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Minimal affine varieties of superalgebras with superinvolution: A characterization

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We exhibit a class $\mathscr{C}$ of finite dimensional algebras with superinvolution over an algebraically closed field of characteristic zero, with the remarkable property that each member of $\mathscr{C}$ generates a minimal variety of algebras with superinvolution. This sums up to the fact that any affine minimal variety of algebras with superinvolution is generated by a suitable member of $\mathscr{C}$, thus providing a complete characterization of the affine minimal varieties of algebras with superinvolution.

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1. Introduction

In the present paper we characterize the minimal affine $\diamond$-varieties, that is those varieties of superalgebras with superinvolutions ($\diamond$-algebras for short) originating from a finitely generated $\diamond$-algebra and such that any proper subvariety has strictly lesser $\diamond$-exponent than the parent variety, thus completing the picture outlined in the former work [4]. More precisely, in [4] it has been proved that any such variety is generated by a suitable finite dimensional $\diamond$-algebra with certain canonical features, denoted $UT^{\diamond}_{\mathbf{g}}(A_1, \dots, A_m)$, depending on a suitable sequence $(A_1, \dots, A_m)$ of finite dimensional $\diamond$-simple algebras and a translation vector $\mathbf{g} \in \mathbb{Z}_2^m$. Moreover, these parameters also determine both a grading and a superinvolution for the algebra. Here we prove that in fact any assignment of these parameters results into a finite dimensional $\diamond$-algebra generating a minimal $\diamond$-variety.

Superalgebras endowed with a superinvolution play a natural and relevant role in the Theory of Lie and Jordan superalgebras [8], [9], and have recently become a subject of increasing interest and extensive investigations in PI-Theory, too. Among the several recent results, we mention that any affine $\diamond$-algebra A admits an integer exponent $\exp(A, \diamond) = \lim_n \sqrt[n]{c_n(A, \diamond)} \in \mathbb{N}$, coinciding with the dimension of a maximal admissible semisimple $\diamond$-subalgebra of A [7]. A complete classification of the $\diamond$-varieties of polynomial growth by means of an exhaustive list of minimal $\diamond$-varieties of exponent 2 has been obtained [5]: besides the $\diamond$-varieties generated by the Grassmann algebra of an infinite dimensional vector space endowed with one between two possible superinvolutions (denoted $\sharp$ and $\star$), there are three affine varieties, generated by the finite dimensional $\diamond$-algebras there denoted D, M, M^{sup} which, in our description, are $UT^{\diamond}_{0}(F)$, $UT^{\diamond}_{(0,0)}(F, F)$ and $UT^{\diamond}_{(0,1)}(F, F)$ respectively.

An even more outstanding result underlines a close connection between $\diamond$-algebras and algebras with an (ordinary) involution, similar to the one linking associative PI-algebras and finite dimensional superalgebras. More precisely, in [1] it has been proved that any variety of algebras with involution can be generated by the Grassmann envelope of a finite dimensional $\diamond$-algebra; that is, in the framework of PI-algebras with involution, superalgebras with superinvolution play the role superalgebras play in the framework of PI-algebras. This, together with the existence of the $*$-exponent obtained in [6], allowed us to describe the $*$-minimal varieties in terms of the $\diamond$-algebras of type $UT^{\diamond}_{\mathbf{g}}(A_1, \dots, A_m)$ and to describe their $*$-identities in terms of the $\diamond$-simple algebras A_i, at least in principle ([4], Theorem 4.7 and Theorem 5.8).

In the present paper, after a brief review of the basic definitions and theoretic tools (Section 2), in Section 3 we recall the classification of the finite dimensional $\diamond$-algebras over an algebraically closed field of characteristic zero, underlining their specific features, with special attention paid to the class of $\diamond$-algebras of type $M_{n,n}$ which, in a loose language, is the forest where the algebras $UT^{\diamond}_{\mathbf{g}}(A_1, \dots, A_m)$ grow. Section 4 is rather technical, and deals with numerical invariants of $\diamond$-simple algebras, namely their $\diamond$-dimensions, embodying in Capelli-type polynomials up to complementary distinguishing polynomials. In fact, this marks a neat difference between graded involutions and superinvolutions: while most of the basic notions about $\diamond$-algebras are much alike the ones about superalgebras endowed with a graded involution, in the latter case non isomorphic simple objects must have different invariants. It turns out, instead, that there exist pairwise non-isomorphic $\diamond$-simple algebras having the same $\diamond$-dimensions (critical $\diamond$-simple algebras in our terminology). The algebras $UT^{\diamond}_{\mathbf{g}}(A_1, \dots, A_m)$ are then presented in Section 5, as a quick survey to what was done in [4]. The main results are presented in Section 6: after a couple of preparatory Propositions, Theorem 6.6 states that any $\diamond$-algebra $UT^{\diamond}_{\mathbf{g}}(A_1, \dots, A_m)$ does generate a minimal $\diamond$-variety. This fact, together with the results in [4] (proving that any affine minimal $\diamond$-variety is generated by a suitable $UT^{\diamond}_{\mathbf{g}}(A_1, \dots, A_m)$), provides a complete characterization of the affine minimal $\diamond$-varieties.

2. Basic background

Throughout the paper, F will denote an algebraically closed field of characteristic zero, and the word algebra will mean a nontrivial associative F-algebra. An algebra A is called a superalgebra if it comes with a vector space decomposition $A = A_0 \oplus A_1$, with indexes $0, 1 \in \mathbb{Z}_2$, such that $A_i A_j \subseteq A_{i+j}$. An element $a \in A_i$ is called homogeneous of degree i, and an F-basis $\mathscr{B}$ of A is called homogeneous if all its members are homogeneous. In this case, we write $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$. We will usually shorten the terminology and speak of even (resp. odd) elements instead of homogeneous elements of degree 0 (resp. 1).

A linear map $\circ \in End_F(A)$ is involutory if $\circ^2 = Id_A$, and $\circ$ is graded if $A_i^{\circ} \subseteq A_i$ for all $i \in \mathbb{Z}_2$. An involutory graded map is called a graded involution if $(ab)^{\circ} = b^{\circ} a^{\circ}$ for all homogeneous elements $a, b \in A$, and a superinvolution if it acts as a graded involution except on pairs of odd elements, in which case $(ab)^{\circ} = -b^{\circ} a^{\circ}$. If $\circ$ is a graded involution on A, we call $(A, \circ)$ a $*$-superalgebra; in case $\circ$ is a superinvolution, $(A, \circ)$ will be called a $\diamond$-algebra. In both cases, a $\circ$-ideal is a (twosided) ideal I of A such that $I = (I \cap A_0) \oplus (I \cap A_1)$ and $I^{\circ} = I$; A is $\circ$-simple if the only $\circ$-ideals of A are the trivial ones.

Since $\circ$ is always diagonalizable, A decomposes into eigenspaces $A = A^+ \oplus A^-$ with respect to the eigenvalues $+1$ and -1; elements from A^+ (resp. A^-) are called symmetric (resp. skew-symmetric). Both decompositions combine into a finer decomposition $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$. If $\delta \in \mathbb{Z}_2$ and $\lambda \in \{+, -\} =: \hat{\mathbb{Z}}_2$, an element $a \in A_\delta^\lambda$ is called $\circ$-homogeneous, and (δ, λ) is called its $\circ$-degree. Collecting elements from an F-basis of each A_δ^λ one gets a $\circ$-homogeneous basis $\mathscr{B} = \mathscr{B}_0^+ \cup \mathscr{B}_0^- \cup \mathscr{B}_1^+ \cup \mathscr{B}_1^-$ of A.

The algebra homomorphisms preserving the $\circ$-algebra structure are called $\circ$-homomorphisms; they consist of all graded algebra homomorphisms commuting with $\circ$. The natural homomorphism theorems hold for $\circ$-homomorphisms too, hence the kernel of a $\circ$-homomorphism is a $\circ$-ideal, we may form factor $\circ$-algebras and so on. In particular, any $\circ$-algebra is an homomorphic image of a free universal $\circ$-algebra, denoted $F\langle X|\circ\rangle$. There are different equivalent ways of presenting $F\langle X \mid \circ\rangle$; the easiest way fitting our concerns is to consider disjoint countable sets of indeterminates Y^+, Y^-, Z^+, Z^-, where the elements of $Y = Y^+ \cup Y^-$ are even indeterminates, the elements in $Z = Z^+ \cup Z^-$ are odd, each set split into subsets of symmetric and skew-symmetric elements. Then take $X := Y \cup Z$ and consider the free associative unitary algebra generated by X endowed with the natural $\mathbb{Z}_2$-grading on the monomials and the formal $\circ$-structure arising from the assigned generators to get a concrete realization of the free $\circ$-algebra $F\langle X|\circ\rangle$. It is free on X in the class of $\circ$-algebras, in the sense that any set theoretic map from X to any $\circ$-algebra A preserving the $\circ$-degrees can be uniquely extended to a $\circ$-homomorphism $F\langle X|\circ\rangle \to A$. Elements of $F\langle X|\circ\rangle$ are called $\circ$-polynomials; if A is any $\circ$-algebra, those $\circ$-polynomials lying in the kernel of all $\circ$-homomorphisms $F\langle X|\circ\rangle \to A$, that is those $\circ$-polynomials $f = f(x_1, \dots, x_n) \in F\langle X|\circ\rangle$ vanishing under all valuations $f(a_1, \dots, a_n)$ obtained replacing the $\circ$-letter x_i of $\circ$-degree (δ, λ) with an element $a_i \in A_\delta^\lambda$, are called the $\circ$-polynomial identities of A (usually shortened in $\circ$-PI of A), forming the set $Id(A, \circ) \subseteq F\langle X|\circ\rangle$.

Actually, $Id(A, \circ)$ is $\circ$-ideal of $F\langle X|\circ\rangle$, which moreover is invariant under all $\circ$-endomorphisms of the free $\circ$-algebra, and therefore called a T°-ideal; more specifically, it is called the T°-ideal of $\circ$-polynomial identities of A.

Although we have so far surveyed notions related to graded involutions and super-involutions following a common path which could go further on, in the present paper we are almost exclusively interested in superinvolutions. Therefore, from now on, we stick to superinvolutions, and consider $\diamond$-polynomials, $\diamond$-algebras, $\diamond$-ideals and so on. In particular, $Id(A, \diamond)$ is the $T^\diamond$-ideal of $\diamond$-PI of the $\diamond$-algebra A.

The object of investigations in PI-Theory is the description of T-ideals, or better $T^\diamond$-ideals of $\diamond$-algebras in our specific settings. An essential tool in this investigation are the so-called multilinear $\diamond$-polynomials:

Definition 2.1. For $1 \leqslant n \in \mathbb{N}$ let $P_n^\diamond$ denote the vector subspace of $F\langle X|\diamond\rangle$ spanned by all monomials $x_{\pi(1)} \dots x_{\pi(n)}$, where $\pi \in S_n$ and, for each $i \in [n]$, it holds $x_i \in \{y_i^{\lambda_i}, z_i^{\lambda_i}\}$, $\lambda_i \in \{+, -\}$. Its elements are called multilinear $\diamond$-polynomials of degree n.

There are two main reasons in considering multilinear $\diamond$-polynomials: the first one is that if $f \in P_n^\diamond$ then $f \in Id(A, \diamond)$ if and only if f vanishes under all assignments of its letters to elements of a $\diamond$-basis of A; the second one is because the slices $P_n^\diamond \cap Id(A, \diamond)$ (for $n \in \mathbb{N}$) determine exactly the whole $Id(A, \diamond)$, in the sense that $Id(A, \diamond)$ is the smallest $T^\diamond$-ideal of $F\langle X|\diamond\rangle$ containing all these slices. Also, the codimension sequence $c_n^\diamond(A) := \dim P_n^\diamond / (P_n^\diamond \cap Id(A, \diamond))$ (for $n \in \mathbb{N}$), called the $\diamond$-codimension sequence of A, provides a kind of measure on how big $Id(A, \diamond)$ is. An important result on $\diamond$-codimensions states that if A is an affine $\diamond$-algebra satisfying an ordinary polynomial identity then the sequence $\sqrt{c_n^\diamond(A)}$ converges to a non-negative integer, called the $\diamond$-exponent of A, denoted $\exp(A, \diamond)$ ([7], Theorem 3.2). This provides also a qualitative selection among $T^\diamond$-ideals, better expressed in the equivalent language of $\diamond$-varieties.

Let us recall that if S is any set of $\diamond$-polynomials, the class $\mathscr{V}$ of $\diamond$-algebras B such that $S \subseteq Id(B, \diamond)$ is called the $\diamond$-variety generated by S, denoted $\mathscr{V} = Var(S, \diamond)$; the intersection $Id(\mathscr{V},\diamond)$ of all $Id(B,\diamond)$ for $B \in \mathscr{V}$ is therefore still a $T^\diamond$-ideal, called the $T^\diamond$-ideal of the variety, and gives rise to the sequence $c_n^\diamond(\mathscr{V})$, and the $\diamond$-exponent $\exp(\mathscr{V},\diamond)$, if it exists. Of course, if $\mathscr{U}$ is any subvariety of $\mathscr{V}$, then $c_n^\diamond(\mathscr{U}) \leqslant c_n^\diamond(\mathscr{V})$ and, when existing, $\exp(\mathscr{U},\diamond) \leqslant \exp(\mathscr{V},\diamond)$, but it may well happen that the $\diamond$-exponent of a proper subvariety equals $\exp(\mathscr{V},\diamond)$. When $\mathscr{U} \subsetneqq \mathscr{V}$ implies the strict inequality $\exp(\mathscr{U},\diamond) \lneqq \exp(\mathscr{V},\diamond)$, one says that $\mathscr{V}$ is a minimal $\diamond$-variety.

Now, let us start from a minimal $\diamond$-variety $\mathscr{V}$ generated by an affine $\diamond$-algebra A. Then $\exp(\mathscr{V},\diamond)$ does indeed exist, and in [4] it has been proved that $\mathscr{V}$ is generated also by a certain algebra $UT^\diamond_{\mathbf{g}}(A_1,\ldots,A_m)$, built on m finite dimensional $\diamond$-simple algebras A_i and a parameter $\mathbf{g} : [m] \to \mathbb{Z}_2$, thus providing a family of finite dimensional $\diamond$-algebras generating all affine minimal $\diamond$-varieties. The natural question is: does *any* $\diamond$-algebra of this family generate a minimal $\diamond$-variety? In the present paper we aim to show that the question has an affirmative answer.

3. Finite dimensional $\diamond$-simple algebras

Finite dimensional $\diamond$-simple algebras are built on simple superalgebras, and a quick reference to their classification is available in [2], Proposition 1. Here, some of them are presented in an alternative version, the same as in [4], based on the so called *antitranspose*, that is the reflection along the secondary diagonal (also called *anti-diagonal* in this paper). For convenience of the reader, and to make the paper self-contained, we begin with a quick survey on this presentation.

For any positive integer m, we will always denote $[m] := \{1,\ldots,m\}$, and M_m will denote the full matrix algebra $M_m(F)$, whose canonical basis is constituted by the usual matrix units e_{ij} $(i,j \in [m])$. Any function $\alpha : [m] \to \mathbb{Z}_2$ defines a so-called elementary grading on M_m by assigning to each e_{ij} the $\mathbb{Z}_2$-degree $|e_{ij}|_\alpha := \alpha(i)+\alpha(j)$, thus turning M_m into a simple superalgebra. The map α may conveniently be identified with a $\mathbb{Z}_2$-word of length m; the function $\mathscr{C}\alpha := \alpha + 1$, namely defined by $\mathscr{C}\alpha(i) = \alpha(i) + 1$, will be called the complementary map/word of α. Both α and $\mathscr{C}\alpha$ provide exactly the same grading on M_m. Then, denoting S_m the symmetric group on m elements, let $\gamma_m \in S_m$ be the reversing bijection defined by $\gamma_m(i) := m + 1 - i$ for all $i \in [m]$.

Definition 3.1. Let $m \geqslant 1$, and let $\vartheta_m \in End_F(M_m)$ be defined by $e_{ij}^{\vartheta_m} := e_{\gamma_m(j)\,\gamma_m(i)}$. The map ϑ_m is an involution on M_m, called the *antitranspose*.

We will simply write γ, ϑ, omitting the index m, whenever this cannot cause misunderstandings. Unlike to what happens for the usual transpose, not all $\alpha : [m] \to \mathbb{Z}_2$ turn ϑ into a graded involution for M_m. More precisely, this happens if and only if $\alpha\gamma = \alpha$ or $\alpha\gamma = \mathscr{C}(\alpha)$; in terms of grading words, since $\alpha\gamma = (\alpha(m),\ldots,\alpha(1)) = Rev(\alpha)$, the reverse word of α, the same cases can be expressed as $Rev(\alpha) = \alpha$ or $Rev(\alpha) = \mathscr{C}(\alpha)$, respectively. In both cases we get a simple $*$-superalgebra, but the latter is of utmost importance in our constructions, so we start surveying its characteristic features.

So, assume $\alpha\gamma = \mathscr{C}(\alpha) = \alpha+1$; then it is easy to see that the fibers $\alpha^{-1}(0)$ and $\alpha^{-1}(1)$ of α have the same cardinality, say n, hence $m = 2n$ and γ_m exchanges the fibers. This allows us to produce a signed version of ϑ_m which turns out to be a superinvolution:

Definition 3.2. For any $n \geqslant 1$ let $\alpha : [2n] \to \mathbb{Z}_2$ be any grading map such that $\alpha\gamma_{2n} = \alpha+1$, and let L, R be the fibers of α. Then define $\varepsilon : [2n]\times[2n] \to \{+1, -1\}$ by assigning $\varepsilon(i,j) = +1$ if and only if $(i,j) \in L \times R$. The F-linear map $\overline{\vartheta}_{2n}$ defined by setting $e_{ij}^{\overline{\vartheta}_{2n}} := \varepsilon(i,j)e_{ij}^{\vartheta_{2n}}$ is called the *supertranspose*.

Remark 3.3. It is immediate to check that $\overline{\vartheta} = \overline{\vartheta}_{2n}$ is a superinvolution on (M_{2n}, α), but of course its definition depends on the chosen fiber L. The "standard" choice is $L := \alpha^{-1}(0)$; the dual choice $L' := \alpha^{-1}(1) = R$ defines a different superinvolution $\overline{\vartheta}'$, indeed, but it does not yield anything really new: the $\diamond$-algebras $(M_{2n}, \alpha, \overline{\vartheta})$ and $(M_{2n}, \alpha, \overline{\vartheta}')$ are $\diamond$-isomorphic, and the radial symmetry $e_{ij} \to e_{\gamma(i)\,\gamma(j)}$ is an explicit $\diamond$-isomorphisms between them.

This is equivalent to inducing the grading on M_{2n} via the complementary map $\beta := \mathscr{C}\alpha$, then proceeding with the standard choice $L_\beta := \beta^{-1}(0)$ in order to define ε_β and, in fact, $\overline{\vartheta}'$.

An important feature of this structure consists in allowing the α-preimages of 0 of being scattered throughout $[2n]$, and in some sense keeping them into an "immaterial" state, since the resulting structure depends only on their number, n, and not on their names. To give a concrete form to this statement, we record the following:

Lemma 3.4. *Let $\alpha : [2n] \to \mathbb{Z}_2$ be such that $\alpha\gamma = \alpha + 1$, and let $u \in [2n]$. Define $g : [2n] \to \mathbb{Z}_2$ by $g^{-1}(1) = \{u, \gamma(u)\}$ and let $\alpha' := \alpha+g$; denote $\overline{\vartheta}$, $\overline{\vartheta}'$ the supertranspose built on the 0-fiber of α, α', respectively. Then $(M_{2n}, \alpha, \overline{\vartheta})$ and $(M_{2n}, \alpha', \overline{\vartheta}')$ are $\diamond$-isomorphic.*

The essence of the previous Lemma is that if $\alpha(u) = 1$ for $u \leqslant n$, then we may safely exchange u with $\gamma_{2n}(u)$ without altering the $\diamond$-structure; notice that this is equivalent to adding $+1$ to both $\alpha(u)$ and $\alpha\gamma(u)$, as in the Lemma.

Corollary 3.5. *Let $M_{n,n}$ denote M_{2n} endowed with grading induced by the word $(0^n, 1^n)$ and supertranspose $\overline{\vartheta}$ associated to the 0-fiber L. Any supertranspose structure on M_{2n} is $\diamond$-isomorphic to $M_{n,n}$.*

Hence, we will denote $M_{n,n}$ the $\diamond$-structure $(M_{2n}, (0^n, 1^n), \overline{\vartheta})$ when concerned just with structural properties. As a matter of fact, there are several circumstances in which allowing the general assignment $[2n] \to \mathbb{Z}_2$ is still preferable. In these cases, it will be enough to fix $\alpha : [n] \to \mathbb{Z}_2$ and then consider $\tilde{\alpha} = (\alpha|\mathscr{C}Rev(\alpha))$, the concatenation of the n-length $\mathbb{Z}_2$-words α, $\mathscr{C}Rev(\alpha) = \alpha\gamma_n + 1$, and choose $L = \tilde{\alpha}^{-1}(0)$, to get a $\diamond$-structure of supertranspose type on M_{2n}, which will be denoted simply by $(M_{2n}, \alpha, \overline{\vartheta})$, or even

(M_{2n}, α). The word α will be called the *grading word* for the supertranspose structure on M_{2n}.

Within this more general setting, a variation on the previous arguments yields a useful result

Proposition 3.6. *Let $\alpha : [n] \to \mathbb{Z}_2$ be any grading word, (M_{2n}, α) be the associated supertranspose structure. If $u, v \in [n]$ belong to different α-fibers, let $g : [n] \to \mathbb{Z}_2$ be defined by $g^{-1}(1) = \{u, v\}$. Finally, define $\alpha' := \alpha + g$. Then (M_{2n}, α) and (M_{2n}, α') are $\diamond$-isomorphic.*

Moreover, setting $\sigma := (u\ v)(\gamma_{2n}(u)\ \gamma_{2n}(v)) \in S_{2n}$, the linear map ψ defined by $\psi(e_{ij}) = e_{\sigma(i)\,\sigma(j)}$ is an explicit $\diamond$-isomorphism between them.

Proof. Denote $\gamma := \gamma_{2n}$ for short, and let $L := \tilde{\alpha}^{-1}(0) = \alpha^{-1}(0) \cup \gamma\alpha^{-1}(1)$. Then $R = \gamma L$, and $\overline{\vartheta}$ is defined through the assignment of ε, that is setting $\varepsilon(i,j) = -1$ if and only if $(i,j) \in L \times R$. The $\tilde{\alpha}'$-fibers then are $L' = \sigma L$ and $R' = \sigma R$. Notice that, by construction, $\sigma\gamma = \gamma\sigma$, hence $R' = \sigma\gamma L = \gamma\sigma L = \gamma L'$, and ε' is defined by $\varepsilon'(i,j) = -1$ if and only if $(i,j) \in L' \times R'$.

Now, the map ψ is for sure an algebra isomorphism, and it is a graded one: indeed one can check it holds $\tilde{\alpha}'\sigma = \tilde{\alpha}$. Therefore, the $\mathbb{Z}_2$-degree of $\psi(e_{ij})$ is $\tilde{\alpha}'\sigma(i) + \tilde{\alpha}'\sigma(j) = \tilde{\alpha}(i) + \tilde{\alpha}(j)$, the degree of e_{ij}, so ψ is indeed graded. Moreover, it holds

$$\psi(e_{ij}^{\overline{\vartheta}}) = (\varepsilon(i,j)e_{\gamma(i)\,\gamma(j)})^{\psi} = \varepsilon(i,j)e_{\sigma\gamma(j)\ \sigma\gamma(i)} = \varepsilon(i,j)e_{\gamma\sigma(j)\ \gamma\sigma(i)}$$
$$(\psi(e_{ij}))^{\overline{\vartheta}'} = e_{\sigma(i)\ \sigma(j)}^{\overline{\vartheta}'} = \varepsilon'(\sigma(i),\sigma(j))e_{\gamma\sigma(j)\ \gamma\sigma(i)},$$

and by definition $\varepsilon(i,j) = -1 \iff (i,j) \in L \times R \iff (\sigma(i),\sigma(j)) \in \sigma L \times \sigma R \iff (\sigma(i),\sigma(j)) \in L' \times R' \iff \varepsilon'(\sigma(i),\sigma(j)) = -1$, that is ψ is a $\diamond$-homomorphism. □

The focus here is on ψ, more than on the given structures being $\diamond$-isomorphic: the Proposition states that we may alter the grading words on pairs of elements belonging to different fibers without affecting the structure. This will be of importance later.

Let now switch briefly to the other case, namely when $\alpha\gamma_m = \alpha$. Here, the fibers L, R are fixed by γ_m so at least one of them, say L, has even size $2l$; let $|R| =: n$, so that $m = 2l + n$. Possibly passing to $\mathscr{C}(\alpha)$, we may assume that $L = \alpha^{-1}(0)$, and similarly to the previous case provide a signature for ϑ_m: choose l elements in L forming the set L_1, write $L_2 := \gamma_m(L_1)$, and fix the signature map $\eta : [m] \times [m] \to \{+1, -1\}$ according to the following table

η	L_1	R	L_2
L_1	$+1$	$+1$	-1
R	-1	$+1$	$+1$
L_2	-1	-1	$+1$

.

So for instance if $(i,j) \in L_1 \times L_1$ then $\eta(i,j) = +1$, if $(i,j) \in R \times L_1$ then $\eta(i,j) = -1$, and so on.

Definition 3.7. Let α be any grading map on $m = n + 2l$ with $\alpha\gamma = \alpha$, let L be a fiber of even size $2l$ and choose $L_1 \subseteq L$ of size l. Denoting η the resulting signature map, let $\tilde{\vartheta}_m$ denote the map defined by $e_{ij}^{\tilde{\vartheta}_m} := \eta(i,j)e_{ij}^{\vartheta_m}$. We call $\tilde{\vartheta}_m$ an *orthosymplectic superinvolution* for (M_m, α).

Again, $\tilde{\vartheta}$ is indeed a superinvolution but depending on the chosen set L_1 in a partially controlled way: denote $M_{l,n,l}$ the orthosymplectic structure arising from the grading $(0^l, 1^n, 0^l)$ and choosing $L_1 = [l]$; then

Proposition 3.8. *Let $(M_{n+2l}, \alpha, \tilde{\vartheta})$ be any orthosymplectic structure with $|L_1| = l$; then it is $\diamond$-isomorphic to $M_{l,n,l}$.*

Hence, $M_{l.n.l}$ will be the canonical representative for the orthosymplectic $\diamond$-algebras, when just structural properties are under considerations. By the way, a word of caution is needed: if $n = 2k$ is even too, but $k \neq l$, the structures $M_{l,2k,l}$ and $M_{k,2l,k}$ are different: the size of the "central" block $R \times R$ really matters.

Example 3.9. The simplest, yet extreme, distinction appears already for $m = 2$.

Consider $M_{0,2,0}$: it is the full matrix algebra M_2 endowed with the trivial grading, the superinvolution turns into ϑ_2 (the antitranspose involution), and its subspace of skew-symmetric elements is spanned by the single vector $e_{11} - e_{22}$. Hence $M_{0,2,0}$ is actually $*$-isomorphic to (M_2, t), t the ordinary transpose.

Then consider $M_{1,0,1}$: it is once again M_2 endowed with the trivial grading, so the superinvolution $\tilde{\vartheta}$ turns again into an ordinary involution; by the way, it coincides precisely with (M_2, s), s the symplectic involution.

In order to present the remaining classes of $\diamond$-simple algebras, recall that

- usually, $M_{k,l}$ denotes the superalgebra structure induced on M_{k+l} by the word $(0^k, 1^l)$;
- if A is any superalgebra, the *superopposite of* A, denoted A^{sop}, is obtained from A by keeping its F-structure (in particular, its grading) but changing its product into $\overline{\odot}$, defined on homogeneous elements $a_i \in A_i$, $b_j \in A_j$ by $a_i \overline{\odot} b_j := (-1)^{ij} b_j a_i$;
- if A is any $\diamond$-algebra, the direct sum $A \oplus A^{\mathrm{sop}}$ is a superalgebra, whose i-graded component is $A_i \oplus A_i$, and the exchange map $(a,b)^{ex} := (b,a)$ is a superinvolution for $A \oplus A^{\mathrm{sop}}$;
- considering the 2-dimensional algebra $Q := FS_2$, which we represent as $Q := F \oplus cF$ (where $c^2 = 1$), it is a simple superalgebra with grading $Q_0 = F$ and $Q_1 = cF$. The $2n^2$-dimensional algebra $M_n(Q) = M_n \oplus cM_n$ inherits a standard superalgebra

structure (actually, a simple superalgebra), and can be $\mathbb{Z}_2$-embedded into $M_{n,n}$ by the map

$$a + cb \in M_n \oplus cM_n \hookrightarrow \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M_{n,n};$$

we will usually identify $M_n(Q)$ with its image inside $M_{n,n}$, but when convenient we will switch to $M_n(Q) = M_n \oplus cM_n$;

- any finite-dimensional simple superalgebra is either isomorphic to $M_{k,l}$ with suitable $k \geqslant l \geqslant 0$, or to $M_n(Q)$ for suitable $n \geqslant 1$.

Then, the remaining classes of $\diamond$-simple algebras are $M_{k,l} \oplus M_{k,l}^{\mathrm{sop}}$, for $k \geqslant l \geqslant 0$, and $M_n(Q) \oplus M_n(Q)^{\mathrm{sop}}$, for $n \geqslant 1$. Their presentation is therefore the same as in [2], unlike the simple ones, and we just recalled them for the sake of completeness. To summarize, we record the classification theorem

Theorem 3.10. *Let A be a finite dimensional $\diamond$-simple algebra. Then A is $\diamond$-isomorphic to one among the following algebras*

(1) *$M_{n,n}$ for suitable $n \geqslant 1$;*
(2) *$M_{l,n,l}$ for suitable $n, l \geqslant 0$ and $(l, n) \neq (0, 0)$;*
(3) *$M_{k,l} \oplus M_{k,l}^{sop}$ for suitable $k \geqslant l \geqslant 0$ and $k > 0$ if $l = 0$;*
(4) *$M_n(Q) \oplus M_n(Q)^{sop}$ for suitable $n \geqslant 1$.*

4. Dimensional invariants of $\diamond$-simple algebras and critical cases

Let $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ be the decomposition of the $\diamond$-simple algebra A into its $\diamond$-homogeneous vector subspaces. To which extent its $\diamond$-dimensions, that is the dimensions $\dim A_\delta^\lambda$ for $(\delta, \lambda) \in \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2$, determine the isomorphism class of A?

We remark that the same question, when formulated for $*$-superalgebras, has a simple and straight answer: if A, B are $*$-simple superalgebras with the same $*$-dimensions then they are isomorphic.

Things go differently when dealing with $\diamond$-algebras. In what follows, of course, we are allowed to work with the canonical representatives of the isomorphism classes of $\diamond$-simple algebras. To begin with, it is easy to construct $\diamond$-homogeneous basis of $\diamond$-simple algebras.

Recall that $\mathscr{E}_m := \{e_{ij} \mid i, j \in [m]\}$ denotes the canonical F-basis of matrix units for M_m; also, if a is an element of an $\diamond$-algebra, let $a^+ := a + a^\diamond$ and $a^- := a - a^\diamond$ be the symmetric and skew-symmetric elements arising from a. Notice that if a is symmetric (resp. skew-symmetric) then $a^- = 0$ (resp. $a^+ = 0$); in this case, $a^+ := a$ (resp. $a^- := a$).

Proposition 4.1. *Let A be a canonical $\diamond$-simple algebra. Then the following are $\diamond$-homogeneous basis of A*

(1) *if* $A = M_{n,n}$, $\mathscr{B} := \{e_{ij} \mid i + j = 2n + 1\} \cup \{e_{ij}^{\lambda} \mid i + j < 2n + 1,\ \lambda \in \hat{\mathbb{Z}}_2\}$*;*
(2) *if* $A = M_{l,n,l}$, $\mathscr{B} := \{e_{ij} \mid i + j = n + 2l + 1\} \cup \{e_{ij}^{\lambda} \mid i + j < n + 2l + 1,\ \lambda \in \hat{\mathbb{Z}}_2\}$*;*
(3) *if* $A = M_{k,l} \oplus M_{k,l}^{sop}$, $\mathscr{B} := \{(e_{ij}, 0)^{\lambda} \mid i, j \in [k + l],\ \lambda \in \hat{\mathbb{Z}}_2\}$*;*
(4) *if* $A = M_n(Q) \oplus M_n(Q)^{sop}$, $\mathscr{B} := \{(e_{ij}, 0)^{\lambda}, (ce_{ij}, 0)^{\lambda} \mid i, j \in [n],\ \lambda \in \hat{\mathbb{Z}}_2\}$*.*

Then it is easy to compute the $\diamond$-dimensions for each of the classes:

Proposition 4.2. *Let A be a $\diamond$-simple algebra.*

(1) *If* $A = M_{n,n}$ *then* $\dim A_{\delta}^{\lambda} = n^2$ *for all* $(\delta, \lambda) \in \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2$*;*
(2) *if* $A = M_{l,n,l}$ *then* $\dim A_1^{+} = 2nl = \dim A_1^{-}$ *and*

$$\dim A_0^{+} = 2l^2 - l + \frac{n^2 + n}{2}, \quad \dim A_0^{-} = 2l^2 + l + \frac{n^2 - n}{2};$$

(3) *if* $A = M_{k,l} \oplus M_{k,l}^{sop}$ *then* $\dim A_0^{\lambda} = k^2 + l^2$, $\dim A_1^{\lambda} = 2kl$ *for all* $\lambda \in \hat{\mathbb{Z}}_2$*;*
(4) *if* $A = M_n(Q) \oplus M_n(Q)^{sop}$ *then* $\dim A_{\delta}^{\lambda} = n^2$ *for all* $(\delta, \lambda) \in \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2$*.*

As already remarked, $*$-simple superalgebras having the same $*$-dimensions have to be isomorphic; while for $\diamond$-simple algebras the general picture is quite different, in some cases having the same $\diamond$-dimensions is still sufficient to conclude they are isomorphic:

Proposition 4.3. *Let A, B be $\diamond$-simple algebras with the same $\diamond$-dimensions. If* $4 \nmid \dim A$ *then* $A \cong B$*.*

For any $n \geqslant 1$ both $M_{n,n}$ and $M_n(Q) \oplus M_n(Q)^{\text{sop}}$ have overall dimension $\equiv 0 \pmod 4$, while this happens for $M_{l,n,l}$ exactly when n is even, say $n = 2k$. It is however easy to check that if $k \neq l$ then $A = M_{l,2k,l}$ and B have the same $\diamond$-dimensions if and only if they are isomorphic, so what remains to discuss are the algebras $M_{n,n}$, $M_n(Q) \oplus M_n(Q)^{\text{sop}}$ and $M_{l,n,l}$ (when $n = 2l$). For all of them it holds $\dim A_{\delta}^{\lambda} = n^2$ for all $(\delta, \lambda) \in \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2$ although being pairwise not isomorphic. We call them *critical algebras*, and we may distinguish among them by means of polynomial identities, more precisely through Capelli polynomials. There is no universally adopted definition for Capelli polynomials; the ones considered in this paper may be called *left controlled* Capelli polynomials:

Definition 4.4. Let $m \geqslant 1$, and let $\mathscr{X} = \{x_1, \ldots, x_m\}$, $\mathscr{U} := \{u_1, \ldots, u_m\}$ be disjoint sets of (arbitrary) noncommuting indeterminates. Then

$$Cap_m(\mathscr{X}, \mathscr{U}) := \sum_{\sigma \in S_m} (-1)^{\sigma} u_1 x_{\sigma(1)} \ldots u_m x_{\sigma(m)}$$

is called the Capelli polynomial in alternating letters $x_1, \ldots, x_m$ controlled by letters $u_1, \ldots, u_m$.

We will often call $\mathscr{X}$ the set of *designed* letters, and omit to indicate the controlling set $\mathscr{U}$, so to shorten $Cap_n(\mathscr{X})$ or, when $n = |\mathscr{X}|$ is clear, even $Cap(\mathscr{X})$. Also, $Cap_n(\mathscr{Y})$ will denote the Capelli polynomial in n *even* indeterminates, $Cap_n(\mathscr{Y}^+)$ the Capelli polynomial in n *even symmetric* indeterminates, and so on. Now consider the following products of Capelli polynomials

Definition 4.5. Let n be a positive integer. Define

(1) $d_{n,n} := Cap_{n^2}(\mathscr{Y}^+)Cap_{2n^2}(\mathscr{Z})$;
(2) $d_{l,n,l} := Cap_{n^2}(\mathscr{Z}^+)Cap_{2n^2}(\mathscr{Z})$ if $n = 2l$;
(3) $d_Q := Cap_{n^2}(\mathscr{Y}^+)Cap_{n^2}(\mathscr{Z}^+)$,

with disjoint sets of designed letters and disjoint sets of *even* control letters in each Capelli polynomial.

Then

Proposition 4.6. *Let n be a positive integer. Then*

(1) *$d_{n,n}$ admits an evaluation ν on $M_{n,n}$ such that $\nu(d_{n,n}) = e_{11}$, but $d_{n,n}$ identically vanishes on $M_n(Q) \oplus M_n(Q)^{sop}$ and, if $n = 2l$, on $M_{l,n,l}$;*
(2) *if $n = 2l$ then $d_{l,n,l}$ admits an evaluation ν on $M_{l,n,l}$ such that $\nu(d_{l,n,l}) = e_{11}$, but $d_{l,n,l}$ identically vanishes on both $M_{n,n}$ and $M_n(Q) \oplus M_n(Q)^{sop}$;*
(3) *d_Q admits an evaluation ν on $M_n(Q) \oplus M_n(Q)^{sop}$ such that $\nu(d_Q) = (e_{11}, 0)$, but d_Q identically vanishes on $M_{n,n}$ and, if $n = 2l$, on $M_{l,n,l}$.*

Hence any critical $\diamond$-simple algebra comes with its *distinguishing* polynomial, strictly related to the $\diamond$-dimensions of the algebra but *also* to its specific structure. These polynomials will be useful later.

In the same spirit, the $\diamond$-dimensions of a $\diamond$-simple algebra may be encoded into a polynomial involving alternating $\diamond$-letters controlled by even letters. First of all, recall that for each $m \geqslant 1$, and each $i \in [m]$, the m^2 matrix units in M_m can be arranged into a sequence $(s_1, \dots, s_{m^2})$ such that $s_1 s_2 \dots s_{m^2} = e_{ii}$ (a so-called *unicursal* sequence e_{ii}-valued). This gives rise to a substitution for the monomial $x_1 \dots x_{m^2}$ such that $x_1 \dots x_{m^2} \to e_{ii}$. Then, inserting control variables $u_1, \dots, u_{m^2}$, one gets a *unique* so called *controlled basic evaluation* ν of the controlled monomial $w := u_1 x_1 \dots u_{m^2} x_{m^2}$, extending the assigned one and sending the control variables to diagonal matrix units, such that $\nu(w) = e_{ii}$. The most remarkable property of ν is the following:

$$\forall 1 \neq \sigma \in S_{m^2} \text{ it holds } \nu(u_1 x_{\sigma(1)} \dots u_{m^2} x_{\sigma(m^2)}) = 0.$$

These arguments can be adapted to $\diamond$-simple algebras. We are going to show the full details just in one case, underlining just the necessary settings in the other ones.

So, let $A = M_{n,n}$ set $m = 2n$ and start with an ordinary controlled basic evaluation ν for M_m such that $\nu(w) = e_{11}$. For any designed letter x occurring in w, let $\nu(x) = e_{uv}$; any matrix unit is $\mathbb{Z}_2$-homogeneous, so let δ be its $\mathbb{Z}_2$-degree and replace x with the corresponding graded variable x_δ. If $u + v = m + 1$ then e_{uv} is already $\diamond$-homogeneous, that is $e_{uv} \in A_\delta^\lambda$ for a suitable $\lambda \in \hat{\mathbb{Z}}_2$, so replace x_δ by x_δ^λ; if $u + v < m + 1$ or $u + v > m + 1$ replace x_δ by x_δ^+ or x_δ^-, respectively. Thus we produced a set $\mathscr{X}$, of designed $\diamond$-homogeneous variables and partitioned into disjoint subsets $\mathscr{X}(\delta, \lambda)$; notice that $|\mathscr{X}(\delta, \lambda)| = \dim A_\delta^\lambda = n^2$ for all δ, λ. Keep the control variables $u \in \mathscr{U}$ as they are, but considering them as *even* variables. Let w_A denote the newborn monomial. In order to build the associated evaluation, keep sending $u \to \nu(u)$ for any control variable $u \in \mathscr{U}$, and simply send $x_\delta^\lambda \to \nu(x)^\lambda$ (but keeping in mind that if $u + v = m + 1$ then $e_{uv}^\lambda = e_{uv}$, since e_{uv} is $\diamond$-homogeneous). Denote ν_A the new $\diamond$-evaluation, and notice that $\nu_A(w_A) = \nu(w) = e_{11}$.

Since this is an easy but central point, we provide

Example 4.7. Consider $A = M_{1,1}$. There are just two unicursal sequences for M_2 factoring e_{11}: $(e_{11}, e_{12}, e_{22}, e_{21})$ and $(e_{12}, e_{22}, e_{21}, e_{11})$; let us choose the former. Its controlled sequence is therefore $(e_{11}, \underline{e_{11}}, e_{11}, \underline{e_{12}}, e_{22}, \underline{e_{22}}, e_{22}, \underline{e_{21}})$ (the underlined entries are those of the unicursal sequence), the controlled monomial is $w = u_1x_1u_2x_2u_3x_3u_4x_4$ and ν is the obvious one.

Now, since $\nu(x_1) = e_{11}$, we replace the former x_1 with y_1^+; then $\nu(x_2) = e_{12}$ is odd and skew, so we replace x_2 by z_2^-, and so on to get $w_A = u_1y_1^+u_2z_2^-u_3y_3^-u_4z_4^+$. Then ν_A coincides with ν on the even variables u_i, while $\nu_A(y_1^+) = e_{11}^+ = e_{11} + e_{22}$, $\nu_A(z_2^-) = e_{12}^- = e_{12}$, $\nu_A(y_3) = e_{22}^- = e_{22} - e_{11}$ and $\nu_A(z_4) = e_{21}$. Notice that $\nu_A w_A = \nu w = e_{11}$. $\square$

For the remaining cases,

- if $A = M_{l,n,l}$, we may repeat exactly the same choices made for $M_{n,n}$ in case $n \leqslant 2l$, getting a controlled pair (w_A, ν_A) with $\nu_A(w_A) = e_{11}$. In case $n > 2l$, instead, start from a basic sequence in M_{n+2l} resulting in $\nu(w) = e_{l+1\,l+1}$ instead than e_{11}. Then repeat the previous construction;
- if $A = M_{k,l} \oplus M_{k,l}^{\mathrm{sop}}$ start from a unicursal sequence in M_{k+l} providing a substitution ν such $\nu(w) = e_{11}$, and consider two copies of w in disjoint variables, getting w_1w_2 and an obvious valuation $\hat{\nu}$ such that $\hat{\nu}(w_1w_2) = e_{11}$. Replace any designed variable x in w_1 by a *symmetric* variable of $\mathbb{Z}_2$-degree of $\nu(x)$, and any designed variable x in w_2 by a *skew-symmetric* one of the same $\mathbb{Z}_2$-degree to get w_A. Then define ν_A by assigning $\nu_A(x)$ to $(\nu(x), \nu(x))$ or $(\nu(x), -\nu(x))$ according to variables occurring in w_1 or w_2, respectively, and assigning any control variable to $(\nu(u), 0)$. This results into a $\diamond$-evaluation such that $\nu_A w_A = (\nu w, 0) = (e_{11}, 0)$;
- if $A = M_n(Q) \oplus M_n(Q)^{\mathrm{sop}}$, all the same start from an (ordinary) evaluation ν on M_n such that $\nu(w) = e_{11}$, then simply produce four copies $w_1, \ldots, w_4$ attributing a different $\diamond$-homogeneous degree (δ, λ) to designed variables in each single w_i to get

w_A, and extend the basic valuation assigning values (e_{ij}, e_{ij}) or $(e_{ij}, -e_{ij})$ to even designed variables according to λ, and similarly values (ce_{ij}, ce_{ij}) or $(ce_{ij}, -ce_{ij})$ to the odd ones, while values $(\nu(u), 0)$ to control variables. The resulting $\diamond$-evaluation ν_A fulfills $\nu_A w_A = (e_{11}, 0)$ once again.

Let $\mathscr{X}$ denote the designed variables of w_A, partitioned into $\diamond$-homogeneous subsets $\mathscr{X}(\delta, \lambda)$. Alternating on each subset $\mathscr{X}(\delta, \lambda)$ produces a polynomial denoted $f_A = f_A(\mathscr{X})$ such that $\nu_A f_A = \nu_A w_A$. For any $t \geqslant 1$ we may replicate t times f_A, each time on disjoint sets $\mathscr{X}^{(i)}$ of designed $\diamond$-variables and disjoint sets $\mathscr{U}^{(i)}$ of control variables, getting $f_A^{(i)} = f_A^{(i)}(\mathscr{X}^{(i)})$. We will denote with abuse of notation $f_A^t := f_A^{(1)} \dots f_A^{(t)}$ their product, and ν_A^t the corresponding obvious evaluation; then $\nu_A^t(f_A^t) = \nu(w)$.

The polynomial f_A^t is alternating on each $\diamond$-homogeneous layer $\mathscr{X}^{(i)}(\delta, \lambda)$, of size $\dim A_\delta^\lambda$; thus the $\diamond$-dimensions of A are embodied in each polynomial f_A and f_A^t, in the sense that if a $\diamond$-simple algebra B has the same $\diamond$-dimensions as A it may be $f_A \notin T^\diamond(B)$ or not, but for sure if A and B do not have the same $\diamond$-dimensions then either $f_A \in T^\diamond(B)$ or $f_B \in T^\diamond(A)$.

5. $\diamond$-embeddings and the algebras $UT^\diamond_{\mathbf{g}}(A_1, \dots, A_m)$

Any $\diamond$-simple algebra A admits a faithful representation as a $\diamond$-subalgebra of a $\diamond$-simple algebra of even size $2s$ with a supertranspose $\overline{\vartheta}_{2s}$. This was Lemma 3.9 in [4] and, for convenience of the reader, we recall what the $\diamond$-embedding φ is:

- let A be a finite dimensional $\diamond$-simple algebra, in its canonical form;
- if A is simple, then it is a full matrix algebra M_s endowed with a grading induced by a *complete* word β and a superinvolution $\diamond$ which is either a supertranspose or an orthosymplectic one. Then consider on M_{2s} the grading word $\alpha := \beta$, choosing $L_\alpha := \tilde{\alpha}^{-1}(0) = L_\beta \cup \gamma_{2s}(R_\beta)$ as defining fiber of $\overline{\vartheta}_{2s}$. The resulting $\diamond$-algebra is $\diamond$-isomorphic to $M_{s,s}$, but not in its canonical form even if A is so: it is more convenient to keep it as it is. Then define $\varphi : A \to M_{2s}$ by setting

$$A \ni a \overset{\varphi}{\to} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a^\diamond & 0 \\ 0 & 0 \end{pmatrix}^{\overline{\vartheta}_{2s}} \in M_{2s};$$

- if $A = M_{k,l} \oplus M_{k,l}^{\mathrm{sop}}$, set $s := k + l$, $\alpha := (0^k, 1^l)$ as grading word for M_{2s} and $L_\alpha := \tilde{\alpha}^{-1}(0)$ as chosen fiber for $\overline{\vartheta}_{2s}$. Then define $\varphi : A \to M_{2s}$ by setting

$$A \ni (a, b) \overset{\varphi}{\to} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}^{\overline{\vartheta}_{2s}} \in M_{2s};$$

- if $A = M_n(Q) \oplus M_n(Q)$, set $s := 2n$, grading word $\alpha := (0^n, 1^n)$ on M_{2s} and $L_\alpha := \tilde{\alpha}^{-1}(0)$. Then define φ just as before:

$$A \ni (a,b) \xrightarrow{\varphi} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}^{\overline{\vartheta}_{2s}} \in M_{2s}.$$

Hence any element $a \in A$ gives rise to a pair of $s \times s$ matrices (a', a'') such that $\varphi(a) = \begin{pmatrix} a' & 0 \\ 0 & a'' \end{pmatrix} \in M_{2s}$ for the appropriate s, and in each case φ turns out to be a $\diamond$-embedding. We call s the *size* of A, and identifying $M_{2s} = M_2 \otimes M_s$ we may write $\varphi(a) = e_{11} \otimes a' + e_{22} \otimes a''$.

Notice that in the former three cases, s equals the size of the simple superalgebra S upon which A is built, while for $S = M_n(Q)$ it equals $2n$ (because, in fact, we identify $M_n(Q)$ with its image inside M_{2n}).

Now let $(A_1, \dots, A_m)$ be a sequence of $\diamond$-simple algebras, s_i the size of A_i, α_i the grading word chosen for the embedding $A_i \hookrightarrow M_{2s_i}$, and consider the $\diamond$-semisimple algebra $A = A_1 \oplus \cdots \oplus A_m$. There is an almost natural embedding of A into an algebra of even size endowed with a supertranspose: set the size to $s = s_1 + \cdots + s_m$ and take $\alpha = \alpha_1 | \dots | \alpha_m$ as grading word, so that the algebra M_{2s} is endowed with the grading induced by $\tilde{\alpha} = (\alpha | \mathscr{C}Rev(\alpha))$ and with the standard supertranspose $\overline{\vartheta}_{2s}$. Then the map $\varphi : A \to M_{2s}$ defined by

$$\varphi(a_1, \dots, a_m) = \sum_{i=1}^{s} (e_{ii} \otimes a_i' + e_{ii}^{\vartheta_{2m}} \otimes a_i'') = \begin{pmatrix} a_1' & 0 & \dots & 0 & & & & \\ 0 & a_2' & \dots & 0 & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 0 & 0 & \dots & a_m' & & & & \\ & & & & a_m'' & \dots & 0 & 0 \\ & & & & \vdots & \ddots & \vdots & \vdots \\ & & & & 0 & \dots & a_2'' & 0 \\ & & & & 0 & \dots & 0 & a_1'' \end{pmatrix}$$

is a $\diamond$-embedding of the $\diamond$-semisimple algebra $A = A_1 \oplus \cdots \oplus A_m$ into (M_{2s}, α) (see Proposition 3.1 in [4]). Of course, this also provides single $\diamond$-embeddings $A_i \hookrightarrow M_{2s}$, with the resulting subalgebra being $\diamond$-isomorphic to the one inside M_{2s_i}. We therefore will not distinguish between the two images, identifying them and writing $\varphi(A_i)$ in both cases, if this does not cause ambiguities. Also, it will sometimes turn useful to write directly A_i instead of $\varphi(A_i)$ in order to keep the notation as simple as possible (if this does not pose troubles).

From now on, when considering a sequence $(A_1, \dots, A_m)$, we will refer to the above construction simply saying (M_{2s}, α) *is the $\diamond$-simple enveloping algebra* of the sequence, assuming implicitly $A = A_1 \oplus \cdots \oplus A_m$, $\alpha = (\alpha_1, \dots, \alpha_m)$, $s = s_1 + \cdots + s_m$, M_{2s}, $\overline{\vartheta}_{2s}$ and φ as assigned according to the construction.

Now, let B be a generic finite dimensional $\diamond$-algebra, with possibly nonzero Jacobson radical $J(B)$. Then $J(B)$ is a $\diamond$-ideal of B admitting a semidirect $\diamond$-complement by the Wedderburn-Malcev Theorem, which is therefore a $\diamond$-semisimple algebra $B_{\text{ss}} = B_1 \oplus \cdots \oplus B_m \cong A_1 \oplus \cdots \oplus A_m = A$ for certain $\diamond$-simple subalgebras $B_i \subseteq B$, each isomorphic to a canonical A_i. Since we can embed A into the suitable (M_{2s}, α) as before, it is natural trying to get a $\diamond$-subalgebra of M_{2s} larger than $\varphi(A)$ in order to represent the whole B, so we need a kind of universal enveloping target for all possible $\diamond$-algebras having semisimple part isomorphic to $A_1 \oplus \cdots \oplus A_m$.

In doing this, a few words of caution are in order: while $J(B)$ is a structural invariant of B, the complement $B_{\rm ss}$ is generally not unique, because $B = B_{\rm ss} + J(B)$ is just a semidirect decomposition. Hence $B_{\rm ss}$ is uniquely determined just up to $\diamond$-isomorphisms, and one should cautiously keep writing $B_{ss} \cong A_1 \oplus \cdots \oplus A_m$, for (uniquely determined) $\diamond$-simple algebras A_i. Moreover, while in writing $B_{ss} \cong A_1 \oplus \cdots \oplus A_m$ the order among the A_i's does not matter, when representing $B \hookrightarrow M_{2s}$ it becomes essential because of the image of $J(B)$: the (often very intricate) interplay between $J(B)$ and the simple components of B_{ss} has to be faithfully represented, so once $J(B) \hookrightarrow M_{2s}$ has been assigned, the order among the images $A_i \hookrightarrow M_{2s}$ has to be fixed, to correctly representing the $\diamond$-structure of B.

Now, for $i, j \in [m]$, let $U_{ij} := e_{ij} \otimes M_{s_i \times s_j} \leqslant_F M_{2s}$, and denote its $\overline{\vartheta}_{2s}$-image by $\overline{U_{ij}} := U_{ij}^{\overline{\vartheta_{2s}}} \leqslant_F M_{2s}$.

If $(A_1, \ldots, A_m)$ is a sequence of finite dimensional $\diamond$-simple algebras, let M_{2s} be its $\diamond$-simple enveloping algebra and define

$$UT^\diamond(A_1, \ldots, A_m) := \varphi(A) \oplus \bigoplus_{\substack{i,j \in [m] \\ i<j}} \left(U_{ij} \oplus \overline{U_{ij}}\right) \subseteq U := \bigoplus_{\substack{i,j \in [m] \\ i \leqslant j}} \left(U_{ij} \oplus \overline{U_{ij}}\right) \subseteq M_{2s}.$$

A direct verification shows that $UT^\diamond(A_1, \ldots, A_m)$ is actually a $\diamond$-subalgebra of M_{2s}, still enveloping A and all its simple summands, and the sequence

$$0 \to J := \bigoplus_{\substack{i,j \in [m] \\ i<j}} \left(U_{ij} \oplus \overline{U_{ij}}\right) \to UT^\diamond(A_1, \ldots, A_m) \to A \to 0$$

is split exact. Actually, it can be easily checked that J is the Jacobson radical of $UT^\diamond(A_1, \ldots, A_m)$. This is quite close to the universal target we were looking for, up to a last ingredient:

Definition 5.1. Let $(A_1, \ldots, A_m)$ be an assigned sequence of $\diamond$-simple algebras, and let $\alpha = (\alpha_1 | \ldots | \alpha_m)$ be the grading word of its $\diamond$-simple enveloping algebra M_{2s}. For any map $\mathbf{g} : [m] \to \mathbb{Z}_2$, let $UT^\diamond_{\mathbf{g}}(A_1, \ldots, A_m)$ denote the $\diamond$-subalgebra of $(M_{2s}, \alpha + \mathbf{g}, \overline{\vartheta}_{2s,\mathbf{g}})$, where $\overline{\vartheta}_{2s,\mathbf{g}}$ is the standard supertranspose defined by the grading word $\alpha + \mathbf{g}$.

The algebras $UT^\diamond_{\mathbf{g}}(A_1, \ldots, A_m)$ are precisely the kind of universal target we needed: if B is a finite dimensional $\diamond$-algebra then there exists a suitable choice of the sequence $(A_1, \ldots, A_m)$ and $\mathbf{g} : [m] \to \mathbb{Z}_2$, such that $B_{ss} \cong \varphi(A_1 \oplus \cdots \oplus A_m)$ and $J(B) \hookrightarrow J(UT^\diamond_{\mathbf{g}}(A_1, \ldots, A_m))$.

Since these $\diamond$-algebras will play a key role, we add some easy but maybe clarifying comments:

- setting $\mathbf{g}(i) := g_i \in \mathbb{Z}_2$, the word $\alpha + \mathbf{g}$ means the concatenations of the words $\alpha_i + g_i$ (for $i \in [m]$), that is the maps $u \in [s_i] \to \alpha_i(u) + g_i$. If $g_i = 0$ then of course $\alpha_i + g_i = \alpha_i$ and nothing changes, but if $g_i = 1$ considering $\alpha_i + 1$ amounts to switching from α_i to the complementary word $\mathscr{C}(\alpha_i)$;
- the essence of $\alpha + \mathbf{g}$ is therefore in changing some of the words α_i into their complement $\mathscr{C}(\alpha_i)$, thus changing the grading word on M_{2s} and, henceforth, both the fibers and the standard supertranspose. This is the very reason why we added $\mathbf{g}$ in $\overline{\vartheta}_{2s,\mathbf{g}}$ in the previous definition. By the way, for notational reasons, in the rest of the paper we will omit this specification and simply write $\overline{\vartheta}_{2s}$ or even more directly $\overline{\vartheta}$, as soon as no confusion may arise;
- not all the 2^m different $\mathbf{g}$ provide different structures: for instance, if $\mathbf{g} = \mathbf{0}$ (the zero function) then $UT^\diamond_{\mathbf{0}}(A_1, \dots, A_m) = UT^\diamond(A_1, \dots, A_m)$, and if $\mathbf{g} = \mathbf{1}$ (the constant 1 function) then $\alpha + \mathbf{1} = \mathscr{C}(\alpha)$, so the resulting algebra is different but isomorphic to $UT^\diamond_{\mathbf{0}}(A_1, \dots, A_m)$. Other instances may arise, depending on the concrete starting sequence. By the way, a key point is in noting that while passing from α to $\alpha+\mathbf{g}$ does not change the semisimple part $\varphi(A)$ of $UT^\alpha(A_1, \dots, A_m)$ up to $\diamond$-isomorphisms, it may effectively change the Jacobson radical, providing a different structure. This is of course not an issue of the construction: it is precisely what was needed in order to get a kind of universal target.

6. Minimal $\diamond$-varieties

We now return to the main question: let $A_1, \dots, A_m$ be an ordered sequence of $\diamond$-simple algebras, let $\mathbf{g} \in \mathbb{Z}_2^m$ be any translation vector and consider the $\diamond$-variety generated by $A := UT^\diamond_{\mathbf{g}}(A_1, \dots, A_m)$. Is $\mathscr{V}$ minimal? Recall this means that for any proper $\diamond$-subvariety $\mathscr{U}$ it has to hold $\exp(\mathscr{U}, \diamond) \lneqq \exp(\mathscr{V}, \diamond)$, and in fact $\exp(\mathscr{U}, \diamond) < \dim A_1 + \dots + \dim A_m$, which is the overall dimension of the semisimple part A_{ss} of A and by construction equals $\exp(\mathscr{V}, \diamond)$.

So, assume $\mathscr{U}$ is a $\diamond$-subvariety of $\mathscr{V}$; since A is finite dimensional, it satisfies an (ordinary) Capelli polynomial, so $\mathscr{U}$ is affine, hence $\mathscr{U}$ is generated by a finite dimensional $\diamond$-algebra ([1], Theorem 1) and contains a $\diamond$-algebra $B = UT^\diamond_{\mathbf{k}}(B_1, \dots, B_n)$, for a suitable sequence $B_1, \dots, B_n$ of $\diamond$-simple algebras and a translation vector $\mathbf{k} \in \mathbb{Z}_2^n$, having the same $\diamond$-exponent of $\mathscr{U}$ (by Proposition 4.4 and Theorem 4.5 in [4]). As a first step, what can be said about n and the sequence $B_1, \dots, B_n$?

This first comparison may be dealt with by means of the $\diamond$-dimensional invariants of A and B, embodied in specific polynomials tailored on the algebras A and B much the same way as for the single $\diamond$-simple algebras up to suitable, decisive, modifications.

Let $t \geqslant m - 1$ be any positive integer, and for each $i \in [m]$ let $f_i^t = f_{A_i}^t$ denote the $\diamond$-polynomial of A_i on t layers of variables, d_i its distinguishing polynomial (in order to keep the notation uniform, in case A_i is not critical then set $d_i = 1$, the constant polynomial) and let ν_i its associated evaluation, obtained by extending $\nu^t_{A_i}$ to the variables of its distinguishing polynomial in order to get the prescribed nonzero value. Recall that if $A_i = M_{l,n,l}$ with $n > 2l$ then A_i is not critical, but ν_i is tailored to produce $\nu_i(f^t_i) = e_{l+1\,l+1}$; in all other cases, $\nu_i(f^t_i) = e_{11}$ or $(e_{11}, 0)$.

For any $j < m$, let $b_j := e_{j\,j+1} \otimes e_{r_j,c_j}$ where $r_j = 1$ if A_j is not of type $M_{l,n,l}$ with $n > 2l$, and $r_j = l + 1$ on the contrary; in a similar way, $c_j = 1$ if A_{j+1} is not of type $M_{l,n,l}$ with $n > 2l$ or otherwise $c_j = l + 1$. Denote $\overline{\delta}_j$ the $\mathbb{Z}_2$-degree of b_j, and create a new symmetric variable $\overline{x}_j$ of $\diamond$-degree $(\overline{\delta}_j, +)$. Also, create new *even* control variables $\overline{u}_j$. Finally, for all $j \in [t]$ let $\mathscr{X}^{(j)}$ be the set of all designed variables occurring in $f^{(j)}_1, \dots, f^{(j)}_m$, together with $\overline{x}_j$ if $j < m$. Then each set $\mathscr{X}^{(j)}$ is partitioned into $\diamond$-homogeneous subsets $\mathscr{X}^{(j)}(\delta, \lambda)$, of size $\dim A_{\mathrm{ss}}(\delta, \lambda)$ but for the cases when $j < m$ and $(\delta, \lambda) = (\overline{\delta}_j, +)$, where the size is $\dim A_{\mathrm{ss}}(\overline{\delta}_j, +)+1$. Thus, so far, we created a polynomial

$$w_{A,t} := f^t_1 d_1 \overline{u}_1 \overline{x}_1 f^t_2 d_2 \overline{u}_2 \overline{x}_2 \dots \overline{u}_{m-1} \overline{x}_{m-1} f^t_m d_m.$$

Definition 6.1. Let $f_{A,t}$ be the polynomial obtained by alternating on each set of designed variables $\mathscr{X}^{(j)}(\delta, \lambda)$, for $j \in [t]$ and $(\delta, \lambda) \in \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2$.

Notice that if $j \geqslant m$ then $\left|\mathscr{X}^{(j)}(\delta, \lambda)\right| = \dim A_{\mathrm{ss}}(\delta, \lambda)$: only the $\diamond$-homogeneous sets $\mathscr{X}^{(j)}$ with $j < m$ have a single $\diamond$-letter more than $\dim A_{\mathrm{ss}}$, grouped together with the other $\dim A_{\mathrm{ss}}(\overline{\delta}_j, +)$ letters in a single set of variables alternating in $f_{A,t}$.

This construction provided us with a polynomial $f_{A,t}$ for any $t \geqslant m - 1$; it is really huge but still multilinear in all its variables and alternating on $4t$ sets of $\diamond$-homogeneous variables partitioning t different layers. Also, it comes with an obvious evaluation $\nu_{A,t}$ inherited from the ν_i's through the compositions $\varphi \circ \nu_i$, and with the new assignments $\overline{x}_j \to b^+_j$. In order to complete $\nu_{A,t}$, just define $\nu_{A,t}(\overline{u}_j) = e_{jj} \otimes e_{r_j r_j}$.

The following result is little surprising, but we record it anyway:

Lemma 6.2. $\nu_{A,t}(f_{A,t}) = e_{1m} \otimes e_{r_1\,c_{m-1}} \neq 0$.

Proof. Of course $\nu := \nu_{A,t}$ has been built up in order to have $\nu(w_{A,t}) \neq 0$, and in fact it is the only summand of $\nu(f_{A,t})$: first notice that if $\sigma w_{A,t}$ is another summand then the control variables are unaffected from σ, hence σ has to fix the variables $\overline{x}_j$ and to stabilize all sets $\mathscr{X}^{(j)}_i(\delta, \lambda)$ (the designed variables of $\diamond$-degree (δ, λ) occurring in $f^{(j)}_i$) in order to produce a nonzero summand. So in fact σ factorizes into a product $\sigma = \sigma^{(1)}_1 \dots \sigma^{(t)}_1 \dots \sigma^{(1)}_m \dots \sigma^{(t)}_m$, where $\sigma^{(j)}_i \in \prod_{(\delta,\lambda)} Sym(\mathscr{X}^{(j)}_i(\delta, \lambda))$, a direct product of symmetric groups. However, by the fundamental property of f_i, it holds $\nu_i(\tau w_i) = 0$ for any nontrivial permutation τ of the $\diamond$-homogeneous designed letters occurring in w_i, because the control variables within each $w^{(j)}_i$ are also fixed once and for all, hence all $\sigma^{(j)}_i$ have to be the identity permutation in order $\nu(\sigma w_{A,t}) \neq 0$ to hold. □

Much more interesting is the next one, closely resembling Lemma 6.1 in [3]:

Proposition 6.3. *Let $A = UT_{\mathbf{g}}^{\diamond}(A_1, \dots, A_n)$ and $B = UT_{\mathbf{k}}^{\diamond}(B_1, \dots, B_m)$ be such that $\dim A_{ss} \leqslant \dim B_{ss}$, and let $t \geqslant m + n - 1$. If the polynomial $f_{B,t}$ is not a $\diamond$-identity for A, then*

(1) *the semisimple parts A_{ss} and B_{ss} have the same $\diamond$-dimensions;*
(2) *$m = n$;*
(3) *$(B_1, \dots, B_m) = (A_1, \dots, A_m)$ or $(B_1, \dots, B_m) = (A_m, \dots, A_1)$.*

Proof. Since $f := f_{B,t}$ is multilinear in all its variables, and it does not vanish on A, f must admit a nonzero A-evaluation η such that $\diamond$-variables assume values in a $\diamond$-basis of A and the control variables are valued on a $\mathbb{Z}_2$-homogeneous basis of A. By the way, since the Jacobson radical $J = J(A)$ is n-nilpotent, at most $n - 1$ elements of J are involved in η; since there are at least $m + n - 1$ layers $\mathscr{X}^{(i)}$ but just $\mathscr{X}^{(1)}, \dots, \mathscr{X}^{(m-1)}$ involve more than $\dim B_{\mathrm{ss}}$ designed variables, there is at least one layer $\mathscr{X}^{(\ell)}$, $m \leqslant \ell \leqslant t$, involving exactly $\dim B_{ss}$ designed variables and such that $\eta(\mathscr{X}^{(\ell)}) \cap J(A) = \varnothing$: in the worst case scenario, all J-valued letters occur in layers of index $i \geqslant m$, so there are still $t - (m - 1) - (n - 1) = t - m - n - 2 \geqslant 1$ layers available. Since by our assumption $\eta(f) \neq 0$ but f is alternating also on each $\mathscr{X}^{(\ell)}(\delta, \lambda)$, of size $\dim B_{\mathrm{ss}}(\delta, \lambda)$, for all $(\delta, \lambda) \in \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2$, we may deduce that $\dim A_{\mathrm{ss}}(\delta, \lambda) \geqslant B_{\mathrm{ss}}(\delta, \lambda)$ for all δ, λ. Hence, summing up those $\diamond$-dimensions, $\dim A_{ss} \geqslant \dim B_{ss}$. This, together with our assumptions, provides $\dim B_{\mathrm{ss}} = \dim A_{\mathrm{ss}}$ and, in fact, $\dim B_{\mathrm{ss}}(\delta, \lambda) = A_{\mathrm{ss}}(\delta, \lambda)$ for all $(\delta, \lambda) \in \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2$.

Since $\eta(f_{B,t}) \neq 0$ and $f_{B,t}$ is a sum of terms $\sigma w_{B,t}$, for σ ranging in the appropriate group of permutations, we may safely assume that $\eta(w_{B,t}) \neq 0$, as well. Since the direct summands of A_{ss} annihilate each other, and all the variables involved in $f_1^{(\ell)}$ have η-images in A_{ss}, all of them must be in the same component; the same holds for all other $f_i^{(\ell)}$, so we have n different labels (the A_h's), each one tagging at least one among m boxes (the polynomials $f_i^{(\ell)}$), hence $n \leqslant m$.

Of course if $m = 1$ then $n = 1$ and the second statement is proved; so assume $m \geqslant 2$. Then there are $m - 1$ variables $\overline{x}_j$, each of them in a different layer $\mathscr{X}^{(j)}$, and precisely $\overline{x}_j \in \mathscr{X}^{(j)}(\overline{\delta}_j, +)$; by the way, $\left|\mathscr{X}^{(j)}(\overline{\delta}_j, +)\right| = \dim B_{\mathrm{ss}}(\overline{\delta}_j, +) + 1 = \dim A_{\mathrm{ss}}(\overline{\delta}_j, +) + 1$ and $f_{B,t}$ is alternating on these variables. Hence at least one of the variables from of $\mathscr{X}^{(j)}(\overline{\delta}_j, +)$ has to be evaluated in an element of J, and this holds for all $j < m$. Therefore there are at least $m-1$ elements of J involved in η, and since the evaluation $\eta(f_{B,t})$ is not zero this implies $m - 1 \leqslant n - 1$. This, together with the previously obtained inequality $n \leqslant m$, implies $m = n$.

Now consider a layer $\mathscr{X}^{(j)}$ with $j < m$: since $\left|\mathscr{X}^{(j)}(\overline{\delta}_j, +)\right| = \dim B_{\mathrm{ss}}(\overline{\delta}_j, +) + 1$ and $f_{B,t}$ is alternating on its variables, at least among them has to be evaluated in a radical element and, in fact, exactly one of them, since there are $m - 1$ layers and $J(A)^m = 0$. As a side effect, all control variables and all variables of the distinguishing polynomials d_i assume values in the semisimple part A_{ss}. This forces the variables $\overline{x}_j$ to be evaluated in radical elements. Since they occur in the relative order $\overline{x}_1 \dots \overline{x}_{m-1}$ and just $\varphi(A_1)J \dots J\varphi(A_m)$, $\varphi(A_m)J \dots J\varphi(A_1)$ are not zero, all the variables of the polynomials $f_1^t, \dots, f_m^t$ assume values in either $\varphi(A_1), \dots, \varphi(A_m)$ respectively or in the reverse order $\varphi(A_m), \dots, \varphi(A_1)$.

Assume the former case. Then denoting $\mathscr{X}_i^{(j)}(\delta,\lambda)$ the (δ,λ)-letters occurring in $f_i^{(j)}$, all of them assume values in $\varphi(A_i)$, and the same happens to d_i (in case B_i is critical). Hence $\dim A_i(\delta,\lambda) = \left|\mathscr{X}_i^{(j)}(\delta,\lambda)\right| = \dim B_i(\delta,\lambda)$ for all $i \in [m]$ and for all $(\delta,\lambda) \in \mathbb{Z}_2 \times \hat{\mathbb{Z}}_2$, that is A_i and B_i have the same $\diamond$-dimensions and the distinguishing polynomial d_i does not vanish on A_i. This means that $(B_1, \dots, B_m) = (A_1, \dots, A_m)$. Similar arguments provide $(B_m, \dots, B_1) = (A_1, \dots, A_m)$ in the latter case. $\square$

Now a more subtle comparison is in order: given algebras with the same simple components $A = UT^\diamond_{\mathbf{g}}(A_1, \dots, A_m)$ and $B = UT^\diamond_{\mathbf{g}'}(A_1, \dots, A_m)$, when are they $\diamond$-isomorphic? Here the translation vectors $\mathbf{g}$, $\mathbf{g}'$ come into play: although a translation does not change the isomorphism class of any simple component, it may decisively change the resulting algebra.

Example 6.4. Let $A = UT^\diamond(F, F)$ with grading word $\beta = (0, 0)$ (hence the trivial grading), let $\mathbf{g} := (0, 1)$ and consider $\beta + \mathbf{g} = (0, 1)$. Then the algebras A and $B := UT^\diamond_{\mathbf{g}}(F, F)$ are clearly not $\diamond$-isomorphic.

To our aims, the critical result is the following:

Proposition 6.5. *Let $A = UT^\diamond_{\mathbf{g}}(A_1, \dots, A_m)$ and $B = UT^\diamond_{\mathbf{g}'}(A_1, \dots, A_m)$ satisfy $Id(A, \diamond) \subseteq Id(B, \diamond)$. Then $A \cong B$.*

Proof. Each A_i has its canonical grading $(\alpha_i | \mathscr{C}Rev(\alpha_i))$, and if $g_i = 1$ then $(\alpha_i + 1 | \mathscr{C}Rev(\alpha_i) + 1) = (\mathscr{C}(\alpha) | Rev(\alpha_i))$, that is translating by $1 \in \mathbb{Z}_2$ means passing to the complementary word. So, relaxing the assumption that the canonical grading is assigned to each A_i, write A as the subalgebra of M_{2s} with grading word $\alpha = (\alpha_1 | \dots | \alpha_m)$ where each α_i is either the canonical or the complementary grading word on A_i. Similarly, B is endowed with a grading word $\beta = (\beta_1 | \dots | \beta_m)$, hence it is the translation of α by the vector $\mathbf{g} := \alpha + \beta$. This way, we may consider a single translation vector altering the grading on A to the grading on B. Of course if $\mathbf{g} = \mathbf{0}$ then $A = B$, and on the other extreme if $\mathbf{g} = \mathbf{1}$ then the grading on B is the complementary of the grading of A, so A and B are $\diamond$-isomorphic. We are interested in all intermediate cases.

Now consider the i-th component A_i, and assume α_i has equipotent fibers (so actually A_i is either a critical algebra or $M_{k,k} \oplus M_{k,k}^{\mathrm{sop}}$, be it endowed with the canonical grading or the complementary one does not really matter), and assume $g_i = 1$. Choosing a pair u, v in different α_i-fibers we know that the map ψ of Proposition 3.6 is a $\diamond$-isomorphisms between (M_{2s}, α) and (M_{2s}, α'), where $\alpha' = \alpha + g_{u,v}$ and $g_{u,v} : [s] \to \mathbb{Z}_2$ is the map defined by $g_{u,v}^{-1}(1) = \{u, v\}$. Since in particular $\psi(A) = A$, $\psi(A)$ is $\diamond$-isomorphic to A, with grading word coinciding with α in all places but u, v, in which it has $\alpha_i(u) + 1$ and $\alpha_i(v) + 1$ instead than $\alpha_i(u)$ and $\alpha_i(v)$. Repeating the process for the remaining pairs of elements in the fibers of α_i results into a $\diamond$-algebra isomorphic to A but with i-th grading word $\alpha_i + 1$.

Therefore we may legitimately alter the words α_i having equipotent fibers by translating them to $\alpha_i + 1$ without altering the isomorphism class. This leave us the $\diamond$-simple components of type $M_{k,l} \oplus M_{k,l}^{\mathrm{sop}}$ $(k > l)$ and $M_{l,n,l}$ $(n \neq 2l)$, to be dealt with. Of course, if just a single algebra of these types occurs among the components $(A_1, \ldots, A_m)$, no real problem is posed: if it is A_i then either $g_i = 0$ or $g_i = 1$, but in the latter case we may consider the algebra with grading $\mathscr{C}(\alpha)$, which is isomorphic to A and has the same i-th grading subword as B. Then just as before we may translate the other grading subwords without affecting the isomorphism class.

So assume there are components A_p, A_q such that $p < q$ and $g_p \neq g_q$; in fact we may safely assume (possibly passing to the complementary grading on B) that $g_p = 0$ and $g_q = 1$. We claim that A and B cannot be $\diamond$-isomorphic, and actually it cannot happen $Id(A, \diamond) \subseteq Id(B, \diamond)$. In order to show this we will produce a polynomial identically vanishing on A but not on B. So, let us start by the polynomial $w_{B,m}$, which we recall is

$$w_{B,m} = f_1^m d_1 \overline{u}_1 \overline{x_1} \ldots f_p^m \overline{u}_p \overline{x}_p \ldots f_1^m \overline{u}_q \overline{x}_q \ldots \overline{u}_{m-1} \overline{x}_{m-1} f_m^m d_m.$$

Notice that since A_p and A_q cannot be critical, their distinguishing polynomials d_p, d_q are 1 and we omitted them in writing $w_{B,m}$. Then, recall there is a valuation $\nu_{B,m}$ in B such that $\nu_{B,m}(f_{B,m}) = \nu_{B,m}(w_{B,m}) \neq 0$. We can modify a bit $w_{B,m}$ in order to single out the features of A_p and A_q.

Consider A_p: if A_p is of type $M_{l,n,l}$ with $n > 2l$, let v_p be the standard polynomial of degree $2n-1$. If A_p is instead still of type $M_{l,n,l}$ but with $n < 2l$, let v_p be the standard polynomial of degree $4l-1$, and finally if A_p is of type $M_{k,l} \oplus M_{k,l}^{\mathrm{sop}}$ with $k > l$ let v_p be the standard polynomial of degree $2k-1$. Then replace the variables in v_p by a new set of *even* variables, and let y_p be a new even (control) variable. Whatever the case, by the Amitsur–Levitzki Theorem v_p is not an identity for the algebra A_p, and $y_p v_p$ admits a nonzero $\mathbb{Z}_2$-graded evaluation resulting in $e_{l+1\,l+1}$, e_{11} or $(e_{11}, 0)$, respectively. Similar considerations hold for A_q: so consider the similar standard polynomial v_q in even variables only, an even control variable y_q, and define

$$w'_{B,m} = \omega_l \overline{w} \omega_r := (f_1^m d_1 \overline{u}_1 \overline{x}_1 \ldots d_{p-1})(y_p v_p f_p^m \overline{u}_p \overline{x}_p \ldots f_q^m y_q v_q)(\overline{u}_q \overline{x}_q \ldots f_m^m d_m).$$

We may extend the evaluation $\nu_{B,m}$ to a new one, $\nu'_{B,m}$, by specializing the new even variables occurring in $y_p v_p$ and $y_q v_q$ according to the above considerations. Then, alternating on the same set of variables affording the passage from $w_{B,m}$ to $f_{B,m}$ we get a polynomial $f'_{B,m} = \sum_\sigma (-1)^\sigma \sigma(\omega_l \overline{w} \omega_r)$, and by construction we get $\nu'_{B,m}(f'_{B,m}) = \nu_{B,m}(f_{B,m}) = \nu_{B,m}(w_{B,m}) \neq 0$; hence $f'_{B,m}$ is still outside $Id(B, \diamond)$, therefore outside $Id(A, \diamond)$. In particular $\nu'_{B,m}(\overline{w}) \neq 0$, say $\nu'_{B,m}(\overline{w}) = e_{pq} \otimes e_{rc}$, for suitable r, c, of degree $\beta_p(r) + \beta_q(c) = \overline{\delta}_p + \cdots + \overline{\delta}_{q-1}$.

The polynomial $f'_{B,m}$ is still multilinear in all its variables, and alternating on the same sets of $\diamond$-designed variables as $f_{B,m}$. Hence there must exist a $\diamond$-substitution η in a $\diamond$-basis of A (for the designed variables) and in a $\mathbb{Z}_2$-basis of A (for the control variables, but also for the even variables occurring in the distinguishing polynomials d_i and in v_p, v_q) such that $\eta(f'_{B,m}) \neq 0$.

As in the proof of Proposition 6.3, we may assume that $\eta(w'_{B,m}) \neq 0$; also, for each $j < m$ there must be a single variable $x_j \in \mathscr{X}^{(j)}(\overline{\delta}_j, +)$ such that $\eta(x_j) \in J(A)$. Therefore all other variables, and in particular the control ones, are evaluated into semisimple elements; hence x_j cannot occur within $\mathscr{X}_i^{(j)}$, or otherwise $\eta(f_i^{(j)}) = 0$ since all control variables of $f_i^{(j)}$ (or, at least, the first control variable of $f_i^{(j+1)}$) have to be evaluated in the same simple component. So in fact $x_j = \overline{x}_j$ is the only possibility, and $\eta(\overline{x}_j) = b_j^+$, the symmetric basis element originated by a suitable matrix unit b_j, for all $j < m$. Then, since in $f'_{B,m}$ the control variables, as well as the variables of the d_i's, v_p and v_q, are fixed, it holds $\eta(f'_{B,m}) = \sum_\sigma (-1)^\sigma \eta(\sigma w'_{B,m}) = \eta(w'_{B,m}) \neq 0$. Thus in particular $\eta(\overline{w}) \neq 0$.

Now: set $e := \sum_{i=1}^s e_{ii}$ and $\overline{e} := \mathbf{1}_{2s} - \overline{e}$, forming a pair of even central orthogonal idempotents of A decomposing its unity $\mathbf{1}_{2s}$. They define a pair of functions π, $\overline{\pi}$ decomposing Id_A, namely $\pi(a) := ea$ and $\overline{\pi}(a) = \overline{e}a$, which in turns are graded homomorphisms (actually projections, but not $\diamond$-homomorphisms). Since $\eta(w'_{B,m}) \neq 0$ then either $\pi(\eta(w'_{B,m})) \neq 0$ or $\overline{\pi}(\eta(w'_{B,m})) \neq 0$.

Assume at first that $\pi(\eta(w'_{B,m})) \neq 0$. Hence $b_j = e_{j\,j+1} \otimes e_{r_j,c_j}$ for $j < m$, and $\pi(\eta(\overline{w})) = e_{pq} \otimes e_{\overline{r},\overline{c}}$ is a matrix unit, of $\mathbb{Z}_2$-degree $\alpha_p(\overline{r}) + \alpha_q(\overline{c})$ equaling the $\mathbb{Z}_2$-degree of $\overline{w}$, that is $\beta_p(r) + \beta_q(c)$. Since $\beta_p = \alpha_p$ and $\beta_q = \alpha_q + 1$, this means that $\alpha_p(r) + \alpha_q(c) + 1 = \alpha_p(\overline{r}) + \alpha_q(\overline{c})$. By the way, the variables of v_p have to be evaluated in the same “maximal square sector” of A_p in order to be not zero, be it $[k] \times [k]$ in case $A_p = M_{k,l} \oplus M_{k,l}^{\mathrm{sop}}$, $L \times L$ in case $A_p = M_{l,n,l}$ with $n < 2l$ or $R \times R$ in case $A_p = M_{l,n,l}$ with $n > 2l$. This means that r and $\overline{r}$ belong to the same α_p-fiber, so $\alpha_p(r) = \alpha_p(\overline{r})$. Similar arguments apply to v_q as well: c and $\overline{c}$ have to be in the same α_q-fiber, corresponding to the “maximal square sector” of A_q, and hence $\alpha_q(c) = \alpha_q(\overline{c})$. But then it should hold $\alpha_p(r) + \alpha_q(c) + 1 = \alpha_p(\overline{r}) + \alpha_q(\overline{c}) = \alpha_p(r) + \alpha_q(c)$, which is an absurd.

So assume $\pi(\eta(f'_{B,m})) = 0$ and $\overline{\pi}(\eta(f'_{B,m})) \neq 0$. Then all the same $\eta(\overline{x}_j) = b_j^+ = (e_{j\,j+1} \otimes e_{r_j,c_j})^+$, but in a sense they occur in the evaluation of $f'_{B,m}$ in the reverse order, namely

$$\overline{\pi}(b_j^+) = \pm e_{\gamma_{2m}(j+1)\,\gamma_{2m}(j)} \otimes e_{c'_j,r'_j} := (e_{j,j+1} \otimes e_{r_j,c_j})^{\overline{\vartheta}_{2s}}.$$

By the way, since $\overline{\pi}$ is a $\mathbb{Z}_2$-graded homomorphism, $\overline{\pi}(b_j^+)$ has the same $\mathbb{Z}_2$-degree as $\pi(b_j^+)$, hence the previous arguments apply, as well, and so we still reach an absurd. Therefore if $Id(A, \diamond) \subseteq Id(B, \diamond)$ then no troubling couple A_p, A_q may in fact appear in the sequence $(A_1, \ldots, A_m)$, so that A and B are $\diamond$-isomorphic. □

Theorem 6.6. *For any sequence $(A_1, \ldots, A_m)$ and any translation vector $\mathbf{g}$ the algebra $UT^\diamond_{\mathbf{g}}(A_1, \ldots, A_m)$ generates a minimal $\diamond$-variety.*

Proof. Starting from $A := UT^{\diamond}_{\mathbf{g}}(A_1, \dots, A_m)$, denote $\mathscr{V}$ the $\diamond$-variety generated by A. Since A is finite dimensional, $\mathscr{V}$ has finite basic rank, and the same holds on all its $\diamond$-subvariety. By Theorem 1 of [1] all of them are generated by suitable finite dimensional $\diamond$-algebras. As a consequence of Prop. 4.4 and Theorem 4.5 of [4], if $\mathscr{U}$ is a $\diamond$-subvariety of $\mathscr{V}$ then it contains a suitable $\diamond$-algebra $B = UT^{\diamond}_{\mathbf{k}}(B_1, \dots, B_n)$ with the same $\diamond$-exponent as $\mathscr{U}$.

Now assume $\mathscr{U} \subseteq \mathscr{V}$ and $\exp(\mathscr{U}, \diamond) = \exp(\mathscr{V}, \diamond)$, that is $\dim B_{ss} = \dim A_{ss}$: since $f_{B,t} \notin Id(B, \diamond)$ for all $t \geqslant m + n - 1$ then in particular $f_{B,t} \notin Id(A, \diamond)$, hence Proposition 6.3 applies, so that $m = n$ and either $(B_1, \dots, B_m) = (A_1, \dots, A_m)$ or $(B_1, \dots, B_m) = (A_m, \dots, A_1)$ holds. In the former case, by Proposition 6.5 we conclude that $A \cong B$, so $\mathscr{U} = \mathscr{V}$. In the latter, a little more work is needed.

Let β denote the grading word of B, so that the complete word (of length $2s$) $\tilde{\beta} = (\beta|\mathscr{C}Rev(\beta))$ turns M_{2s} into a $\diamond$-algebra with fibers L_β and R_β and supertranspose $\overline{\vartheta}_\beta$, having B as a $\diamond$-subalgebra. Let τ be the product of all transpositions $(i\ s+i)$, for $i \in [s]$. Then $\tilde{\beta}' := \tilde{\beta}\tau = (\mathscr{C}Rev(\beta)|\beta)$ is still a (complete) grading word, originating from the grading word $\beta' = \mathscr{C}Rev(\beta)$, and with fibers $L_{\beta'} = \tau L_\beta$ and $R_{\beta'} = \tau R_\beta$, endowing M_{2s} with a supertranspose $\overline{\vartheta}_{\beta'}$. Moreover, as a general fact, the linear map $\hat{\tau}$ defined by $\hat{\tau}(e_{ij}) := e_{\tau(i)\tau(j)}$ turns out to be a $\diamond$-isomorphism from (M_{2s}, β) to (M_{2s}, β'). Since $\hat{\tau}(B) = B$, $\hat{\tau}$ induces in fact a $\diamond$-isomorphism between $UT^{\diamond}_{\mathbf{k}}(B_1, \dots, B_m)$ and $UT^{\diamond}_{\mathbf{k}'}(B_m, \dots, B_1)$, where $\mathbf{k}' = \mathscr{C}Rev(\mathbf{k})$.

Therefore $\tau(B) \in \mathscr{U}$ as well, has the same exponent of B but, now, the sequence of its $\diamond$-simple constituents coincides with $(A_1, \dots, A_m)$, so $\mathscr{U} = \mathscr{V}$ all the same. □

Gluing this result together with Theorem 4.6 in [4] we get the complete characterization of the affine $\diamond$-varieties:

Corollary 6.7. *An affine $\diamond$-variety of algebras is $\diamond$-minimal if and only if it is generated by an algebra $UT^{\diamond}_{\mathbf{g}}(A_1, \dots, A_m)$ for a suitable sequence $(A_1, \dots, A_m)$ of finite dimensional $\diamond$-simple algebras and vector $\mathbf{g} \in \mathbb{Z}_2^m$.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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