Of course, any $\overline{*}$-algebra structure on the full matrix algebra $M_n(F)$ yields a $\overline{*}$-simple algebra, but the classification of finite-dimensional $\overline{*}$-simple algebras involves also full matrix algebras over the polynomial algebra $F[x]/(x^2 - 1)$, which we represent as

$$Q := F \oplus cF$$

(where $c^2 = 1$), endowed with the grading $Q_0 = F$ and $Q_1 = cF$. We denote it explicitly by $M_n(Q)$, but for short we will write M_n instead of $M_n(F)$; accordingly, $M_n(Q) = M_n \oplus cM_n$ provides both a description of $M_n(Q)$ and its natural grading.

While the grading on $M_n(Q)$ will be fixed, M_n will be endowed with a so-called elementary grading. Setting from now on $[n] := \{1, 2, \ldots, n\}$, and denoting e_{ij} the usual (i, j)-matrix unit of M_n, we recall

Definition 3.1: Let n be a positive integer, and let $\mathscr{E} := \{e_{ij} \mid i, j \in [n]\}$ be the canonical basis of matrix units of M_n. For any set theoretic map $\alpha : [n] \to \mathbb{Z}_2$, define

$$|e_{ij}|_\alpha := \alpha(i) + \alpha(j).$$

The resulting grading on M_n is called the **elementary grading induced by** α, and the resulting superalgebra structure will be denoted (M_n, α).

We may equivalently represent the map α as a $\mathbb{Z}_2$-word of length n, namely $\alpha = (\alpha(1), \dots, \alpha(n))$; hence we may consider the **reverse** map $\mathrm{Rev}(\alpha)$ corresponding to the reverse word of α, and the **complementary** map $\mathscr{C}(\alpha)$ corresponding to the shift $\alpha + 1$ of α. In case $\alpha(i) = 0$ iff $1 \leqslant i \leqslant k \leqslant n$, we write $\alpha = (0^k, 1^{n-k})$, and setting $l := n - k$ the resulting superalgebra (M_n, α) is commonly denoted by $M_{k,l}$. When $k = n$, it gives rise to the trivial grading on M_n.

A quick survey on the classification of finite-dimensional $\overline{*}$-simple algebras is available in [Ba&T&T]; in particular, the classification is summarized in its Proposition 1, and in our settings any finite-dimensional $\overline{*}$-simple algebra A is $\overline{*}$-isomorphic to one of the following:

- $M_{k,k}$ endowed with the so-called **supertranspose**, **trp**;
- $M_{k,2l}$ endowed with the so-called **orthosymplectic superinvolution**, **osp**;
- $M_{k,l} \oplus M_{k,l}^{\mathrm{sop}}$;
- $M_n(Q) \oplus M_n(Q)^{\mathrm{sop}}$.

Both trp, osp rely on the usual transpose involution, with suitable modifications providing the necessary superinvolution properties (see also [Ka], [Ra&Ze], [GA&Sh]).

Hence the only finite-dimensional $\overline{*}$-simple algebras are built upon the finite-dimensional simple superalgebras $M_{k,l}$ and $M_n(Q)$. Each of them is a subalgebra of a full matrix algebra M_s endowed with an elementary grading: this is clear in the former case; in the latter, after setting $s = 2n$, we may consider the graded embedding

$$a + cb \in M_n(Q) \to \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M_{n,n}.$$

We can actually turn the whole M_s into a $\overline{*}$-algebra with respect to a superinvolution arising from an involution different from the transpose. With this in mind,

Definition 3.2: Let $\gamma \in S_n$ denote the permutation defined by $\gamma(i) := n+1-i$ and let ϑ be the linear endomorphism of M_n defined by $e_{ij}^{\vartheta} := e_{\gamma(j),\gamma(i)}$ for all $i, j \in [n]$. We call it the **anti-transpose** (also called **reflection** in literature) on M_n.

It is easy to verify that ϑ is indeed an involution, and a^{ϑ} is obtained by reflecting the matrix a along the secondary diagonal (the **anti-diagonal**). When M_n is endowed with an elementary $\mathbb{Z}_2$-grading, the usual transpose involution preserves the grading; this is no longer true for the anti-transpose involution. More precisely, it is easy to see that the following holds

Lemma 3.3: *Let $\alpha : [n] \to \mathbb{Z}_2$ be any map, and consider the induced elementary superalgebra structure on M_n. Then ϑ is a graded involution for (M_n, α) if and only if either $\alpha\gamma = \alpha$ or $\alpha\gamma = \alpha + 1$.*

We may also impose a signature to ϑ, it being graded or not:

Definition 3.4: Let $(\mathscr{L}, \mathscr{R})$ be a partition of $[n]$, and define

$$\varepsilon : [n] \times [n] \to \{+1, -1\}$$

setting

$$\varepsilon(i,j) = -1 :\Longleftrightarrow (i,j) \in \mathscr{L} \times \mathscr{R}.$$

We will denote by $\overline{\vartheta}$ the linear endomorphism of M_n defined on the matrix units by

$$e_{ij}^{\overline{\vartheta}} := \varepsilon(i,j) e_{ij}^{\vartheta}.$$

In case $\alpha\gamma = \alpha + 1$, only occurring when n is even, the fibers $\mathscr{L}, \mathscr{R}$ of α partition $[n]$ and are exchanged by γ: $\gamma(\mathscr{L}) = \mathscr{R}$. We may use these facts to define a superinvolution on the superalgebra (M_n, α):

Lemma 3.5: *Let $n = 2k$, and let $\alpha : [n] \to \mathbb{Z}_2$ fulfill $\alpha\gamma = \alpha + 1$. Let $\mathscr{L}, \mathscr{R}$ be the fibers of α, and define $\varepsilon : [n] \times [n] \to \{-1, +1\}$ setting $\varepsilon(i,j) = -1$ if and only if $(i,j) \in \mathscr{L} \times \mathscr{R}$. Then $(M_n, \alpha, \overline{\vartheta})$ is a $\overline{*}$-algebra, $\overline{*}$-isomorphic to $(M_{k,k}, \mathrm{trp})$.*

Hence the only difference between $\overline{\vartheta}$ and trp is in allowing the elements of $\alpha^{-1}(0)$, constituting one of the fibers of α, to be scattered throughout $[n]$, instead of being consecutive. For this reason, we still call $\overline{\vartheta}$ the **supertranspose** on (M_n, α).

Remark 3.6: It is obvious that $\overline{\vartheta}$ depends on ε, that is on the choice of $\mathscr{L} \in \{\alpha^{-1}(0), \alpha^{-1}(1)\}$. One could consider the alternative choice, picking $\mathscr{L}' := \mathscr{R}$, and define a new superinvolution $\overline{\vartheta}'$ setting

$$\varepsilon'(i,j) = -1 :\iff (i,j) \in \mathscr{L}' \times \mathscr{R}' = \mathscr{R} \times \mathscr{L}.$$

It turns out that this choice does not affect the overall $\overline{*}$-structure on M_n: the $\overline{*}$-algebras $(M_n, \alpha, \overline{\vartheta})$ and $(M_n, \alpha, \overline{\vartheta}')$ are $\overline{*}$-isomorphic, since the central symmetry map $\rho : M_n \to M_n$, defined by

$$\rho(e_{ij}) := e_{\gamma(i),\gamma(j)},$$

is a $\overline{*}$-isomorphism.

Notice that passing from $\mathscr{L}$ to $\mathscr{L}'$ is equivalent to passing from the grading word α to the grading word $\mathscr{C}(\alpha)$: this not only will keep the grading on M_n unchanged, but the whole $\overline{*}$-structure, as well.

The case $\gamma(\mathscr{L}) = \mathscr{L}$ (and $\gamma(\mathscr{R}) = \mathscr{R}$) can be dealt with in a similar way: since γ fixes at most a single element of $[n]$, at least one of the fibers of α must have even cardinality, and we denote it by $\mathscr{L}$. Then $\mathscr{L}$ can be decomposed into a disjoint union $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2$ such that $\mathscr{L}_2 = \gamma(\mathscr{L}_1)$. Therefore, γ stabilizes $\mathscr{L}$, but exchanges $\mathscr{L}_1$ and $\mathscr{L}_2$. We employ this fact in attributing a signature to ϑ:

Lemma 3.7: *Let $\alpha : [n] \to \mathbb{Z}_2$ fulfill $\alpha\gamma = \alpha$, and let $\mathscr{L}$ be a fixed fiber of α of cardinality $2l$; let $\mathscr{L}$ be partitioned in $\mathscr{L} = \mathscr{L}_1 \cup \gamma(\mathscr{L}_1) =: \mathscr{L}_1 \cup \mathscr{L}_2$. Define $\eta : [n] \times [n] \to \{-1, +1\}$ according to the following table:*

η	$\mathscr{L}_1$	$\mathscr{R}$	$\mathscr{L}_2$
$\mathscr{L}_1$	$+$	$+$	$-$
$\mathscr{R}$	$-$	$+$	$+$
$\mathscr{L}_2$	$-$	$-$	$+$

Then the linear operator on M_n defined by $e_{ij}^{\tilde{\vartheta}} := \eta(i,j) e_{ij}^{\vartheta}$ is a superinvolution. Moreover, setting $k := n - 2l$, $(M_n, \alpha, \tilde{\vartheta})$ is $\overline{*}$-isomorphic to $(M_{k,2l}, \mathrm{osp})$.

Hence we will still use the term **orthosymplectic** in referring to $\tilde{\vartheta}$.

Remark 3.8: In this case, passing from α to $\mathscr{C}(\alpha)$ does not alter the grading nor the signature. The only way to obtain the (in a loose sense) "transpose" signature table consists in switching the roles of $\mathscr{L}_1$ and $\mathscr{L}_2$, hence in choosing $\mathscr{L}_1' := \mathscr{L}_2$.

Therefore, if A is any finite-dimensional $\overline{*}$-simple algebra, then either

$$A \cong S = M_n,$$

graded by a suitable β and endowed with a supertranspose or an orthosymplectic superinvolution, or

$$A \cong S \oplus S^{\text{sop}},$$

where $S \in \{M_n, M_n(Q)\}$, endowed with the exchange superinvolution **ex**. We call S the **constituent** of A. If A is built upon $S \subseteq M_s$ (recall that $s = 2n$ when $S = M_n(Q)$), we can conveniently trade the size s in favor of a unique kind of superinvolution on a larger enveloping matrix algebra. In different words, if A is built on $S \subseteq M_s$, we can $\overline{*}$-embed A into M_{2s} instead of M_s, but endowing M_{2s} with a supertranspose.

More explicitly, if A is isomorphic to $(M_s, \beta, \overline{*})$, with $\overline{*}$ being either a supertranspose or an orthosymplectic superinvolution, we may define a grading word $\alpha = (\beta | \mathscr{C}\operatorname{Rev}(\beta))$ on M_{2s}, turning M_{2s} into a $\overline{*}$-algebra with supertranspose $\overline{\vartheta}_{2s}$, and consider the $\overline{*}$-embedding

$$a \in M_s \to \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a^{\overline{*}} & 0 \\ 0 & 0 \end{pmatrix}^{\overline{\vartheta}_{2s}} \in M_{2s};$$

if instead A is isomorphic to $S \oplus S^{\text{sop}}$ for $S \subseteq (M_s, \beta)$, we may all the same consider $(M_{2s}, \alpha, \overline{\vartheta}_{2s})$, and the $\overline{*}$-embedding

$$(u, v) \in S \oplus S^{\text{sop}} \to \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}^{\overline{\vartheta}_{2s}} \in M_{2s}.$$

The common idea is the following:

- If A is built on the constituent $S \subseteq M_s$, there is a canonical graded epimorphism $p : A \to S$, that is the projection on the first (and possibly, only) component of the isomorphic image of A. This means p is the identity map if $A = M_n$, and $p(u, v) = u$ if $A = S \oplus S^{\text{sop}}$.

- If $S \subseteq M_s$ is endowed with the grading induced by the $\mathbb{Z}_2$-word β of length s, endow the algebra M_{2s} with the grading induced by the word $\alpha := (\beta | \mathscr{C}\,\mathrm{Rev}(\beta))$, concatenation of the word β with the complement of its reverse word. Then α satisfies $\alpha\gamma_{2s} = \alpha + 1$, and we can turn M_{2s} into a $\overline{*}$-algebra by assigning $\overline{\vartheta_{2s}}$ on it.
- Let $\overline{*}$ denote the superinvolution of A. For $a \in A$, let
$$\zeta(a) := \begin{pmatrix} p(a) & \\ & 0 \end{pmatrix} \in M_{2s},$$
and define a map $\varphi : A \to M_{2s}$, $\varphi := \zeta(a) + (\zeta(a^{\overline{*}}))^{\overline{\vartheta_{2s}}}$.

Lemma 3.9: *The map φ is a $\overline{*}$-embedding.*

A sensible way of looking at this embedding is through the glasses of block decomposition of M_{2s}: considering it as the tensor product $M_2 \otimes M_s$, the map ζ is simply the one sending $a \in A$ to $e_{11} \otimes p(a)$.

Remark 3.10: Notice that in both cases $A \cong S$ (which forces $S = M_s$ endowed with a supertranspose or orthosymplectic superinvolution) and $A \cong S \oplus S^{\mathrm{sop}}$, the target matrix algebra has size $2s$, the same as the matrix algebra enveloping S. Hence, if A is built on $S \subseteq M_s$, we can unambiguously call s the **size of** A, and for short we can say that A embeds into a full matrix algebra of twice its size.

Viewing φ as a $\overline{*}$-embedding of the $\overline{*}$-simple algebra A into a block-matrix algebra M_{2s}, with two diagonal blocks represented as $e_{11} \otimes p(a)$ and
$$(e_{11} \otimes p(a^{\overline{*}}))^{\overline{\vartheta_{2s}}} = e_{11}^{\vartheta_2} \otimes (p(a^{\overline{*}}))^{\overline{\vartheta_s}},$$
lends itself to an easy generalization in embedding a direct sum of $\overline{*}$-simple algebras $A = A_1 \oplus \cdots \oplus A_m$ into a block-matrix algebra with $2m$ diagonal blocks:

Proposition 3.11: *Let $A_1, \ldots, A_m$ be finite-dimensional $\overline{*}$-simple algebras with superinvolutions $\overline{*}_i$, constituents S_i of size s_i, grading words β_i on $M_{s_i} \supseteq S_i$, and projections $p_i : A_i \to S_i$ respectively. Let $A = A_1 \oplus \cdots \oplus A_m$ and set $s := \sum_{i=1}^{m} s_i$, $\beta := (\beta_1 | \ldots | \beta_m)$; consider M_{2s} endowed with the grading induced by $\alpha := (\beta \mid \mathscr{C}\,\mathrm{Rev}(\beta))$ and supertranspose $\overline{\vartheta_{2s}}$. Finally, define $\zeta : A \to M_{2s}$ by*
$$\zeta(a_1, \ldots, a_m) := \sum_{i=1}^{m} (e_{ii} \otimes p_i(a_i)).$$

Then the map φ*, defined by*

$$\varphi(a_1, \dots, a_m) := \zeta(a_1, \dots, a_m) + (\zeta(a_1^{\overline{*}_1}, \dots, a_m^{\overline{*}_m}))^{\overline{\vartheta_{2s}}},$$

is a $\overline{*}$*-embedding of* A *in* $(M_{2s}, \alpha, \overline{\vartheta_{2s}})$*.*

Notice that $(e_{ii} \otimes p_i(a_i^{\overline{*}_i}))^{\overline{\vartheta_{2s}}} = e_{ii}^{\vartheta_{2m}} \otimes (p_i(a_i^{\overline{*}_i}))^{\overline{\vartheta_{s_i}}}$. Hence, the essence of the map φ is to insert the blocks of the matrix $\varphi_i(a_i) \in M_{2s_i}$ along the diagonal of the block-matrix algebra M_{2s}, symmetrically with respect to the anti-diagonal. Of course, composing the canonical injections $A_i \hookrightarrow A$ with φ provides $\overline{*}$-embeddings of the single components of A inside the big algebra M_{2s}.

The embedding φ works well in describing $\overline{*}$-semisimple algebras, but is not sufficient in representing the general finite dimensional ones. In fact, in case of a non-zero Jacobson radical, even the order in listing the $\overline{*}$-simple components, that is the order of the diagonal blocks in the representation matrix space, becomes relevant, unlike the $\overline{*}$-semisimple case, since it has to depict how the Jacobson radical interacts with them. For this task, we need to consider a larger subalgebra of M_{2s} as a general enveloping target.

Notation: For $i, j \in [m]$, let $V_{ij} := M_{s_i \times s_j}$ denote the F-vector space of $s_i \times s_j$-matrices; let $U_{ij} := e_{ij} \otimes V_{ij} \leqslant_F M_{2s}$. Finally, denote $\overline{U_{ij}} := U_{ij}^{\overline{\vartheta_{2s}}} \leqslant_F M_{2s}$. If $A = A_1 \oplus \cdots \oplus A_m$ is a finite-dimensional $\overline{*}$-semisimple algebra A, define

$$\begin{aligned} UT^{\overline{*}}(A_1, \dots, A_m) &:= \varphi(A) \oplus \bigoplus_{\substack{i,j \in [m] \\ i<j}} (U_{ij} \oplus \overline{U_{ij}}) \subseteq U \\ &:= \bigoplus_{\substack{i,j \in [m] \\ i \leqslant j}} (U_{ij} \oplus \overline{U_{ij}}) \subseteq M_{2s}. \end{aligned}$$

A direct verification shows that $UT^{\overline{*}}(A_1, \dots, A_m)$ is actually a $\overline{*}$-subalgebra of M_{2s} enveloping A and all its simple summands, and the sequence

$$0 \to J := \bigoplus_{\substack{i,j \in [m] \\ i<j}} (U_{ij} \oplus \overline{U_{ij}}) \to UT^{\overline{*}}(A_1, \dots, A_m) \to A \to 0$$

is split exact. Actually, it can be easily checked that J is the Jacobson radical of $UT^{\overline{*}}(A_1, \dots, A_m)$. A standard basis for $UT^{\overline{*}}(A_1, \dots, A_m)$ can be obtained by considering homogeneous bases $\mathscr{B}_i$ of the algebras A_i, then completing $\bigcup_i \varphi(\mathscr{B}_i)$ by joining the matrix units $e_{rs} \in J$. By the way, two considerations deserve attention:

(1) if $b \in \mathscr{B}_i \subseteq A_i$, $\varphi(b)$ need not to be a matrix unit in $UT^{\overline{*}}(A_1, \dots, A_m)$. In fact, just in the embedding of $M_{k,l} \oplus M_{k,l}^{\mathrm{sop}}$ the image of $(e_{ij}, 0)$ or $(0, e_{ij})$ is a single matrix unit of M_{2s}, up to a sign; in all other cases, it is a sum of two scalar multiples of matrix units of M_{2s};

(2) despite the previous remark, any matrix unit of $U_{ii} \oplus \overline{U_{ii}}$ is a summand of the image of exactly one basis element of the canonical basis of S_i (or $S_i \oplus S_i^{\mathrm{sop}}$). This is so because the maps $p_i : A_i \to S_i$ are surjective.

More explicitly, since the canonical bases $\mathscr{E}_n := \{e_{uv} \mid u, v \in [n]\}$ and $\mathscr{Q}_n := \{te_{uv} \mid t \in \{1, c\}, u, v \in [n]\}$ are always homogeneous, we get the corresponding images into $UT^{\overline{*}}(A_1, \dots, A_m)$:

- if $A_i = M_{s_i}$ is endowed with a supertranspose superinvolution, then

$$\varphi(e_{uv}) = e_{ii} \otimes e_{uv} + e_{ii}^{\vartheta_{2m}} \otimes e_{uv} \in \varphi(A_i);$$

- if $A_i = M_{s_i}$ is endowed with an orthosymplectic superinvolution, then

$$\varphi(e_{uv}) = e_{ii} \otimes e_{uv} + e_{ii}^{\vartheta_{2m}} \otimes \delta(u, v) e_{uv} \in \varphi(A_i),$$

  where $\delta : [s_i] \times [s_i] \to \{+1, -1\}$ is a suitable signature built on the fiber-partition defining η;

- if $A = M_{s_i} \oplus M_{s_i}^{\mathrm{sop}}$ then

$$\varphi(e_{uv}, 0) = e_{ii} \otimes e_{uv}, \quad \varphi(0, e_{uv}) = e_{ii}^{\vartheta_{2m}} \otimes \varepsilon_{s_i}(u, v) e_{uv}^{\vartheta_{s_i}};$$

- if $A = M_{s_i}(Q) \oplus M_{s_i}(Q)^{\mathrm{sop}}$ then

$$\varphi(e_{uv}, 0) = e_{ii} \otimes (e_{uv} + e_{s_i+u, s_i+v}), \quad \varphi(0, e_{uv}) = e_{ii}^{\vartheta_{2m}} \otimes (e_{uv} + e_{u+s_i, v+s_i})^{\vartheta_{s_i}},$$
$$\varphi(ce_{uv}, 0) = e_{ii} \otimes (e_{u+s_i, v} + e_{u, v+s_i}), \varphi(0, ce_{uv}) = e_{ii}^{\vartheta_{2m}} \otimes (e_{u+s_i, v} - e_{u, v+s_i})^{\vartheta_{s_i}}.$$

By the second part of the previous remark, we may agree on the

Definition 3.12: Let $e_{rs} \in U_{ii} \cup \overline{U_{ii}}$. We will denote by $\mathbf{e}_i(r, s)$ the unique element of $\varphi(\mathscr{B}_i)$ having e_{rs} or $-e_{rs}$ among its components.

Since the canonical bases $\mathscr{E}_n$, $\mathscr{Q}_n$ of the constituents $M_n, M_n(Q)$ are strongly multiplicative (that is: if b_1, b_2 are basis elements and $b_1 b_2 \neq 0$ then $b_1 b_2$ is a basis element, too), the standard basis of $\varphi(A)$ is still strongly multiplicative. Actually, it is well behaved with respect to permutations and superinvolutions, in the sense of the following

Lemma 3.13: *Let A be a $\overline{*}$-simple matrix superalgebra endowed with a superinvolution $\diamond$, let $\mathscr{B}$ be its canonical basis, and let $b_1, \dots, b_n \in \mathscr{B}$, $b := b_1 \cdots b_n$. For any $\sigma \in S_n$ and any set-theoretic map $\lambda : [n] \to \{1, \diamond\}$ denote*

$$b_{\sigma}^{\lambda} := b_{\sigma(1)}^{\lambda(1)} \cdots b_{\sigma(n)}^{\lambda(n)}.$$

If $b, b_{\sigma}^{\lambda} \neq 0$ then

1. *if b is diagonal, then b_{σ}^{λ} is diagonal, too;*
2. *if b is off-diagonal, then $b_{\sigma}^{\lambda} \in \{\pm b, \pm b^{\diamond}\}$.*

Proof. If $A = M_k$, then $\diamond$ is either a supertranspose or an orthosymplectic superinvolution. In the former case, for all $i, j \in [k]$ we have $e_{ij}^{\diamond} = \varepsilon(i,j) e_{ij}^{\vartheta_k}$, in the latter $e_{ij}^{\diamond} = \eta(i,j) e_{ij}^{\vartheta_k}$. So, up to a sign, b_{σ}^{λ} equals $b_{\sigma}^{\lambda'}$ where $\lambda' : [k] \to \{1, \vartheta_k\}$ is defined by $\lambda'(i) = 1$ if and only if $\lambda(i) = 1$; hence the analogous result [DV&Va], Lemma 1, about involutions applies and we may conclude that if $b = e_{ii}$ (if b is diagonal) then there exists $j \in [k]$ such that $b_{\sigma}^{\lambda} = \pm e_{jj}$, while if $b = e_{ij}$, for $i \neq j$, then $b_{\sigma}^{\lambda} = \pm b_{\sigma}^{\lambda'} \in \{\pm e_{ij}, \pm e_{ij}^{\vartheta_k}\}$.

If $A = S \oplus S^{\mathrm{sop}}$ then $b \neq 0$ implies that all the factors b_i belong to the same component S or S^{sop}. Therefore, $b_{\sigma}^{\lambda} \neq 0$ implies that λ has to be constant, that is $\lambda(i) = 1$ for all i or $\lambda(i) = \diamond$ for all i. In the former case, $b_{\sigma}^{\lambda} = b_{\sigma}$ and the result follows from the analogous result about products of matrix units [Le], Lemma 1; in the latter case, instead, $b_{\sigma}^{\lambda} = b_{\sigma(1)}^{\diamond} \cdots b_{\sigma(n)}^{\diamond} = \pm (b_{\sigma(n)} \cdots b_{\sigma(1)})^{\diamond}$ and again the statement follows. ∎

In fact, $UT^{\overline{*}}(A_1, \dots, A_m)$ is just the basic one among several $\overline{*}$-algebras enveloping the direct sum $A = A_1 \oplus \cdots \oplus A_m$:

Definition 3.14: Let

$$A := A_1 \oplus \cdots \oplus A_m$$

be a $\overline{*}$-semisimple finite-dimensional algebra, and let $\mathbf{g} : [m] \to \mathbb{Z}_2$ be any function (or, equivalently, $\mathbf{g} = (g_1, \dots, g_m) \in \mathbb{Z}_2^m$). With the same notation as in Proposition 3.11, define $\beta_{\mathbf{g}} := \beta + \mathbf{g}$, and let $\alpha_{\mathbf{g}} := (\beta_{\mathbf{g}} \mid \mathscr{C}\,\mathrm{Rev}(\beta_{\mathbf{g}}))$. Then denote $UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)$ the $\overline{*}$-subalgebra of $(M_{2s}, \alpha_{\mathbf{g}}, \overline{\vartheta}_{2s})$.

According to this extended notation, $UT^{\overline{*}}(A_1, \dots, A_m) = UT_{\mathbf{0}}^{\overline{*}}(A_1, \dots, A_m)$. For any $\mathbf{g} \in \mathbb{Z}_2^m$, the resulting $\overline{*}$-algebra $UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)$, while being in general not $\overline{*}$-isomorphic to $UT^{\overline{*}}(A_1, \dots, A_m)$, still $\overline{*}$-envelopes A, since the effect of the translation $\beta \rightsquigarrow \beta_{\mathbf{g}} = \beta + \mathbf{g}$ on the diagonal components $\varphi(A_i)$ consists either in leaving β_i unchanged (if $g_i = 0$) or in switching β_i into $\mathscr{C}(\beta_i)$ (when $g_i = 1$), hence in substituting A_i by a $\overline{*}$-isomorphic algebra.

The very idea behind these constructions is to produce enough targets for embedding a finite-dimensional $\overline{*}$-algebra A into a matrix algebra endowed with a supertranspose, preserving the $\overline{*}$-semisimple component of a Wedderburn–Malcev decomposition of A. This will become clearer in the following section.

4. Triangular $\overline{*}$-algebras

We recall that any finite-dimensional unitary algebra A comes with a set of primitive idempotents decomposing the unit of A. They are referred to as minimal idempotents, because they generate minimal left ideals of A (as well as minimal right ideals), and are pairwise orthogonal. Both in case $A = M_n$ and $A = M_n(Q)$, the natural minimal idempotents are the unit matrices e_{ii}, for $i \in [n]$. Notice that they are also homogeneous of degree 0.

Any finite-dimensional $\overline{*}$-algebra A admits a Wedderburn–Malcev decomposition $A = A_{ss} + J$, that is a semidirect decomposition of A into a $\overline{*}$-semisimple subalgebra A_{ss} and the Jacobson radical J of A ([Gi&Io&LM], Theorem 4.1). The interplay between A_{ss} and J is generally very intricate, even in the ordinary case, and it is essential in our study to isolate a class of algebras where this interplay is quite under control:

Definition 4.1: Let A be a finite-dimensional $\overline{*}$-algebra, and let

$$A_{ss} = A_1 \oplus \cdots \oplus A_m$$

be its $\overline{*}$-semisimple part; A is **triangular** if it is either $\overline{*}$-simple (that is: $m = 1$ and $J = 0$) or $m \geqslant 2$, there exist homogeneous elements $w_{1,2}, \dots, w_{m-1\,m} \in J$, and for each $i \in [m]$ there exists a minimal idempotent $e_i \in A_i$ such that

1. $w_{1,2} \cdots w_{m-1,m} \neq 0$ (the sequence $(w_{1,2}, \dots, w_{m-1,m})$ is **regular**);
2. $e_i w_{i,i+1} = w_{i,i+1} = w_{i,i+1} e_{i+1}$ for all $i \in [m-1]$ (the element $w_{i,i+1}$ **connects** the components A_i and A_{i+1});
3. if I is the (ordinary) two-sided ideal generated by $w_{1,2}, \dots, w_{m-1,m}$, then $J = I \oplus I^{\overline{*}}$.

We call the $w_{i,i+1}$'s the **(homogeneous) connectors** of the simple components of A.

Remark 4.2: As anticipated before, while the $\overline{*}$-simple summands of A are uniquely determined, when choosing connectors and idempotents the order among the A_i's becomes important. So in the following when writing

$$A = A_1 \oplus \cdots \oplus A_m + J$$

we will actually assume that the m-ple $(A_1, \dots, A_m)$ has been fixed.

Example 4.3: Of course, if A is $\overline{*}$-simple, then by definition it is triangular, and the set of connectors is the empty set. A class of non-trivial examples is provided by the algebras $UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)$, for $m \geqslant 2$. A sequence of connectors is $w_{1,2} := e_{12} \otimes e_{11}$, $w_{2,3} := e_{23} \otimes e_{11}, \dots, w_{m-1,m} := e_{m-1,m} \otimes e_{11}$, and a related set of idempotents is provided by the elements $\mathbf{e}_1(1,1)$, $\mathbf{e}_2(1,1), \dots, \mathbf{e}_m(1,1)$ of the standard basis of $UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)$. Note that while the idempotents are all homogeneous of degree 0, the degrees of the connectors $w_{i,i+1}$ may vary depending on the translation $\mathbf{g}$.

A fundamental property of triangular $\overline{*}$-algebras is the following:

Proposition 4.4: *Let A be a finite-dimensional $\overline{*}$-algebra, and let $\mathscr{V}$ be the $\overline{*}$-variety generated by A. Then $\mathscr{V}$ contains a triangular $\overline{*}$-algebra R with the same $\overline{*}$-exponent of A. In particular: if $\mathscr{V}$ is $\overline{*}$-minimal, then $\mathscr{V}$ is generated by R.*

Proof. Essentially, the proof is a modification of the one in Theorem 4.5 of [DV&LS] in order to work with superinvolutions instead of involutions. So here we just underline the necessary changes and sketch the main ideas, addressing the reader to the mentioned Theorem for the technical details and computations.

Let B be a $\overline{*}$-semisimple complement to the Jacobson radical $J = J(A)$ in A. Then there are $\overline{*}$-simple components $B_1, \dots, B_k$ of B such that $B_1 J \cdots J B_k \neq 0$ and

$$\exp^{\overline{*}}(A) = \dim(B_1 \oplus \cdots \oplus B_k).$$

Since the product $B_1 J \cdots J B_k$ is not zero, there exist elements $b_i \in B_i$ and $x_1, \dots, x_{k-1} \in J$ such that $b_1 x_1 \cdots x_{k-1} b_k \neq 0$. Of course, neither the b_i nor the x_i are necessarily homogeneous. By the way, since $b_i = 1_{B_i} b_i$ and the unit 1_{B_i} decomposes into a sum of homogeneous (of degree 0) primitive idempotents, there exists a sequence $(e_1, \dots, e_k) \in B_1 \times \cdots \times B_k$ of primitive idempotents, e_i occurring in the decomposition of 1_{B_i}, such that

$$e_1(b_1x_1)\cdots e_{k-1}(b_{k-1}x_{k-1})e_kb_k \neq 0.$$

Moreover, the radical elements b_ix_i (for $i<k$) are a sum of a 0-degree element and a 1-degree one, since J is a graded ideal. Hence there exists a sequence $(y_1,\dots,y_{k-1}) \in J^{k-1}$ of homogeneous elements such that

$$e_1y_1\cdots e_{k-1}y_{k-1}e_k \neq 0.$$

Now set $v_{i,i+1} := e_iy_ie_{i+1}$ for $i<k$. The elements $v_{i,i+1}$ are homogeneous elements in J, and $v_{i,i+1}^2 = 0$ for all i, since $e_{i+1}e_i = 0$. Of course,

$$v_{1,2}\cdots v_{k-1,k} \neq 0$$

and $e_iv_{i,i+1} = v_{i,i+1} = v_{i,i+1}e_{i+1}$ for all $i<k$, but the elements $v_{i,i+1}$ do not generate, in general, J as a $\bar{*}$-ideal.

Let

$$C := F[t_1,\dots,t_{k-1}]/(t_1^2,\dots,t_{k-1}^2),$$

and extend the scalars passing to $A^C := C \otimes A$. We can turn A^C into a $\bar{*}$-algebra by setting $(A^C)_i := C \otimes A_i$, and for homogeneous elements $a \in A$ defining $(c \otimes a)^\star := c \otimes a^{\bar{*}}$, that is considering on the commutative algebra C the identity involution. The $\bar{*}$-simple components B_i of A are embedded into A^C by the natural maps $B_i \hookrightarrow 1 \otimes B_i =: B_i'$. Then define $w_{i,i+1} := t_i \otimes v_{i,i+1}$ for all $i \in [k-1]$. This produces a set of idempotents $e_i' = 1 \otimes e_i \in B_i'$ and a set of nilpotent elements $w_{i,i+1}$ such that $w_{1,2}\cdots w_{k-1,k} \neq 0$,

$$e_i'w_{i,i+1} = w_{i,i+1} = w_{i,i+1}e_{i+1}'$$

for all $i \in [k-1]$ and, moreover, $w_{i,i+1}w_{i,i+1}^\star = 0$.

Let R be the $\bar{*}$-superalgebra generated by $B_1',\dots,B_k'$ and the elements $w_{i,i+1}$'s, and denote I the (ordinary) two-sided ideal generated by the $w_{i,i+1}$'s. Then the same arguments used in the proof of Theorem 4.5 in [DV&LS] show that R is a triangular $\bar{*}$-algebra, whose Jacobson radical is $I \oplus I^{\bar{*}}$ and whose semisimple part is $B_1' \oplus \cdots \oplus B_k'$. This proves that

$$\exp^{\bar{*}}(R) = \dim(B_1' \oplus \cdots \oplus B_k') = \exp^{\bar{*}}(A).$$

Since R is a subalgebra of A^C, $T^{\bar{*}}(A^C) \subseteq T^{\bar{*}}(R)$ and, since $T^{\bar{*}}(A^C) = T^{\bar{*}}(A)$ because F is an infinite field, it follows that $R \in \mathscr{V} = \mathrm{Var}^{\bar{*}}(A)$, that is R is a triangular $\bar{*}$-algebra in $\mathscr{V}$, with the same $\bar{*}$-exponent of A. ∎

Actually, the class of triangular $\overline{*}$-algebras is very close to the class of algebras of kind $UT_{\mathbf{g}}^{\overline{*}}$, and in some sense the latter are an approximation of the former, or provide a basic model for the former. In a more precise sense, in our settings, we have

Theorem 4.5: *Let A be a triangular $\overline{*}$-algebra with $\overline{*}$-semisimple part $A_1 \oplus \cdots \oplus A_m$, and let $\mathscr{V}$ be the $\overline{*}$-variety generated by A. Then there exists $\mathbf{g} \in \mathbb{Z}_2^m$ such that $UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m) \in \mathscr{V}$ and*

$$\exp^{\overline{*}}(A) = \exp^{\overline{*}}(UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)) = \sum_i \dim(A_i).$$

In particular, if $\mathscr{V}$ is $\overline{*}$-minimal, then it is generated by $UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)$.

Proof. Let $\diamond$ denote the superinvolution on A, and let $w_{1,2}, \dots, w_{m-1,m}$ be a set of homogeneous connectors of A with respect to the minimal homogeneous idempotents $e_1 \in A_1, \dots, e_m \in A_m$. According to Proposition 3.11, let $S_i \subseteq M_{s_i}$ be the constituent of A_i and β_i the corresponding grading word. Define recursively $g_1 := 0$ and, for $2 \leqslant i \leqslant m$,

$$g_i = \beta_{i-1}(1) + \beta_i(1) + g_{i-1} + |w_{i-1,i}|_A,$$

thus getting the $\mathbb{Z}_2$-vector $\mathbf{g} = (g_1, \dots, g_m)$. Then let $R := UT_{\mathbf{g}}^{\overline{*}}(A_1, \dots, A_m)$. We are going to prove that $R \in \mathscr{V}$, that is any $\overline{*}$-polynomial identity of A vanishes on R as well. Actually, we are going to prove the contrapositive form of this statement, that is if a $\overline{*}$-polynomial f does not vanish on R then f does not vanish on A. Since F has characteristic zero, it will be enough to consider multilinear polynomials only.

We work inductively on the number m of $\overline{*}$-simple components of A. If $m = 1$, that is if A is $\overline{*}$-simple, the statement is clearly true. Hence we assume $m \geqslant 2$, and consider any multilinear $\overline{*}$-polynomial $f = f(x_1, \dots, x_n)$ not vanishing on R. Then there exist elements $b_1, \dots, b_n$ of the standard basis of R such that the $\overline{*}$-substitution $\mathscr{S} = (b_1, \dots, b_n)$ is admissible and $f_{\mathscr{S}} = f(b_1, \dots, b_n) \neq 0_R$. Since $J(R)$ is m-nilpotent, the number t of the b_i's belonging to $J(R)$ has to fulfill $t < m$ and, up to renaming the indeterminates, assume that they are $b_1, \dots, b_t$. Namely, let $b_1 = e_{i_1 j_1} \otimes e_{u_1,v_1}, \dots, b_t = e_{i_t,j_t} \otimes e_{u_t,v_t}$, with $i_\ell < j_\ell$ for all $\ell \leqslant t$, and all other b_ℓ's are in diagonal blocks. We have to distinguish between two cases:

Case $t < m-1$. Here the statement follows almost at once from the inductive hypothesis: there must be an index $i_0 < m$ such that none among the elements $b_1, \dots, b_t$ belongs to $\bigoplus_{j > i_0}(U_{i_0,j} \oplus \overline{U_{i_0,j}})$, and two subcases may arise:

- There exists $i \in [n]$ such that $b_i \in U_{\ell,i_0} \oplus \overline{U_{\ell,i_0}}$, for $\ell \leqslant i_0$. This means that all elements $b_1, \dots, b_n$ belong to the subalgebra
$$\bigoplus_{i \in [i_0]} \varphi(A_i) \oplus \bigoplus_{1 \leqslant i < i_0} (U_{ii_0} \oplus \overline{U_{ii_0}}),$$
a $\overline{*}$-algebra isomorphic to $R' := UT^{\overline{*}}_{\mathbf{g}'}(A_1, \dots, A_{i_0})$ for $\mathbf{g}' = (g_1, \dots, g_{i_0})$, built upon the triangular $\overline{*}$-subalgebra A' of A with $\overline{*}$-simple components $A_1, \dots, A_{i_0}$ and connectors $w_{1,2}, \dots, w_{i_0-1,i_0}$.
- The $\overline{*}$-simple component $\varphi(A_{i_0})$ is not involved in $\mathscr{S}$, that is after setting $\mathscr{S} := [m] \backslash \{i_0\}$, all the basis elements b_i belong to the subalgebra
$$\bigoplus_{i \in \mathscr{S}} \varphi(A_i) \oplus \bigoplus_{\substack{i,j \in \mathscr{S} \\ i<j}} (U_{ij} \oplus \overline{U_{ij}}),$$
isomorphic to the $\overline{*}$-algebra
$$R' := UT^{\overline{*}}_{\mathbf{g}'}(A_1, \dots, \hat{A}_{i_0}, \dots, A_m)$$
(that is, with missing A_{i_0} component) for $\mathbf{g}' = (g_1, \dots, \hat{g}_{i_0}, \dots, g_m)$, built upon the triangular $\overline{*}$-subalgebra A' of A with $m-1$ simple components forming the ordered sequence $A_1, \dots, A_{i_0-1}, A_{i_0+1}, \dots, A_m$ and system of connectors obtained from the previous one by melding $w_{i_0-1,i_0}, w_{i_0,i_0+1}$ into the single connector $w_{i_0-1,i_0}w_{i_0,i_0+1}$, that is $w_{12}, \dots, (w_{i_0-1,i_0}w_{i_0,i_0+1}), \dots, w_{m-1,m}$.

In both cases, R' is built upon a triangular $\overline{*}$-algebra A' with no more than $m-1$ $\overline{*}$-simple components and related system of homogeneous connectors. Since $f_{\mathscr{S}} \neq 0$ and $\mathscr{S}$ is an admissible substitution in R' then $f \notin T^{\overline{*}}(R')$; by inductive hypothesis, it follows that $f \notin T^{\overline{*}}(A')$ and, since $A' \subseteq A$, it turns out that $f \notin T^{\overline{*}}(A)$.

Case $t = m-1$. The critical case to be worked out. Denote $\overline{\vartheta} := \overline{\vartheta_{2s}}$.

Let us define $e := \sum_{i=1}^{s} e_{ii} \in M_{2s}$, and denote $\overline{e} := e^{\overline{\vartheta}}$; e is a homogeneous idempotent of degree 0, and $\mathbf{1}_{2s} := 1_{M_{2s}} = e + \overline{e}$ is a decomposition of the unity of M_{2s} into a sum of homogeneous orthogonal idempotents. Let $\pi : U \to U$ be the $\mathbb{Z}_2$-graded linear transformation defined by $\pi(a) := eae$ and denote $\overline{\pi}$ the similar map induced by $\overline{e}$. Then it is easy to see that π is a superalgebra homomorphism, and $\pi(U) \lhd U$ is complemented by $\overline{\pi}(U) = \pi(U)^{\overline{\vartheta}}$. Unfortunately, π is not a $\overline{*}$-homomorphism, more precisely we have: $\pi(a^{\overline{\vartheta}}) = \overline{\pi}(a)^{\overline{\vartheta}}$. By the way, $f_{\mathscr{S}} = \pi(f_{\mathscr{S}}) + \overline{\pi}(f_{\mathscr{S}}) \neq 0$ implies that at least one summand is not zero, and we may safely assume $\pi(f_{\mathscr{S}}) \neq 0$. Now, let $f = \sum_{\lambda,\sigma} a_{\lambda,\sigma} x_{\sigma(1)}^{\lambda(1)} \cdots x_{\sigma(n)}^{\lambda(n)}$. We get

$$0 \neq \pi(f_{\mathscr{S}}) = \sum_{\lambda,\sigma} a_{\lambda,\sigma} \pi(b_{\sigma(1)}^{\lambda(1)}) \cdots \pi(b_{\sigma(n)}^{\lambda(n)}).$$

Claim 1: *We may assume that $x_1, \dots, x_{m-1}$ occur with λ-value 1 in any monomial $\mathbf{x}_\sigma^\lambda := x_{\sigma(1)}^{\lambda(1)} \cdots x_{\sigma(n)}^{\lambda(n)}$ contributing in $\pi(f_{\mathscr{S}})$, and all of them evaluate to the same matrix unit $e_{1m} \otimes e_{uv}$ up to a sign depending on (σ, λ) only.*

Proof of Claim. Let us pick a monomial $\mathbf{x}_\sigma^\lambda := x_{\sigma(1)}^{\lambda(1)} \cdots x_{\sigma(n)}^{\lambda(n)}$ contributing in $\pi(f_{\mathscr{S}})$. Certain requirements have to be fulfilled:

- For each $i \in [m-1]$ the λ-value of x_i is prescribed, since $\pi(b_i) \in \{0, b_i\}$: if $\pi(b_i) = 0$ then x_i must occur in $\mathbf{x}_\sigma^\lambda$ with λ-value $\diamond$. Notice that the same holds for any other contributing monomial, hence the indeterminates $x_1, \dots, x_{m-1}$ occur in all contributing monomials with the same λ-value they have in $\mathbf{x}_\sigma^\lambda$, precisely $\lambda(i) = \diamond$ if and only if $\pi(b_i) = 0$.
- For each $i \in [m-1]$, there must be exactly one element b among $b_1, \dots, b_{m-1}$ such that $\pi(b^\lambda) =: e_{ii+1} \otimes e_{u_i,v_i} \in U_{i,i+1}$, and the relative order among these elements in the evaluated monomial must be the natural one, that is $e_{12} \otimes e_{u_1,v_1}, e_{23} \otimes e_{u_2,v_2}, \dots, e_{m-1,m} \otimes e_{u_{m-1},v_{m-1}}$.
- All other elements $b_m, \dots, b_n$ belong to diagonal blocks

  $$\varphi(A_i) \subseteq U_{ii} \oplus \overline{U_{ii}},$$

  but for each b among them the element $0 \neq \pi(b^\lambda) \in U_{ii}$ must occur in the evaluated monomial between the radical elements $e_{i-1,i} \otimes e_{u_{i-1},v_{i-1}}$ and $e_{i,i+1} \otimes e_{u_i,v_i}$.

Hence, if $0 \neq \pi(\mathscr{S}(\mathbf{x}_\sigma^\lambda)) = \pi(b_{\sigma(1)}^{\lambda(1)}) \cdots \pi(b_{\sigma(n)}^{\lambda(n)})$, then it is of kind

$$w_1(e_{1,2} \otimes e_{u_1,v_1}) w_2 (e_{2,3} \otimes e_{u_2,v_2}) w_3 \cdots w_{m-1} (e_{m-1,m} \otimes e_{u_{m-1},v_{m-1}}) w_m,$$

where w_i is a (possibly empty) product involving all elements $\pi(b_j)$ belonging to U_{ii}. Without loss of generality, therefore, we may assume that

$$b_1 = e_{12} \otimes e_{u_1,v_1}, \dots, b_{m-1} = e_{m-1,m} \otimes e_{u_{m-1},v_{m-1}},$$

and the indeterminates $x_1, \dots, x_{m-1}$ occur in f with λ-value 1 in all monomials contributing to $\pi(f_{\mathscr{S}})$.

Moreover, we can partition $\mathbf{x}_\sigma^\lambda$ in subwords $[\mathbf{x}_\sigma^\lambda]_i$, getting

$$\mathbf{x}_\sigma^\lambda = [\mathbf{x}_\sigma^\lambda]_1 x_1 [\mathbf{x}_\sigma^\lambda]_2 \cdots [\mathbf{x}_\sigma^\lambda]_{m-1} x_{m-1} [\mathbf{x}_\sigma^\lambda]_m,$$

and $[\mathbf{x}_\sigma^\lambda]_i$ is the subword of $\mathbf{x}_\sigma^\lambda$ involving all x_j's such that $\mathscr{S}(x_j) \in \varphi(A_i)$.

For any other monomial $\mathbf{x}_{\sigma'}^{\lambda'}$ contributing to $\pi(f_{\mathscr{S}})$, the subwords $[\mathbf{x}_\sigma^\lambda]_i$ and $[\mathbf{x}_{\sigma'}^{\lambda'}]_i$ have the same lengths for all $i = 1, \dots, m$ and involve the same indeterminates, in a possibly different order and under possibly different $\diamond$-action. Up to a sign, let $\pi(\mathscr{S}(\mathbf{x}_\sigma^\lambda)) =: e_{1m} \otimes e_{u,v}$. This forces $0 \neq \mathscr{S}([\mathbf{x}_\sigma^\lambda]_1)$ to be equal to $\mathbf{e}_1(u, u_1)$ up to a sign depending on (σ, λ) only, and the same holds for

$$\mathscr{S}([\mathbf{x}_\sigma^\lambda]_2) = \pm \mathbf{e}_2(v_1, u_2), \dots, \mathscr{S}([\mathbf{x}_\sigma^\lambda]_m) = \pm \mathbf{e}_m(v_{m-1}, v).$$

By Lemma 3.13 these are the same values yielded by the subwords $[\mathbf{x}_{\sigma'}^{\lambda'}]_i$ of any other contributing monomial $\mathbf{x}_{\sigma'}^{\lambda'}$ under the substitution $\mathscr{S}$, up to a sign depending on (σ', λ') only. ∎

Hence, by Claim 1

$$0 \neq \pi(f_{\mathscr{S}}) = a(e_{1m} \otimes e_{uv})$$

for a non-zero scalar $a \in F$. We can normalize the evaluation: consider the homogeneous elements $b_0 := \mathbf{e}_1(1, u)$, $b_{n+1} := \mathbf{e}_m(v, 1)$, and consider the polynomial $f' := x_0 f x_{n+1}$ with new graded indeterminates x_0, x_{n+1} of degree $|b_0|_R$ and $|b_{n+1}|_R$ respectively. Then in the extended evaluation $\mathscr{S}' := (b_0, b_1, \dots, b_n, b_{n+1})$ we get $\pi(f'_{\mathscr{S}'}) = a(e_{1m} \otimes e_{11})$. The next step in our proof is to employ the selected elements $b_0, b_1, \dots, b_{n+1}$ in order to provide a non-vanishing admissible A-substitution for the polynomial f'.

Since $\bigoplus_i A_i \cong \bigoplus_i \varphi(A_i) \subseteq R$ as $\overline{*}$-algebras, any $\overline{*}$-isomorphism maps the minimal homogeneous idempotent $e_i \in A_i$ into a minimal homogeneous idempotent of $\varphi(A_i)$, and without loss of generality we may choose an isomorphism $\psi : A_1 \oplus \cdots \oplus A_m \to \varphi(A_1) \oplus \cdots \oplus \varphi(A_m)$ such that $\psi(e_i) = \mathbf{e}_i(1,1)$. This provides us the elements $a_0 := \psi^{-1}(b_0) \in A_1$, and $a_j := \psi^{-1}(b_j) \in A_j$ for $j \geqslant m$. For the indexes $i \in [m-1]$, instead, define $c_i := \psi^{-1}(\mathbf{e}_i(u_i, 1))$, $d_i := \psi^{-1}(\mathbf{e}_{i+1}(1, v_i))$, then set $a_i := c_i w_{i,i+1} d_i$, built on c_i, d_i and the connector $w_{i,i+1}$.

The substitution $\mathscr{T}' := (a_0, a_1, \dots, a_{n+1})$ is admissible for f': clearly

$$|a_0|_A = |x_0| = |b_0|_R \quad \text{and} \quad |a_j|_A = |x_j| = |b_j|_R$$

for all $j \geqslant m$, because ψ is $\mathbb{Z}_2$-graded. Moreover, for $i \in [m-1]$ one has

$$\begin{aligned}|a_i|_A &= |c_i|_A + |d_i|_A + |w_{i\,i+1}|_A = |\mathbf{e}_i(u_i,1)|_R + |\mathbf{e}_{i+1}(1,v_i)|_R + |w_{i\,i+1}|_A \\ &= |e_{ii}\otimes e_{u_i,1}|_U + |e_{i+1\,i+1}\otimes e_{1,v_i}|_U + |w_{i\,i+1}|_A \\ &= |e_{i\,i+1}\otimes e_{u_i\,v_i}|_R = |b_i|_R.\end{aligned}$$

Recall that by assumption $J(A) = I \oplus I^\diamond$, where I is the ordinary two-sided ideal generated by the connectors $w_{12},\dots,w_{m-1,m}$, and let $\pi_I : J(A) \to I$ denote the canonical projection of the radical on its first component.

Claim 2: *A monomial of f' contributes to $\pi_I(f'_{\mathscr{T}'})$ if and only if it contributes to $\pi(f'_{\mathscr{S}'})$.*

Proof of Claim. Assume at first that for assigned $\sigma \in S_n$ and $\lambda : [n] \to \{1,\diamond\}$ it is $\pi(\mathscr{S}'(x_0\mathbf{x}_\sigma^\lambda x_{n+1})) \neq 0$; then $\mathbf{x}_\sigma^\lambda = [\mathbf{x}_\sigma^\lambda]_1 x_1 [\mathbf{x}_\sigma^\lambda]_2 \cdots [\mathbf{x}_\sigma^\lambda]_{m-1} x_{m-1} [\mathbf{x}_\sigma^\lambda]_m$ and

$$\begin{aligned}&\mathscr{T}'(x_0\mathbf{x}_\sigma^\lambda x_{n+1}) \\ &\qquad = \psi^{-1}(\mathbf{e}_1(1,u))\psi^{-1}(\mathscr{S}([\mathbf{x}_\sigma^\lambda]_1)\psi^{-1}(\mathbf{e}_1(u_1,1))w_{1,2}\psi^{-1}(\mathbf{e}_2(1,v_1))\cdots \\ &\qquad = (-1)^{\sigma,\lambda}\psi^{-1}(\mathbf{e}_1(1,u)\mathbf{e}_1(u,u_1)\mathbf{e}_1(u_1,1))w_{1,2}\cdots \\ &\qquad = (-1)^{\sigma,\lambda}\psi^{-1}(\mathbf{e}_1(1,1))w_{1,2}\cdots w_{m-1,m}\psi^{-1}(\mathbf{e}_m(1,1)) \\ &\qquad = (-1)^{\sigma,\lambda}e_1 w_{1,2} e_2 \cdots e_{m-1} w_{m-1,m} e_m = (-1)^{\sigma,\lambda} w_{1,2}\cdots w_{m-1,m} \neq 0,\end{aligned}$$

where the sign $(-1)^{\sigma,\lambda}$ depends on (σ,λ) only. This holds for any $x_0\mathbf{x}_\sigma^\lambda x_{n+1}$ contributing to $\pi(f_{\mathscr{S}'})$. Conversely, let $x_0\mathbf{x}_\sigma^\lambda x_{n+1}$ be a monomial of f' such that $\pi_I(a_0\mathscr{T}(\mathbf{x}_\sigma^\lambda)a_{n+1}) \neq 0$. Since each one of the indeterminates $x_1,\dots,x_{m-1}$ is mapped to the element $a_i = c_i w_{i,i+1} d_i \in I$, then they have to appear in $\mathbf{x}_\sigma^\lambda$ with λ-value 1. Moreover, since $w_{i,i+1} = e_i w_{i,i+1} e_{i+1}$ and the e_i are orthogonal idempotents, the natural relative order $x_1,\dots,x_{m-1}$ must be preserved by σ. As before we may partition the monomial in

$$\mathbf{x}_\sigma^\lambda = [\mathbf{x}_\sigma^\lambda]_1 x_1 \cdots [\mathbf{x}_\sigma^\lambda]_{m-1} x_{m-1} [\mathbf{x}_\sigma^\lambda]_m,$$

and $[\mathbf{x}_\sigma^\lambda]_i$ is the subword of $\mathbf{x}_\sigma^\lambda$ involving all the x_j's $\mathscr{T}$-evaluated in $a_j \in A_i$. By assumption, $\mathscr{T}'(x_0 f x_{n+1}) \neq 0$, hence

$$\begin{aligned}&0 \neq a_0\mathscr{T}([\mathbf{x}_\sigma^\lambda]_1)(c_1 w_{1,2} d_1)\cdots(c_{m-1}w_{m-1,m}d_{m-1})\mathscr{T}([\mathbf{x}_\sigma^\lambda]_m))a_{n+1} \\ &\quad = \psi^{-1}(\mathbf{e}_1(1,u))\mathscr{T}([\mathbf{x}_\sigma^\lambda]_1)(\psi^{-1}(\mathbf{e}_1(u_1,1))w_{1,2}\psi^{-1}(\mathbf{e}_2(1,v_1))) \\ &\qquad \cdots(\psi^{-1}(\mathbf{e}_{m-1}(u_{m-1},1))w_{m-1,m}\psi^{-1}(\mathbf{e}_m(1,v_{m-1})))\mathscr{T}([\mathbf{x}_\sigma^\lambda]_m))\psi^{-1}(\mathbf{e}_m(v,1)).\end{aligned}$$

Actually, $\mathscr{T}([\mathbf{x}^{\lambda}_{\sigma}]_i) = \psi^{-1}(\mathscr{S}([\mathbf{x}^{\lambda}_{\sigma}]_i))$ for all $i = 1, \ldots, m$, so

$$\begin{aligned}
&0 \neq \mathbf{e}_1(1,u)\mathscr{S}([\mathbf{x}^{\lambda}_{\sigma}]_1)\mathbf{e}_1(u_1,1)\\
&0 \neq \mathbf{e}_i(1,v_{i-1})\mathscr{S}([\mathbf{x}^{\lambda}_{\sigma}]_i)\mathbf{e}_i(u_i,1) \text{ for } 1 < i < m,\\
&0 \neq \mathbf{e}_m(1,v_{m-1})\mathscr{S}([\mathbf{x}^{\lambda}_{\sigma}]_m)\mathbf{e}_m(v,1).
\end{aligned}$$

Since it is not zero, $\mathscr{S}([x^{\lambda}_{\sigma}]_1) = \mathbf{e}_1(u,u_1)$ must hold: in fact, it is enough to check that $\pi(\mathbf{e}_1(1,u))\pi(\mathscr{S}([\mathbf{x}^{\lambda}_{\sigma}]_1))\pi(\mathbf{e}_1(u_1,1)) \neq 0$. This inequality holds because if $A_1 \cong M_{s_1}$ then $\pi(\mathbf{e}_1(1,u)) = e_{11} \otimes e_{1,u}$ and $\pi(\mathbf{e}_1(u_1,1)) = e_{11} \otimes e_{u_1,1}$, and all the same it is true in case $A_1 \cong M_{s_1} \oplus M_{s_1}^{\mathrm{sop}}$; in case $A_1 \cong M_{s_1}(Q) \oplus M_{s_1}(Q)^{\mathrm{sop}}$ then simply $\pi(\mathscr{S}([\mathbf{x}]^{\lambda}_{\sigma}))$ cannot be zero because $\pi(\mathbf{e}_1(1,u)) = \mathbf{e}_1(1,u) \neq 0$, hence $\pi(\mathscr{S}([\mathbf{x}]^{\lambda}_{\sigma})) = \mathscr{S}([\mathbf{x}]^{\lambda}_{\sigma})$, so $e_{11} \otimes e_{u,u_1}$ must be a summand of that basis vector. Similar arguments work for all other i, so $\pi(\mathscr{S}([\mathbf{x}^{\lambda}_{\sigma}]_i)) = \mathbf{e}_i(v_{i-1},u_i)$ for $1 < i < m$, and $\pi(\mathscr{S}([\mathbf{x}^{\lambda}_{\sigma}]_m)) = \mathbf{e}_m(v_{m-1},v)$. Then

$$\pi(\mathscr{S}(\mathbf{x}^{\lambda}_{\sigma})) = (-1)^{\sigma,\lambda} e_{1m} \otimes e_{uv}$$

and

$$\pi(\mathscr{S}'(x_0\mathbf{x}^{\lambda}_{\sigma}x_{n+1})) = (-1)^{\sigma,\lambda} e_{1m} \otimes e_{11} \neq 0,$$

that is $x_0\mathbf{x}^{\lambda}_{\sigma}x_{n+1}$ is a monomial contributing in $\pi(f'_{\mathscr{S}'})$. ■

Finally, notice that the proof of this claim shows that

$$\mathscr{T}'(x_0\mathbf{x}^{\lambda}_{\sigma}x_{n+1}) = (-1)^{\sigma,\lambda} w_{1,2}\cdots w_{m-1,m} \Leftrightarrow \mathscr{S}'(x_0\mathbf{x}^{\lambda}_{\sigma}x_{n+1}) = (-1)^{\sigma,\lambda} e_{1m} \otimes e_{11}$$

with sign $(-1)^{\sigma,\lambda}$ depending on σ and λ only. It follows that $\mathscr{T}'(f') \neq 0$ and, a fortiori, $f \notin T^{\overline{*}}(A)$. This concludes the inductive argument and we obtain $R \in \mathscr{V}$. ■

We can now state the main results of this section:

Theorem 4.6: *Let $\mathscr{V}$ be a $\overline{*}$-minimal affine variety. Then there exists a sequence $(A_1, \ldots, A_m)$ of finite-dimensional $\overline{*}$-simple algebras and a vector $\mathbf{g} \in \mathbb{Z}_2^m$ such that $\mathscr{V}$ is generated by $UT^{\overline{*}}_{\mathbf{g}}(A_1, \ldots, A_m)$.*

Proof. In virtue of Theorem 1 in [Al&Gi&Ka], any affine variety of PI $\overline{*}$-algebras is generated by a finite-dimensional one. Then, tunnelling Proposition 4.4 and Theorem 4.5, it follows that $\mathscr{V}$ contains a $\overline{*}$-algebra $UT^{\overline{*}}_{\mathbf{g}}(A_1, \ldots, A_m)$, for suitable $\overline{*}$-simple algebras $A_1, \ldots, A_m$ and $\mathbf{g} \in \mathbb{Z}_2^m$, with the same $\overline{*}$-exponent of $\mathscr{V}$. In case $\mathscr{V}$ is $\overline{*}$-minimal, it must be equal to the $\overline{*}$-variety generated by $UT^{\overline{*}}_{\mathbf{g}}(A_1, \ldots, A_m)$. ■

Although the previous result deepens its roots in the ground of $\overline{*}$-superalgebras and $\overline{*}$-varieties, its fruits are available for ordinary (non-graded) $*$-algebras and $*$-varieties:

Theorem 4.7: *Let $\mathscr{V}$ be any $*$-variety. Then there exist finite-dimensional $\overline{*}$-simple algebras $A_1, \dots, A_m$ and an element $\mathbf{g} \in \mathbb{Z}_2^m$ such that the Grassmann envelope $E\hat{\otimes}UT^{\overline{*}}_{\mathbf{g}}(A_1, \dots, A_m)$ belongs to $\mathscr{V}$ and has the same $*$-exponent as $\mathscr{V}$. In particular, if $\mathscr{V}$ is $*$-minimal then $\mathscr{V}$ is generated by $E\hat{\otimes}UT^{\overline{*}}_{\mathbf{g}}(A_1, \dots, A_m)$.*

Proof. Let $\mathscr{V}$ be generated by a (non-graded) $*$-algebra A. By [Al&Gi&Ka], Theorem 4, there exists a finite-dimensional $\overline{*}$-algebra B such that

$$\mathrm{Id}^*(A) = \mathrm{Id}^*(\mathscr{G}(B)),$$

where $\mathscr{G}(B)$ is the Grassmann envelope of the superalgebra B endowed with the involution $\star := \sharp \otimes \overline{*}$ (it is actually more than this: $\star$ is a graded involution). Moreover, by [Gi&PM&Va], Theorem 3, $\exp^{\overline{*}}(\mathscr{G}(B)) = d$ does not only exist, but can be concretely interpreted as the maximum dimension of the admissible $\overline{*}$-subalgebras of B (see Definition 1 in the same paper). Since B is finite-dimensional, Proposition 4.4 and Theorem 4.5 imply that there exists a suitable sequence $(A_1, \dots, A_m)$ of finite-dimensional $\overline{*}$-simple algebras and $\mathbf{g} \in \mathbb{Z}_2^n$ such that $R = UT^{\overline{*}}_{\mathbf{g}}(A_1, \dots, A_m)$ belongs to the $\overline{*}$-variety generated by B, and $A_1 \oplus \cdots \oplus A_m$ is $\overline{*}$-isomorphic to an admissible subalgebra of B of maximum dimension d. Hence, employing the $\tilde{\ }$-map and its properties (see [Al&Gi&Ka], Lemma 2) it follows that $\mathrm{Id}^*(\mathscr{G}(B)) \subseteq \mathrm{Id}^*(\mathscr{G}(R))$, and

$$\exp^*(\mathscr{G}(B)) = \exp^*(\mathscr{G}(R)) = d.$$

Since $\mathscr{V}$ is by assumption $*$-minimal and $\mathscr{G}(R)$ generates a $*$-subvariety of $\mathscr{V}$ with the same $*$-exponent d, it follows that $\mathscr{V} = \mathrm{Var}^*(\mathscr{G}(B))$. ■

5. Identities

Let A be a superalgebra. Recall that A is supercommutative if for homogeneous elements $a, b \in A$ we have $ba = (-1)^{|a|\cdot|b|}ab$, that is homogeneous elements commute unless both have $\mathbb{Z}_2$-degree one, in which case they anticommute. The Grassmann algebra E is an immediate and relevant example of supercommutative superalgebra. One can define a free object in the class of supercommutative superalgebras (see [Be]; a brief survey is also available in [GZBook], Section 3.8):

Definition 5.1: Let U, V be disjoint countable sets of indeterminates, and denote $\mathscr{S} := \mathscr{S}_{U,V}$ the unitary superalgebra presented on the even central variables of U and the odd grassmannian variables of V, that is satisfying the relations $u_i u_j = u_j u_i$, $u_i v_j = v_j u_i$, $v_i v_j = -v_j v_i$ for all $u_i, u_j \in U$ and $v_i, v_j \in V$. The superalgebra $\mathscr{S}$ is called the **free supercommutative superalgebra** on U, V.

Any homogeneous map out of $U \cup V$ to a supercommutative superalgebra A extends uniquely into a graded homomorphism $\mathscr{S} \to A$. A convenient way of representing $\mathscr{S}$ is simply $\mathscr{S} = F[U] \otimes E_{FV}$, that is the scalar extension to $F[U]$ of the exterior algebra of the infinite-dimensional F-vector space spanned by the variables of V. As for the usual Grassmann envelope, any superalgebra A gives rise to the graded tensor product

$$\mathscr{S} \hat{\otimes} A := \mathscr{S}_0 \otimes A_0 + \mathscr{S}_1 \otimes A_1;$$

here we denote it $\mathscr{S}(A)$, and call it the supercommutative envelope of A, by similarity with the Grassmann envelope, denoted $\mathscr{G}(A)$ in Remark 2.3 and occurring in the proof of Theorem 4.7. Since the superinvolution $\sharp$ on the Grassmann algebra can be extended to a superinvolution on $\mathscr{S} = F[U] \otimes E_{FV}$ (still denoted $\sharp$ with abuse of notation), it turns out that in case A is a $\overline{*}$-algebra then $\mathscr{S}(A)$ is a $*$-algebra with respect to the (graded) involution $\star := \sharp \otimes \overline{*}$.

A remarkable property of $\mathscr{S}(A)$ is that it shares the same ordinary identities of $\mathscr{G}(A)$ (Proposition 3.8.4 of [GZBook]), and this can be easily generalized taking into account the involution, getting $\mathrm{Id}^*(\mathscr{S}(A)) = \mathrm{Id}^*(\mathscr{G}(A))$ for a $\overline{*}$-algebra A. In case A is finite-dimensional, another remarkable property, arising from the ordinary case as well, is the following (see also [Be], Section 2 and [DV&Ko], Section 6):

Lemma 5.2: *Let A be a finite-dimensional $\overline{*}$-algebra, $\mathscr{B} = \mathscr{B}_0 \uplus \mathscr{B}_1$ a homogeneous basis of A, and define the countable sets of variables*

$$U = U_{\mathscr{B}} := \{u_b^{(i)} \mid i \geqslant 1,\, b \in \mathscr{B}_0\}, \quad V = V_{\mathscr{B}} := \{v_b^{(i)} \mid i \geqslant 1,\, b \in \mathscr{B}_1\}.$$

Let $\mathscr{S} = \mathscr{S}_{U,V}$ be the free supercommutative algebra on U, V, and define for all $i \geqslant 1$

$$\tilde{x}_i := \sum_{b \in \mathscr{B}} \xi_b^{(i)} \otimes b \in \mathscr{S}(A), \quad \textit{where } \xi_b^{(i)} = \begin{cases} u_b^{(i)} & \textit{if } b \in \mathscr{B}_0, \\ v_b^{(i)} & \textit{if } b \in \mathscr{B}_1. \end{cases}$$

The $*$-subalgebra of $\mathscr{S}(A)$ generated by the elements $\tilde{x}_1, \tilde{x}_2, \dots$ is denoted $\tilde{A}$, and is a relatively free $*$-algebra in the $*$-variety generated by $\mathscr{G}(A)$.

Proof. Just small adjustments in the proof of Proposition 3.8.5 in [GZBook] are needed in order to get the result. ∎

Particularly, the natural $*$-epimorphism $\psi : F\langle X,*\rangle \to \tilde{A}$, defined by $\psi(x_i) = \tilde{x}_i$, has kernel $\mathrm{Id}^*(\mathscr{G}(A))$, so that $\tilde{A} \cong_* F\langle X,*\rangle / \mathrm{Id}^*(\mathscr{G}(A))$.

In order to describe the Id^*-ideal of identities of a minimal $*$-variety, the notion of supercommutative envelope and the relatively free $*$-algebras it contains are a key ingredient; the other one is the intervention of Lewin's Theorem, which we recall immediately:

Theorem 5.3 (Lewin's Theorem, [Lw]): *Let I, J be twosided ideals of $F\langle X\rangle$, and let $\{\delta_i \mid i \geqslant 1\}$ be a countable free subset of a $(F\langle X\rangle/I, F\langle X\rangle/J)$-bimodule. Then the map*

$$x_i \in X \to \begin{pmatrix} x_i + I & \delta_i \\ 0 & x_i + J \end{pmatrix}$$

uniquely defines an algebra homomorphism whose kernel is the product IJ.

As a second step, we rephrase [DV&LS, Lemma 3.2] according to our settings:

Proposition 5.4: *Let $\mathscr{S}_1, \mathscr{S}_2$ be free supercommutative superalgebras on disjoint pairs of variables (U_1, V_1) and (U_2, V_2) respectively, and let A, B be subalgebras of $M_m(\mathscr{S}_1)$ and $M_n(\mathscr{S}_2)$ respectively. Let*

$$U := \{u_{ij}^{(h)} \mid h \geqslant 1,\, i \in [m],\, j \in [n]\}, \quad V := \{v_{ij}^{(h)} \mid h \geqslant 1, i \in [m], j \in [n]\}$$

be countable sets disjoint from $U_1 \cup U_2 \cup V_1 \cup V_2$, and let $\mathscr{S}$ be the free supercommutative superalgebra on the central variables of $U_1 \cup U_2 \cup U$ and on the grassmannian variables of $V_1 \cup V_2 \cup V$.

The vector space $M_{m\times n}(\mathscr{S})$ is an (A, B)-bimodule, and its elements

$$\tilde{x}_h := \sum_{\substack{i\in[m]\\ j\in[n]}} (u_{ij}^{(h)} + v_{ij}^{(h)}) \otimes e_{ij} \in M_{m\times n}(\mathscr{S}) \quad (h \geqslant 1)$$

are (A, B)-free.

Proof. Any $\tilde{x}_h$ involves different variables from $U \cup V$ for different $h \geqslant 1$, hence showing each $\tilde{x}_h$ is (A, B)-torsion free will suffice. So, let us pick a single $\tilde{x} = \sum_{h,k} \xi_{hk} \otimes e_{hk}$ and let $a_r = (a_{ih}^{(r)}) \in M_m(\mathscr{S}_1)$, $b_r = (b_{kj}^{(r)}) \in M_n(\mathscr{S}_2)$, $r = 1, \ldots, l$ be such that $\sum_r a_r \tilde{x} b_r = 0$, with b_r's F-linearly independent. We then want to show that $a_r = 0$ for each $r = 1, \ldots, l$.

Since $a_{ih}^{(r)} \in (\mathscr{S}_1)_0 \oplus (\mathscr{S}_1)_1$, let us write $a_{ih}^{(r)} = \alpha_{ih}^{(r)} + \beta_{ih}^{(r)}$, where $\alpha_{ih}^{(r)} \in (\mathscr{S}_1)_0$ and $\beta_{ih}^{(r)} \in (\mathscr{S}_1)_1$, and denote

$$\overline{a}_{ih}^{(r)} := \alpha_{ih}^{(r)} - \beta_{ih}^{(r)}.$$

The (i, j)-entry of $\sum_r a_r \tilde{x} b_r$ is

$$\sum_{h,k} u_{hk} \sum_r a_{ih}^{(r)} b_{kj}^{(r)} + \sum_{h,k} v_{hk} \sum_r \overline{a}_{ih}^{(r)} b_{kj}^{(r)}.$$

Since the variables u_{hk}, v_{hk} are pairwise distinct, for each $h \in [m]$ and $k \in [n]$ it has to be $\sum_r a_{ih}^{(r)} b_{kj}^{(r)} = 0$ and $\sum_r \overline{a}_{ih}^{(r)} b_{kj}^{(r)} = 0$. Hence $\sum_r a_{ih}^{(r)} b_r = 0 = \sum_r \overline{a}_{ih}^{(r)} b_r$ in $M_n(\mathscr{S})$. Moreover,

$$\sum_r a_{ih}^{(r)} b_r + \sum_r \overline{a}_{ih}^{(r)} b_r = 2 \sum_r \alpha_{ih}^{(r)} b_r, \quad \sum_r a_{ih}^{(r)} b_r - \sum_r \overline{a}_{ih}^{(r)} b_r = 2 \sum_r \beta_{ih}^{(r)} b_r, \tag{1}$$

hence it is enough to prove that all $\alpha_{ih}^{(r)}$ and $\beta_{ih}^{(r)}$ are zero.

Since the field F is infinite, usual Vandermonde arguments allow us to assume that all $\alpha_{ih}^{(r)}$'s are multihomogeneous polynomials in the commutative indeterminates of U_1 and the grassmannian indeterminates of V_1. Therefore each $\alpha_{ih}^{(r)}$ is a scalar multiple $\gamma_{ih}^{(r)} w_u w_v$ of the monomial $w_u w_v$, where w_u is the semistandard monomial in the U_1-indeterminates occurring in $\alpha_{ih}^{(r)}$, w_v is the standard monomial in the V_1-indeterminates occurring in $\alpha_{ih}^{(r)}$, and $\gamma_{ih}^{(r)} \in F$. Hence

$$\sum_r \alpha_{ih}^{(r)} b_r = w_u w_v \left(\sum_r \gamma_{ih}^{(r)} b_r \right) = 0.$$

Now: $w_u w_v$ belongs to the canonical monomial F-basis of $\mathscr{S}_1$, while the entries of b_r are polynomials in $U_2 \cup V_2$. Hence the product $(w_u w_v)(\sum_r \gamma_{ih}^{(r)} b_r)$ is zero if and only if $\sum_r \gamma_{ih}^{(r)} b_r = 0$. By the way, since $\gamma_{ih}^{(r)} \in F$ and $b_1, \dots, b_l$ are F-linearly independent by assumption, it follows that if $\gamma_{ih}^{(r)} = 0$ for all r then $\alpha_{ih}^{(r)} = 0$ for each r, and this must hold for all i, h. Similar arguments show that $\beta_{ih}^{(r)} = 0$ for all i, h, r. Therefore $a_{ih}^{(r)} = 0$ for all i, h and r, that is $a_r = 0$ for all $r = 1, \dots, l$. ∎

For our aims, a weaker version of this statement will suffice:

PROPOSITION 5.5: *Let $M_{m \times n}$ be endowed with an elementary grading, and let us set $\tilde{x}_h := \sum_{i,j} \xi_{ij}^{(h)} e_{ij}$, where $\xi_{ij} = u_{ij}$ if $|e_{ij}| = 0$, and $\xi_{ij} = v_{ij}$ otherwise. Then the elements $\tilde{x}_h$, for $h \geqslant 1$, are a basis of a free (A, B)-bimodule.*

Proof. The elements $\tilde{x}_h$ are obtained by those of the previous Proposition by specializing either $u_{ij}^{(h)}$ or $v_{ij}^{(h)}$ to zero, according to the actual degree of e_{ij}. By the way, the statement $\sum_r a_r \tilde{x} b_r = 0$, equivalent to

$$\sum_r \alpha_{ij}^{(r)} b_r = \sum_r \beta_{ij}^{(r)} b_r = 0$$

(consequence of equation (1) in the preceding proof), does not distinguish between $\xi_{ij} = u_{ij}$ and $\xi_{ij} = v_{ij}$. Hence $\tilde{x}_1, \tilde{x}_2, \dots$ still constitute a free system. ∎

A last easy remark is the following:

Lemma 5.6: *Let S be any superalgebra. Then $\mathscr{G}(S^{\mathrm{sop}}) \cong_{\mathbb{Z}_2} \mathscr{G}(S)^{\mathrm{op}}$ and*

$$(\mathscr{G}(S \oplus S^{\mathrm{sop}}), \star) \cong_* (\mathscr{G}(S) \oplus \mathscr{G}(S)^{\mathrm{op}}, ex).$$

Proof. Just define $a \otimes s \to a^\sharp \otimes s$ on a homogeneous element $a \otimes s \in E_i \otimes S_i$ in order to get a graded isomorphism between $\mathscr{G}(S^{\mathrm{sop}})$ and $\mathscr{G}(S)^{\mathrm{op}}$. Then, the map $\nu : \mathscr{G}(S \oplus S^{\mathrm{sop}}) \to \mathscr{G}(S) \oplus \mathscr{G}(S)^{\mathrm{op}}$ defined on homogeneous elements by

$$\nu(a \otimes (s,t)) = (a \otimes s, a^\sharp \otimes t)$$

is readily checked to be a $*$-isomorphism. ∎

Now, let A be a finite-dimensional $\overline{*}$-simple algebra built on $S \subseteq M_s$, and consider the relatively free $*$-algebra $\tilde{A} \subseteq \mathscr{S}(A)$. The graded projection $p : A \twoheadrightarrow S$ can be extended to an algebra epimorphism $\tilde{p} : \tilde{A} \to \tilde{S}$, by setting

$$\tilde{p}(\tilde{x}_i) = \sum_{b \in \mathscr{B}} \xi_b^{(i)} \otimes p(b), \quad \tilde{p}(\tilde{x}_i^\star) = \sum_{b \in \mathscr{B}} (\xi_b^{(i)})^\sharp \otimes p(b^{\overline{*}}).$$

Of course $\tilde{p}$ is not a $*$-homomorphism in general, nor a graded one. By the way, $\tilde{p}\psi : F\langle X \cup X^* \rangle \to \tilde{S}$ is an algebra homomorphism out of the free algebra $F\langle X \cup X^* \rangle \cong F\langle X \rangle$, and clearly

$$\mathrm{Id}^*(\mathscr{G}(A)) = \mathrm{Id}^*(\tilde{A}) = \ker \psi \subseteq \ker(\tilde{p}\psi).$$

We are going to prove that, in fact, these are all equalities:

Proposition 5.7: $\ker(\tilde{p}\psi) = \mathrm{Id}^*(\mathscr{G}(A))$.

Proof. Let $f \in \ker(\tilde{p}\psi)$, $S = p(A)$ the constituent of A and let

$$\varphi : F\langle X, * \rangle \to \mathscr{G}(A)$$

be any $*$-homomorphism, defined by

$$\varphi(x_i)=\sum_b \alpha_b^{(i)}\otimes b\in\mathscr{G}(A).$$

There exists a unique $\overline{*}$-homomorphism $\eta:\mathscr{S}\to E$ such that $\eta(\xi_b^{(i)})=\alpha_b^{(i)}$, since $|\xi_b^{(i)}|=|b|=|\alpha_b^{(i)}|$, and $\eta\otimes\mathrm{Id}_S:\mathscr{S}\otimes S\to E\otimes S$ is a well defined algebra homomorphism. The restriction to $\mathscr{S}(S)$ is a graded homomorphism $\overline{\varphi}:\mathscr{S}(S)\to\mathscr{G}(S)$, and denoting $p_{\mathscr{G}}$ the E-linear extension of p, we have

$$p_{\mathscr{G}}\varphi(x_i)=\sum_b \alpha_b^{(i)}\otimes p(b)=\overline{\varphi}\Big(\sum_b \xi_b^{(i)}\otimes p(b)\Big)=\overline{\varphi}\tilde{p}\psi(x_i),$$

$$p_{\mathscr{G}}\varphi(x_i^*)=\sum_b (\alpha_b^{(i)})^{\sharp}\otimes p(b^{\overline{*}})=\overline{\varphi}\Big(\sum_b (\xi_b^{(i)})^{\sharp}\otimes p(b^{\overline{*}})\Big)=\overline{\varphi}\tilde{p}\psi(x_i^*)$$

$\Rightarrow p_{\mathscr{G}}\varphi=\overline{\varphi}\tilde{p}\psi$, hence $p_{\mathscr{G}}\varphi(f)=0$. Although this is always true, in order to show that $\varphi(f)=0$ we must confront two possible situations:

1. A is a simple algebra, that is $A\cong_{\overline{*}} M_s=S$ endowed with a superinvolution. In this case $p:A\to S$ is a $\overline{*}$-isomorphism indeed, so $p_{\mathscr{G}}$ is injective and since $p_{\mathscr{G}}\varphi(f)=0$ it follows that $\varphi(f)=0$. This means that $f\in\mathrm{Id}^*(\mathscr{G}(A))$.
2. A is $\overline{*}$-simple but not simple, that is $A\cong_{\overline{*}} S\oplus S^{\mathrm{sop}}$. From $p_{\mathscr{G}}\varphi(f)=0$ we cannot deduce $\varphi(f)=0$ any longer. By the way, there exists a homogeneous F-basis $\mathscr{B}=\mathscr{B}_1\uplus\mathscr{B}_2$ of A such that $p(\mathscr{B}_1)$ is a homogeneous F-basis of S and $p(\mathscr{B}_1^{\overline{*}})=0$, and similarly $p(\mathscr{B}_2)=0$ while $p(\mathscr{B}_2^{\overline{*}})$ is a homogeneous F-basis of S too. We are then going to prove that $f\in T(\mathscr{G}(S))$.

So, let ρ be any $\mathscr{G}(S)$-valued algebra homomorphism out of $F\langle X\cup X^*\rangle$, namely defined on the free generators $X\cup X^*$ by

$$\rho(x_i)=\alpha_i:=\sum_{b\in\mathscr{B}_1}\alpha_b^{(i)}\otimes p(b)\in\mathscr{G}(S),$$

$$\rho(x_i^*)=\beta_i:=\sum_{b\in\mathscr{B}_2}(\beta_b^{(i)})^{\sharp}\otimes p(b^{\overline{*}})\in\mathscr{G}(S),$$

(we are employing the fact that both $p(\mathscr{B}_1)$ and $p(\mathscr{B}_2^{\overline{*}})$ are homogeneous F-bases for S and, moreover, $\sharp$ is in particular a graded F-automorphism of E). We claim $\rho(f)=0$.

To prove the claim, let us define a $*$-homomorphism $\varphi : F\langle X, *\rangle \to \mathscr{G}(A)$ by assigning

$$\varphi(x_i) := \sum_{b\in\mathscr{B}_1} \alpha_b^{(i)} \otimes b + \sum_{b\in\mathscr{B}_2} \beta_b^{(i)} \otimes b \in \mathscr{G}(A),$$

and of course notice

$$\varphi(x_i^*) = \varphi(x_i)^\star = \sum_{b\in\mathscr{B}_1} (\alpha_b^{(i)})^\sharp \otimes b^{\overline{*}} + \sum_{b\in\mathscr{B}_2} (\beta_b^{(i)})^\sharp \otimes b^{\overline{*}} \in \mathscr{G}(A).$$

Since $f \in \ker(\tilde{p}\psi)$ and φ is a $*$-homomorphism, we know that $p_{\mathscr{G}}\varphi(f) = 0$. By the way, $p_{\mathscr{G}}\varphi$ is an algebra homomorphism out of the free associative algebra $F\langle X \cup X^*\rangle$, coinciding with ρ on the free generators. Hence $\rho = p_{\mathscr{G}}\varphi$, and therefore $\rho(f) = 0$. Since ρ was arbitrarily chosen, it follows that $f \in \mathrm{Id}(\mathscr{G}(S))$.

Since $(\mathscr{G}(A), \star) \cong_* (\mathscr{G}(S) \oplus \mathscr{G}(S)^{\mathrm{op}}, ex)$, it follows that

$$\mathrm{Id}^*(\mathscr{G}(A)) = \mathrm{Id}(\mathscr{G}(S)) \cap \mathrm{Id}(\mathscr{G}(S)^{\mathrm{op}}) = \mathrm{Id}(\mathscr{G}(S)),$$

hence $\ker(\tilde{p}\psi) \subseteq \mathrm{Id}^*(\mathscr{G}(A))$ in this case, too. ∎

Now let $R := UT^{\overline{*}}_{\mathbf{g}}(A_1, \dots, A_m)$, where A_i has constituent $S_{s_i} \subseteq M_{s_i}$, set $d := \sum_{i=1}^m s_i$ and define

$$S = UT_{\mathbb{Z}_2,\mathbf{g}}(S_1, \dots, S_m) = \bigoplus_{i=1}^{m} (e_{ii} \otimes S_i) \oplus \bigoplus_{\substack{i,j\in[m]\\ i<j}} (e_{ij} \otimes V_{ij}).$$

While $R \subseteq M_{2d}$ is a superalgebra with respect to the elementary grading $\alpha_{\mathbf{g}} = (\beta_{\mathbf{g}} | \mathscr{C}\,\mathrm{Rev}(\beta_{\mathbf{g}}))$, S is $\mathbb{Z}_2$-subalgebra of M_d with elementary grading $\beta_{\mathbf{g}}$, and one can define the natural generalization of the map p by setting

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R \xrightarrow{\pi} a \in S.$$

Just as p, the map π is a superalgebra epimorphism, and if $\mathscr{S}$ is the free supercommutative superalgebra associated to the standard homogeneous basis $\mathscr{R}$ of R, just as p the map π admits an $\mathscr{S}$-linear extension $\tilde{\pi} : \tilde{R} \to \mathscr{S}(S)$. More precisely, $\tilde{\pi}(\tilde{R})$ is the subalgebra of $\mathscr{S}(S)$ generated by the elements

$$\tilde{\pi}(\tilde{x}_i) = \sum_{b\in\mathscr{R}} \xi_b^{(i)} \otimes \pi(b), \quad \tilde{\pi}(\tilde{x}_i^\star) = \sum_{b\in\mathscr{R}} (\xi_b^{(i)})^\sharp \otimes \pi(b^{\overline{\vartheta}}),$$

where $\overline{\vartheta} = \overline{\vartheta}_{2d}$ is the superinvolution defined on M_{2d}. Notice that, in fact, when $m = 1$ the maps π and $\tilde{\pi}$ coincide with the maps p and $\tilde{p}$, respectively.

After this brief introduction, we can now state and prove the main result of this section:

Theorem 5.8: *Let $\mathscr{V}$ be the $*$-variety generated by the Grassmann envelope of $UT^{\overline{*}}_{\mathbf{g}}(A_1,\dots,A_m)$, for a suitable sequence $(A_1,\dots,A_m)$ of finite-dimensional $\overline{*}$-simple algebras and a suitable $\mathbf{g}=(g_1,\dots,g_m)\in\mathbb{Z}_2^m$. Then*

$$\mathrm{Id}^*(\mathscr{V})=\mathrm{Id}^*(\mathscr{G}(A_1))\cdots\mathrm{Id}^*(\mathscr{G}(A_m))\cap\mathrm{Id}^*(\mathscr{G}(A_m))\cdots\mathrm{Id}^*(\mathscr{G}(A_1)).$$

Proof. With the same notation adopted above, so that

$$R=UT^{\overline{*}}_{\mathbf{g}}(A_1,\dots,A_m)\subseteq M_{2d}$$

and $S=UT_{\mathbb{Z}_2,\mathbf{g}}(S_1,\dots,S_m)=\pi(R)\subseteq M_d$, the kernel of the algebra homomorphism $\tilde{\pi}\psi:F\langle X\cup X^*\rangle\to\mathscr{S}(S)$ satisfies

$$\mathrm{Id}^*(\mathscr{G}(R))=\mathrm{Id}^*(\tilde{R})=\ker\psi\subseteq\ker\tilde{\pi}\psi,$$

and inducting on $m\geqslant 1$ we aim to prove that in fact

$$\ker\tilde{\pi}\psi=\mathrm{Id}^*(\mathscr{G}(A_1))\cdots\mathrm{Id}^*(\mathscr{G}(A_m)).$$

Actually, the case $m=1$ (so that $\pi=p$ and $\tilde{\pi}=\tilde{p}$) has been already verified in Proposition 5.7, and $\ker\tilde{\pi}\psi=\mathrm{Id}^*(\mathscr{G}(A_1))$, so let us assume $m\geqslant 2$.

Define $s:=\sum_{i=1}^{m-1}s_i$, $t:=d-s$,

$$\mathbf{g}':=(g_1,\dots,g_{m-1}),\quad A:=UT^{\overline{*}}_{\mathbf{g}'}(A_1,\dots,A_{m-1})\subseteq M_{2s}$$

and let $\pi_A:A\to S_A\subseteq M_s$. It immediately follows that $\tilde{R}$ contains a copy of $\tilde{A}$, a copy of $\tilde{A}_m$ and, after setting $M:=M_{s\times t}$, we have

$$\tilde{\pi}(\tilde{R})\cong\begin{pmatrix}\tilde{\pi}_A(\tilde{A}) & \tilde{M}\\ & \tilde{p}(\tilde{A}_m)\end{pmatrix},$$

where $\tilde{M}$ is a $(\tilde{\pi}_A(\tilde{A}),\tilde{p}(\tilde{A}_m))$-bimodule spanned by the elements

$$\tilde{\delta}_i:=\sum_{h,k}(\xi^{(i)}_{hk}+\varepsilon(h,k)(\xi^{(i)}_{\gamma(k)\gamma(h)})^\sharp)\otimes e_{hk}\quad(h\in[s],\ k\in[t],\ i\geqslant 1).$$

Since M is endowed with an elementary grading, and the (U,V)-variables involved in $\tilde{A},\tilde{A}_m$ and in the spanning set of $\tilde{M}$ are from pairwise disjoint sets, Proposition 5.5 applies and it follows that

$$\ker\tilde{\pi}\psi=(\ker\tilde{\pi}_A\psi)(\ker\tilde{p}\psi)=\mathrm{Id}^*(\mathscr{G}(A_1))\cdots\mathrm{Id}^*(\mathscr{G}(A_m))$$

by Lewin's theorem and inductive hypotheses.

So far, we just proved

$$\mathrm{Id}^*(\mathscr{G}(R)) \subseteq \mathrm{Id}^*(\mathscr{G}(A_1)) \cdots \mathrm{Id}^*(\mathscr{G}(A_m));$$

by the way, since $\mathrm{Id}^*(\mathscr{G}(R))$ is a $*$-ideal,

$$\mathrm{Id}^*(\mathscr{G}(R)) \subseteq \mathrm{Id}^*(\mathscr{G}(A_m)) \cdots \mathrm{Id}^*(\mathscr{G}(A_1))$$

holds as well. Hence $\mathrm{Id}^*(\mathscr{G}(R))$ is contained in the intersection of the two product ideals. Since the reverse inclusion is clearly true, the proof of the theorem is complete. ∎

The involved T^*-ideals $\mathrm{Id}^*(\mathscr{G}(A))$ arising from the $\overline{*}$-simple algebras A can be presented in a concrete way:

Theorem 5.9: *Let A be a finite-dimensional simple algebra, endowed with a superinvolution $\overline{*}$.*

1. *If $(A, \overline{*}) \cong (M_{2k}, \overline{\vartheta}_{2k})$ then $\mathrm{Id}^*(\mathscr{G}(A)) = \mathrm{Id}^*(M_{k,k}(E), \star)$;*
2. *if $(A, \overline{*}) \cong (M_{k+2l}, \tilde{\vartheta}_{k+2l})$ then $\mathrm{Id}^*(\mathscr{G}(A)) = \mathrm{Id}^*(M_{k,2l}(E), \star)$.*

Proof. In case M_{2k} is endowed with a supertranspose, we can rearrange the grading word up to $\overline{*}$-isomorphisms obtaining $(M_{k,k}, \overline{\vartheta})$. Hence

$$\mathscr{G}(M_{k,k}) = M_{k,k}(E)$$

is endowed with the involution $\star$ defined by $(s \otimes a)^\star = s^\sharp \otimes a^{\overline{\vartheta}_{2k}}$. In particular, in case i, j are both $\leqslant k$ or both $> k$ then $|e_{ij}| = 0$ and $(s \otimes e_{ij})^\star = s \otimes e_{ij}^{\vartheta_{2k}}$, while in the remaining cases $|e_{ij}| = 1$ and $(s \otimes e_{ij})^\star = -\varepsilon(i,j)(s \otimes e_{ij}^{\vartheta_{2k}})$. Viewing $M_{k,k}(E)$ decomposed in $(k \times k)$-blocks as usual, that is the set of the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (a, d \in M_k(E_0),\ b, d \in M_k(E_1)),$$

it follows that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\star = \begin{pmatrix} d^{\vartheta_k} & b^{\vartheta_k} \\ -c^{\vartheta_k} & a^{\vartheta_k} \end{pmatrix}.$$

Similar considerations hold in case A is isomorphic to M_{k+2l} endowed with an orthosymplectic superinvolution, getting $\mathscr{G}(A) \cong_* M_{k,2l}(E)$. ∎

Remark 5.10: The description of the $*$-identities of $(M_{k,k}(E), \star)$ is far from trivial: for $k = 1$, that is for $(M_{1,1}(E), \star)$, the $*$-identities have been determined in [DV&Ko], and their $*$-proper multilinear structure has been described in [DV&dS].

A further remark is in order: the usual $*$-polynomial identities of M_n under the transpose and symplectic involution can be recollected from $(M_{k,2l}(E),\star)$. More precisely, if $l=0$ then the resulting algebra is isomorphic to $M_k(E_0)$, hence $*$-PI equivalent to M_k endowed with the transpose involution, while if $k=0$ for the same reasons the resulting algebra is $*$-PI equivalent to M_{2l} endowed with the symplectic involution.

Before describing the remaining cases, notice that if A is any $*$-algebra then we can view $\mathrm{Id}(A)\subseteq\mathrm{Id}^*(A)$: the difference between the free algebra $F\langle X\cup X^*\rangle$ and the free $*$-algebra $F\langle X,*\rangle$ is marked just in the morphisms selection. More precisely, while any algebra homomorphism $F\langle X\cup X^*\rangle\to A$ is uniquely defined by an arbitrary assignment of the free variables $x\in X$, $x^*\in X^*$, a $*$-homomorphism is uniquely determined by the arbitrary assignment of the free variables $x\in X$, because if $\varphi(x)=a\in A$ then by definition $\varphi(x^*)=a^*$. Thus, viewing $F\langle X\cup X^*\rangle$ as the free associative algebra, it makes perfectly sense considering $\mathrm{Id}(A)\subseteq F\langle X\cup X^*\rangle$, and $\mathrm{Id}(A)\subseteq\mathrm{Id}^*(A)$ is plainly true.

The following easy result should now be clear:

Lemma 5.11: *Let A be any algebra, and consider the algebra $A\oplus A^{\mathrm{op}}$ endowed with the exchange involution. Then in $F\langle X\cup X^*\rangle$ we have that*

$$\mathrm{Id}^*(A\oplus A^{\mathrm{op}})=\mathrm{Id}(A)\cap\mathrm{Id}(A^{\mathrm{op}}).$$

About the remaining $\overline{*}$-simple algebras, not listed in Theorem 5.9, we can now state the last result:

Theorem 5.12: *Let $(A,\overline{*})\cong(S\oplus S^{\mathrm{sop}},ex)$, for $S\in\{M_{k,l},M_n(Q)\}$.*

1. *If $S=M_{k,l}$ then $\mathrm{Id}^*(\mathscr{G}(A))=\mathrm{Id}(M_{k,l}(E))$;*
2. *if $S=M_n(Q)$ then $\mathrm{Id}^*(\mathscr{G}(A))=\mathrm{Id}(M_n(E))$.*

Proof. Since $(\mathscr{G}(S\oplus S^{\mathrm{sop}}),\star)\cong_*(\mathscr{G}(S)\oplus\mathscr{G}(S)^{\mathrm{op}},ex)$, it follows at once that

$$\mathrm{Id}^*(\mathscr{G}(A))=\mathrm{Id}(\mathscr{G}(S))\cap\mathrm{Id}(\mathscr{G}(S)^{\mathrm{op}}).$$

If $S=M_{k,l}$ then $\mathscr{G}(S)=M_{k,l}(E)\cong M_{k,l}(E)^{\mathrm{op}}$, and the first statement is proved. If $S=M_n(Q)$ then $\mathscr{G}(M_n(Q))\cong M_n(E)$ is readily checked, and $M_n(E)\cong M_n(E)^{\mathrm{op}}$ is true, too. ∎

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References

[Al&Gi&Ka] E. Aljadeff, A. Giambruno and Y. Karasik, *Polynomial identities with involution, superinvolutions and the Grassmann envelope*, Proceedings of the American Mathematical Society **145** (2017), 1843–1857.

[Ba&T&T] Yu. Bahturin, M. Tvalavadze and T. Tvalavadze, *Group gradings on superinvolution simple superalgebra*, Linear Algebra and its Applications **431** (2009), 1054–1069.

[Be] A. Berele, *Generic verbally prime PI-algebras and their GK-dimensions*, Communications in Algebra **21** (1993), 1487–1504.

[DV&dS] O. M. Di Vincenzo and V. R. T. da Silva, *On the $*$-cocharacters of $M_{1,1}(E)$*, Journal of Pure and Applied Algebra **217** (2013), 1740–1753.

[DV&Ko] O. M. Di Vincenzo and P. Koshlukov, *On the $*$-polynomial identities of $M_{1,1}(E)$*, Journal of Pure and Applied Algebra **215** (2011), 262–275.

[DV&LS] O.M. Di Vincenzoa and R. La Scala, *Minimal algebras with respect to their $*$-exponent*, Journal of Algebra **317** (2007), 642–657.

[DV&N] O. M. Di Vincenzo and V. Nardozza, *$*$-polynomial identities of a nonsymmetric $*$-minimal algebra*, Algebras and Representation Theory **17** (2014), 181–198.

[DV&Sp] O. M. Di Vincenzo and E. Spinelli, *A characterization of $*$-minimal algebras with involution*, Israel Journal of Mathematics **186** (2011), 381–400.

[DV&Va] O. M. Di Vincenzo and A. Valenti, *$*$-multilinear polynomials with invertible values*, Rendiconti del Seminario Matematico della Università di Padova **86** (1991), 47–55.

[GA&Sh] C. Gómez-Ambrosi and I. P. Shestakov, *On the Lie structure of the skew elements of a simple superalgebra with superinvolution*, Journal of Algebra **208** (1998), 43–71.

[Gi&Io&LM] A. Giambruno, A. Ioppolo and D. La Mattina, *Varieties of algebras with superinvolution of almost polynomial growth*, Algebras and Representation Theory **19** (2016), 599–611.

[Gi&Mi] A. Giambruno and S. Mishchenko, *On $*$-varieties with almost polynomial growth*, Algebra Colloquium **8** (2001), 33–42.

[Gi&PM&Va] A. Giambruno, C. Polcino Milies and A. Valenti, *Star-polynomial identities: computing the exponential growth of the codimensions*, Journal of Algebra **469** (2017), 302–322.

[Gi&Za1] A. Giambruno and M. Zaicev, *On codimension growth of finitely generated associative algebras*, Advances in Mathematics **140** (1998), 145–155.

[Gi&Za2] A. Giambruno and M. Zaicev, *Exponential codimension growth of PI-Algebras: an exact estimate*, Advances in Mathematics **142** (1999), 221–243.

[Gi&Za3] A. Giambruno and M. Zaicev, *Involution codimensions of finite dimensional algebras and exponential growth*, Journal of Algebra **222** (1999), 471–484.

[Gi&Za4] A. Giambruno and M. Zaicev, *Codimension growth and minimal superalgebras*, Transactions of the American Mathematical Society **355** (2003), 5091–5117.

[GZBook] A. Giambruno and M. Zaicev, *Polynomial Identities and Asymptoyic Methods*, Mathematical Surveys and Monographs, vol. 122, American Mathematical Society, 2005.

[Io] A. Ioppolo, *The exponent for superalgebras with superinvolution*, Linear Algebra and Its Applications **555** (2018), 1–10.

[Ka] V. G. Kac, *Lie superalgebras*, Advances in Mathematics **26** (1977), 8–96.

[Le] U. Leron, *Nil and power-central polynomials in rings*, Transactions of the American Mathematical Society **202** (1975), 97–103.

[Lw] J. Lewin, *A matrix representation for associative algebras, I*, Transactions of the American Mathematical Society **188** (1974), 293–308.

[Mi&Va] S. Mishchenko and A. Valenti, *A star-variety with almost polynomial growth*, Journal of Algebra **223** (2000), 66–84.

[Ra&Ze] M. L. Racine and E. I. Zelmanov, *Simple Jordan superalgebras with semisimple even part*, Journal of Algebra **270** (2003), 374-444.

[Sv] I. Sviridova, *Finitely generated algebras with involution and their identities*, Journal of Algebra **383** (2013), 144–167.