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The interplay between fractional damping and nonlinear memory for the plate equation

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In this paper, we study the interplay between a fractional damping $(-\Delta)^\theta u_t$ with $\theta \in [0,1/2)$, and a nonlinear memory term applied to a plate equation:

$$u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u + (-\Delta)^\theta u_t = \int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds, \qquad t>0, x \in \mathbb{R}^n,$$

in space dimension $n = 1,2,3,4$. In different scenarios, we prove the global existence of small data solutions in $\mathcal{C}([0,\infty),H^2) \cap \mathcal{C}^1([0,\infty),H^1)$ for supercritical powers, where the critical exponent is determined by the fractional power of the damping term and by the order of the fractional integration in the memory term. In one case, we also feel the influence of the regularity-loss type decay, which is due to the presence of the rotational inertia term $-\Delta u_{tt}$ in the plate equation.

KEYWORDS

critical exponent, global small data solutions, nonlinear memory, plate equation

MSC CLASSIFICATION

35L71; 26A33; 74K20

1 | INTRODUCTION

In this work, we study the global existence of solutions in $\mathcal{C}([0,\infty),H^2) \cap \mathcal{C}^1([0,\infty),H^1)$ to the problem

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u + (-\Delta)^\theta u_t = \int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds, \ t \geq 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

for $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in (0,1)$ and $p > 1$, assuming small data in L^1 and in $H^s \times H^{s-1}$, where $s = 2 + 2\gamma(1-\theta)$.

The fractional Laplace operator appearing in the damping term $(-\Delta)^\theta u_t$ is defined as the extension by density of the operator $f \in \mathcal{S} \mapsto (-\Delta)^\theta f = \mathfrak{F}^{-1}(|\xi|^{2\theta} \hat{f})$, where $\mathfrak{F} f = \hat{f}$ denotes the Fourier transform with respect to the x variable.

The integral on the right-hand side of the differential equation in (1) is sometimes called *nonlinear memory term*, and it may be written in the form $\Gamma(1-\gamma)I^{1-\gamma}|u|^p$, where $I^{1-\gamma}$ is the Riemann–Liouville fractional integral of order $1-\gamma$.

Since the Riemann–Liouville Integral $I^\alpha f$ converges to f almost everywhere when $\alpha \to 0$, it is natural to expect a relation between problem (1) and problem

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u + (-\Delta)^\theta u_t = |u|^p, & t \geq 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{2}$$

as $\gamma \to 1$.

We will look for the *critical exponent* $\bar{p} := \bar{p}(n,\gamma,\theta)$, $\bar{p} > 1$, for (1), that is, a positive value such that

- If $p > \bar{p}$, then there exist global in-time small data solutions to (1), for a suitable choice of data and solution spaces;
- If $1 < p < \bar{p}$, there exist arbitrarily small initial data such that there is no global in-time solution to (1).

H. Fujita proved in 1966[1] that the critical exponent for the classical semilinear heat equation with nonlinearity $F(u) = u^p$ is $p_F = 1 + \frac{2}{n}$. This is widely known as the *Fujita exponent*. In 2001, G. Todorova and B. Yordanov proved[2] that the *critical exponent* is still Fujita exponent for the classical damped wave equation with nonlinearity $F(u) = |u|^p$, with the nonexistence result for the critical case $p = p_F$ being proved by Qi S. Zhang.[3] The case of critical regularity for a nonlinearity of type $|u|^{1+2/n}\mu(|u|)$ has been recently investigated in Ebert et al.[4]

In space dimensions $n = 1, 2$ we prove the existence of global small data solutions for $p > p_c$ (Theorem 1) where p_c is given by

$$p_c(n,\gamma,\theta) := 1 + \frac{2\,(1 + (1-\gamma)(1-\theta))}{(n - 2 + 2\gamma(1-\theta))_+}. \tag{3}$$

As $\gamma \to 1,\ p_c \to 1 + 2/(n - 2\theta)$, consistently with the result obtained in D'Abbicco and Ebert[5] for evolution equations with structural damping and, previously, in D'Abbicco and Reissig[6] for wave models.

Since the study of plate models has a special interest in space dimension $n = 2$, due to its physical background (see Section 1.1), we stress that in space dimension $n = 2$, we may write p_c in its simpler form

$$p_c = \frac{1}{\gamma}\frac{2-\theta}{1-\theta}.$$

In particular, p_c increases from $2\gamma^{-1}$ to $3\gamma^{-1}$ as θ goes from 0 to 1/2.

The influence of the nonlinear memory becomes stronger in space dimension $n \geq 3$. In this case, for small values of γ the critical exponent becomes γ^{-1}. We study this phenomenon in space dimension $n = 3$, proving the global existence of small data solutions for $p > \bar{p}$ (Theorems 2 and 3), where

$$\bar{p}(3,\gamma,\theta) = \max\left\{p_c, \gamma^{-1}\right\} = \begin{cases} p_c & \text{if } 1/3 \leq \gamma < 1, \\ \gamma^{-1} & \text{if } 0 < \gamma \leq 1/3. \end{cases}$$

This phenomenon was first investigated in 2008 by T. Cazenave, F. Dickstein, and F. Weissler,[7] who proved that the *critical exponent* for the heat equation with nonlinear memory

$$\begin{cases} v_t - \Delta v = \int\limits_0^t (t-s)^{-\gamma} v(s,\cdot)^p\, ds \\ v(0,x) = v_0(x) \geq 0, \end{cases} \tag{4}$$

is $\bar{p}(n,\gamma) := \max\{p_\gamma(n), \gamma^{-1}\}$, where

$$p_\gamma(n) := 1 + \frac{2(2-\gamma)}{(n - 2(1-\gamma))_+},$$

with the usual convention for indexes that $\frac{k}{0} = \infty$, for any $k \in \mathbb{R}$.

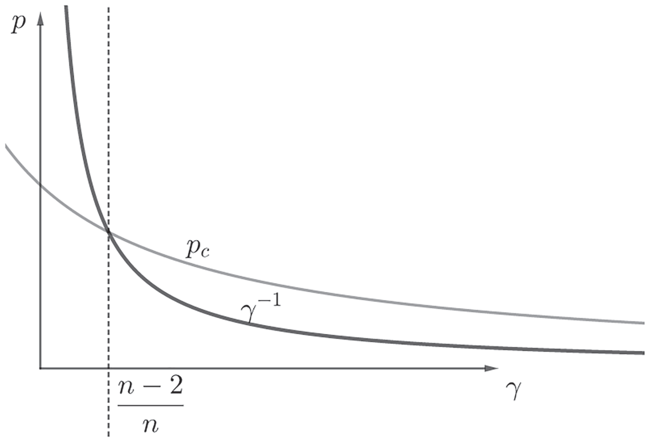

FIGURE 1 Critical exponent

In 2014, the first author[8] proved that the same effect occurs for the damped wave equation with nonlinear memory,

$$\begin{cases} u_{tt} - \Delta u + u_t = \int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds, \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \end{cases} \tag{5}$$

and later extended this analysis to the case of a wave equation with structural damping.[9] The same phenomenon appears if one consider a memory type nonlinearity with $|u_t(s,\cdot)|^p$ in place of $|u(s,\cdot)|^p$; for instance, in[10] the authors studied the nonlinear Cauchy problem

$$\begin{cases} u_{tt} + (-\Delta)^{\sigma} u + (-\Delta)^{\frac{\sigma}{2}} u_t = \int_0^t (t-s)^{-\gamma} |u_t(s,\cdot)|^p \, ds \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \end{cases}$$

and they proved the global existence of small data solutions for $p > \max\{\gamma^{-1}, 1 + (2-\gamma)\sigma/(n - \sigma(1-\gamma))\}$ for any $\sigma > 0$ and $\gamma \in (0,1)$.

In this paper we do not investigate the nonexistence of global solutions in the subcritical and critical case $1 < p \leq \bar{p}$, but these results can be easily obtained using the test function method in D'Abbicco and Fujiwara[11] to treat the fractional power of the Laplace operator and extend the approach in D'Abbicco.[9]

In Figure 1 we show the critical exponent $\bar{p}$ obtained by the maximum between p_c and γ^{-1}, when $n \geq 3$.

We mention that in space dimension $n = 3$, for large values of γ, namely, $\gamma > \frac{3-2\theta}{4(1-\theta)}$, the critical exponent (3) becomes smaller than 2. To deal with critical exponents smaller than 2, we employ decay estimates of the $\|u(t,\cdot)\|_{L^p}$ norm of the solution, which are, in general, more difficult to obtain than energy estimates. The solution space is so resctricted from $\mathcal{C}([0,\infty), H^2) \cap \mathcal{C}^1([0,\infty), H^1)$ to $\mathcal{C}([0,\infty), H^2) \cap \mathcal{C}^1([0,\infty), H^1) \cap L^\infty([0,\infty), L^p)$.

In particular, we shall consider high frequencies $L^p - L^p$ estimates for the linear equation associated to our problem. In this case, the phenomenon of regularity-loss decay adds to the loss of decay rate for L^p estimates, due to the presence of oscillations at high frequencies. Although the high frequencies decay rate does suffer some loss, this loss is not large enough to interfere with the critical exponent, which remains $\bar{p}$, as we prove in Theorem 4, if we assume initial data smallness in $W^{3,p} \times W^{2,p}$.

So far, up to space dimension $n = 3$, this paper extends the interplay between the fractional power of the damping term θ and the nonlinear memory term, which was previously investigated for the wave equation in D'Abbicco.[9] The results obtained in this paper for the plate equation in space dimension $n = 1, 2, 3$, are consistent with the results obtained for the wave equation in Theorems 5.1 and 5.3 in D'Abbicco,[9] in space dimension $n \leq 4$. That is, the critical exponent for

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^{\theta} u_t = \int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds, \;\; t \geq 0, \, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x), \end{cases} \tag{6}$$

for $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in (0,1)$, is also given by $\max\{p_c(n,\gamma,\theta), \gamma^{-1}\}$, with p_c as in (3).

The lack of influence from the rotational inertia term, to the critical exponent, in dimension $n \leq 3$ motivates us to go one step further and investigate the case $n = 4$. Here, the interplay between the parameters p, γ and θ becomes very delicate, and therefore for simplicity we assume that $\theta = 0$, since our intention is only to show that some new effect may occur. Indeed, it does: Assuming $n = 4$ and $\theta = 0$, the loss of decay when $\gamma \in \left(\frac{7}{8}, 1\right)$ leads us to find existence of global small data solutions for $p > \tilde{p}_c$, where

$$\tilde{p}_c := 1 + \frac{7 - 2\gamma}{7 + 2\gamma}, \tag{7}$$

in particular, $\tilde{p}_c > p_c$ if, and only if, $\gamma > 7/8$. That is, in space dimension $n = 4$ we prove the existence of global in-time small data solutions to (1) if $p > \bar{p}(4,\gamma,0)$, where

$$\bar{p}(4,\gamma,0) = \max\left\{\gamma^{-1}, p_c, \tilde{p}_c\right\}. \tag{8}$$

Besides the assumption that the initial data of (1) are in L^1, the regularity-loss decay structure of the equation makes natural the choice of the space $H^{s_c}(\mathbb{R}^n) \times H^{s_c - 1}(\mathbb{R}^n)$, for the regularity of initial data, where

$$s_c = s_c(\gamma, \theta) := 2 + 2\gamma(1 - \theta). \tag{9}$$

This condition allows us to produce enough decay rate at high frequencies, to match the desired decay rate at low frequencies, for p close to the critical exponent p_c. For larger values of p, this condition can be relaxed.

In this paper, we do not deal with the case of stronger damping $\theta \in [1/2, 2]$. We expect that the case $\theta = 1/2$ could be dealt in a relatively simple way, whereas in the case $\theta \in (1/2, 2]$ the oscillations in the low-frequency region make more difficult to obtain estimates for the linear problem (19) which can be effectively applied to find the critical exponent for the semilinear problem[12,13]; moreover, the regularity-loss effect disappears for $\theta \in [1, 2]$, and the oscillations are canceled at high frequencies in the case $\theta \in [3/2, 2]$; on the other hand, adding also a strong damping $(-\Delta)^{\delta} u_t$ with $\delta > 1/2$ in (19) one would feel the benefit of both the dissipation terms, in the low-frequencies and large-frequencies regions, in the study of estimates for the linear problem (see Chen et al.[14]).

1.1 | Background for plate models

Fourth-order evolution partial differential equations arise in problems of solid mechanics as, for example, in the theory of thin plates and beams. Also, in particular formulations of problems related with the Navier–Stokes equations (see Teman[15]) appear elliptic equations of fourth-order. Models to study the vibrations of thin plates ($n = 2$) given by the full von Kármán system have been studied by several authors, in particular by Ciarlet,[16] Sánches,[17] Lasiecka,[18] Lasiecka–Benabdallah,[19] Koch–Lasiecka,[20] Puel–Tucsnak.[21] Perla Menzala–Zuazua[22] considered the full von Kármán system and they proved that the Timoshenko's model

$$u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + u = 0, \text{in } \mathbb{R}^2 \times (0, \infty) \tag{10}$$

may be obtained as limit of a full von Kármán system when suitable parameters tend to zero. The term $-\Delta u_{tt}$ is to absorb in the system the rotational inertia effects at the point x of the plate in a positive time t. It is well known that the plate Eqution (10) with $\gamma > 0$ is a hyperbolic equation with finite speed of propagation, whereas the non-rotational plate model with $\gamma = 0$ has infinite speed of propagation. The hyperbolic model with $\gamma > 0$ is more complicated to be analyzed than the non-hyperbolic one. In particular, for the dissipative plate equation

$$u_{tt} - \gamma \Delta u_{tt} + (-\Delta)^{\theta} u_t + \Delta^2 u = 0,$$

with $\theta \in [0, 1)$ and $\gamma > 0$, new difficulties arise, due to the property of regularity-loss decay.[23,24] This fact can be observed by analyzing the structure of the eigenvalues associated to the plate equation in the Fourier space (see previous works[23,24]). Due to that special structure, when we get estimates in the region of high frequencies it is necessary to impose additional regularity on the initial data to obtain the same decay estimates as in the region of low frequencies. This special decay rate structure also influences the application to semilinear problems.[25,26] The additional regularity necessary to obtain the result appears in a theorem in the next section. This effect does not appear if $\gamma = 0$, since the solution exponentially decays in the zone of high frequency of the Fourier space, even with fractional damping; in this latter case, rather, a smoothing effect appear if one considers $\theta > 1$ (see, for instance,[27] where the case $\theta = 2$ and $\gamma = 0$ is considered).

A more general equation to model the vibrations of a thin plate is given by

$$u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + g_0(u_t) - \operatorname{div} g_1(\nabla u_t) = 0. \tag{11}$$

Such models have been studied by several authors.[28–31] Sugitani–Kawashima[24] obtained decay rates for a semilinear plate equation in $\mathbb{R}^n$ with $g_1 = 0$ and $g_0 = \mathrm{Id} - f$. The term u_t represents a frictional dissipation in the plate, and the nonlinear term $f(v)$ is a smooth function of v satisfying $f(v) = O(|v|^2)$ for $v \to 0$.

Moreover, Andrade–Silva–Ma[32] proved exponential stability for a plate equation with p-Laplacian and memory terms. To get this result they considered a structural damping of type $-\triangle u_t$. Furthermore, there are some papers in which a strong damping of type $(-\Delta)^2 u_t$ is considered in model (11), in place of the damping given in $g_0(u_t) - \operatorname{div} g_1(\nabla u_t)$ (see, e.g., previous works,[33–35] and references therein).

1.2 | Notation

We list some notation used in this paper:

- the expression $(a)_+$ denotes the positive part of a, that is, $(a)_+ = \max\{a, 0\}$;
- the expression $f(t) \lesssim g(t)$ denotes that there exists a constant $C > 0$, such that $f(t) \leq Cg(t)$, uniformly with respect to t;
- the expression $\mathcal{F}f$ or $\hat{f}$ denotes the Fourier transform with respect to the x variable of f;
- the notation $\|f\|_{\dot{H}^s}$, when $s > 0$, denotes the quantity $\left\| |\xi|^s \hat{f} \right\|_{L^2}$ for a function $f \in H^s$;
- we put $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$;
- the operator $(I - \Delta)^{-1}$ denotes the Bessel potential of order 2, whose action may be defined by $(I - \Delta)^{-1} f = \mathcal{F}^{-1}(\langle \xi \rangle^{-2} \hat{f})$ for any $f \in \mathcal{S}$, and then extended by density.

2 | RESULTS

We first deal with dimensions $n = 1$ and $n = 2$.

Here, we observe that the definition of p_c implicitly requires that $\gamma \in \left(\frac{1}{2(1-\theta)}, 1 \right)$ in the case $n = 1$. Otherwise, the denominator in (3) becomes non-positive.

Theorem 1. *Assume* $n = 1, 2$, $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in (0, 1)$, $p > p_c$ *and* $s = s_c$, *with* p_c *as in (3) and* s_c *as in (9).*

Then, there exists $\varepsilon > 0$ *such that, for initial data*

$$(u_0, u_1) \in \mathcal{A} := \left(H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \right) \times \left(H^{s-1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \right), \tag{12}$$

with $\|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon$, *there exists a unique global solution to the problem (1),* $u \in C\left([0, \infty), H^2\right) \cap C^1\left([0, \infty), H^1\right)$. *Moreover, the following estimates hold:*

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{n-4\theta}{4(1-\theta)}+1-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } n = 2, \text{ or } n = 1 \text{ and } \theta \in \left[0, \frac{1}{4}\right), \\ (1+t)^{2-\frac{1}{4\theta}-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } n = 1 \text{ and } \theta \in \left[\frac{1}{4}, \frac{1}{2}\right). \end{cases}$$

$$\|u(t,\cdot)\|_{\dot{H}^1} \lesssim \begin{cases} (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } n = 1, \\ \log(2+t)(1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } n = 2, \end{cases}$$

$$\|u(t,\cdot)\|_{\dot{H}^2} \lesssim (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } \theta \in [0, n/4), \\ (1+t)^{-\gamma} \log(2+t) \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if}, n = 1 \text{ and } \theta = 1/4, \\ (1+t)^{1-\frac{1}{4\theta}-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } n = 1 \text{ and } \theta \in (1/4, 1/2), \end{cases}$$

$$\|u_t(t,\cdot)\|_{\dot{H}^1} \lesssim (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}.$$

In space dimension $n = 3$ we shall distinguish different scenarios. First of all, we show that the new critical exponent γ^{-1} appears for small values of γ. This effect also shows up in space dimension $n = 4$, so that we merge these two cases, for the sake of brevity. The regularity of the solution and the initial data is the same as given by Theorem 1.

Theorem 2. *Assume that* $n = 3, 4$, $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in \left(0, \frac{n-2}{n}\right]$. *Also, assume* $p > \gamma^{-1}$, *and* $s = s_c$, *with* s_c *as in (9).*

Then, there exists $\varepsilon > 0$ *such that, for initial data as in (12), there exists a unique global solution to the problem (1),* $u \in C\left([0, \infty), H^2\right) \cap C^1\left([0, \infty), H^1\right)$. *Moreover, the following estimates hold:*

$$\|u(t, \cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } n = 3, \\ \log(2+t)(1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } n = 4, \end{cases} \tag{13}$$

$$\|u(t, \cdot)\|_{\dot{H}^2} \lesssim \begin{cases} (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0, \\ (1+t)^{-\gamma} \log(2+t) \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0, \end{cases} \tag{14}$$

$$\|u_t(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, \tag{15}$$

$$\|u_t(t, \cdot)\|_{\dot{H}^1} \lesssim \begin{cases} (1+t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0, \\ (1+t)^{-\gamma} \log(2+t) \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0, \end{cases} \tag{16}$$

Assuming larger values for γ, the candidate p_c for *critical exponent* decreases. If $\gamma \leq \frac{3-2\theta}{4-4\theta}$, then $p_c \geq 2$ and the initial data and solution regularity may be easily taken as in Theorems 1 and 2. For $\gamma > \frac{3-2\theta}{4-4\theta}$, though, p_c becomes smaller than 2, therefore we could have $p < 2$. We break this case in two possibilities. First, we impose the condition $p \geq 2$, for which we can prove our results with no additional difficulties.

Theorem 3. *Assume that* $n = 3$ *and* $\theta \in \left[0, \frac{1}{2}\right)$. *Let* $\gamma \in \left(\frac{1}{3}, \frac{3-2\theta}{4(1-\theta)}\right]$ *and* $p > p_c$, *with* p_c *as in (3); or* $\gamma \in \left(\frac{3-2\theta}{4(1-\theta)}, 1\right)$ *and* $p \geq 2$. *Also, let* $s = s_c$, *with* s_c *as in (9). Then, there exists* $\varepsilon > 0$ *such that, for initial data as in (12), there exists a unique global solution to the problem (1),* $u \in C\left([0, \infty), H^2\right) \cap C^1\left([0, \infty), H^1\right)$. *Moreover, estimates (13), (14), (15), and (16) hold.*

To deal with the case $p < 2$, we use an approach based on $(L^1 \cap L^p) - L^p$ estimates for $p < 2$, see Narazaki[36] (see also D'Abbicco[8]). That is, we additionally ask the solution to be in $L^\infty([0, \infty), L^p)$.

Theorem 4. *Assume that* $n = 3$ *and* $\theta \in \left[0, \frac{1}{2}\right)$. *Let* $\gamma \in \left(\frac{3-2\theta}{4(1-\theta)}, 1\right)$, $p \in (p_c, 2]$ *and* $s = s_c$, *with* p_c *as in (3) and* s_c *as in (9). Then, there exists* $\varepsilon > 0$ *such that, for initial data* $(u_0, u_1) \in \mathcal{A}$, *where*

$$\mathcal{A} := \left(H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \cap W^{3,p}(\mathbb{R}^3)\right) \times \left(H^{s-1}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \cap W^{2,p}(\mathbb{R}^3)\right), \tag{17}$$

with $\|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon$, *there exists a unique global solution to the problem (1),* $u \in C\left([0, \infty), H^2\right) \cap C^1\left([0, \infty), H^1\right) \cap L^\infty\left([0, \infty), L^p\right)$. *Moreover, estimates (13), (14), (15), (16) hold, as well as the following estimate:*

$$\|u(t, \cdot)\|_{L^p} \leq C_\delta (1+t)^{\delta - \frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+1-\gamma} \|(u_0, u_1)\|_{\mathcal{A}},$$

for any small $\delta > 0$.

For dimension $n = 4$ the reasoning is the same: for large values of γ, we first consider $p \geq 2$.

Theorem 5. *Assume that* $n = 4$, *and* $\theta \in \left[0, \frac{1}{2}\right)$. *Let* $\gamma \in \left(\frac{1}{2}, 1\right)$, $p \geq 2$ *and* $s = s_c$, *with* s_c *as in (9). Then, there exists* $\varepsilon > 0$ *such that, for initial data as in (12), there exists a unique global solution to the problem (1),* $u \in C\left([0, \infty), H^2\right) \cap C^1\left([0, \infty), H^1\right)$. *Moreover, estimates (13), (14), (15), (16) hold.*

For $\gamma \in \left(\frac{1}{2}, 1\right)$ and $p < 2$, a new threshold appears. Here, for simplicity, we assume $\theta = 0$. Then, if $\gamma \in \left(\frac{1}{2}, \frac{7}{8}\right]$, we can still prove existence of solutions for $p > p_c$, using the aforementioned $(L^1 \cap L^p) - L^p$ estimates.

Theorem 6. *Assume $n = 4$ and $\theta = 0$. Let $\gamma \in \left(\frac{1}{2}, \frac{7}{8}\right]$, $p \in (p_c, 2)$ and $s = s_c = 2 + 2\gamma$. Then, there exists $\varepsilon > 0$ such that, for initial data*

$$\begin{aligned} &(u_0, u_1) \in \mathcal{A} := \left(H^s(\mathbb{R}^4) \cap L^1(\mathbb{R}^4) \cap W^{3,p}(\mathbb{R}^4)\right) \times \left(H^{s-1}(\mathbb{R}^4) \cap L^1(\mathbb{R}^4) \cap W^{2,p}(\mathbb{R}^4)\right), \\ &\quad \textit{with} \quad \|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon, \end{aligned} \tag{18}$$

there exists a unique global solution to the problem (1), $u \in C\left([0, \infty), H^2\right) \cap C^1\left([0, \infty), H^1\right) \cap L^\infty\left([0, \infty), L^p\right)$. Moreover, estimates (13), (14), (15), and (16) hold, as well as

$$\|u(t, \cdot)\|_{L^p} \leq C_\delta (1 + t)^{\delta - 2\left(1 - \frac{1}{p}\right) + 1 - \gamma} \|(u_0, u_1)\|_{\mathcal{A}},$$

for any small $\delta > 0$.

Lastly, in the case $n = 4$, for $\gamma > 7/8$, the regularity loss structure of decay may influence the critical exponent. In the following result, we prove global existence of small data solutions for $p > \tilde{p}_c$, where $\tilde{p}_c > p_c$. We do not know if the estimates we obtained are sharp, and consequently the associated exponent $\tilde{p}_c$.

Theorem 7. *Assume $n = 4$ and $\theta = 0$. Let $\gamma \in \left(\frac{7}{8}, 1\right)$, $p \in (\tilde{p}_c, 2)$, with $\tilde{p}_c$ as in (7), and $s = s_c = 2 + 2\gamma$. Then, there exists $\varepsilon > 0$ such that, for initial data as in (18), there exists a unique global solution to the problem (1), $u \in C\left([0, \infty), H^2\right) \cap C^1\left([0, \infty), H^1\right) \cap L^\infty\left([0, \infty), L^p\right)$. Moreover, estimates (13), (14), (15), (16) hold, as well as*

$$\|u(t, \cdot)\|_{L^p} \lesssim \begin{cases} (1 + t)^{-\frac{3}{2} - 6\left(\frac{1}{p} - \frac{1}{2}\right) + 1 - \gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \textit{if } p \in \left(\tilde{p}_c, \frac{8}{5}\right), \\ C_\delta (1 + t)^{\delta - 2\left(1 - \frac{1}{p}\right) + 1 - \gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \textit{if } p \in \left[\frac{8}{5}, 2\right). \end{cases}$$

Incidentally, we notice that H^s and H^{s-1} may be omitted in (17) and (18) when $W^{3,p} \hookrightarrow H^s$, that is, $2(n - (3 - s)p) \leq np$.

3 | ESTIMATES FOR THE LINEAR PROBLEM

The main argument in all of our proofs consists in a construction of solutions to nonlinear problems, based on applying Duhamel's Principle and a fixed point argument. To this end, we make use of the decay rate of solutions to the linear problem associated to (1), that is, the following linear Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u + (-\Delta)^\theta u_t = 0 \\ u(0, x) = u_0(x) \\ u_t(0, x) = u_1(x), \end{cases} \tag{19}$$

with $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in (0, 1)$ and $p > 1$.

If u^{lin} is the solution to the linear associated problem (19), then it satisfies the following estimates:

Lemma 1. *Let $n \in \mathbb{N}$, $\theta \in \left[0, \frac{1}{2}\right)$, $k \geq 0$, $j = 0, 1$, $s \geq 1$, and assume that $u^{lin}(t, x)$ is the (global) solution to the Cauchy problem (19) with initial data $(u_0, u_1) \in (L^1(\mathbb{R}^4) \cap W^{3,p}(\mathbb{R}^4)) \times (L^1(\mathbb{R}^4) \cap W^{2,p}(\mathbb{R}^4))$. Then, $u^{lin}(t, x)$ satisfies*

$$\left\|\partial_t^j u^{lin}(t, \cdot)\right\|_{\dot{H}^k} \lesssim (1 + t)^{-\frac{s-k-j}{2(1-\theta)}} \|(u_0, u_1)\|_{H^s \times H^{s-1}} + \begin{cases} (1 + t)^{-\frac{n+2k-4\theta}{4(1-\theta)} - j} \|(u_0, u_1)\|_{L^1 \times L^1}, & \textit{if } \frac{n}{2} + k - 2\theta > 0, \\ (1 + t)^{1 - \frac{n+2k}{4\theta} - j} \|(u_0, u_1)\|_{L^1 \times L^1}, & \textit{if } \frac{n}{2} + k - 2\theta < 0, \\ a_j(t) \|(u_0, u_1)\|_{L^1 \times L^1}, & \textit{if } \frac{n}{2} + k - 2\theta = 0, \end{cases} \tag{20}$$

with

$$a_j(t) = \begin{cases} \log(e+t), & j = 0 \\ (1+t)^{-1}, & j = 1. \end{cases} \tag{21}$$

The proof of Lemma 1 is a special case of an already treated problem, so we omit its proof, addressing the interested reader to the work by M. Ebert, C. da Luz, and M. Palma,[37] and to the cited references.

We stress that in the term with no derivatives in time in (19), that is, $-\Delta u + \Delta^2 u$, the first part, $-\Delta u$ influences the solution in the low-frequency region, whereas the second part, $\Delta^2 u$, influences the solution in the high-frequency region. This is due to behavior of its symbol $|\xi|^2 \jmath 2$, that is, $|\xi|^2 \jmath 2 \approx |\xi|^2$ for small $|\xi|$ and $|\xi|^2 \jmath 2 \approx |\xi|^4$ for large $|\xi|$.

To deal with small powers $p<2$ of the power nonlinearity, we supplement Lemma 1 with additional $(L^1 \cap L^p) - L^p$ estimates, whose proof we postpone to the Appendix 3.

Lemma 2. *Let* $n \in \mathbb{N}$, $\theta \in \left[0, \frac{1}{2}\right)$, $p \in \left(\frac{n}{n-2\theta}, 2\right)$, *and assume that* $u^{lin}(t,x)$ *is the (global) solution to the Cauchy problem (19) with initial data in* $(L^1(\mathbb{R}^4) \cap W^{3,p}(\mathbb{R}^4)) \times (L^1(\mathbb{R}^4) \cap W^{2,p}(\mathbb{R}^4))$. *Then, for any small* $\delta > 0$, $u^{lin}(t,x)$ *satisfies*

$$\left\| u^{lin}(t,\cdot) \right\|_{L^p} \leq C_\delta (1+t)^{\delta - \frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}} \|(u_0,u_1)\|_{L^1 \times L^1} + C(1+t)^{-\frac{3-n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|(u_0,u_1)\|_{W^{3,p} \times W^{2,p}}. \tag{22}$$

The regularity $W^{2,p}$ in (22) is related to the fact that in Duhamel's principle, the Bessel potential $(I-\Delta)^{-1}$ will be applied to the data regularity, effectively lowering the regularity to L^p.

We remark that the assumption $p > \frac{n}{n-2\theta}$ is trivially verified in our statements in Section 2, since this value is smaller than p_c, for any $\gamma \in (0,1)$.

Now, another important set of estimates are the ones involving the operator associated with the application of Duhamel's principle to the corresponding semilinear equation. These estimates are obtained taking $u_0 = 0$ and applying the Bessel potential $(I-\Delta)^{-1}$ to u^{lin}. Indeed, the presence of the rotational inertia term $-\Delta u_{tt}$ and the application of Duhamel's principle, leads to gain two derivatives in the regularity-loss decay rate structure at high frequencies. These estimates will be crucial in the nonlinear existence argument:

Lemma 3. *Let* $n \in \mathbb{N}$, $\theta \in \left[0, \frac{1}{2}\right)$, $k \geq 0, j = 0, 1$, $s \geq 3$, *and assume that* $u^{lin}(t,x)$ *is the (global) solution to the Cauchy problem (19) with* $u_0 = 0$ *and* $u_1 = \varphi$, *where* $\varphi \in H^{s-3} \cap L^1$. *Then, we have the estimates*

$$\left\| \partial_t^j (I-\Delta)^{-1} u^{lin}(t,\cdot) \right\|_{\dot{H}^k} \lesssim (1+t)^{-\frac{s-k-j}{2(1-\theta)}} \|\varphi\|_{H^{s-3}} + \begin{cases} (1+t)^{-\frac{n+2k-4\theta}{4(1-\theta)}-j} \|\varphi\|_{L^1}, & \text{if } \frac{n}{2}+k-2\theta > 0, \\ (1+t)^{1-\frac{n+2k}{4\theta}-j} \|\varphi\|_{L^1}, & \text{if } \frac{n}{2}+k-2\theta < 0, \\ a_j(t) \|\varphi\|_{L^1}, & \text{if } \frac{n}{2}+k-2\theta = 0, \end{cases} \tag{23}$$

with $a_j(t)$ *as in (21).*

Lemma 4. *Let* $n \in \mathbb{N}$, $\theta \in \left[0, \frac{1}{2}\right)$, $p \in \left(\frac{n}{n-2\theta}, 2\right)$, *and assume that* $u^{lin}(t,x)$ *is the (global) solution to the Cauchy problem (19) with* $u_0 = 0$ *and* $u_1 = \varphi$, *where* $\varphi \in L^1 \cap L^p$. *Then, for any* $\delta > 0$, *we have the estimates*

$$\left\| (I-\Delta)^{-1} u^{lin}(t,\cdot) \right\|_{L^p} \leq C_\delta (1+t)^{\delta - \frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}} \|\varphi\|_{L^1} + C(1+t)^{-\frac{3}{2(1-\theta)} + \frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|\varphi\|_{L^p}. \tag{24}$$

We postpone the proofs of Lemmas 2 and 4, which are fairly technical, to Appendix 3.

4 | SCHEME OF THE GLOBAL EXISTENCE ARGUMENT

We may write the solution to the linear Cauchy problem (19) in the following form of convolution in the spatial variable

$$u^{\text{lin}}(t,x) = K_0(t,x) *_{(x)} u_0(x) + K_1(t,x) *_{(x)} u_1(x), \tag{25}$$

where $K_0(t,x), K_1(t,x)$ are the fundamental solutions to (19). Let $\lambda_{\pm}$ be the characteristic roots of (19), i.e.,

$$\lambda_{\pm} := \frac{-\mathbf{x}2\theta \pm \sqrt{\mathbf{x}4\theta - 4\jmath 4\mathbf{x}2}}{2\jmath 2}. \tag{26}$$

Then $K_j(t,x) = \mathcal{F}^{-1}\left(\hat{K}_j(t,\xi)\right)$, $j = 0, 1$, where

$$\hat{K}_0(t,\xi) := \frac{\lambda_+ e^{t\lambda_-} - \lambda_- e^{t\lambda_+}}{\lambda_+ - \lambda_-}, \; \hat{K}_1(t,\xi) := \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-}. \tag{27}$$

Now, for $T > 0$, we define the Banach space

$$X(T) := \begin{cases} \mathcal{C}\left([0,T], H^2\right) \cap \mathcal{C}^1\left([0,T], H^1\right), & \text{if } p \geq 2; \\ \mathcal{C}\left([0,T], H^2\right) \cap \mathcal{C}^1\left([0,T], H^1\right) \cap L^\infty\left([0,T], L^p\right), & \text{if } p < 2, \end{cases} \tag{28}$$

equipped with the norm

$$\|v\|_{X(T)} := \begin{cases} \sup\limits_{t\in[0,T]} \left(\sum\limits_{j=0}^{m} h_j(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^{k_j}} + \sum\limits_{l=0}^{1} \tilde{h}_l(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^{l}} \right), & \text{if } p \geq 2; \\ \sup\limits_{t\in[0,T]} \left(\sum\limits_{j=0}^{m} h_j(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^{k_j}} + \sum\limits_{l=0}^{1} \tilde{h}_l(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^{l}} + h_p^*(t)^{-1} \|v(t,\cdot)\|_{L^p} \right), & \text{if } p < 2, \end{cases} \tag{29}$$

for some $m \geq 1$ and suitable functions $h_j, \tilde{h}_l, h_p^*$ where $0 = k_0 < \ldots < k_m = 2$. (here $\dot{H}^0 = L^2$). We fix an adequate space $\mathcal{A}$ for the initial data (u_0, u_1) such that the solution u^{lin} to the associated linear problem (19) satisfies the estimates

$$\begin{aligned} \left\|u^{\text{lin}}(t,\cdot)\right\|_{\dot{H}^{k_j}} &\lesssim h_j(t) \|(u_0,u_1)\|_{\mathcal{A}}, \text{ for any } j = 0, \ldots, m, \\ \left\|u^{\text{lin}}(t,\cdot)\right\|_{\dot{H}^{l}} &\lesssim \tilde{h}_l(t) \|(u_0,u_1)\|_{\mathcal{A}}, \text{ for } l = 0, 1, \\ \left\|u^{\text{lin}}(t,\cdot)\right\|_{L^p} &\lesssim h_p^*(t) \|(u_0,u_1)\|_{\mathcal{A}}, \text{ if } p < 2. \end{aligned} \tag{30}$$

Therefore, $u^{\text{lin}} \in X(T)$ for any $T > 0$, and it satisfies

$$\left\|u^{\text{lin}}\right\|_{X(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}}, \text{ uniformly with respect to } T. \tag{31}$$

Next, we define the operator G on $X(T)$ as

$$Gu(t,x) := \int_0^t \int_0^\tau (\tau - s)^{-\gamma} E_1(t-\tau, x) *_{(x)} |u(s,x)|^p \, ds\, d\tau, \tag{32}$$

where

$$E_1(t,x) = (I - \Delta)^{-1} K_1(t,x). \tag{33}$$

By Duhamel's Principle, a function $u \in X(T)$ is the solution to (1) if, and only if, it satisfies the identity

$$u(t,\cdot) = u^{\text{lin}}(t,\cdot) + Gu(t,x) \text{ in } X(T). \tag{34}$$

Then we need to prove the following estimates:

$$\begin{gathered}\|Gu\|_{X(T)} \lesssim \|u\|_{X(T)}^{p}, \\ \|Gu - Gv\|_{X(T)} \lesssim \|u - v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right),\end{gathered} \tag{35}$$

also uniformly with respect to T. By a fixed point argument, it follows that $u^{\mathrm{lin}} + G$ maps balls of $X(T)$ into balls of $X(T)$ for small data in $\mathcal{A}$, and it is also a contraction map in $X(T)$, hence its iterative limit is the unique fixed point for $u^{\mathrm{lin}} + G$ (at least inside some sufficiently small ball in $X(T)$), that is, the unique solution to (1), $u = u^{\mathrm{lin}} + Gu$. The uniformity with respect to T of our estimates makes it possible to take the limit $T \to \infty$, yielding a global in-time existence result. In order to prove these estimates, we rely heavily on the following

Lemma 5. *Let* $\omega \in \mathbb{R}$, $\alpha > 1$, $\gamma \in (0,1)$. *Then, it holds*

$$\int_0^t (1+t-\tau)^{-\omega} \int_0^{\tau} (\tau - s)^{-\gamma} (1+s)^{-\alpha} \, ds d\tau \lesssim \begin{cases} (1+t)^{-\gamma} & \omega > 1 \\ (1+t)^{-\gamma} \log(2+t) & \omega = 1 \\ (1+t)^{1-\omega-\gamma} & \omega < 1. \end{cases}$$

The proof of the Lemma follows by standard arguments. See, for instance, Cui.[38]

We also make use of the Gagliardo–Nirenberg inequality[39] (in particular, also a fractional version of it):

$$\|u\|_{L^q} \lesssim \|u\|_{L^2}^{1-\theta_\kappa(q)} \|u\|_{\dot{H}^\kappa}^{\theta_\kappa(q)}, \theta_\kappa(q) = \frac{n}{\kappa}\left(\frac{1}{2} - \frac{1}{q}\right), \tag{36}$$

where $q \in [2, \infty]$ if $\kappa > n/2$, $q \in [2, \infty)$ if $\kappa = n/2$ and

$$2 \leq q \leq \frac{2n}{n - 2\kappa}$$

if $\kappa \in (0, n/2)$. In the limit case $q = \frac{2n}{n-2\kappa}$ when $\kappa \in (0, n/2)$, the inequality (36) reduces to the critical Sobolev embedding

$$H^\kappa \hookrightarrow L^q, \text{ with } \quad \|u\|_{L^q} \lesssim \|u\|_{\dot{H}^\kappa}, \kappa = n\left(\frac{1}{2} - \frac{1}{q}\right), \tag{37}$$

for any $q \in (2, \infty)$.

5 | PROOF OF THEOREM 1

Even if Theorem 1 addresses the easiest case $n = 1, 2$, we see that some interesting effects appear in the decay rate profile of the solution:

- the regularity-loss structure influences the decay rate of $\|u(t,\cdot)\|_{\dot{H}^2}$;
- the regularity-loss structure also influences the decay rate of $\|u_t(t,\cdot)\|_{\dot{H}^1}$ if $n = 2$, or $n = 1$ and $\theta < 1/4$, but not if $n = 1$ and $\theta \geq 1/4$;
- the asymptotic profile of the norm $\|u(t,\cdot)\|_{L^2}$ of the solution changes if $n = 1$ and $\theta \geq 1/4$ (see D'Abbicco and Ebert[5]).

None of the above effects, however, influences the critical exponent of the problem. In order to achieve this goal, fractional Sobolev embedding (37) is used, when needed.

Proof of Theorem 1. We first consider the cases where $n = 1$ and $\theta \in \left[0, \frac{1}{4}\right)$, or $n = 2$ and $\theta \in \left[0, \frac{1}{2}\right)$. The remaining case (that is, $n = 1, \theta \geq \frac{1}{4}$), must be dealt separately since the decay rate in the linear estimates is different. We fix $m = 2$, $k_0 = 0, k_1 = 1, k_2 = 2$,

$$
\begin{aligned}
h_0(t) &= (1+t)^{-\frac{n-4\theta}{4(1-\theta)}+1-\gamma},\\
h_1(t) &= l_1(t)(1+t)^{-\frac{n+2-4\theta}{4(1-\theta)}+1-\gamma},\\
h_2(t) &= \tilde{h}_1(t) = \tilde{h}_0(t) = (1+t)^{-\gamma},
\end{aligned}
$$

in (29), where the logarithmic term

$$
l_1(t) = \begin{cases} 1, & \text{if } n = 1,\\ \log(2+t), & \text{if } n = 2. \end{cases}
$$

may appear, although it doesn't actually interfere on the final existence results. Setting $j = 0, k = 0, 1, 2$ and $s = 2 + 2\gamma(1-\theta)$ in (20) and using the fact that $\gamma \in (0,1)$, we get

$$
\left\| u^{\mathrm{lin}}(t,\cdot) \right\|_{\dot{H}^k} \lesssim h_k(t) \|(u_0, u_1)\|_{\mathcal{A}}, \quad k = 0, 1, 2.
$$

Also, setting $j = 1$, $k = 0, 1$ and $s = 2 + 2\gamma(1-\theta)$ in (20), we get

$$
\left\| u_t^{\mathrm{lin}}(t,\cdot) \right\|_{\dot{H}^k} \lesssim \tilde{h}_k(t) \|(u_0, u_1)\|_{\mathcal{A}}, \quad k = 0, 1.
$$

In particular, we stress that, due to $s = s_c = 2 + 2\gamma(1-\theta)$, the decay rate related to the high frequencies regularity-loss decay structure $(1+t)^{-\frac{s-k-j}{2(1-\theta)}}$ produces a decay rate not slower than $(1+t)^{-\gamma}$ when $j + k \leq 2$. Indeed, the speed $(1+t)^{-\gamma}$ is the fastest possible, as a consequence of the presence of the nonlinear memory term, so that the assumption $s = s_c$ for the initial data guarantees that there is no benefit in the assumption of more regular small data.

Combining the previous five estimates, (31) follows immediately. To prove (35), let's estimate each norm of Gu that composes the norm in $X(T)$. Now setting $j = 0, k = 0, 1$ and $s = 3$; and $j = 0, k = 2$, $s = 4$ in (23), we have

$$
\|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t (1+t-\tau)^{-\omega} \int_0^\tau (\tau - s)^{-\gamma} \|u(s,\cdot)\|^p_{L^p \cap L^{2p}} \, ds \, d\tau, \tag{38}
$$

with

$$
\omega = \omega_k = \min\left\{ \frac{n+2k-4\theta}{4(1-\theta)}, \frac{3-k}{2(1-\theta)} \right\} = \frac{n+2k-4\theta}{4(1-\theta)} \text{ if } k = 0, 1,
$$

and

$$
\|Gu(t,\cdot)\|_{\dot{H}^2} \lesssim \int_0^t (1+t-\tau)^{-\omega} \int_0^\tau (\tau - s)^{-\gamma} \| |u(s,\cdot)|^p \|_{L^1 \cap H^1} \, ds \, d\tau, \tag{39}
$$

with

$$
\omega = \omega_2 = \min\left\{ \frac{n+4-4\theta}{4(1-\theta)}, \frac{2}{2(1-\theta)} \right\} = \frac{2}{2(1-\theta)} > 1.
$$

Recalling that $\|\cdot\|_{H^1} = \|\cdot\|_{L^2} + \|\cdot\|_{\dot{H}^1}$, we now estimate the norms $\|u(s,\cdot)\|^p_{L^p}$, $\|u(s,\cdot)\|^p_{L^{2p}}$ and $\||u(s,\cdot)|^p\|_{\dot{H}^1}$. Applying Gagliardo–Nirenberg inequality (36) with $\kappa = 1$ and $q = p > p_c$, we obtain, recalling that $u \in X(T)$,

$$
\|u(s,\cdot)\|^p_{L^p} \lesssim h_0(s)^{(1-\theta_1(p))p} h_1(s)^{\theta_1(p)p} \|u\|^p_{X(T)} = (1+s)^{-\alpha} \ell_1(s)^{\theta_1(p)p} \|u\|^p_{X(T)},
$$

with

$$
\alpha = \frac{(n-2\theta)p - n}{2(1-\theta)} - (1-\gamma)p. \tag{40}
$$

We remark that $\alpha > 1$ if, and only if, $p > p_c$. In fact, on the estimate above we may ignore the logarithmic term that appears if $n = 2$; this term doesn't change the critical exponent, since it can be bounded by $C_\delta(1+t)^\delta$, for any arbitrarily small $\delta > 0$. For the norm in L^{2p}, we apply Gagliardo–Nirenberg with $\kappa = 1$ and $q = 2p$, obtaining

$$
\|u(s,\cdot)\|^p_{L^{2p}} \lesssim (1+s)^{-\tilde{\alpha}} \ell_1(s)^{\theta_1(2p)p} \|u\|^p_{X(T)},
$$

with

$$\tilde{\alpha} = \alpha + \frac{n}{4(1-\theta)} > \alpha, \tag{41}$$

hence the condition for $\tilde{\alpha} > 1$ is achieved automatically for $p > p_c$. With these estimates, we can apply Lemma 5 in (38), thus obtaining

$$\|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim h_k(t)\, \|u\|_{X(T)}^p, \quad k = 0, 1, \tag{42}$$

for $\theta \in \left[0, \frac{1}{4}\right)$ and $n = 1$, or for $\theta \in [0, 1/2)$ and $n = 2$. We remark that for $k = 0, 1$, the decay rate is the same as for the linear problem in the low-frequency region. For $n = 2$ and $k = 1$, we find $\omega = 1$, which is the reason why the logarithmic term $\ell_1(t)$ appears.

For the norm $\||u(s,\cdot)|^p\|_{\dot{H}^1}$, the low dimensions $n = 1, 2$ and the assumption $p > p_c$ imply $p > n$, so that the Sobolev embedding is done into the much more regular spaces $C^{0,1-n/p}(\mathbb{R}^n)$. In fact, we can estimate the desired norm as follows.

If $n = 1$, from chain rule we get,

$$\||u(s,\cdot)|^p\|_{\dot{H}^1} \approx \|\nabla |u(s,\cdot)|^p\|_{L^2} \approx \left\||u(s,\cdot)|^{p-1}\nabla u(s,\cdot)\right\|_{L^2} \lesssim \|u(s,\cdot)\|_{L^\infty}^{p-1}\, \|\nabla u(s,\cdot)\|_{L^2}, \tag{43}$$

Applying Gagliardo–Nirenberg inequality (36) to estimate $\|u(s,\cdot)\|_{L^\infty}^{p-1}$, we obtain

$$\|u(s,\cdot)\|_{L^\infty} \leq \|u(s,\cdot)\|_{L^2}^{\frac{1}{2}}\, \|u(s,\cdot)\|_{\dot{H}^1}^{\frac{1}{2}} \leq (1+t)^{-\frac{2-4\theta}{4(1-\theta)}+1-\gamma}\, \|u\|_{X(T)},$$

so that we get

$$\||u(s,\cdot)|^p\|_{\dot{H}^1} \lesssim (1+s)^{\frac{2p-1}{4(1-\theta)}-p\gamma}\, \|u\|_{X(T)}. \tag{44}$$

Due to

$$p\gamma - \frac{2p-1}{4(1-\theta)} > p\gamma - \frac{p+1}{2(1-\theta)} = \alpha,$$

we can apply Lemma 5 in (39) and conclude

$$\|Gu(t,\cdot)\|_{\dot{H}^2} \lesssim h_2(t)\, \|u\|_{X(T)}^p. \tag{45}$$

If $n = 2$, from chain rule and Hölder inequality,

$$\||u(s,\cdot)|^p\|_{\dot{H}^1} \approx \|\nabla |u(s,\cdot)|^p\|_{L^2} \approx \left\||u(s,\cdot)|^{p-1}\nabla u(s,\cdot)\right\|_{L^2} \lesssim \|u(s,\cdot)\|_{L^{q(p-1)}}^{p-1}\, \|\nabla u(s,\cdot)\|_{L^r}, \tag{46}$$

with $q \in (2, \infty)$ as large as we please and $r = \frac{2q}{q-2}$. We apply Gagliardo–Nirenberg inequality 36, obtaining

$$\|u(s,\cdot)\|_{L^{q(p-1)}(\mathbb{R}^n)} \lesssim \|u(s,\cdot)\|_{L^2(\mathbb{R}^n)}^{1-\theta_1(q(p-1))}\, \|u(s,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)}^{\theta_1(q(p-1))} \lesssim (1+t)^{-\frac{\left(1-\frac{1}{q(p-1)}\right)-\theta}{1-\theta}+1-\gamma}\, (\log(2+t))^{\theta_1(q(p-1))}\, \|u\|_{X(T)},$$

as well as

$$\|\nabla u(s,\cdot)\|_{L^r(\mathbb{R}^n)} \lesssim \|\nabla u(s,\cdot)\|_{L^2(\mathbb{R}^n)}^{1-\theta_1(r)}\, \|\nabla u(s,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)}^{\theta_1(r)} \lesssim \|u(s,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)}^{1-\theta_1(r)}\, \|u(s,\cdot)\|_{\dot{H}^2(\mathbb{R}^n)}^{\theta_1(r)} \lesssim (1+t)^{-\gamma}\, (\log(2+t))^{1-\theta_1(r)}\, \|u\|_{X(T)}.$$

In particular, for any $\varepsilon > 0$, there exists a sufficiently large q such that

$$\||u(s,\cdot)|^p\|_{\dot{H}^1} \lesssim (1+s)^{\varepsilon-\gamma p}\, \|u\|_{X(T)}. \tag{47}$$

Therefore, we can still apply Lemma 5 in (39) and conclude

$$\|Gu(t,\cdot)\|_{\dot{H}^2} \lesssim h_2(t)\, \|u\|_{X(T)}^p. \tag{48}$$

Next, we estimate the two norms that concern the first order time derivative. Since $E_1(0,\cdot)\equiv 0$, we get

$$\|\partial_t(Gu)(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t \int_0^t (\tau-s)^{-\gamma}\,\|\partial_t E_1(t-\tau,\cdot) * |u(s,\cdot)|^p\|_{\dot{H}^k}\, ds\, d\tau.$$

Now, setting $j=1$, $k=0,1$ and $s=4$ on (23), we get

$$\|\partial_t(Gu)(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t (1+t-\tau)^{-\omega} \int_0^\tau (\tau-s)^{-\gamma}\,\||u(s,\cdot)|^p\|_{L^1\cap\dot{H}^1}\, dsd\tau, \tag{49}$$

with

$$\omega = \min\left\{\frac{n+2k-4\theta}{4(1-\theta)}-1,\ \frac{3-k}{2(1-\theta)}\right\} > 1.$$

Since we have already estimated the norms $\|u(s,\cdot)\|_{L^p}^p$ and $\||u(s,\cdot)|^p\|_{\dot{H}^1}$, we can apply Lemma 5 again and find

$$\|\partial_t(Gu)(t,\cdot)\|_{\dot{H}^k} \lesssim \tilde{h}_k(t)\ \|u\|_{X(T)}^p\,,\ \ k=0,1.$$

Combining the derived estimates, we obtain

$$\|Gu\|_{X(T)} \lesssim \|u\|_{X(T)}^p$$

in the case $n=1$ and $\theta\in[0,1/4)$ or $n=2$ and $\theta\in[0,1/2)$. Now, for the case where $n=1$ and $\theta\in\left[\frac{1}{4},\frac{1}{2}\right)$, the estimates for $\|u(t,\cdot)\|_{L^2}$ and $\|u_t(t,\cdot)\|_{L^2}$ change, because condition $\frac{n}{2}+k-2\theta>0$ in (20) doesn't hold anymore when $k=0$. We must then change the decay rate on the L^2-norms and also, include another norm on $X(T)$ so this loss on decay won't interfere on the critical exponent p_c. We fix $m=3$, $k_0=0$, $k_1=\frac{1}{2}-\frac{1}{p}$, $k_2=1$, $k_3=2$,

$$\begin{aligned}
h_0(t) &= (1+t)^{2-\frac{1}{4\theta}-\gamma},\\
h_1(t) &= (1+t)^{-\frac{1-\frac{1}{p}-2\theta}{2(1-\theta)}+1-\gamma},\\
h_2(t) &= (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma},\\
\tilde{h}_0(t) &= l_0(t)(1+t)^{1-\frac{1}{4\theta}-\gamma}\\
h_3(t) &= \tilde{h}_1(t) = (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma},
\end{aligned}$$

in (29), where

$$l_0(t) = \begin{cases} \log(e+t), & \text{if } \theta=\frac{1}{4}\\ 1, & \text{otherwise.}\end{cases}$$

The linear estimates give us again (31), and to prove (35), we set $j=0, k=0,k_1,1\ s=3$; and $j=0,k=2,s=4$ in (23), getting

$$\|Gu(t,\cdot)\|_{L^2} \lesssim \int_0^t l_0(t-\tau)(1+t-\tau)^{1-\frac{1}{4\theta}} \int_0^t (\tau-s)^{-\gamma}\ \|u(s,\cdot)\|_{L^p\cap L^{2p}}^p\ \ dsd\tau \tag{50}$$

$$\|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t (1+t-\tau)^{-\omega} \int_0^t (\tau-s)^{-\gamma}\ \|u(s,\cdot)\|_{L^p\cap L^{2p}}^p\ \ dsd\tau,\ \ k=k_1,1, \tag{51}$$

with

$$\omega = \min\left\{ \frac{1+2k-4\theta}{4(1-\theta)}, \frac{3-k}{2(1-\theta)} \right\},$$

and

$$\|Gu(t,\cdot)\|_{\dot H^2} \lesssim \int_0^t (1+t-\tau)^{-\omega} \int_0^\tau (\tau-s)^{-\gamma} \| |u(s,\cdot)|^p \|_{L^1\cap \dot H^1} \, ds d\tau, \tag{52}$$

with

$$\omega = \min\left\{ \frac{5-4\theta}{4(1-\theta)}, \frac{2}{2(1-\theta)} \right\} = \frac{5-4\theta}{4(1-\theta)} > 1.$$

To estimate the L^p norm, we use $\dot H^{k_1}$ directly,

$$\|u(s,\cdot)\|_{L^p}^p \lesssim \|u(s,\cdot)\|_{\dot H^{k_1}}^p \lesssim (1+s)^{-\alpha} \, \|u\|_{X(T)}^p,$$

with $\alpha := \frac{(1-2\theta)p-1}{2(1-\theta)} - (1-\gamma)p$, as in (40). Then $\alpha > 1$ if, and only if, $p > p_c$. For the L^{2p} norm, we use again critical Sobolev's Embedding (37) and interpolate $\|\cdot\|_{\dot H^{k_1}}$ and $\|\cdot\|_{\dot H^1}$, so that

$$\|u(s,\cdot)\|_{L^{2p}} \lesssim (1+s)^{-\alpha} \, \|u\|_{X(T)}^p,$$

as well, due to the fact that $h_2(t) \le h_1(t)$.

For the norm $\| |u(s,\cdot)|^p \|_{L^1\cap \dot H^1}$, we proceed similarly to (46), with the small change of interpolating between $\|\cdot\|_{\dot H^{k_1}}$ and $\|\cdot\|_{\dot H^1}$ instead of $\|\cdot\|_{L^2}$ and $\|\cdot\|_{\dot H^1}$. We then obtain

$$\| |u(s,\cdot)|^p \|_{\dot H^1} \lesssim (1+s)^{-\alpha} \|u\|_{X(T)}. \tag{53}$$

Using these estimates back in (50) and (51), as we did in the previous case, we get

$$\|Gu(t,\cdot)\|_{\dot H^k} \lesssim h_k(t) \, \|u\|_{X(T)}^p, k = 0, k_1, 1, 2, \tag{54}$$

the only difference being the decay rate in $h_0(t)$, and the new estimate which leads to $h_1(t)$.

We now consider the two norms of $\partial_t(Gu)(t,\cdot)$. For the $\|\cdot\|_{\dot H^1}$ norm we may proceed as before. On the other hand, for $\|\partial_t(Gu)(t,\cdot)\|_{L^2}$, due to $n=1$ and $\theta \in [1/4, 1/2)$, we find

$$\frac{1}{4\theta} \begin{cases} = 1 & \text{if } \theta = 1/4, \\ < 1 & \text{if } = \theta > 1/4, \end{cases} \qquad \frac{1}{1-\theta} > 1,$$

so that, applying Lemma 5 with

$$\omega = \min\left\{ \frac{1}{4\theta}, \frac{1}{1-\theta} \right\} = \frac{1}{4\theta},$$

we find

$$\|\partial_t(Gu)(t,\cdot)\|_{L^2} \lesssim \tilde h_0(t) \, \|u\|_{X(T)}^p.$$

Combining the previous estimates, we get

$$\|Gu\|_{X(T)} \lesssim \|u\|_{X(T)}^p.$$

Now it remains to show that G is a contraction in $X(T)$, that is, the second estimate in (35). This final part of the argument concerns all the cases together. We will use the Mean Value Inequality to obtain

$$||u(t,\cdot)|^p - |v(t,\cdot)|^p| \lesssim |u(t,\cdot) - v(t,\cdot)| \left(|u(t,\cdot)|^{p-1} + |v(t,\cdot)|^{p-1} \right).$$

Hence, by Hölder inequality,

$$\begin{aligned}
&\| |u(s,\cdot)|^p - |v(s,\cdot)|^p \|_{L^1} \\
&\quad \lesssim \left\| |u(s,\cdot) - v(s,\cdot)| \left(|u(s,\cdot)|^{p-1} + |v(s,\cdot)|^{p-1} \right) \right\|_{L^1} \\
&\quad \lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^p} \left(\|u(s,\cdot)\|_{L^p}^{p-1} + \|v(s,\cdot)\|_{L^p}^{p-1} \right) \\
&\quad = (1+s)^{\delta-\alpha} \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right)
\end{aligned}$$

for a sufficiently small $\delta \in (0, \alpha - 1)$, and

$$\| |u(s,\cdot)|^p - |v(s,\cdot)|^p \|_{L^2} \lesssim (1+s)^{\delta-\alpha} \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right).$$

Using these estimates, one can proceed as we did for $\|Gu(t,\cdot)\|_{X(T)}$, to get

$$\|Gu - Gv\|_{X(T)} \lesssim \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right).$$

Therefore, (35) is proved. ☐

6 | PROOF OF THEOREM 2

In the proof of Theorem 2, we consider the case $\gamma \le (n-2)/n$, in which the influence of the nonlinear memory is so strong that the critical exponent switches to the value γ^{-1}, as first investigated for the heat equation with nonlinear memory in.[7] The possible influence of the regularity-loss decay on the critical exponent is neglected by estimating the power nonlinearity at high frequencies in H^1, as we have done in Theorem 1 to improve the decay rates of u in H^2 and u_t in L^2 and H^1 to the best possible profile $(1+t)^{-\gamma}$.

Proof of Theorem 2. We fix $m = 2$, $k_0 = 0$, $k_1 = n\left(\frac{1}{2} - \frac{1}{p}\right)$, $k_2 = 2$, and

$$\begin{aligned}
h_0(t) &= l_0(t)(1+t)^{-\frac{n-4\theta}{4(1-\theta)}+1-\gamma}, \\
\tilde{h}_0(t) &= (1+t)^{-\gamma}, \\
h_1(t) &= (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+1-\gamma}, \\
h_2(t) = \tilde{h}_1(t) &= \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0, \\ (1+t)^{-\gamma}\log(2+t) & \text{if } \theta = 0, \end{cases}
\end{aligned}$$

where

$$l_0(t) := \begin{cases} 1, & n = 3 \\ \log(2+t), & n = 4. \end{cases}.$$

Setting $j = 0$, $k = 0, n\left(\frac{1}{2} - \frac{1}{p}\right), 2$ and $s = 2 + 2\gamma(1-\theta)$ in (20), we get

$$\left\| u^{\mathrm{lin}}(t,\cdot) \right\|_{\dot{H}^{k_i}} \lesssim h_i(t) \|(u_0, u_1)\|_{\mathcal{A}}, \quad i = 0, 1, 2,$$

and setting $j = 1$, $k = 0, 1$ and $s = 2 + 2\gamma(1-\theta)$ in (20) we get

$$\left\| u_t^{\mathrm{lin}}(t,\cdot) \right\|_{\dot{H}^k} \lesssim \tilde{h}_k(t) \|(u_0, u_1)\|_{\mathcal{A}}, \quad k = 0, 1.$$

Combining these five estimates, (31) follows immediately. Let's prove (35). Setting $j = 0$, $k = 0$ and $s = 3$ in (23), we obtain again (38), so we must estimate the norms $\|u(s,\cdot)\|_{L^p}^p$ and $\|u(s,\cdot)\|_{L^{2p}}^p$. Since $p > \gamma^{-1} \geq \frac{n}{n-2} \geq 2$, we can estimate

$$\|u(s,\cdot)\|_{L^p}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^{k_1}}^p \lesssim (1+s)^{-\gamma p}\|u\|_{X(T)}. \tag{55}$$

For the $L^{2p}-$norm, we set

$$\tilde{\kappa} = n\left(\frac{1}{2} - \frac{1}{2p}\right). \tag{56}$$

Notice that $\tilde{\kappa} \in (k_1, 2)$ in (56), and therefore, we can interpolate between $\|\cdot\|_{\dot{H}^{n\left(\frac{1}{2}-\frac{1}{p}\right)}}$ and $\|\cdot\|_{\dot{H}^2}$ to obtain

$$\|u(s,\cdot)\|_{L^{2p}}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^{\tilde{\kappa}}}^p \lesssim (1+s)^{-\gamma p}(\log(2+s))^p \ \|u\|_{X(T)}^p\,. \tag{57}$$

In the above estimate, we considered the worst case scenario, in which $\theta = 0$ so that $h_1(t) = (1+t)^{-\gamma}\log(2+t)$. As discussed in the proof of Theorem 1, a logarithmic loss does not influence the critical exponent.

The exponent γp that appears on (55) and (57) is larger than 1 if, and only if, $p > \gamma^{-1}$. Using the estimates (55) and (57) in (38), and applying Lemma 5 with

$$\omega = \min\left\{\frac{n-4\theta}{4(1-\theta)}, \frac{3}{2(1-\theta)}\right\} = \frac{n-4\theta}{4(1-\theta)}\begin{cases} < 1 & \text{if } n = 3, \\ = 1 & \text{if } n = 4, \end{cases}$$

we get

$$\|Gu(t,\cdot)\|_{L^2} \lesssim h_0(t)\,\|u\|_{X(T)}^p\,.$$

For the $\dot{H}^{k_1}$ and $\dot{H}^{k_2}$ norms, we set $j = 0$, $k = k_1, k_2$ and $s = 4$ in (23), yielding

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot{H}^{k_i}} \lesssim &\int_0^t (1+t-\tau)^{-\frac{n+2k_i-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\,\|u(s,\cdot)\|_{L^p}^p\ dsd\tau \\ &+ \int_0^t (1+t-\tau)^{-\frac{4-k_i}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\||u(s,\cdot)|^p\|_{H^1}\ dsd\tau.\end{aligned} \tag{58}$$

So, we must then estimate the norm $\|\nabla|u(s,\cdot)|^p\|_{L^2}$. From chain rule and Hölder Inequality, we have

$$\|\nabla|u|^p\|_{L^2} = \left\|\left(p|u|^{p-2}u\right)\nabla u\right\|_{L^2} \lesssim \|u\|_{L^{n(p-1)}}^{p-1}\,\|\nabla u\|_{L^{\frac{2n}{n-2}}}\,. \tag{59}$$

On the one hand,

$$\|\nabla u(s,\cdot)\|_{L^{\frac{2n}{n-2}}} \lesssim \|\nabla u(s,\cdot)\|_{\dot{H}^1} \approx \|u(s,\cdot)\|_{\dot{H}^2} \lesssim (1+s)^{-\gamma}\ \log(2+s)\,\|u\|_{X(T)}; \tag{60}$$

on the other hand, setting

$$\kappa = n\left(\frac{1}{2} - \frac{1}{n(p-1)}\right), \tag{61}$$

and due to $\kappa \in (k_1, k_2)$ for $p > \frac{n}{n-2}$, we get

$$\|u(s,\cdot)\|_{L^{n(p-1)}}^{p-1} \lesssim \|u(s,\cdot)\|_{\dot{H}^\kappa}^{p-1} \lesssim (1+s)^{-\gamma(p-1)}(\log(2+s))^{p-1}\ \|u\|_{X(T)}^{p-1}\,. \tag{62}$$

Using (60), and (62) in (59), we obtain

$$\||u(s,\cdot)|^p\|_{\dot{H}^1} \lesssim (1+s)^{-\gamma p}(\log(2+s))^p\ \|u\|_{X(T)}^p\,. \tag{63}$$

By applying Lemma 5 with

$$\omega = \min\left\{ \frac{n+2k_i-4\theta}{4(1-\theta)}, \frac{4-k_i}{2(1-\theta)} \right\} \begin{cases} > 1 & \text{if } i=1, \\ > 1 & \text{if } i=2 \quad \text{and} \quad \theta > 0, \\ = 1 & \text{if } i=2 \quad \text{and} \quad \theta = 0, \end{cases}$$

and using (55), (57) and (63) in (58), we get

$$\|Gu(t,\cdot)\|_{\dot{H}^{k_i}} \lesssim h_i(t)\, \|u\|^p_{X(T)}\,, i = 1, 2.$$

We may use the same strategy to estimate $\|\partial_t(Gu)\|_{\dot{H}^k}$, with $k = 0, 1$, and we obtain

$$\|\partial_t(Gu)(t,\cdot)\|_{\dot{H}^k} \lesssim \tilde{h}_k(t)\, \|u\|^p_{X(T)}\,, \quad k = 0, 1.$$

We then conclude the proof as we did in the proof of Theorem 1. □

7 | PROOF OF THEOREM 3

In the proof of Theorem 3, the condition $p \geq 2$ is a consequence of the cases considered in the statement. Due to this property, the proof is similar to the proof of Theorem 1.

Proof of Theorem 3. Assume $n = 3$. We fix $m = 2$, $k_0 = 0$, $k_1 = \frac{1}{2}$, $k_2 = 2$,

$$\begin{aligned} h_0(t) &= (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}, \\ h_1(t) &= \log(2+t)(1+t)^{-\gamma} \\ \tilde{h}_0(t) &= (1+t)^{-\gamma}, \\ h_2(t) = \tilde{h}_1(t) &= \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0, \\ (1+t)^{-\gamma}\log(2+t) & \text{if } \theta = 0, \end{cases} \end{aligned}$$

in (29). Once more, (31) follows immediately after setting $j = 0, k = 0, \frac{1}{2}, 2$ and $j = 1, k = 0, 1$ with $s = 2 + 2\gamma(1-\theta)$ in (20).

Now, estimating the H^k norms ($\kappa = 0, \frac{1}{2}$), we have

$$\|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t (1+t-\tau)^{-\omega} \int_0^\tau \tau(\tau-s)^{-\gamma}\, \|u(s,\cdot)\|^p_{L^p\cap L^{2p}}\, d\tau, \tag{64}$$

with

$$\omega = \min\left\{ \frac{3+2k-4\theta}{4(1-\theta)}, \frac{3-k}{2(1-\theta)} \right\}.$$

First, we consider the case $p \in [2,3]$. In order to estimate the L^p−norm, we will use Gagliardo–Nirenberg inequality (36)

$$\begin{aligned} \|u(t,\cdot)\|^p_{L^p} &\lesssim \|u(t,\cdot)\|_{L^2}^{\left(1-\theta_{\frac{1}{2}}\right)p}\, \|u(t,\cdot)\|_{\dot{H}^{\frac{1}{2}}}^{\theta_{\frac{1}{2}}p} \\ &\lesssim (1+t)^{\left(-\frac{3-4\theta}{4(1-\theta)}+1-\gamma\right)\left(1-\theta_{\frac{1}{2}}\right)p} (1+t)^{-\gamma\theta_{\frac{1}{2}}p} (\log(2+t))^{\theta_{\frac{1}{2}}p}\, \|u\|^p_{X(T)} \\ &\lesssim (1+t)^{-\alpha_1}\, \|u\|^p_{X(T)}\,, \end{aligned} \tag{65}$$

with

$$\alpha_1 := \frac{3p - 2\theta p - 3}{2(1-\theta)} - (1-\gamma)p. \tag{66}$$

We remark that the condition $p \in [2, 3]$ implies that

$$2 \leq p \leq \frac{6}{3 - 2\kappa,} \ \kappa = \frac{1}{2}, \tag{67}$$

and we also notice that

$$\alpha_1 > 1 \iff p > 1 + \frac{2(1 + (1-\theta)(1-\gamma))}{1 + 2\gamma(1-\theta)} =: p_c.$$

(Incidentally, we notice that $p > p_c$ holds as a consequence of the assumption $p \geq 2$, when $\gamma > (3-2\theta)/(4(1-\theta))$).

If $p > 3$, condition (67) doesn't hold, so we use critical Sobolev's embedding (37) with $q = p$ and

$$\kappa = 3\left(\frac{1}{2} - \frac{1}{p}\right). \tag{68}$$

For $p > 3$, one has κ in (68) satisfying $\kappa \in \left(\frac{1}{2}, \frac{3}{2}\right]$, hence

$$\|u\|_{L^p}^p \lesssim \|u\|_{\dot{H}^k}^p \lesssim (\log(2+t))^p (1+t)^{-\gamma p} \|u\|_{X(T)}^p. \tag{69}$$

In this case, since $\gamma > \frac{1}{3}$ and $p > 3$, one has automatically $\gamma p > 1$.

Now, for the $L^{2p}-$norm, we use critical Sobolev's Embedding (37) to estimate $\|\cdot\|_{H^\kappa(\mathbb{R}^3)} \lesssim \|\cdot\|_{L^q(\mathbb{R}^3)}$, with $q = 2p$ and $\kappa = 3\left(\frac{1}{2} - \frac{1}{2p}\right)$. We observe that $\kappa \geq \frac{3}{4}$, and thus,

$$\|u(t,\cdot)\|_{L^{2p}}^p \lesssim \|u(t,\cdot)\|_{\dot{H}^\kappa}^p \lesssim (\log(2+t))^p (1+t)^{-\gamma p} \|u\|_{X(T)}^p. \tag{70}$$

Returning to the estimate (64) and using the two last estimates, due to Lemma 5 it follows

$$\|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim \begin{cases} (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} \|u\|_{X(T)}^p, & \kappa = 0, \\ (\log(2+t))^p (1+t)^{-\gamma} \|u\|_{X(T)}^p, & \kappa = \frac{1}{2}. \end{cases} \tag{71}$$

For the H^2-norm, one can proceed as in Theorem 2, relying on (59) and (60), but replacing (62) by

$$\|u(s,\cdot)\|_{L^{3(p-1)}}^{p-1} \lesssim \|u(s,\cdot)\|_{\dot{H}^\kappa}^{p-1} \lesssim (\log(2+t))^{p-1} (1+s)^{-\gamma(p-1)} \|u\|_{X(T)}^{p-1}, \tag{72}$$

where we used that κ is given by (61), so that $\kappa = 3\left(\frac{1}{2} - \frac{1}{3(p-1)}\right) \geq \frac{1}{2}$. In turn, we obtain

$$\|Gu(t,\cdot)\|_{\dot{H}^2} \lesssim \begin{cases} (1+t)^{-\gamma} \|u\|_{X(T)}^p & \text{if } \theta > 0, \\ (1+t)^{-\gamma} \log(2+t) \|u\|_{X(T)}^p & \text{if } \theta = 0. \end{cases} \tag{73}$$

The norms $\|\partial_t Gu(t,\cdot)\|_{L^2}$ and $\|\partial_t Gu(t,\cdot)\|_{\dot{H}^1}$ are estimated in a similar way, and we conclude the proof as in the proof of Theorem 1. □

8 | PROOF OF THEOREM 4

In this case, we deal with power nonlinearities $|u|^p$, with $p < 2$, so we employ $(L^1 \cap L^p) - L^p$ estimates for the linear problem to obtain an estimate for $\|u(t,\cdot)\|_{L^p}$.

Proof of Theorem 4. We fix $m = 1$, $k_0 = 0$, $k_1 = 2$, and

$$\begin{aligned} h_0(t) &= (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}, \\ h_1(t) &= \tilde{h}_1(t) = l_2(t)(1+t)^{-\gamma}, \\ \tilde{h_0}(t) &= (1+t)^{-\gamma} \end{aligned}$$

in (29), with

$$l_2(t) = \begin{cases} \log(2+t), & \text{if } \theta = 0 \\ 1, & \text{if } \theta \in \left(0, \frac{1}{2}\right). \end{cases}$$

Moreover, we fix

$$h_p^*(t) = (1+t)^{\delta - \frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+1-\gamma}$$

for a sufficiently small $\delta > 0$. Setting first $j = 0$, $k = 0, 2$, and then $j = 1$, $k = 0, 1$, with $s = 2 + 2\gamma(1-\theta)$ in (20), and using (22), estimate (31) follows. We stress that the decay rate of u^{lin} in (22) is determined by the low-frequency region, since

$$\begin{aligned} \omega &= \min\left\{ -\delta + \frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}, \frac{3}{2(1-\theta)} - \frac{3(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)} \right\} \\ &= -\delta + \frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}, \end{aligned} \tag{74}$$

for a sufficiently small $\delta > 0$, for any $\theta \geq 0$ and $\gamma \leq 1$, due to $p > p_c \geq 4/3$.

To prove (35), let $T > 0$. For the norms of $Gu(t,\cdot)$ in L^2 and H^2, we would need again to estimate the norms $\|u(s,\cdot)\|_{L^p}$, $\|u(s,\cdot)\|_{L^{2p}}$ and $\||u(s,\cdot)|^p\|_{\dot{H}^1}$, as in the proof of Theorem 2. However, for the first one, we may directly use the $X(T)$ norm of u, obtaining:

$$\|u\|_{L^p}^p \lesssim h_p^*(t)\, \|u\|_{X(T)}^p = (1+s)^{-\alpha}\, \|u\|_{X(T)}^p,$$

with $\alpha := \frac{(3-2\theta)p-3}{2(1-\theta)} - (1+\delta-\gamma)p$. We remark that $\alpha > 1$ for a sufficiently small $\delta > 0$ (here we fix the smallness of δ once and for all), if, and only if, $p > p_c$. For the L^{2p} norm, we use critical Sobolev's Embedding (37) with $\|\cdot\|_{\dot{H}^\kappa(\mathbb{R}^3)} \lesssim \|\cdot\|_{L^q(\mathbb{R}^3)}$, with $q = 2p$ and $\kappa = 3\left(\frac{1}{2} - \frac{1}{2p}\right)$, and for the norm $\||u(s,\cdot)|^p\|_{\dot{H}^1}$, we proceed as in Theorem 2.

Setting $n = 3$, $j = 0$, $k = 0$, $s = 3$ in (23) and applying Lemma 5, we find

$$\begin{aligned} \|Gu(t,\cdot)\|_{L^2} &\lesssim \left[\int_0^t (1+t-\tau)^{-\frac{3-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, ds d\tau \right. \\ &\quad \left. + \int_0^t (1+t-\tau)^{-\frac{3}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\gamma p}\, ds d\tau \right] \|u\|_{X(T)}^p \\ &\lesssim h_0(t)\, \|u\|_{X(T)}^p \end{aligned} \tag{75}$$

Setting $n = 3$, $j = 0$, $k = 2$, and $s = 4$ in (23) and applying Lemma 5, we get

$$\begin{aligned} \|Gu(t,\cdot)\|_{\dot{H}^2} &\lesssim \left[\int_0^t (1+t-\tau)^{-\frac{7-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, ds d\tau \right. \\ &\quad \left. + \int_0^t (1+t-\tau)^{-\frac{1}{1-\theta}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\gamma p}\, ds d\tau \right] \|u\|_{X(T)}^p \\ &\lesssim h_1(t)\, \|u\|_{X(T)}^p \end{aligned} \tag{76}$$

For the L^p−norm, we will have to estimate also the norm $\|u(s,\cdot)\|_{L^{p^2}}$. In fact, setting $n=3$ and $\varphi=|u(s,\cdot)|^p$ in (24),

$$\begin{aligned}\|Gu(t,\cdot)\|_{L^p} \lesssim & \int_0^t (1+t-\tau)^{\delta-\frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\, \|u(s,\cdot)\|_{L^p}^p \; ds d\tau \\ & + \int_0^t (1+t-\tau)^{-\frac{3}{2(1-\theta)}+\frac{3(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\, \|u(s,\cdot)\|_{L^{p^2}}^p \; ds d\tau \end{aligned} \tag{77}$$

Now, since for $n=3$ we have $p>p_c>\frac{5}{3}$, it holds that $p^2>2$, and we can apply critical Sobolev's Embedding (37) with $\|\cdot\|_{H^\kappa(\mathbb{R}^3)} \lesssim \|\cdot\|_{L^q(\mathbb{R}^3)}$, where $q=p^2$ and $\kappa=3\left(\frac{1}{2}-\frac{1}{p^2}\right)\in\left(\frac{7}{50},\frac{3}{2}\right)$:

$$\|u(s,\cdot)\|_{L^{p^2}}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^\kappa}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^2}^p \lesssim (1+s)^{-\gamma p}\,\ell_2(s)^p\,\|u\|_{X(T)}^p .$$

Using this estimate in (77) and applying Lemma 5 with ω as in (74), noticing that $\omega<1$, we get

$$\|Gu(t,\cdot)\|_{L^p} \lesssim h_p^*(t)\,\|u\|_{X(T)}^p .$$

We stress that the minimum in (74) is attained by the first term for any $\theta\geq 0$, due to $p>p_c\geq 4/3$.

Once more, the norms of $\partial_t(Gu)(t,\cdot)$ can be estimated with no further difficulties, and we conclude the proof as in the previous cases. ☐

9 | PROOF OF THEOREM 5

As in the proof of Theorem 3, also in Theorem 5, the condition $p\geq 2$ is an assumption of the statement. Due to this property, the proof is similar to the proof of Theorems 1 and 3. We stress that in this theorem the condition $p\geq 2$ never corresponds to the critical exponent, which is smaller than 2 for $\gamma>1/2$.

Proof of Theorem 5. We fix $m=1$, $k_0=0$, $k_1=2$, and

$$\begin{aligned} h_0(t) &= (1+t)^{-\gamma}\log(2+t), \\ h_1(t) &= \tilde{h}_1(t) = l_2(t)(1+t)^{-\gamma}, \\ \tilde{h}_0(t) &= (1+t)^{-\gamma}, \end{aligned}$$

in (29), with

$$l_2(t) = \begin{cases} \log(2+t) & \text{if } \theta=0, \\ 1 & \text{if } \theta\in(0,\frac{1}{2}). \end{cases}$$

Setting $j=0$, $k=0,2$; and $j=1$, $k=0,1$; with $s=2+2\gamma(1-\theta)$ in (20), estimate (31) follows. To prove (35), we set $j=0$, $k=0,2$, $\psi=|u|^p$ and $s=4$ in (23), getting

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim & \int_0^t (1+t-\tau)^{-1-\frac{k}{2(1-\theta)}} \int_0^t (\tau-s)^{-\gamma}\, \|u(s,\cdot)\|_{L^p}^p \, ds d\tau \\ & + \int_0^t (1+t-\tau)^{-\frac{4-k}{2(1-\theta)}} \int_0^t (\tau-s)^{-\gamma}\, \||u(s,\cdot)|^p\|_{\dot{H}^1} ds d\tau . \end{aligned} \tag{78}$$

To bound the L^p–norm, we use critical Sobolev's Embedding (37) with $\|\cdot\|_{\dot{H}^{\kappa}(\mathbb{R}^4)} \lesssim \|\cdot\|_{L^q(\mathbb{R}^4)}$, with $q=p$ and $\kappa = 4\left(\frac{1}{2}-\frac{1}{p}\right) \in [0,2)$. Therefore,

$$\|u(s,\cdot)\|_{L^p}^p \lesssim \|u\|_{\dot{H}^{\kappa}}^p \lesssim (1+s)^{-\gamma p}(\log(2+s))^p\ \|u\|_{X(T)}^p\,.$$

To estimate $\||u(s,\cdot)|^p\|_{\dot{H}^1}$, one can simply set $n=4$ in (59)-(63) to obtain again

$$\||u(s,\cdot)|^p\|_{\dot{H}^1} \lesssim (1+s)^{-\gamma p}(\log(2+s))^p\ \|u\|_{X(T)}^p\,.$$

Therefore, using these estimates in (78), and applying Lemma 5 with $\omega=1$ if $k=0$ and

$$\omega = \min\left\{1+\frac{1}{1-\theta}, \frac{1}{1-\theta}\right\} = \frac{1}{1-\theta}\begin{cases} >1 & \text{if } \theta>0,\\ =1 & \text{if } \theta=0,\end{cases}$$

if $k=2$, we get

$$\|Gu(t,\cdot)\|_{\dot{H}^{k_i}} \lesssim h_i(t)\,\|u\|_{X(T)}^p\,.$$

The rest of the proof follows as in the proof of the previous theorems. □

10 | PROOF OF THEOREMS 6 AND 7

In Theorems 6 and 7, we assume $\theta=0$ for simplicity. Indeed, Theorem 6 is analogous to Theorem 4, but there is a range of γ near to 1, for which the regularity-loss structure seems to influence the critical exponent of global small data solutions. This range is treated in Theorem 7.

Proof of Theorem 6. We fix $m=1$, $k_0=0$, $k_1=2$, and

$$h_0(t) = h_2(t) = \tilde{h}_1(t) = (1+t)^{-\gamma}\log(2+t),$$
$$\tilde{h}_0(t) = (1+t)^{-\gamma},$$

in (29), as in the proof of Theorem 5. Since $p<2$, we also fix

$$h_p^*(t) = (1+t)^{\delta-2\left(1-\frac{1}{p}\right)+1-\gamma},$$

for a sufficiently small $\delta>0$. Setting $j=0$, $k=0,2$, and $j=1$, $k=0,1$, with $s=2+2\gamma$ in (20), we get

$$\left\|u^{\text{lin}}(t,\cdot)\right\|_{L^2} \lesssim h_0(t)\|(u_0,u_1)\|_{\mathcal{A}},$$
$$\left\|u^{\text{lin}}(t,\cdot)\right\|_{\dot{H}^2} \lesssim h_2(t)\|(u_0,u_1)\|_{\mathcal{A}},$$
$$\left\|u_t^{\text{lin}}(t,\cdot)\right\|_{L^2} \lesssim \tilde{h}_0(t)\|(u_0,u_1)\|_{\mathcal{A}},$$
$$\left\|u_t^{\text{lin}}(t,\cdot)\right\|_{\dot{H}^1} \lesssim \tilde{h}_1(t)\|(u_0,u_1)\|_{\mathcal{A}},$$

and using (22), we obtain

$$\left\|u^{\text{lin}}(t,\cdot)\right\|_{L^p} \lesssim h_p^*(t)\|(u_0,u_1)\|_{\mathcal{A}}.$$

Here, the fact that $\gamma \leq \frac{7}{8}$ is important, so that $p_c \geq \frac{8}{5}$. This means that for $p>p_c$, the decay rate from the low-frequency will be worse than the high-frequency one, namely, that

$$\omega = \min\left\{-\delta+2\left(1-\frac{1}{p}\right), \frac{3}{2}-6\left(\frac{1}{p}-\frac{1}{2}\right)\right\} = -\delta+2\left(1-\frac{1}{p}\right). \tag{79}$$

Therefore, estimate (31) follows. Let $T > 0$ and let's prove (35). Setting $j = 0, k = 0, 2$ $\psi = |u|^p$ and $s = 4$ in (23), we get

$$\|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t (1+t-\tau)^{-1-\frac{k}{2}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^p}^p \, dsd\tau \tag{80}$$

$$+\int_0^t (1+t-\tau)^{-\frac{4-k}{2}} \int_0^\tau (\tau-s)^{-\gamma} \| |u(s,\cdot)|^p \|_{H^1} \, dsd\tau \quad k = 0, 2. \tag{81}$$

Recalling that $u \in X(T)$, we obtain

$$\|u(s,\cdot)\|_{L^p}^p \lesssim h_p^*(s)^p \, \|u\|_{X(T)}^p = (1+s)^{-\alpha} \|u\|_{X(T)}^p, \alpha := (1-\delta+\gamma)p - 2 \tag{82}$$

with $\alpha > 1$ for a sufficiently small $\delta > 0$ (here we fix the smallness of δ once and for all) if, and only if, $p > \frac{3}{1+\gamma} = p_c$.

Now, we estimate $\| |u(s,\cdot)|^p \|_{\dot{H}^1}$ using once again (59), (60), (61) and since, for $p > p_c > \frac{3}{2}$, we have $\kappa \in (0,2)$, we get

$$\|u(s,\cdot)\|_{L^{4(p-1)}}^{p-1} \lesssim \|u(s,\cdot)\|_{\dot{H}^\kappa}^{p-1} \lesssim (1+s)^{-\gamma(p-1)} (\log(2+s))^{p-1} \, \|u\|_{X(T)}^{p-1}, \tag{83}$$

so that

$$\| |u(s,\cdot)|^p \|_{\dot{H}^1} \lesssim (1+s)^{-\gamma p} (\log(2+s))^p \, \|u\|_{X(T)}^p. \tag{84}$$

Notice that $\gamma > \frac{1}{2}$ and $p > \frac{3}{1+\gamma}$, hence $\gamma p > 1$. We can then apply Lemma 5 and obtain

$$\|Gu\|_{\dot{H}^k} \lesssim h_k(t) \, \|u\|_{X(T)}^p, \quad k = 0, 2.$$

For the L^p−norm of $Gu(t,\cdot)$, we set $n = 4$ in (24), thus getting

$$\begin{aligned} \|Gu(t,\cdot)\|_{L^p} &\lesssim \int_0^t (1+t-\tau)^{\delta-2\left(1-\frac{1}{p}\right)} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^p}^p \, dsd\tau \\ &+ \int_0^t (1+t-\tau)^{-\frac{3-12\left(\frac{1}{p}-\frac{1}{2}\right)}{2}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^{p^2}}^p \, dsd\tau. \end{aligned} \tag{85}$$

We have already estimated the L^p−norm. For the L^{p^2}−norm, since $p > p_c > \frac{3}{2}$, we have $p^2 > 2$ and therefore we can apply critical Sobolev's Embedding (37) with $q = p^2$ and

$$\kappa = 4\left(\frac{1}{2} - \frac{1}{p^2}\right), \tag{86}$$

which, since $\kappa \in \left(\frac{2}{9}, 2\right)$ gives us

$$\|u(s,\cdot)\|_{L^{p^2}}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^\kappa}^p \lesssim (1+s)^{-\gamma p} (\log(2+s))^p \, \|u\|_{X(T)}^p.$$

Therefore, applying Lemma 5 with ω as in (79), we obtain

$$\|Gu(t,\cdot)\|_{L^p} \lesssim h_p^*(t) \, \|u\|_{X(T)}^p.$$

The norms of time-derivatives of $Gu(t,\cdot)$ using 23 bring no additional difficulties:

$$\begin{aligned} \|\partial_t(Gu)(t,\cdot)\|_{\dot{H}^k} &\lesssim \int_0^t (1+t-\tau)^{-2-\frac{k}{2}} \int_0^\tau (\tau-s)^{-\gamma}\, \|u(s,\cdot)\|_{L^p}^p \; ds d\tau \\ &\quad + \int_0^t (1+t-\tau)^{-\frac{4-k}{2}} \int_0^\tau (\tau-s)^{-\gamma} \| |u(s,\cdot)|^p \|_{\dot{H}^1} \; ds d\tau \\ &\lesssim \tilde{h}_k(t)\, \|u\|_{X(T)}^p . \end{aligned} \tag{87}$$

We then conclude the proof as in the previous cases. ☐

In the proof of Theorem 7 we finally provide a model in which the regularity loss structure may influence the critical exponent of the problem. Even if we do not expect the threshold $\tilde{p}_c$ to be sharp, the model below shows an interplay which can be object of future investigations.

Proof of Theorem 7. We fix $m = 1$, $k_0 = 0$, $k_1 = 2$, and

$$\begin{aligned} &h_0(t) = h_2(t) = \tilde{h}_1(t) = (1+t)^{-\gamma} \log(2+t), \\ &\tilde{h}_0(t) = (1+t)^{-\gamma}, \end{aligned}$$

in (29), as in the proof of Theorems 5 and 6. Since $p < 2$, we also fix

$$h_p^*(t) = (1+t)^{-\omega+1-\gamma},$$

where

$$\omega = \begin{cases} -\delta + 2\left(1-\frac{1}{p}\right) & \text{if } \; p \geq 8/5, \\ \frac{3}{2} - 6\left(\frac{1}{p}-\frac{1}{2}\right) & \text{if } \; p < 8/5, \end{cases} \tag{88}$$

for a sufficiently small $\delta > 0$. As in the proof of Theorem 6, we conclude (31). We proceed as in the proof of Theorem 6 to prove (35), but with two differences. First of all,

$$\|u(s,\cdot)\|_{L^p}^p \lesssim h_p^*(s)^p\, \|u\|_{X(T)}^p = (1+s)^{-\alpha}\, \|u\|_{X(T)}^p ,$$

where

$$\alpha = (\omega - 1 + \gamma)p = \left(\frac{7}{2} + \gamma\right)p - 6, \text{ if } \; p < 8/5.$$

In particular, $\alpha > 1$ if, and only if, $p > \frac{14}{7+2\gamma} = \tilde{p}_c$, due to the assumption $\gamma > \frac{7}{8}$.

The second difference is that we apply Lemma 5 with ω as in (88), obtaining

$$\begin{aligned} \|Gu(t,\cdot)\|_{L^p} &\lesssim \int_0^t (1+t-\tau)^{\delta-2\left(1-\frac{1}{p}\right)} \int_0^\tau (\tau-s)^{-\gamma}\, \|u(s,\cdot)\|_{L^p}^p \; ds d\tau \\ &\quad + \int_0^t (1+t-\tau)^{-\frac{3-12\left(\frac{1}{p}-\frac{1}{2}\right)}{2}} \int_0^\tau (\tau-s)^{-\gamma}\, \|u(s,\cdot)\|_{L^{p^2}}^p \; ds d\tau \\ &\lesssim h_p^*(t)\, \|u\|_{X(T)}^p . \end{aligned} \tag{89}$$

The rest of the proof follows as in the proof of Theorem 6. ☐

11 | BRIEF COMPILATION OF RESULTS AND STRATEGIES

In order to summarize the results and strategies employed in the several cases considered, we provide a table that displays the different ranges for the parameters θ, γ and p, as well as the correspondent spaces required for the initial data u_0 and u_1. Here,

$$p_c = p_c(n,\gamma,\theta) := 1 + \frac{2\,(1+(1-\gamma)(1-\theta))}{n-2+2\gamma(1-\theta)}, s_c = s_c(\gamma,\theta) := 2+2\gamma(1-\theta)$$

$$\tilde{p}_c := \frac{14}{7+2\gamma}.$$

n	θ	γ	p	u_0	u_1
1	$\left[0,\frac{1}{2}\right)$	$\left(\frac{1}{2(1-\theta)},1\right)$	(p_c,∞)	$H^{s_c}\cap L^1$	$H^{s_c-1}\cap L^1$
2	$\left[0,\frac{1}{2}\right)$	$(0,1)$	(p_c,∞)	$H^{s_c}\cap L^1$	$H^{s_c-1}\cap L^1$
3	$\left[0,\frac{1}{2}\right)$	$\left(0,\frac{1}{3}\right]$	$\left(\frac{1}{\gamma},\infty\right)$	$H^{s_c}\cap L^1$	$H^{s_c-1}\cap L^1$
3	$\left[0,\frac{1}{2}\right)$	$\left(\frac{1}{3},\frac{3-2\theta}{4(1-\theta)}\right]$	(p_c,∞)	$H^{s_c}\cap L^1$	$H^{s_c-1}\cap L^1$
3	$\left[0,\frac{1}{2}\right)$	$\left(\frac{3-2\theta}{4(1-\theta)},1\right)$	$(p_c,2]$	$H^{s_c}\cap L^1\cap W^{3,p}$	$H^{s_c-1}\cap L^1\cap W^{2,p}$
3	$\left[0,\frac{1}{2}\right)$	$\left(\frac{3-2\theta}{4(1-\theta)},1\right)$	$[2,\infty)$	$H^{s_c}\cap L^1$	$H^{s_c-1}\cap L^1$
4	$\left[0,\frac{1}{2}\right)$	$\left(0,\frac{1}{2}\right]$	$\left(\frac{1}{\gamma},\infty\right)$	$H^{s_c}\cap L^1$	$H^{s_c-1}\cap L^1$
4	$\left[0,\frac{1}{2}\right)$	$\left(\frac{1}{2},1\right)$	$[2,\infty)$	$H^{s_c}\cap L^1$	$H^{s_c-1}\cap L^1$
4	0	$\left(\frac{1}{2},\frac{7}{8}\right]$	$(p_c,2]$	$H^{s_c}\cap L^1\cap W^{3,p}$	$H^{s_c-1}\cap L^1\cap W^{2,p}$
4	0	$\left(\frac{7}{8},1\right)$	$(\tilde{p}_c,2]$	$H^{s_c}\cap L^1\cap W^{3,p}$	$H^{s_c-1}\cap L^1\cap W^{2,p}$

The strategies we used to estimate norms in each theorem are briefly described below:

- In Theorem 1, we used Gagliardo–Nirenberg, with $\kappa = 1$; For the special case $n = 1, \theta \geq \frac{1}{4}$, we used critical Sobolev Embedding into L^p instead. We also applied critical Sobolev Embedding into $L^{q(p-1)}$ and L^r with $q \to \infty$ and $r \to 2$, in order to improve the decay rate.
- In Theorem 2, we used critical Sobolev embedding into L^p, as well as critical Sobolev embeddings into $L^{n(p-1)}$ and $L^{2n/(n-2)}$.
- In Theorem 3, we used fractional Gagliardo–Nirenberg with $\kappa = \frac{1}{2}$ to reach the critical exponent, and critical Sobolev embedding into L^p for larger values of p. Critical Sobolev embeddings into $L^{3(p-1)}$ and L^6 were also employed (as in Theorem 2).
- In Theorem 4, we asked for regularity in L^p for the initial data, and applied critical Sobolev embedding into L^{2p} and L^{p^2}.
- In Theorem 5, we used critical Sobolev embedding into L^p and critical Sobolev embeddings into $L^{4(p-1)}$ and L^4 (by the same reason as in Theorems 2 and 3).
- In Theorems 6 and 7, we again required additional L^p regularity for initial data, and applied critical Sobolev embedding into L^{p^2} and into $L^{4(p-1)}$ and L^4.

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REFERENCES

1. Fujita H. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J Fac Sci Univ Tokyo*. 1966;1966:109-124.
2. Todorova G, Yordanov B. Critical exponent for a nonlinear wave equation with damping. *J Differ Equ*. 2001;174:464-489.
3. Qi S. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case. *C R Acad Sci Paris Sér I Math*. 2001;333:109-114.
4. Ebert MR, Girardi G, Reissig M. Critical regularity of nonlinearities in semilinear classical damped wave equations. *Math Ann*. 2020;378(3–4):1311-1326.
5. D'Abbicco M, Ebert MR. A new phenomenon in the critical exponent for structurally damped semi-linear evolution equations. *Nonlinear Anal Theory Methods Appl*. 2017;149:1-40.
6. D'Abbicco M, Reissig M. Semilinear structural damped waves. *Math Meth Appl Sci*. 2014;37:1570-1592.
7. Cazenave T, Dickstein F, Weissler F. An equation whose Fujita critical exponent is not given by scaling. *Nonlinear Anal*. 2008;68:862-874.
8. D'Abbicco M. The influence of a nonlinear memory on the damped wave equation. *Nonlinear Anal*. 2014;95:130-145.
9. D'Abbicco M. A wave equation with structural damping and nonlinear memory. *Nonlinear Differ Equ Appl*. 2014;21(5):751-773.
10. D'Abbicco M, Girardi G. A structurally damped σ-evolution equation with nonlinear memory. *Math Meth Appl Sci*. 2020:1-19. doi:10.1002/mma.6633
11. D'Abbicco M, Fujiwara K. A test function method for evolution equations with fractional powers of the Laplace operator. *Nonlinear Anal*. 2021;202:112114.
12. D'Abbicco M, Ebert MR. $L^p - L^q$ estimates for a parameter-dependent multiplier with oscillatory and diffusive components. *J Math Anal Appl*. 2021;504(1):125393. doi:10.1016/j.jmaa.2021.125393
13. D'Abbicco M, Ebert MR. The critical exponent for semilinear sigma-evolution equations with a strong non-effective damping. *Nonlinear Anal*. 2022;215:112637. doi:10.1016/j.na.2021.112637
14. Chen W, D'Abbicco M, Girardi G. Global small data solutions for semilinear waves with two dissipative terms. *Annali di Matematica Pura ed Applicata*. 2022;201:529-560. doi:10.1007/s10231-021-01128-z
15. Temam R. *Navier-Stokes Equations Theory and Numerical Analysis Studies in Mathematics and its Applications*, Vol. 2. Amsterdam-New York: North-Holland Publishing Co.; 1979.
16. Ciarlet PG. A justification of the von Kármán equations. *Arch Rational Mech Anal*. 1980;73:349-389.
17. Luyo Sánchez JR, Sistema Dinâmico de Von Kármán em Domínios Não Limitados é Globalmente Bem Posto no Sentido de Hadamard O. Análise do seu Limite Singular (Portuguese). *Ph.D thesis*: Institute of Mathematics, Federal University of Rio de Janeiro Rio de Janeiro Brazil; 2003.
18. Lasiecka I. Uniform stability of a full von Kármán system with nonlinear boundary feedback. *SIAM J Control Optim*. 1998;36:1376-1422.
19. Lasiecka I, Benabdallah A. Exponential decay rates for a full von Kármán thermoelasticity system with nonlinear thermal coupling. *ESAIM: Control Optim Cal Variations*. 2000;8:13-38.
20. Koch H, Lasiecka I. Hadamard wellposedness of weak solutions in nonlinear dynamical elasticity - full von Kármán systems. *Progress Nonlin Diff Eqs Appl*. 2002;50:197-216.
21. Puel JP, Tucsnak M. Global existence for full von Kármán system. *Appl Math Optim*. 1996;34:139-160.
22. Menzala GP, Zuazua E. Timoshenko's plate equations as a singular limit of the dinamical von Kármán system. *J Math Pures Appl*. 2000;79:73-94.
23. Coimbra Charão R, da Luz CR, Ikehata R. New decay rates for a problem of plate dynamics with fractional damping. *J Hyperbolic Differ Equ*. 2013;10:563-575.
24. Sugitani Y, Kawashima S. Decay estimates of solutions to a semi-linear dissipative plate equation. *J Hyperbolic Differ Equ*. 2010;7:471-501.
25. Charão RC, da Luz CR. Asymptotic properties for a semilinear plate equation in unbounded domains. *J Hyperbolic Differ Equ*. 2009;6:269-294.
26. Liu Y. Decay of solutions to an inertial model for a semilinear plate equation with memory. *J Math Anal Appl*. 2012;394:616-632.
27. D'Abbicco M, Girardi G, Liang J. $L^1 - L^1$ estimates for the strongly damped plate equation. *J Math Anal Appl*. 2019;478:476-498.
28. Chueshov I, Lasiecka I. Long-time behavior of second order evolution equations with nonlinear damping. *Memoirs Am Math Soc*. 2008;195:viii+183.
29. Denk R, Schnaubelt R. A structurally damped plate equation with Dirichlet-Neumann boundary conditions. *J Differ Equ*. 2015;259:1323-1353.
30. Geredeli PG, Lasiecka I. Asymptotic analysis and upper semicontinuity with respect to rotational inertia of attractors to von Kármán plates with geometrically localized dissipation and critical nonlinearity. *Nonlinear Anal*. 2013;91:72-92.
31. Schnaubelt R, Veraar M. Structurally damped plate and wave equations with random point force in arbitrary space dimensions. *Differ Integr Equ*. 2010;23:957-988.
32. Andrade D, Jorge Silva MA, Ma TF. Exponential stability for a plate equation with p-Laplacian and memory terms. *Math Methods Appl Sci*. 2012;35:417-426.
33. Ma Q, Yang Y, Zhang X. Existence of exponential attractors for the plate equations with strong damping. *Electron J Differ Equ*. 2013;2013:1-10.
34. Wang Y-Z. Asymptotic behavior of solutions to the damped nonlinear hyperbolic equation. *J Appl Math*. 2013;2013:353757.
35. Xu L, Ma Q. Existence of random attractors for the floating beam equation with strong damping and white noise. *Bound Value Probl*. 2015;2015:126.

36. Narazaki T. $L^p - L^q$ estimates for damped wave equations and their applications to semilinear problem. *J Math Soc Japan*. 2004;56:586-626.
37. Ebert M, da Luz C, Palma M. The influence of data regularity in the critical exponent for a class of semilinear evolution equations. *Nonlinear Differ Equ Appl*. 2020;27:44.
38. Cui S. Local and global existence of solutions to semilinear parabolic initial value problems. *Nonlinear Anal Theory Methods Appl*. 2001;43:293-323.
39. Nirenberg L. On elliptic partial differential equations. *Ann Scuola Normale Sup Pisa, Classe di Scienze 3e, serie*. 1959;13(2):115-162.
40. D'Abbicco M, Ebert MR. Diffusion phenomena for the wave equation with structural damping in the $L^p - L^q$ framework. *J Differ Equ*. 2014;256:2307-2336. doi:10.1016/j.jde.2014.01.002
41. Miyachi A. On some Fourier multipliers for $H^p(\mathbb{R}^n)$. *J Faculty Sci Univer Tokyo*. 1980;27:157-179.

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APPENDIX A: TECHNICAL CALCULATIONS

We recall from vector calculus that, for any $\alpha = (\alpha_1, \dots \alpha_v)$ multi-index,

$$\left|\partial_\xi^\alpha |\xi|^k\right| \lesssim |\xi|^{k-|\alpha|}, \; k \neq 0, \xi \neq 0. \tag{A1}$$

We also recall the multivariate version of the Leibniz's rule,

$$\partial_\xi^\alpha (fg) = \sum_{\beta \leq \alpha} C_\beta \left(\partial_\xi^\beta f\right) \left(\partial_\xi^{\alpha-\beta} g\right), \tag{A2}$$

where $\beta \leq \alpha$ is in the sense of multi-indexes, that is, $\beta_i \leq \alpha_i$ for all i. Another useful identity is Faà di Bruno's formula for higher order derivatives: Let $y = g(\xi)$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Then,

$$\partial_\xi^\alpha f(y) := \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}\xi_1 \dots \partial^{\alpha_n}\xi_n} f(y) = \sum_{\pi \in \Pi} f^{(|\pi|)}(y) \cdot \prod_{B \in \pi} \frac{\partial^{|B|}}{\prod_{j \in B} \partial \xi_j} y, \tag{A3}$$

where π runs through the set Π of all the partitions of the set $\{1, \dots, |\alpha|\}$ and $\}\}B \in \pi''$ means that B runs through the list of all blocks of the partition π. Observe that $|\alpha| = \alpha_1 +, \dots, +\alpha_n$, while $|\pi|$ is the number of blocks in the partition π and $|B|$ is the size of the block B.

Applying Faà di Bruno with $y(\xi) = \mathbf{x}2$ and $f(y) = (1+y)^{\frac{k}{2}}$, we get

$$\left|\partial_\xi^\alpha \langle \xi \rangle^k\right| \lesssim \begin{cases} |\xi|^{k-|\alpha|}, & |\xi| \geq 1, \\ |\xi|^{-|\alpha|}, & 0 < |\xi| \leq 1. \end{cases} \tag{A4}$$

Lastly, we'll need Young's Inequality for products: If $a, b \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{A5}$$

A.1 | Estimates for the low-frequency region

Following as in previous works,[5,40] with several applications of (A2) and (A3), one can easily prove that, for the low-frequency region $\{\xi : |\xi| \leq \varepsilon_0\}$, the following estimates hold:

- $\left|\partial_\xi^\alpha \lambda_-\right| \lesssim |\xi|^{2\theta-|\alpha|}$;
- $\left|\partial_\xi^\alpha \lambda_+\right| \lesssim |\xi|^{2(1-\theta)-|\alpha|}$;

- $\left|\partial_\xi^\alpha e^{t\lambda_-}\right| \lesssim |\xi| - |\alpha| e^{-\frac{t}{2}|\xi|^{2\theta}}$;
- $\left|\partial_\xi^\alpha e^{t\lambda_+}\right| \lesssim |\xi| - |\alpha| e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$;
- $\left|\partial_\xi^\alpha ((\lambda_+ - \lambda_-)^{-1})\right| \lesssim |\xi| - 2\theta - |\alpha|$,

where, we recall, the eigenvalues $\lambda_\pm$ are defined as

$$\lambda_\pm = -\frac{1}{2}|\xi|^{2\theta}\langle\xi\rangle^{-2} \pm \frac{1}{2}(\mathbf{x}4\theta_J - 4 - 4\mathbf{x}2)^{\frac{1}{2}}, \qquad |\xi| \leq \varepsilon_0, \tag{A6}$$

where, for a given $\theta \in [0, 1/2)$, $\varepsilon_0 = \varepsilon_0(\theta)$ is sufficiently small to get

$$\varepsilon_0^{4\theta}(1+\varepsilon_0^2)^{-2} \geq 8\varepsilon_0^2.$$

Since $\hat{K}_0$, and $\hat{K}_1$ are defined as products of the three terms above and $\hat{E}_1 = \langle\xi\rangle^{-2}\hat{K}_1$, by applying (A2) again we get the following estimates in the low-frequency region:

Lemma 6. *Let $\lambda_\pm$ as in (A6), α any multi-index and $\xi \in \mathbb{R}^n \setminus \{0\}$ with $|\xi| \leq \varepsilon_0$. Then,*

(i) $\left|\partial_\xi^\alpha \hat{K}_0(t,\xi)\right| \lesssim |\xi|^{-|\alpha|} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$;
(ii) $\left|\partial_\xi^\alpha \hat{K}_1(t,\xi)\right| \lesssim |\xi|^{-2\theta-|\alpha|} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$;
(iii) $\left|\partial_\xi^\alpha \hat{E}_1(t,\xi)\right| \lesssim |\xi|^{-2\theta-|\alpha|} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$.

We will combine these estimates for $\hat{K}_0$, $\hat{K}_1$ and $\hat{E}_1$ with a lemma that derives pointwise estimates for their inverse Fourier transforms:

Lemma 7 (Lemma 8.5 in[12]). *Assume that $f \in C_c^\kappa(\mathbb{R}^n)$, for some $\kappa \geq 0$ integer, and that it verifies the estimates*

$$\left|\partial_\xi^\alpha f(\xi)\right| \lesssim |\xi|^{-a} \forall |\alpha| \leq \kappa,$$

for some $a < n$. Then, $g = \mathcal{F}^{-1}f$ satisfies the estimate $|g(x)| \lesssim (1+|x|)^{-\kappa}$.

Moreover, if $f \in C_c^{\kappa+1}$, and

$$\left|\partial_\xi^\alpha f(\xi)\right| \lesssim |\xi|^{-a_1} \forall |\alpha| = \kappa + 1,$$

for some $a_1 \in [n, n+1)$, then

$$|g(x)| \lesssim \begin{cases} (1+|x|)^{-\kappa-(n-a)}, & \textit{if } a > a_1 - 1, \\ (1+|x|)^{-\kappa-(n+1-a_1)}, & \textit{if } a \leq a_1 - 1 \textit{ and } a_1 \in (n, n+1) \\ (1+|x|)^{-\kappa-1}\log(e+|x|), & \textit{if } a \geq n-1 \textit{ and } a_1 = n. \end{cases}$$

Lemma 8. *Assume $n \in \mathbb{N}$, $\theta \in [0, \frac{1}{2})$ and consider the localization in the low-frequency region of the operators $K_0(t,x), K_1(t,x), E_1(t,x)$ as in (27) and (33). Then, the following estimates hold:*

(i) $\left\|\mathcal{F}^{-1}(\hat{K}_0(t,\cdot)\chi_0)\right\|_{L^p} \lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)}{2(1-\theta)}+\delta}, \qquad p > 1,$
(ii) $\left\|\mathcal{F}^{-1}(\hat{K}_1(t,\cdot)\chi_0)\right\|_{L^p} \lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta}, \qquad p > n/(n-2\theta),$
(iii) $\left\|\mathcal{F}^{-1}(\hat{E}_1(t,\cdot)\chi_0)\right\|_{L^p} \lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta}, \qquad p > n/(n-2\theta),$

where $\chi_0 \in C_c^\infty$ is a cut-off function supported in the ball $B(0, \varepsilon_0)$.

Proof. To shorten notation, we may use, throughout this whole proof, only f to address the low-frequency localization of the function f, that is, $\mathcal{F}^{-1}\left(\hat{f}(\xi)\chi_0(\xi)\right)$.

The proof consists on applying Lemma 7, using the estimates obtained in Lemma 6. We will strike both cases (i) and (ii) at the same time by writing

$$\left|\partial_\xi^\alpha \hat{K}_\ell(t,\xi)\right| \lesssim |\xi|^{-2\ell\theta-|\alpha|}e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}, \ \ell = 0, 1.$$

Also, the estimates for $\hat{K}_1$ and $\hat{E}_1$ are the same, so the result obtained for $\hat{K}_1$ will also hold for $\hat{E}_1$.

First of all, we notice that

$$\|K_\ell(t,\cdot)\|_\infty \leq \left\|\hat{K}_\ell(t,\cdot)\right\|_{L^1} \lesssim \int_{B(0,\varepsilon_0)} |\xi|^{-2\ell\theta}e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}d\xi \lesssim (1+t)^{-\frac{n-2\ell\theta}{2(1-\theta)}}.$$

Indeed, for $t \in [0,1]$ the estimate above follows due to $e^{-\frac{t}{2}|\xi|^{2(1-\theta)}} \leq 1$ and

$$\int_{B(0,\varepsilon_0)} |\xi|^{-2\ell\theta}d\xi \leq C_{n,\theta},$$

whereas, for $t \geq 1$, it follows by the change of variable $\xi \mapsto t^{-\frac{1}{2(1-\theta)}}\xi$:

$$\int_{\mathbb{R}^n} |\xi|^{-2\ell\theta}e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}d\xi = t^{-\frac{n-2\ell\theta}{2(1-\theta)}}\int_{\mathbb{R}^n} |\xi|^{-2\ell\theta}e^{-\frac{1}{2}|\xi|^{2(1-\theta)}}d\xi = \tilde{C}_{n,\theta}t^{-\frac{n-2\ell\theta}{2(1-\theta)}}.$$

On the other hand, we may apply Lemma 7 with $\kappa = n-1$, $a = n-1+2\ell\theta$, $a_1 = n+2\ell\theta$, that is, we get

$$|K_0(t,x)| \lesssim (1+|x|)^{-n}\log(e+|x|), \qquad |K_1(t,x)| \lesssim (1+|x|)^{-n+2\theta}.$$

By interpolation, it follows that $K_0(t,\cdot) \in L^p$ for any $p > 1$, and (i) holds for any $\delta > 0$, and that $K_1(t,\cdot), E_1(t,\cdot) \in L^p$ for any $p > n/(n-2\theta)$, and (ii), (iii) hold for any $\delta > 0$. ☐

Since $u^{\text{lin}}(t,x) = K_0(t,x) * u_0(x) + K_1(t,x) * u_1(x)$, the previous lemma can be used together with Young's theorem for convolution to prove the low-frequency part of Lemmas 2 and 4.

A.2 | Estimates for the high-frequency region

As in the low-frequency region, one can apply several instances of (A2) and (A3) to obtain the following estimates in the high-frequency region $\{\xi : |\xi| \geq 1\}$:

- $\left|\partial_\xi^\alpha \lambda_\pm\right| \lesssim |\xi|^{1-|\alpha|}$;
- $\left|\partial_\xi^\alpha e^{t\lambda_\pm}\right| \lesssim \left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}$;
- $\left|\partial_\xi^\alpha (\lambda_+ - \lambda_-)^{-1}\right| \lesssim |\xi|^{-1-|\alpha|}$,

where

$$\lambda_\pm = -\frac{1}{2}|\xi|^{2\theta}\langle\xi\rangle^{-2} \pm \frac{i}{2}\left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{\frac{1}{2}}, \qquad |\xi| \geq 1.$$

Therefore, from the definitions of $\hat{K}_0$, $\hat{K}_1$, and $\hat{E}_1$, we may apply once more (A2) to collect the following estimates in the high-frequency region. It is easy to prove that those estimates are, also valid in the intermediate frequency region $\{\xi : \varepsilon_0/2 \leq |\xi| \leq 1\}$, since it is a compact subset of $\mathbb{R}^n \setminus \{0\}$, so that we have the following.

With this remark, the proof of Lemmas 2 and 4 immediately follow by combining Lemma 8 in low-frequency region with the next results in intermediate and high-frequency region. Indeed, it is sufficient to assume that $\chi_0 = 1$ in the ball $B(0, \varepsilon_0/2)$ and define $\chi_1 = 1 - \chi_0$, so that χ_1 is supported in the region $\{\xi : |\xi| \geq \varepsilon_0/2\}$.

Lemma 9. *Let* $\lambda_\pm$ *as in (A6),* α *any multi-index,* $\theta \in \left[0, \frac{1}{2}\right)$, $\xi \in \mathbb{R}^n$ *with* $|\xi| \geq \varepsilon_0/2$ *and* $s \in \mathbb{R}$. *Also, let* $\hat{K}_0, \hat{K}_1, \hat{E}_1$ *be as in (27) and (33). Then,*

(i) $\left|\partial_\xi^\alpha \hat{K}_0(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-s}\left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right) e^{-t|\xi|^{2(\theta-1)}}$;

(ii) $\left|\partial_\xi^\alpha \hat{K}_1(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-s-1}\left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right) e^{-t|\xi|^{2(\theta-1)}}$;

(iii) $\left|\partial_\xi^\alpha \hat{E}_1(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-s-3}\left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right) e^{-t|\xi|^{2(\theta-1)}}$.

Using the boundedness of $P(x)e^{-x}$ for $P(x)$ a polynomial in x, we can rewrite these estimates in a more fit way to apply the desired multiplier theorem:

Lemma 10. *Let* $k \geq 0$, α *any multi-index,* $\theta \in \left[0, \frac{1}{2}\right)$, $\xi \in \mathbb{R}^n$ *with* $|\xi| \geq \varepsilon_0/2$. *Assume that*

$$\left|\partial_\xi^\alpha \hat{f}(\xi)\right| \lesssim |\xi|^{-s-k}\left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right) e^{-t|\xi|^{2(\theta-1)}}.$$

Then, for every $b \leq s+k$, *the following estimate holds*

$$\left|\partial_\xi^\alpha \hat{f}(\xi)\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s+k-b}{2(1-\theta)}}.$$

Proof. We have, using the fact that $x \mapsto (2x+1)^m e^{-x}$ and $x \mapsto x^m e^{-x}$ are bounded for $m > 0$,

$$\begin{aligned}
\left|\partial_\xi^\alpha \hat{f}(\xi)\right| &\lesssim |\xi|^{-s-k}\left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right) e^{-t|\xi|^{2(\theta-1)}} \\
&\lesssim |\xi|^{-s-k}\left(t + |\xi|^{-1}\right)^{|\alpha|} e^{-t|\xi|^{2(\theta-1)}} \\
&\lesssim |\xi|^{-s-k} e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}\left(t|\xi|^{2(\theta-1)} + |\xi|^{2\theta-3}\right)^{|\alpha|} e^{-\frac{t}{2}|\xi|^{2(\theta-1)}} \\
&\lesssim |\xi|^{-s-k} e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}\underbrace{\left(t|\xi|^{2(\theta-1)} + 1\right)^{|\alpha|} e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}}_{\text{bounded}} \\
&\lesssim |\xi|^{-s-k} e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}.
\end{aligned} \tag{A7}$$

Now, let $b \leq s+k$. The desired estimate is trivial if $t \leq 1$, due to $|\xi|^{-s-k} \lesssim |\xi|^{-b}$, since $|\xi| \geq \varepsilon_0/2$. Let $t \geq 1$. Then,

$$\begin{aligned}
\left|\partial_\xi^\alpha \hat{f}(\xi)\right| &\lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}\left[|\xi|^{-s-k+b} e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\right] \\
&\lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}\left[\left(\frac{t}{2}|\xi|^{2(\theta-1)}\right)^{\frac{-s-k+b}{2(\theta-1)}} e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\right] t^{-\frac{-s-k+b}{2(\theta-1)}} \\
&\lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|} t^{-\frac{s+k-b}{2(1-\theta)}},
\end{aligned} \tag{A8}$$

provided that the exponent of the term inside the square brackets is nonnegative, that is,

$$\frac{-s-k+b}{2(\theta-1)} \geq 0 \iff s+k-b \geq 0 \iff b \leq s+k.$$

☐

Combining the results of Lemmas 9 and 10, we have the following estimates:

(i) $\left|\partial_\xi^\alpha \hat{K}_0(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s-b}{2(1-\theta)}}, \ b \leq s$;

(ii) $\left|\partial_\xi^\alpha \hat{K}_1(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s+1-b}{2(1-\theta)}}, \ b \leq s+1$;

(iii) $\left|\partial_\xi^\alpha \hat{E}_1(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s+3-b}{2(1-\theta)}}, \ b \leq s+3$.

With these three estimates, we can use the following multiplier theorem to obtain $L^p - L^p$ estimates on the high-frequency region:

Theorem 8 (Theorem 1 in[41]). *Let* $a \geq 0$, $b \geq 0$, $1 < p_0 < 2$, $na\left(\frac{1}{p_0}-\frac{1}{2}\right) = b$, $k = \max\left\{\left[n\left(\frac{1}{p_0}-\frac{1}{2}\right)\right]+1, [\frac{n}{2}]+1\right\}$. *If* $m \in C^k(\mathbb{R}^n)$, $m(\xi) = 0$ *in a neighborhood of 0 and*

$$\left|\partial_\xi^\alpha m(\xi)\right| \leqslant |\xi|^{-b}\left(A|\xi|^{a-1}\right)^{|\alpha|}, |\alpha| \leqslant k$$

with $A \geqslant 1$, *then for any* $f \in L^p(\mathbb{R}^n)$, *it holds* $\mathcal{F}^{-1}(m\hat{f}) \in L^p(\mathbb{R}^n)$ *and*

$$\|\mathcal{F}^{-1}(m\hat{f})\|_{L^p(\mathbb{R}^n)} \leqslant CA^{n\left(\frac{1}{p}-\frac{1}{2}\right)}\|f\|_{L^p(\mathbb{R}^n)}, \text{ for } p \in [p_0, 2].$$

We are finally ready to collect the $L^p - L^p$ estimates in the high-frequency region:

Lemma 11. *Let* $n \in \mathbb{N}$, $\theta \in \left[0, \frac{1}{2}\right)$, *α any multi-index, and* $p \in (1,2)$. *Assume that* $u_0 \in W^{3,p}(\mathbb{R}^n)$, $u_1 \in W^{2,p}(\mathbb{R}^n)$ *and* $\psi \in L^p(\mathbb{R}^n)$, *and that*

$$n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right) \leq 3.$$

Then, the localization in the intermediate and high-frequency regions of $u^{lin}(t,x)$ *and* $E_1(t,x)$ *defined in (25) and (33), satisfy the following estimates:*

(i) $\left\|\mathcal{F}^{-1}(\hat{u}^{lin}(t,\cdot)\chi_1)\right\|_{L^p} \lesssim (1+t)^{-\frac{3}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}}\|(u_0,u_1)\|_{W^{3,p}\times W^{2,p}}$,

(ii) $\left\|\mathcal{F}^{-1}(\hat{E}_1(t,\cdot)\hat{\psi}\chi_1)\right\|_{L^p} \lesssim (1+t)^{-\frac{3}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}}\|\psi\|_{L^p}$,

where $\chi_1 \in C^\infty$ *is supported in the region* $\{\xi : |\xi| \geq \varepsilon_0/2\}$.

Proof. We recall that $|\xi| \approx \langle|\xi|\rangle$, since $|\xi| \geq \varepsilon_0/2$. The proof follows applying Theorem 8 with $m = \hat{K}_0(t,\xi)\langle\xi\rangle^{-3}\hat{\chi}_1$, $m = \hat{K}_1(t,\xi)\langle\xi\rangle^{-2}\hat{\chi}_1$, $m = \hat{E}_1(t,\xi)\hat{\chi}_1$, and, respectively, $\hat{f} = \hat{u}_0\langle\xi\rangle^3$, $\hat{f} = \hat{u}_1\langle\xi\rangle^2$ and $f = \psi$.

(i) For $\hat{K}_0(t,\xi)$, we use the estimate

$$\left|\partial_\xi^\alpha \hat{K}_0(t,\xi)\langle\xi\rangle^{-3}\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{3-b}{2(1-\theta)}},$$

and apply Theorem 8 with $m(t,\xi) = (1+t)^{\frac{3-b}{2(1-\theta)}}\hat{K}_0(t,\xi)\langle\xi\rangle^{-3}\chi_1$, $\hat{f} = \hat{u}_0\langle\xi\rangle^3$, $A = 1$, $a = 3-2\theta$ and $b = n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)$. Hence, for any $u_0 \in W^{3,p}(\mathbb{R}^n)$,

$$\|m(t,\cdot) * f\|_{L^p} \lesssim (1+t)^{-\frac{3}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}}\|u_0\|_{W^{3,p}}, \tag{A9}$$

provided that $n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right) \leq 3$.

Similarly, applying Theorem 8 with $m(t,\xi) = (1+t)^{\frac{3-b}{2(1-\theta)}}\hat{K}_1(t,\xi)\langle\xi\rangle^{-2}\chi_1$ and $\hat{f} = \hat{u}_1\langle\xi\rangle^2$, we get

$$\|m(t,\cdot) * f\|_{L^p} \lesssim (1+t)^{-\frac{3}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}}\|u_1\|_{W^{2,p}}, \tag{A10}$$

provided that $n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right) \leq 3$. Recalling that $u^{\text{lin}}(t,x) = K_0(t,x) * u_0(x) + K_1(t,x) * u_1(x)$, estimates (A8) and (A9) give us the desired result.

(ii) Again from Lemmas 9 and 10, we have the estimate

$$\left|\partial_\xi^\alpha \hat{E}_1(t,\xi)\right| \lesssim |\xi|^{-b}\left(|\xi||\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{3-b}{2(1-\theta)}},$$

for $b \leq 3$. Applying Theorem 8 with $m(\xi) = (1+t)^{\frac{3-b}{2(1-\theta)}}\hat{E}_1(t,\xi)\chi_1$, $f = \psi$ $A = 1$, $a = 3-2\theta$ and $b = n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)$, we obtain

$$\|E_1(t,\cdot) * \psi\|_{L^p} \lesssim (1+t)^{-\frac{3}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}}\|\psi\|_{L^p}, \tag{A11}$$

provided that $n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right) \leq 3$. $\square$